Companion theories

For a fixed language we put the ‘companion’ equivalence relation on the set of all theories.

We look for a special member of each block.

Some blocks don’t have such a special member.

So we begin to look for a possible substitute.
Companion theories

Two theories $T_1, T_2$ (in the same language) are companions if each model of the one can be embedded in a model of the other.

That is, if

$$S(T_1) = S(T_2)$$

equivalently, when

$$T_1 \cap \forall_1 = T_2 \cap \forall_1$$

This imposes an equivalence relation of the set of all theories.
Lemma. Suppose $T^*$ and $T$ are companion theories with $T^*$ model complete. Then $T \cap \forall_2 \subseteq T^*$.

Consider any $\mathfrak{A} \models T^*$. By the companion property we obtain

$$\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}' \quad \mathfrak{A}, \mathfrak{A}' \models T^* \quad \mathfrak{B} \models T$$

Since $T^*$ is model complete we have

$$\mathfrak{A} \prec \mathfrak{A}' \quad \mathfrak{A} \prec_1 \mathfrak{B}$$

and hence

$$\mathfrak{A} \models T \cap \forall_2$$

for the required result.

Corollary. Suppose $T_1$ and $T_2$ are model complete companion theories. Then $T_1 = T_2$. 

Model companion

A model companion of a theory $T$ is a companion $T^*$ of $T$ which is model complete.

Each theory has at most one model companion.

Chapter 4 produces theories $T$ with a model companion $T^*$.

There are theories with no model companion. (Examples later.)

When does a theory have a model companion?

What can we do when a theory doesn’t have a model companion?
Companion operator

For a fixed language $L$, a companion operator is an assignment

$$T \rightarrow T^a$$

between theories (of the underlying language) such that

(i) $T$ and $T^a$ are companions

(ii) if $T_1$ and $T_2$ are companions, then $T_1^a = T_2^a$

(iii) $T \cap \forall_2 \subseteq T^a$

hold for all theories $T, T_1, T_2$.

Theorem. Let $(\cdot)^a$ be a companion operator. If the theory $T$ has a model companion $T^*$, then $T^a = T^*$. 
\textbf{∀}_2\text{-uniqueness}

Lemma. Let \((\cdot)^a\) and \((\cdot)^b\) be a pair of companion operators (for a given language). Then for each theory \(T\) we have

\[ T^a \cap \forall_2 = T^b \cap \forall_2 \]

The companion operator \((\cdot)^a\) gives

\[ T^b \cap \forall_2 \subseteq T^{ba} \]

Since \(T^b\) and \(T\) are companions we have to give

\[ T^{ba} = T^a \]

and hence

\[ T^b \cap \forall_2 \subseteq T^a \cap \forall_2 \]
The minimum companion operator

Lemma. For each companion operator \((\cdot)^a\), the assignment

\[
T \mapsto T^m = (T^a \cap \forall_2)^\dagger
\]

is a companion operator, and is independent of \((\cdot)^a\).

However, as yet we have not produced any example of a companion operator.

We now construct this minimum companion operator by syntactic methods.
0-tame sentences

A sentence $\sigma$ is 0-tame over a (consistent) theory $T$ if it is a $\forall_2$-sentence and

$$T \cap \forall_1 \vdash (\sigma \rightarrow \alpha) \implies T \vdash \alpha$$

for each $\forall_1$-sentence $\alpha$.

Equivalently if

$$(T \cap \forall_1) \cup \{\sigma\}$$

axiomatizes a companion of $T$.

Let $0(T)$ be the set of all 0-tame sentences over $T$.

Notice that $T \cap \forall_2 \subseteq 0(T)$. 
0(\(T\)) is closed under conjunction and is consistent

Consider \(\sigma, \tau \in 0(T)\). Consider any \(\mathfrak{A} \in \mathcal{S}(T)\). We show that \(\mathfrak{A}\) embeds into a model of \(T \cup \{\sigma \land \tau\}\)

Using first \(\sigma\) and then \(\tau\) we we obtain

\[
\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \quad \mathfrak{B} \models \sigma \quad \mathfrak{C} \models \tau \quad \mathfrak{B}, \mathfrak{C} \in \mathcal{S}(T)
\]

By iteration we generate two interlacing chains in \(\mathcal{S}(T)\).

\[
\mathcal{B} = \{\mathfrak{B}_i \mid i < \omega\} \quad \mathcal{B} \models \sigma \quad \mathcal{C} = \{\mathfrak{C}_i \mid i < \omega\} \quad \mathcal{C}_i \models \tau
\]

Let \(\mathcal{U}\) be the common union of these two chains. Then

\[
\mathfrak{A} \subseteq \mathcal{U} \in \mathcal{S}(T) \quad \mathcal{U} \models \sigma \land \tau
\]

to give the required result.
0(\(T\)) axiomatizes a companion of \(T\)

For each \(\forall_1\)-sentence we show

\[
0(T) \vdash \alpha \implies \alpha \in T
\]

Observe that \(0(T) \vdash \alpha\) gives some \(\sigma \in 0(T)\) with

\[
\vdash \sigma \rightarrow \alpha
\]

Consider any \(\mathcal{A} \models T\). Since \(T \cap \forall_1 \cup \{\sigma\}\) axiomatizes a companion of \(T\) we have some structure \(\mathcal{B}\) with

\[
\mathcal{A} \subseteq \mathcal{B} \models \sigma
\]

But now \(\mathcal{B} \models \alpha\) and hence \(\mathcal{A} \models \alpha\) (since \(\alpha\) is \(\forall_1\)).
The minimum companion operator \((\cdot)^0\)

For each (consistent) theory \(T\) we set \(T^0 = 0(T)^\vdash\)

- \(T\) and \(T^0\) are companions – previous slide.
- \(0(T)\) depends only on \(T \cap \forall_1\).
- By construction \(T \cap \forall_2 \subseteq 0(T)\).

- Thus we have a companion operator.

- For each companion operator \((\cdot)^a\) we have \(T^0 \subseteq T^{0a} = T^a\).
A companion operator \((\cdot)^f\)

By lifting this construction up all quantifier levels we can produce another companion operator.

\[
T \rightarrow T^f
\]

This was originally produced by a method known as finite forcing, and worked only for countable languages.

For that reason \(T^f\) is often called the finite forcing companion.

We (probably) won’t look at this operator.