Existentially closed v. Existentially universal

We strengthen the property of being existentially closed to produce structures that can handle $\exists_1$-types, not just $\exists_1$-formulas.
Another characterization of being e.c.

Fix a theory $T$.

A structure $\mathfrak{A}$ is e.c. for $T$ if $\mathfrak{A} \in \mathcal{S}(T)$ and: For each $\exists_1$-formula $\theta(a, v)$ with parameters $a$ from $\mathfrak{A}$, if

$$\mathfrak{B} \models (\exists v)\theta(a, v)$$

for some $\mathfrak{A} \subseteq \mathfrak{B} \models T$, then [This just says $\mathfrak{A} \prec_1 \mathfrak{B}$]

$$\mathfrak{A} \models (\exists v)\theta(a, v)$$

If $\theta(a, v)$ is $T$-consistent over $\mathfrak{A}$ then $\theta(a, v)$ is realized in $\mathfrak{A}$. 
Definition of being e.u.

A structure $\mathcal{M}$ is e.u. for $T$ if $\mathcal{M} \in \mathcal{S}(T)$ and:

For each set $\Theta(a, v)$ of $\exists_1$-formulas with finitely many parameters $a$ from $\mathcal{M}$ and a batch $v$ of finitely many free variables, if

$$\mathcal{B} \models (\exists v)[\wedge \Theta(a, v)]$$

for some $\mathcal{M} \subseteq \mathcal{B} \models T$, then

$$\mathcal{M} \models (\exists v)[\wedge \Theta(a, v)]$$

If $\Theta(a, v)$ is $T$-consistent over $\mathcal{M}$ then $\Theta(a, v)$ is realized in $\mathcal{M}$. 
Some observations

The set $\Theta(a, v)$ is an $\exists_1$-type over $\mathcal{A}$.

Let $\mathcal{U}(T)$ be the class of structure that are e.u. for $T$.

Trivially we have $\mathcal{U}(T) \subseteq \mathcal{E}(T)$.

The major problem is to show that $\mathcal{U}(T)$ is non-empty, in fact cofinal in $\mathcal{S}(T)$.

This is the first genuine saturation process. [See next block]
A forth construction

Suppose

\[(\mathfrak{A}, a) \equiv (\exists_1) (\mathfrak{M}, b)\]

where \(\mathfrak{A} \in \mathcal{S}(T), \mathfrak{M} \in \mathcal{U}(T)\) with matching points \(a, b\).

Then for each \(x \in \mathfrak{A}\) there is some \(y \in \mathfrak{M}\) with

\[(\mathfrak{A}, a \vdash x) \equiv (\exists_1) (\mathfrak{M}, b \vdash y)\]

Let \(\Theta(a, v)\) be the \(\exists_1\)-type of \(x\) in \((\mathfrak{A}, a)\). That is

\(\Theta(a, x) = Th(\mathfrak{A}, a, x) \cap \exists_1\).

The type \(\Theta(b, v)\) is finitely satisfiable in \((\mathfrak{M}, b)\), and hence satisfiable in \((\mathfrak{M}, b)\)
Some consequences

For $\mathfrak{N} \in \mathcal{U}(T)$ and countable $\mathfrak{A} \in \mathcal{S}(T)$

$$(\mathfrak{A}, a) \equiv(\exists_1)(\mathfrak{N}, b) \implies (\mathfrak{A}, a) \text{ is embeddable in } (\mathfrak{N}, b)$$

If the base theory $T$ has $JEP$ then each countable $\mathfrak{A} \in \mathcal{M}d(T)$ is embeddable in each $\mathfrak{N} \in \mathcal{U}(T)$.

For each $\mathfrak{M}, \mathfrak{N} \in \mathcal{U}(T)$

$$(\mathfrak{M}, a) \equiv(\exists_1)(\mathfrak{N}, b) \implies (\mathfrak{M}, a) \equiv_p (\mathfrak{N}, b)$$

If the base theory $T$ has $JEP$ then

$$(\mathfrak{M}, a) \equiv_0 (\mathfrak{N}, b) \implies (\mathfrak{M}, a) \equiv_p (\mathfrak{N}, b)$$

Recall that since $\mathfrak{M}, \mathfrak{N}$ are e.c. we have

$$(\mathfrak{M}, a) \equiv(\exists_1)(\mathfrak{N}, b) \implies (\mathfrak{M}, a) \equiv_2 (\mathfrak{N}, b)$$