Companion theories

For a fixed language we put the ‘companion’ equivalence relation on the set of all theories.

We look for a special member of each block.

Some blocks don’t have such a special member.

So we begin to look for a possible substitute.

Companion theories

Two theories T_1, T_2 (in the same language) are companions if each model of the one can be embedded in a model of the other.

That is, if

$$\mathcal{S}(T_1) = \mathcal{S}(T_2)$$

equivalently, when

$$T_1 \cap \forall_1 = T_2 \cap \forall_1$$

This imposes an equivalence relation of the set of all theories.

A special companion?

Lemma. *Suppose T^* and T are companion theories with T^* model complete. Then $T \cap \forall_2 \subseteq T^*$.*

Consider any $\mathfrak{A} \models T^*$. By the companion property we obtain

$$\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}' \qquad \mathfrak{A}, \mathfrak{A}' \models T^* \quad \mathfrak{B} \models T$$

Since T^* is model complete we have

$$\mathfrak{A} \prec \mathfrak{A}' \qquad \mathfrak{A} \prec_1 \mathfrak{B}$$

and hence

$$\mathfrak{A} \models T \cap \forall_2$$

for the required result.

Corollary. *Suppose T_1 and T_2 are model complete companion theories. Then $T_1 = T_2$.*

Model companion

A model companion of a theory T is a companion T^* of T which is model complete.

Each theory has at most one model companion.

Chapter 4 produces theories T with a model companion T^*.

There are theories with no model companion. (Examples later.)

When does a theory have a model companion?

What can we do when a theory doesn't have a model companion?

Companion operator

For a fixed language L, a companion operator is an assignment

$$T \longmapsto T^a$$

between theories (of the underlying language) such that

(i) T and T^a are companions

(ii) if T_1 and T_2 are companions, then $T_1^a = T_2^a$

(iii) $T \cap \forall_2 \subseteq T^a$

hold for all theories T, T_1, T_2.

Theorem. *Let $(\cdot)^a$ be a companion operator. If the theory T has a model companion T^*, then $T^a = T^*$.*

$\forall_2$-uniqueness

Lemma. *Let $(\cdot)^a$ and $(\cdot)^b$ be a pair of companion operators (for a given language). Then for each theory T we have*

$$T^a \cap \forall_2 = T^b \cap \forall_2$$

The companion operator $(\cdot)^a$ gives $\quad T^b \cap \forall_2 \subseteq T^{ba}$

Since T^b and T are companions we have $\quad T^{ba} = T^a$

to give

$$T^b \cap \forall_2 \subseteq T^{ba} = T^a$$

and hence

$$T^b \cap \forall_2 \subseteq T^a \cap \forall_2$$

The minimum companion operator

Lemma. *For each companion operator* $(\cdot)^a$, *the assignment*

$$T \longmapsto T^m = (T^a \cap \forall_2)^{\vdash}$$

is a companion operator, and is independent of $(\cdot)^a$.

However, as yet we have not produced any example of a companion operator.

We now construct this minimum companion operator by syntactic methods.

0-tame sentences

A sentence σ is 0-tame over a (consistent) theory T if it is a $\forall_2$-sentence and

$$T \cap \forall_1 \vdash (\sigma \to \alpha) \Longrightarrow T \vdash \alpha$$

for each $\forall_1$-sentence α.

Equivalently if

$$(T \cap \forall_1) \cup \{\sigma\}$$

axiomatizes a companion of T.

Let $0(T)$ be the set of all 0-tame sentences over T

Notice that $T \cap \forall_2 \subseteq 0(T)$.

$0(T)$ is closed under conjunction and is consistent

Consider $\sigma, \tau \in 0(T)$. Consider any $\mathfrak{A} \in \mathcal{S}(T)$. We show that $\mathfrak{A}$ embeds into a model of $T \cup \{\sigma \wedge \tau\}$

Using first σ and then τ we we obtain

$$\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \qquad \mathfrak{B} \models \sigma \quad \mathfrak{C} \models \tau \quad \mathfrak{B}, \mathfrak{C} \in \mathcal{S}(T)$$

By iteration we generate two interlacing chains in $\mathcal{S}(T)$.

$$\mathcal{B} = \{\mathfrak{B}_i \mid i < \omega\} \quad \mathcal{B} \models \sigma \qquad \mathcal{C} = \{\mathfrak{C}_i \mid i < \omega\} \quad \mathcal{C}_i \models \tau$$

Let $\mathfrak{U}$ be the common union of these two chains. Then

$$\mathfrak{A} \subseteq \mathfrak{U} \in \mathcal{S}(T) \qquad \mathfrak{U} \models \sigma \wedge \tau$$

to give the required result.

$0(T)$ axiomatizes a companion of T

For each $\forall_1$-sentence we show

$$0(T) \vdash \alpha \implies \alpha \in T$$

Observe that $0(T) \vdash \alpha$ gives some $\sigma \in 0(T)$ with

$$\vdash \sigma \to \alpha$$

Consider any $\mathfrak{A} \models T$. Since $T \cap \forall_1 \cup \{\sigma\}$ axiomatizes a companion of T we have some structure $\mathfrak{B}$ with

$$\mathfrak{A} \subseteq \mathfrak{B} \models \sigma$$

But now $\mathfrak{B} \models \alpha$ and hence $\mathfrak{A} \models \alpha$ (since α is $\forall_1$).

The minimum companion operator $(\cdot)^0$

For each (consistent) theory T we set $\qquad T^0 = 0(T)^{\vdash}$

- T and T^0 are companions – previous slide.
- $0(T)$ depends only on $T \cap \forall_1$.
- By construction $T \cap \forall_2 \subseteq 0(T)$.

- Thus we have a companion operator.

- For each companion operator $(\cdot)^a$ we have $T^0 \subseteq T^{0a} = T^a$.

A companion operator $(\cdot)^f$

By lifting this construction up all quantifier levels we can produce another companion operator.

$$T \longmapsto T^f$$

This was originally produced by a method known as finite forcing, and worked only for countable languages.

For that reason T^f is often called the finite forcing companion.

We (probably) won't look at this operator.