Miniature arithmetic

We wish to analyse the following theory of arithmetic.

$$Th(\mathbb{N}, \leq, S, 0)$$

To do that we produce a subtheory

$$T(\leq, S, 0) \subseteq Th(\mathbb{N}, \leq, S, 0)$$

and show:

- $T(\leq, S, 0)$ is $\forall_2$-axiomatizable.
- The class of models of $T(\leq, S, 0)$ has a nice shape.
- In particular, $T(\leq, S, 0)$ has a special countable model (which is not $\mathbb{N}$).
- The theory is complete, and hence is just $Th(\mathbb{N}, \leq, S, 0)$.
- The theory is model complete and, in fact, has EQ.

The axioms

Axioms for

Linear order

(L1) $(\forall v)[v \leq v]$

(L2) $(\forall u, v, y)[u \leq v \leq w \rightarrow u \leq w]$

(L3) $(\forall u, v)[u \leq v \leq u \rightarrow u \mathrel{\hat{=}} v]$

(L4) $(\forall u, v)[u \leq v \vee v \leq u]$

Simple arithmetic

(S0) $(\forall v)[Sv \not\mathrel{\hat{=}} 0]$

(S1) $(\forall u, v)[Su \mathrel{\hat{=}} Sv \rightarrow u \mathrel{\hat{=}} v]$

(S2) $(\forall v)[v \not\mathrel{\hat{=}} 0 \rightarrow (\exists u)[Su = v]]$

($S3_k$) $(\forall v)[S^{k+1}v \not\mathrel{\hat{=}} v]$ (for all $k \in \mathbb{N}$)

The interaction

(I) $(\forall v)[v \leq Sv]$

(M) $(\forall u, v)[u \leq v \rightarrow Su \leq Sv]$

(Z) $(\forall v)[0 \leq v]$

(D) $(\forall u, v)[u \leq Sv \rightarrow u \leq v \vee u = Sv]$

Structure of a model

Each model is $\mathbb{N}$ together with several non-overlapping copies of $\mathbb{Z}$ arranged in linear order.

Thus each model is

$$\mathbb{N} + \mathbb{Z} \cdot \alpha$$

for some linear order type.

Setting $\alpha = 0$ gives $\mathbb{N}$, the minimal model.

Setting $\alpha = \mathbb{Q}$ gives

$$\mathfrak{Q} = \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$$

the 'universal' countable model.

How is $\mathfrak{Q}$ universal

Each countable model

$$A \text{ of order type } \mathbb{N} + \mathbb{Z} \cdot \alpha$$

is elementarily embeddable in $\mathfrak{Q}$.

By attacking its type α we enlarge A bit by bit until we get $\mathbb{Q}$.

We produce a long enough chain of elementary embeddings where the type approaches $\mathbb{Q}$, and then take the union.

Let's look at a 1-step.

$$\begin{array}{ccc} \mathbb{N} + \mathbb{Z} \cdot \alpha & & \mathbb{N} + \mathbb{Z} \cdot \beta \\ A & \prec & B \end{array}$$

where β is more like $\mathbb{Q}$ than α is.

The 1-step construction

Consider A of type α. Let l, r be non-standard elements with the $\mathbb{Z}$-block of l to the left of the $\mathbb{Z}$-block of r.

We insert three new blocks

to the left of l between l, r to the right of r

Let a enumerate A. Add three new constants x, y, z. We require a consistent set

$$Th(A, \mathsf{a}) \cup \Sigma$$

where Σ is all sentences (for $n \in \mathbb{N}$)

$$x + n \leq l \qquad l + n \leq y \qquad y + n \leq r \qquad r + n \leq z$$

We check that this is finitely satisfiable in A.

Partial isomorphisms of finite width

Consider a pair of models

$$A = \mathbb{N} + \sum \{\mathbb{Z}_i \mid i \in \eta\} \qquad B = \mathbb{N} + \sum \{\mathbb{Z}_j \mid j \in \zeta\}$$

where each of η, ζ is a dlowep.

Take a pair of list of indexes of equal length.

$$\mathsf{i} = \big(i(1) < i(2) < \cdots < i(l)\big) \qquad \mathsf{j} = \big(j(1) < j(2) < \cdots < j(l)\big)$$

These give a pair of submodels

$$A_{\mathsf{i}} = \mathbb{N} + \mathbb{Z}_{i(1)} + \cdots + \mathbb{Z}_{i(l)} \qquad B_{\mathsf{j}} = \mathbb{N} + \mathbb{Z}_{j(1)} + \cdots + \mathbb{Z}_{j(l)}$$

Any isomorphism between these is a typical
partial isomorphism of finite width.

These form a b&f system

Going forth: Given

$$\begin{array}{ccc} A & & B \\ \uparrow & & \uparrow \\ A_{\mathsf{i}} & \xrightarrow[f]{} & B_{\mathsf{j}} \end{array}$$

and an element x of $A - A_{\mathsf{i}}$

- Locate the block in which x lives.
- Locate the position of the block relative to the list i.
- Go to the same position in j.
- Choose a block of B in that position.
- Match these two new blocks.

The theory is complete and model complete

Consider any two models A, B.
There are countable elementary submodels $A^+ \prec A, B^+ \prec B$.
For each of these we have $A^+ \prec \mathfrak{Q}, B^+ \prec \mathfrak{Q}$.
Hence $A \equiv B$.

Now suppose $A \subseteq B$ and consider any point a in A, and let b be the corresponding (same) point of B.
Select $a \in A^+$ and $b \in B^+$, so it suffices to show $(\mathfrak{Q}, a) \equiv (\mathfrak{Q}, b)$.

With the submodels of finite width given by a and b we have

$$\begin{array}{ccc} \mathfrak{Q} & & \mathfrak{Q} \\ \uparrow & & \uparrow \\ A_a & \xrightarrow[f]{} & B_b \end{array}$$

and hence $(\mathfrak{Q}, a) \equiv_p (\mathfrak{Q}, b)$, for the required result.