

Miniature arithmetic

We wish to analyse the following theory of arithmetic.

$$Th(\mathbb{N}, \leq, S, 0)$$

To do that we produce a subtheory

$$T(\leq, S, 0) \subseteq Th(\mathbb{N}, \leq, S, 0)$$

and show:

- ▶ $T(\leq, S, 0)$ is \forall_2 -axiomatizable.
- ▶ The class of models of $T(\leq, S, 0)$ has a nice shape.
- ▶ In particular, $T(\leq, S, 0)$ has a special countable model (which is not \mathbb{N}).
- ▶ The theory is complete, and hence is just $Th(\mathbb{N}, \leq, S, 0)$.
- ▶ The theory is model complete and, in fact, has EQ .

The axioms

- (L1) $(\forall v)[v \leq v]$
- (L2) $(\forall u, v, w)[u \leq v \leq w \rightarrow u \leq w]$
- (L3) $(\forall u, v)[u \leq v \leq u \rightarrow u \simeq v]$
- (L4) $(\forall u, v)[u \leq v \vee v \leq u]$

Axioms for

Linear order

- (S0) $(\forall v)[Sv \neq 0]$
- (S1) $(\forall u, v)[Su \simeq Sv \rightarrow u \simeq v]$
- (S2) $(\forall v)[v \neq 0 \rightarrow (\exists u)[Su = v]]$
- (S3_k) $(\forall v)[S^{k+1}v \neq v] \quad (\text{for all } k \in \mathbb{N})$

Simple arithmetic

- (I) $(\forall v)[v \leq Sv]$
- (M) $(\forall u, v)[u \leq v \rightarrow Su \leq Sv]$
- (Z) $(\forall v)[0 \leq v]$
- (D) $(\forall u, v)[u \leq Sv \rightarrow u \leq v \vee u = Sv]$

The interaction

Structure of a model

Each model is \mathbb{N} together with several non-overlapping copies of \mathbb{Z} arranged in linear order.

Thus each model is

$$\mathbb{N} + \mathbb{Z} \cdot \alpha$$

for some linear order type.

Setting $\alpha = 0$ gives \mathbb{N} , the minimal model.

Setting $\alpha = \mathbb{Q}$ gives

$$\Omega = \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$$

the ‘universal’ countable model.

How is \mathfrak{Q} universal

Each countable model

A of order type $\mathbb{N} + \mathbb{Z} \cdot \alpha$

is elementarily embeddable in \mathfrak{Q} .

By attacking its type α we enlarge A bit by bit until we get \mathbb{Q} .

We produce a long enough chain of elementary embeddings where the type approaches \mathbb{Q} , and then take the union.

Let's look at a 1-step.

$$\begin{array}{ccc} \mathbb{N} + \mathbb{Z} \cdot \alpha & & \mathbb{N} + \mathbb{Z} \cdot \beta \\ A & \prec & B \end{array}$$

where β is more like \mathbb{Q} than α is.

The 1-step construction

Consider A of type α . Let l, r be non-standard elements with the \mathbb{Z} -block of l to the left of the \mathbb{Z} -block of r .

We insert three new blocks

to the left of l between l, r to the right of r

Let \mathbf{a} enumerate A . Add three new constants x, y, z . We require a consistent set

$$Th(A, \mathbf{a}) \cup \Sigma$$

where Σ is all sentences (for $n \in \mathbb{N}$)

$$x + n \leq l \quad l + n \leq y \quad y + n \leq r \quad r + n \leq z$$

We check that this is finitely satisfiable in A .

Partial isomorphisms of finite width

Consider a pair of models

$$A = \mathbb{N} + \sum \{\mathbb{Z}_i \mid i \in \eta\} \quad B = \mathbb{N} + \sum \{\mathbb{Z}_j \mid j \in \zeta\}$$

where each of η, ζ is a dlowep.

Take a pair of list of indexes of equal length.

$$i = (i(1) < i(2) < \cdots < i(l)) \quad j = (j(1) < j(2) < \cdots < j(l))$$

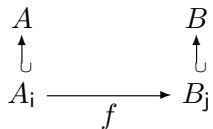
These give a pair of submodels

$$A_i = \mathbb{N} + \mathbb{Z}_{i(1)} + \cdots + \mathbb{Z}_{i(l)} \quad B_j = \mathbb{N} + \mathbb{Z}_{j(1)} + \cdots + \mathbb{Z}_{j(l)}$$

Any isomorphism between these is a typical
partial isomorphism of finite width.

These form a b&f system

Going forth: Given



and an element x of $A - A_i$

- ▶ Locate the block in which x lives.
- ▶ Locate the position of the block relative to the list i .
- ▶ Go to the same position in j .
- ▶ Choose a block of B in that position.
- ▶ Match these two new blocks.

The theory is complete and model complete

Consider any two models A, B .

There are countable elementary submodels $A^+ \prec A, B^+ \prec B$.

For each of these we have $A^+ \prec \mathfrak{Q}, B^+ \prec \mathfrak{Q}$.

Hence $A \equiv B$.

Now suppose $A \subseteq B$ and consider any point a in A , and let b be the corresponding (same) point of B .

Select $a \in A^+$ and $b \in B^+$, so it suffices to show $(\mathfrak{Q}, a) \equiv (\mathfrak{Q}, b)$.

With the submodels of finite width given by a and b we have

$$\begin{array}{ccc} \mathfrak{Q} & & \mathfrak{Q} \\ \uparrow & & \uparrow \\ A_a & \xrightarrow{f} & B_b \end{array}$$

and hence $(\mathfrak{Q}, a) \equiv_p (\mathfrak{Q}, b)$, for the required result.