

Existence of e.u. structures

We refine the construction of ec structures.

There is a 1-step construction which give a partial improvement.

We then iterated this to accumulate the improvements.

This is the first genuine example of a **saturation** process.

We now have to deal with \exists_1 -types, not just formulas.

We also have to worry about certain cardinalities.

The result

Let T be a theory in a language L .

For each $\mathfrak{A} \in \mathcal{S}(T)$ with

$$|\mathfrak{A}| \leq \text{CARDINALITY RESTRICTION}$$

there is some

$$\mathfrak{A} \subseteq \mathfrak{M} \in \mathcal{U}(T)$$

with

$$|\mathfrak{M}| \leq \text{CARDINALITY RESTRICTION}$$

We use a 1-step + accumulation construction.

We first do a crude version, and then refine it.

The crude version

(The 1-step) For each $\mathfrak{A} \in \mathcal{S}(T)$ there is some

$$\mathfrak{A} \subseteq \mathfrak{A}' \in \mathcal{S}(T)$$

such that:

For each \exists_1 -type $\Theta(a, v)$ with parameters from \mathfrak{A} , if this type is T -consistent over \mathfrak{A}' , then it is already realized in \mathfrak{A}' .

(Accumulation) Given $\mathfrak{A} \in \mathcal{S}(T)$ we iterate the 1-step construction to generate

$$\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{A}'' \subseteq \dots \subseteq \mathfrak{A}^{(i)} \subseteq \dots \quad (i < \omega)$$

and then set

$$\mathfrak{M} = \bigcup \{ \mathfrak{A}^{(i)} \mid i < \omega \}$$

The 1-step construction

Given $\mathfrak{A} \in \mathcal{S}(T)$ let

$$\Theta = \{\Theta_i \mid i < \tau\}$$

be an enumeration of all the relevant types.

We generate

$$\mathcal{A} = \{\mathfrak{A}_i \mid i < \tau\}$$

satisfying

- (0) $\mathfrak{A}_0 = \mathfrak{A}$.
- (1) $i < j < \tau \implies \mathfrak{A}_i \subseteq \mathfrak{A}_j$.
- (2) $|\mathfrak{A}_i| \leq (?)$ for each $i < \tau$. [Need to calculate (?) later]
- (3) If $l < \tau$ is a limit ordinal, then $\mathfrak{A}_l = \bigcup \{\mathfrak{A}_i \mid i < l\}$.
- (4) For each $i < \tau$, if the type Θ_i is T -consistent over \mathfrak{A}_i , then it is realized in \mathfrak{A}_{i+1} .

The recursion

$$(0) \mathfrak{A}_0 = \mathfrak{A}$$

$(i \mapsto i + 1)$ Look at Θ_i .

If Θ_i is not T -consistent over \mathfrak{A}_i , then we set $\mathfrak{A}_{i+1} = \mathfrak{A}_i$.

If Θ_i is T -consistent over \mathfrak{A}_i , then we select $\mathfrak{A} \subseteq \mathfrak{A}_{i+1}$ in which Θ_i is realized. By taking an elementary substructure we may arrange that $|\mathfrak{A}_i| = |\mathfrak{A}_{i+1}|$.

$$(l) \mathfrak{A}_l = \bigcup \{ \mathfrak{A}_i \mid i < l \}.$$

Note that if $|\mathfrak{A}_i| \leq \sigma$ for each $i < l$, then

$$|\mathfrak{A}_l| \leq \sigma \cdot |l| \leq \sigma \cdot \tau$$

so that $|\mathfrak{A}_l| \leq \tau$ if $\sigma \leq \tau$.

What is τ ?

There are $2^{|L|}$ pure types.

The structure \mathfrak{A} provides $|\mathfrak{A}|$ points.

Thus $\tau \leq |\mathfrak{A}| \cdot 2^{|L|}$.

If $|L| \leq \kappa$ and $|\mathfrak{A}| \leq 2^\kappa$ then

$$\tau \leq 2^{\kappa+|L|} = 2^\kappa$$

When L is countable, only $|\mathfrak{A}|$ matters.

The refined version

(The 1-step) Let $|L| \leq \kappa$ and $|\mathfrak{A}| \leq 2^\kappa$. There is some

$$\mathfrak{A} \subseteq \mathfrak{A}' \in \mathcal{S}(T) \quad \text{with } |\mathfrak{A}'| \leq 2^\kappa$$

such that for each \exists_1 -type $\Theta(a, v) \dots$

(Accumulation) Given $\mathfrak{A} \in \mathcal{S}(T)$ we iterate the 1-step construction to generate

$$\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{A}'' \subseteq \dots \subseteq \mathfrak{A}^{(i)} \subseteq \dots \quad (i < \omega)$$

and then set

$$\mathfrak{M} = \bigcup \{ \mathfrak{A}^{(i)} \mid i < \omega \}$$

to get

$$\mathfrak{A} \subseteq \mathfrak{M} \in \mathcal{U}(T) \quad \text{with } |\mathfrak{M}| \leq 2^\kappa$$

A tighter calculation

Suppose $|L| = \aleph_0$ and $|\mathfrak{A}| = \aleph_0$.

Then, in general, $|\mathfrak{M}| \leq 2^{\aleph_0}$ is the best we can do.

Observe that it is sufficient to realize the maximal \exists_1 -types.

For each list w of variables let

$\exists_1^{\max}(T, w)$ = set of \exists_1 -types in w each maximally consistent over T

$$m(T, w) = |\exists_1^{\max}(T, w)| \leq 2^{\aleph_0}$$

$$m(T) = \sup\{m(T, w) \mid w \text{ a batch}\}$$

We may take $\tau = m(T)$.

A tighter result

Let T be a theory in a countable language, and suppose $m(T, w) \leq \aleph_0$ for each list w of variable.

Then for each countable $\mathfrak{A} \in \mathcal{S}(T)$ there is a countable $\mathfrak{M} \in \mathcal{U}(T)$ with $\mathfrak{A} \subseteq \mathfrak{M}$.

In general $m(T, w)$ is not countable.

But there are reasonable conditions which ensure this.