1 JEP and AP

After sorting out the notion of an embedding we look at two structural properties of theories involving embeddings.

Elementary equivalence, isomorphism

Let $\mathfrak{A}, \mathfrak{B}$ be structures (for the same language).

We write

$$\mathfrak{A} \equiv \mathfrak{B}$$

and say $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences.

We write

$$\mathfrak{A} \cong \mathfrak{B}$$

and say $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if there is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, a surjective embedding.

Almost trivially

$$\mathfrak{A} \cong \mathfrak{B} \Longrightarrow \mathfrak{A} \equiv \mathfrak{B}$$

but in general the converse doesn't hold.

Embedding

An embedding $\mathfrak{A} \xrightarrow{f} \mathfrak{B}$ is a function $f : A \longrightarrow B$ between the carriers which is injective and

- For each constant symbol K

$$f(\mathfrak{A}[\![K]\!]) = \mathfrak{B}[\![K]\!]$$

- For each n-placed relation symbol R

$$\mathfrak{A}[\![R]\!]a_1 \cdots a_n \iff \mathfrak{B}[\![R]\!]f(a_1) \cdots f(a_n)$$

  for each $a_1, \ldots, a_n$ from $\mathfrak{A}$.

- For each n-placed relation symbol O

$$f(\mathfrak{A}[\![O]\!]a_1 \cdots a_n) = \mathfrak{B}[\![O]\!]f(a_1) \cdots f(a_n)$$

  for each $a_1, \ldots, a_n$ from $\mathfrak{A}$.

A neater version

Lemma *Let* $\mathfrak{A}, \mathfrak{B}$ *be a pair of structures, and let* $f : A \longrightarrow B$ *be a function between the carriers. Then* f *is an embedding if and only if*

$$\mathfrak{A} \models \delta(a_1, \ldots, a_n) \iff \mathfrak{B} \models \delta(b_1, \ldots, b_n)$$

holds for each quantifier-free formula $\delta(v_1, \ldots, v_n)$ *and elements* $a_1, \ldots, a_n$ *from* $\mathfrak{A}$ *and where* $b_i = f(a_i)$ *for each* $1 \leq i \leq n$.

By replacing δ by more complicated formulas we obtain special kinds of embeddings. For instance an elementary embedding works for all formulas.

In many places in model theory we can pretend than an embedding is an insertion $\mathfrak{A} \subseteq \mathfrak{B}$, but sometimes we can't.

5 The diagram technique produces embeddings

Lemma *Let* $\mathfrak{A}, \mathfrak{B}$ *be two* L*-structures. Let* a *be an enumeration of the whole of* $\mathfrak{A}$*. Then* $\mathfrak{A}$ *is*

embeddable *elementarily embeddable*

in $\mathfrak{B}$ *if and only if*

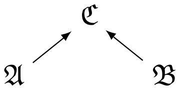
$$(\mathfrak{B}, \mathsf{b}) \models Diag(\mathfrak{A}, \mathsf{a}) \qquad (\mathfrak{B}, \mathsf{b}) \models Th(\mathfrak{A}, \mathsf{a})$$

for some enumeration b *of a part of* $\mathfrak{B}$*.*

There are many obvious refinements of this idea.

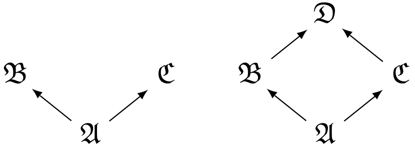
Two structural properties — Section 4.2

(jep) A theory T has JEP the joint embedding property if for each pair $\mathfrak{A}, \mathfrak{B}$ of models of T, there is a wedge of embeddings



to some model $\mathfrak{C}$ of T.

(ap) A theory T has AP the amalgamation property if for each wedge of embedding between models of T, there is a model $\mathfrak{D}$



of T and a commuting square of embeddings.

Two characterizations

Theorem *A (consistent) theory T has JEP if and only if*

$$T \vdash \alpha \vee \beta \Longrightarrow T \vdash \alpha \text{ or } T \vdash \beta$$

for each pair α, β of $\forall_1$-sentences.

Theorem *A (consistent) theory T has AP if and only if for each pair ψ, ϕ of $\forall_1$-formulas with*

$$T \vdash \psi \vee \phi$$

we have

$$T \vdash \lambda \vee \rho \qquad T \vdash \lambda \to \psi \qquad T \vdash \rho \to \phi$$

for some pair λ, ρ of $\exists_1$-formulas. [**Make correction**]

Here we (usually) have to deal with embeddings, not just insertions.

From syntax to JEP

Let $\mathfrak{A}, \mathfrak{B}$ be any pair of models. Enumerate each separately. It suffices to show the following is consistent.

$$T \cup Diag(\mathfrak{A}, \mathsf{a}) \cup Diag(\mathfrak{B}, \mathsf{b})$$

By compactness, if this is not consistent then we have

$$\mathfrak{A} \models \gamma(a) \quad \mathfrak{B} \models \delta(b) \quad T \vdash \neg\gamma(a) \vee \neg\delta(b)$$

for two QF formulas. The last of these gives

$$T \vdash \neg\gamma(v) \vee \neg\delta(w)$$

so that

$$T \vdash (\forall v)\neg\gamma(v) \vee (\forall w)\neg\delta(w)$$

hence

$$T \vdash (\forall v)\neg\gamma(v) \quad \text{or} \quad T \vdash (\forall w)\neg\delta(w)$$

which leads to a contradiction.

From syntax to AP

Consider a wedge of models, as above. By taking isomorphic copies we may suppose

$$\mathfrak{A} \subseteq \mathfrak{B} \quad \mathfrak{A} \subseteq \mathfrak{C} \quad \mathfrak{B} \cap \mathfrak{C} = \mathfrak{A}$$

Take enumerations and check the following is consistent.

$$T \cup Diag(\mathfrak{B}, \mathsf{a}, \mathsf{b}) \cup Diag(\mathfrak{C}, \mathsf{a}, \mathsf{c})$$

If not consistent then get QF formulas with

$$\mathfrak{B} \models \beta(a, b) \quad \mathfrak{C} \models \gamma(a, c) \quad T \vdash \neg\beta(u, v) \vee \neg\gamma(u, w)$$

to get $\exists_1$-formulas with

$$T \vdash \lambda(u) \to (\forall v)\neg\beta(u, v) \quad T \vdash \rho(u) \to (\forall w)\neg\gamma(u, w) \quad T \vdash \lambda \vee \rho$$

A contradiction follows from the appropriate one of

$$\mathfrak{A} \models \lambda(a) \text{ hence } \mathfrak{B} \models \lambda(a) \qquad \mathfrak{A} \models \rho(a) \text{ hence } \mathfrak{C} \models \rho(a)$$

From AP to syntax

This involves a new notion, that of a type – a set of formulas containing only finitely many variables in total.

Suppose

$$T \vdash \psi \vee \phi$$

and let v be the batch of variables occurring in these formulas.

Consider the sets of $\forall_1$ consequences in v, as indicated.

$$\Pi(T, \psi) \qquad \Pi(T, \phi)$$

$$T \vdash \neg\psi \rightarrow \alpha \qquad T \vdash \neg\phi \rightarrow \beta$$

We show that

$$T \cup \Pi(T, \psi) \cup \Pi(T, \phi)$$

is inconsistent, hence . . . [**On board**]

A preliminary observation

Suppose we have a structure $\mathfrak{A}$ and a point a with

$$\mathfrak{A} \models T \qquad \mathfrak{A} \models \Pi(T, \psi)(a)$$

Let a enumerate $\mathfrak{A}$ and check the following is consistent.

$$T \cup Diag(\mathfrak{A}, \mathsf{a}) \cup \{\neg\psi(a)\}$$

If not consistent then get some QF $\delta(v, w)$ and part x of a with

$$\mathfrak{A} \models \delta(a, x) \qquad T \vdash \neg\psi(a) \to \neg\delta(a, x)$$

and hence

$$(\forall w)\neg\delta(v, w) \in \Pi(T, \psi)$$

which leads to a contradiction.

Final step

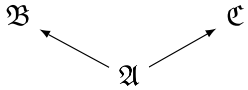
Suppose

$$T \cup \Pi(T,\psi) \cup \Pi(T,\phi)$$

is consistent with $(\mathfrak{A}, a)$ as a model.

Two uses of the preliminary observation gives a wedge of embeddings between models of T.

$\mathfrak{B} \models \neg\psi(x)$ $\qquad$ $\mathfrak{C} \models \neg\phi(x)$



A use of AP produces a single model $\mathfrak{D}$ of T with

$$\mathfrak{D} \models \neg\psi(a) \wedge \neg\phi(a)$$

which contradicts the hypothesis on ψ, ϕ.