

The back-and-forth technique

A method of producing isomorphisms between countable structures, and showing that arbitrary structures are elementarily equivalent.

Let $\mathfrak{A}, \mathfrak{B}$ be structures for some language L .

A **partial isomorphism** between \mathfrak{A} and \mathfrak{B} is a bijection

$$f : U \longrightarrow V$$

from a subset U of \mathfrak{A} to a subset V of \mathfrak{B} which is an isomorphism ‘as far as it goes’

To make this precise we take an enumeration \mathbf{a} of U and the corresponding enumeration \mathbf{b} of V . Then we must have

$$(\mathfrak{A}, \mathbf{a}) \equiv_0 (\mathfrak{B}, \mathbf{b})$$

A b&f-system

Between two structures $\mathfrak{A}, \mathfrak{B}$ is a non-empty set P of partial isomorphisms

$$f : U \longrightarrow V$$

with the following two properties.

(forth) For each $f \in P$ and each element x of \mathfrak{A} , there is an element y of \mathfrak{B} together with an extension $f^+ \in P$

(back) For each $f \in P$ and each element y of \mathfrak{B} , there is an element x of \mathfrak{A} together with an extension $f^+ \in P$

$$f^+ : U \cup \{x\} \longrightarrow V \cup \{y\}$$

with $f^+(x) = y$.

We write

$$\mathfrak{A} \equiv_p \mathfrak{B}$$

to indicate the existence of a b&f-system.

Lines – part 1

Let A, B be a pair of lines (dloewp). Consider the set of all pairs (a, b) of finite lists with

$$(A, a) \equiv_0 (B, b)$$

To go forth consider $x \in A$. Locate its position relative to the list a . There must be some $y \in B$ in the same position relative to b .

As a particular example we have

$$\mathbb{Q} \equiv_p \mathbb{R}$$

$$\underline{\mathfrak{A} \cong \mathfrak{B} \implies \mathfrak{A} \equiv_p \mathfrak{B} \implies \mathfrak{A} \equiv \mathfrak{B}}$$

First implication is trivial. The trick for the second implication:
Suppose we have a collection Φ of formulas with

$$(a, b) \in P \implies [\mathfrak{A} \models \phi(a) \iff \mathfrak{B} \models \phi(a)]$$

Then we can bump up the quantifier complexity.
Consider $\phi(u, v) \in \Phi$.

$$\begin{array}{ll} \mathfrak{A} \models (\exists u)\phi(u, a) & (a, b) \in P \\ \mathfrak{A} \models \phi(x, a) & \text{some } x \text{ from } \mathfrak{A} \\ (x \frown a, y \frown b) \in P & \text{by going forth} \\ \mathfrak{B} \models \phi(y, b) & \text{since } \phi \in \Phi \\ \mathfrak{B} \models (\exists u)\phi(u, b) & \end{array}$$

In fact this shows $(a, b) \in P \implies (\mathfrak{A}, a) \equiv (\mathfrak{B}, b)$

Lines – part 2

Let A, B be a pair of lines, with P as before. For $(a, b) \in P$ we have

$$(A, a) \equiv (B, b)$$

and hence the theory of lines is complete.

The same result shows that

$$A \subseteq B \implies A \prec B$$

and hence the theory of lines is model complete.

As a particular example we have $\mathbb{Q} \prec \mathbb{R}$.

For countable structures $\mathfrak{A} \equiv_p \mathfrak{B} \implies \mathfrak{A} \cong \mathfrak{B}$

There are full enumerations of $\mathfrak{A}, \mathfrak{B}$

$$(x_i \mid i < \omega) \quad (y_i \mid i < \omega)$$

Starting from any $f_0 \in P$ (the given b&f-system) we generate an ascending ω -chain of members of P .

$$f_0 \subseteq f_1 \subseteq \cdots \subseteq f_i \subseteq \quad f_i : U_i \longrightarrow V_i$$

At each step we ensure

$$x_i \in U_{i+1} \quad y_i \in V_{i+1}$$

The union is a full isomorphism. $f = \bigcup \{f_i \mid i < \omega\}$

Lines – part 3

The theory of lines is

- (0) \aleph_0 -categorical
- (1) complete
- (2) model complete
- (3) has *EQ*

To prove (3) we verify directly that the \forall_1 -part of the theory has *AP*.