Existentially closed .v. Existentially universal

We strengthen the property of being existentially closed to produce structures that can handle $\exists_1$-types, not just $\exists_1$-formulas.

Another characterization of being e.c.

Fix a theory T.

A structure $\mathfrak{A}$ is e.c. for T if $\mathfrak{A} \in \mathcal{S}(T)$ and:

For each $\exists_1$-formula $\theta(a, v)$ with parameters a from $\mathfrak{A}$, if

$$\mathfrak{B} \models (\exists v)\theta(a, v)$$

for some $\mathfrak{A} \subseteq \mathfrak{B} \models T$, then [This just says $\mathfrak{A} \prec_1 \mathfrak{B}$]

$$\mathfrak{A} \models (\exists v)\theta(a, v)$$

If $\theta(a, v)$ is T-consistent over $\mathfrak{A}$ then $\theta(a, v)$ is realized in $\mathfrak{A}$.

Definition of being e.u.

A structure $\mathfrak{M}$ is e.u. for T if $\mathfrak{M} \in \mathcal{S}(T)$ and:

For each set $\Theta(a, v)$ of $\exists_1$-formulas with finitely many parameters a from $\mathfrak{M}$ and a batch v of finitely many free variables, if

$$\mathfrak{B} \models (\exists v)[\bigwedge \Theta(a, v)]$$

for some $\mathfrak{M} \subseteq \mathfrak{B} \models T$, then

$$\mathfrak{M} \models (\exists v)[\bigwedge \Theta(a, v)]$$

If $\Theta(a, v)$ is T-consistent over $\mathfrak{M}$ then $\Theta(a, v)$ is realized in $\mathfrak{M}$.

Some observations

The set $\Theta(a, v)$ is an $\exists_1$-type over $\mathfrak{A}$.

Let $\mathcal{U}(T)$ be the class of structure that are e.u. for T.

Trivially we have $\mathcal{U}(T) \subseteq \mathcal{E}(T)$.

The major problem is to show that $\mathcal{U}(T)$ is non-empty, in fact cofinal in $\mathcal{S}(T)$.

This is the first genuine saturation process. [See next block]

A forth construction

Suppose

$$(\mathfrak{A}, a) \equiv\rangle(\exists_1)\, (\mathfrak{N}, b)$$

where $\mathfrak{A} \in \mathcal{S}(T), \mathfrak{N} \in \mathcal{U}(T)$ with matching points a, b.

Then for each $x \in \mathfrak{A}$ there is some $y \in \mathfrak{N}$ with

$$(\mathfrak{A}, a \frown x) \equiv\rangle(\exists_1)\, (\mathfrak{N}, b \frown y)$$

Let $\Theta(a, v)$ be the $\exists_1$-type of x in $(\mathfrak{A}, a)$. That is $\Theta(a, x) = Th(\mathfrak{A}, a, x) \cap \exists_1$.

The type $\Theta(b, v)$ is finitely satisfible in $(\mathfrak{N}, b)$, and hence satisfiable in $(\mathfrak{N}, b)$

Some consequences

For $\mathfrak{N} \in \mathcal{U}(T)$ and countable $\mathfrak{A} \in \mathcal{S}(T)$

$$(\mathfrak{A}, a) \equiv\rangle(\exists_1)\,(\mathfrak{N}, b) \Longrightarrow (\mathfrak{A}, a) \text{ is embeddable in } (\mathfrak{N}, b)$$

If the base theory T has JEP then each countable $\mathfrak{A} \in \mathcal{M}d(T)$ is embeddable in each $\mathfrak{N} \in \mathcal{U}(T)$.

For each $\mathfrak{M}, \mathfrak{N} \in \mathcal{U}(T)$

$$(\mathfrak{M}, a) \equiv\rangle(\exists_1)\,(\mathfrak{N}, b) \Longrightarrow (\mathfrak{M}, a) \equiv_p (\mathfrak{N}, b)$$

If the base theory T has JEP then

$$(\mathfrak{M}, a) \equiv_0 (\mathfrak{N}, b) \Longrightarrow (\mathfrak{M}, a) \equiv_p (\mathfrak{N}, b)$$

Recall that since $\mathfrak{M}, \mathfrak{N}$ are e.c. we have

$$(\mathfrak{M}, a) \equiv\rangle(\exists_1)\,(\mathfrak{N}, b) \Longrightarrow (\mathfrak{M}, a) \equiv_2 (\mathfrak{N}, b)$$