Companion theories from eu structures

Let us suppose we haven't seen any companion theories, but we have constructed $\mathcal{U}(T)$ for an arbitrary theory T.

These are the crucial properties of $\mathcal{U}(T)$.

(i) $\mathcal{U}(T)$ is cofinal in $\mathcal{S}(T)$.

(ii) For $\mathfrak{M}, \mathfrak{N} \in \mathcal{U}(T)$ we have

$$(\mathfrak{M}, a) \equiv\rangle(\exists_1)\,(\mathfrak{N}, b) \Longrightarrow (\mathfrak{M}, a) \equiv_p (\mathfrak{N}, b)$$

for all matching points a, b.

(iii) In particular, for all $\mathfrak{M}, \mathfrak{N} \in \mathcal{U}(T)$, we have

$$\mathfrak{M} \subseteq \mathfrak{N} \Longrightarrow \mathfrak{M} \prec \mathfrak{N}$$

Two 'companion classes'

Let

$$\mathcal{E}(T) \qquad \mathcal{G}(T)$$

be those classes of structures $\mathfrak{A}$ for which there is some $\mathfrak{M} \in \mathcal{U}(T)$ with

$$\mathfrak{A} \prec_1 \mathfrak{M} \qquad \mathfrak{A} \prec \mathfrak{M}$$

respectively. Observe that

$$\mathcal{U}(T) \subseteq \mathcal{G}(T) \subseteq \mathcal{E}(T)$$

Let's see what these two classes tell us about T.

We know that $\mathcal{E}(T)$ is the class of stuctures ec for T. Let's pretend we don't know that.

We have learnt that it is not companion theories that are important, but the generating classes.

Properties of $\mathcal{E}(T)$ and $\mathcal{G}(T)$

$\mathcal{E}(T)$ has the three properties

(i) $\mathcal{E}(T)$ is cofinal in $\mathcal{S}(T)$

(ii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{E}(T) \quad \mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}$

(iii) $\mathfrak{A} \prec_1 \mathfrak{B} \in \mathcal{E}(T) \Longrightarrow \mathfrak{A} \in \mathcal{E}(T)$

$\mathcal{G}(T)$ has the three properties

(i) $\mathcal{G}(T)$ is cofinal in $\mathcal{S}(T)$

(ii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{E}(T) \quad \mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec \mathfrak{B}$

(iii) $\mathfrak{A} \prec \mathfrak{B} \in \mathcal{E}(T) \Longrightarrow \mathfrak{A} \in \mathcal{E}(T)$

In both cases (i, iii) are trivial (from the current definition).
Property (ii) take a little more work, same proof in both cases.

Uniqueness of $\mathcal{E}(T)$ and $\mathcal{G}(T)$

$\mathcal{E}(T)$ is the unique class with (i, ii, iii).

$\mathcal{G}(T)$ is the unique class with (i, ii, iii).

The proof for $\mathcal{E}(T)$ is a simplified version of that for $\mathcal{G}(T)$.

The proof for $\mathcal{G}(T)$ has two variants.

From the proofs we see that both $\mathcal{E}(T)$ and $\mathcal{G}(T)$ are closed under unions of directed families.

The 'closed' property of $\mathcal{E}(T)$

We find that $\mathfrak{A} \in \mathcal{E}(T)$ precisely when $\mathfrak{A} \in \mathcal{S}(T)$ and

$$\mathfrak{A} \subseteq \mathfrak{B} \models T \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

Warning: It is temping to assume that each $\mathfrak{A} \in \mathcal{G}(T)$ has the property

$$\mathfrak{A} \subseteq \mathfrak{B} \models T \Longrightarrow \mathfrak{A} \prec \mathfrak{B}$$

This is not true, even when T is $\forall_2$-axiomatizable.

Such a class of structures can exists when the language is countable. But this class can be very different from $\mathcal{G}(T)$.

A generalization

For each $n \in \mathbb{N}$ let $\mathcal{E}_n(T)$ be the class of those structures $\mathfrak{A}$ for which there is some $\mathfrak{M} \in \mathcal{U}(T)$ with

$$\mathfrak{A} \prec_{n+1} \mathfrak{M}$$

We have

$$\mathcal{E}(T) = \mathcal{E}_0(T) \supseteq \mathcal{E}_1(T) \supseteq \cdots \supseteq \mathcal{E}_n(T) \supseteq \mathcal{E}_{n+1}(T) \supseteq \cdots$$

with

$$\mathcal{G}(T) = \bigcap \{\mathcal{G}(T) \mid n \in \mathbb{N}\}$$

Each $\mathcal{E}_n(T)$ has a (i, ii, iii) characterization.

Ex: Show that

$$\mathcal{E}_n(T) = \mathcal{E}_{n+1}(T) \Longrightarrow \mathcal{E}_n(T) = \mathcal{G}(T)$$