Existentially closed structures

Another method of producing a companion operator is to select a suitable class

$$\mathcal{C}(T) \subseteq \mathcal{S}(T)$$

for each theory T and set

$$T^c = Th(\mathcal{C}(T))$$

This often gives more information because we select ‘special’ submodels.

We look at the easiest example of this method (and glance at a variant of this).

$\mathcal{E}(T)$

A structure $\mathfrak{A}$ is existentially closed for a theory T if $\mathfrak{A} \in \mathcal{S}(T)$ and we have

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

for each model $\mathfrak{B} \models T$, or even each $\mathfrak{B} \in \mathcal{S}(T)$.

Let $\mathcal{E}(T)$ be the class of such structures.

Quite soon we look at an intrinsic characterization of $\mathcal{E}(T)$, and later we show that $\mathcal{E}(T)$ is non-empty.

Why $\mathcal{E}(T)$ is important

Theorem. *Let T be a theory with a model companion T^*. Then $\mathcal{E}(T) = \mathcal{M}d(T^*)$.*

$\mathcal{E}(T) \subseteq \mathcal{M}d(T^*)$ since T^* is $\forall_2$-axiomatizable.

$\mathcal{M}d(T^*) \subseteq \mathcal{E}(T)$ since T^* model complete.

Theorem. *Let T be a theory for which $\mathcal{E}(T)$ is elementary. Then $Th(\mathcal{E}(T))$ is the model companion of T.*

This needs some more information about $\mathcal{E}(T)$ before it can be proved.

Characterization of $\mathcal{E}(\cdot)$

For each theory T the class $\mathcal{E}(T)$ is uniquely characterized by the following three properties.

(i) $\mathcal{E}(T)$ is cofinal in $\mathcal{S}(T)$.

(ii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{E}(T)$ we have

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

(iii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{S}(T)$ we have

$$\mathfrak{A} \prec_1 \mathfrak{B} \in \mathcal{E}(T) \Longrightarrow \mathfrak{A} \in \mathcal{E}(T)$$

(i) means that for each $\mathfrak{A} \in \mathcal{S}(T)$ there is some $\mathfrak{A} \subseteq \mathcal{B} \in \mathcal{E}(T)$.
This should be proved now, but we postpone it.
Watch out for when it is used, and check there is no circularity.

A companion operator $(\cdot)^e$

For each theory T we set

$$T^e = Th(\mathcal{E}(T))$$

- $\mathcal{E}(T)$ is cofinal in $\mathcal{S}(T)$.
- $\mathcal{E}(T)$ depends only on $\mathcal{S}(T)$.
- For each $\mathfrak{A} \in \mathcal{E}(T)$ there is some $\mathfrak{A} \subseteq \mathfrak{B} \models T$, and then $\mathfrak{A} \prec_1 \mathfrak{B}$, so that $\mathfrak{A} \models T \cap \forall_2$, and hence $\mathcal{E}(T) \models T \cap \forall_2$.

Observe that if $\mathcal{E}(T)$ is elementary then

$$\mathcal{M}d(T^e) = \mathcal{E}(T)$$

to show that T^e is model complete, and hence is the model companion of T.

A refinement $\mathcal{G}(\cdot)$ and $(\cdot)^g$

For each theory T there is a unique class $\mathcal{E}(T)$ with the following three properties.

(i) $\mathcal{G}(T)$ is cofinal in $\mathcal{S}(T)$.

(ii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{G}(T)$ we have

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec \mathfrak{B}$$

(iii) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{S}(T)$ we have

$$\mathfrak{A} \prec \mathfrak{B} \in \mathcal{G}(T) \Longrightarrow \mathfrak{A} \in \mathcal{G}(T)$$

We set $T^g = Th(\mathcal{G}(T))$ to obtain another companion operator.

This was originally produced by a method known as infinite forcing. In fact, it has a straight forward standard construction.