Complete, model complete, submodel complete

We look for a model theoretic characterization of those theories with EQ.

Complete theories, Exercise 1.24

Lemma *For each consistent theory T the following are equivalent.*

(i) $T = Th(\mathfrak{A})$ *for some structure* $\mathfrak{A}$.

(ii) *We have*

$$\sigma \in T \quad or \quad \neg\sigma \in T$$

for each sentence σ.

(iii) *We have*

$$\mathfrak{A} \equiv \mathfrak{B}$$

for each pair of models $\mathfrak{A}, \mathfrak{B}$ *of* T.

Such a theory is complete.
By (ii) it decides every sentence and can not be made any larger without becoming inconsistent.

Classes of submodels

Let T be a theory in some language L.
We set

$$\begin{array}{lcl} \mathcal{M}d(T) & = & \text{the class of models of } T \\ \mathcal{S}(T) & = & \text{the class of submodels of } T \\ & = & \text{the class of those structures } \mathfrak{A} \text{ with} \\ & & \mathfrak{A} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \models T \qquad = \quad \mathcal{M}(T \cap \forall_1) \end{array}$$

In due course we look at several 'special' classes $\mathcal{K} \subseteq \mathcal{S}(T)$ which give us information about T.

Being in $\mathcal{S}(T)$

For an arbitrary L-structure $\mathfrak{A}$, when is $\mathfrak{A} \in \mathcal{S}(T)$?

Let $L(\mathsf{a}) = L(\mathfrak{A})$ be the $\mathfrak{A}$-enrichment of L.
We set

$$T[\mathfrak{A}] = T \cup Diag(\mathfrak{A}, \mathsf{a})$$

to produce a set of $L(\mathsf{a})$-sentences.

We have

$$\mathfrak{A} \in \mathcal{S}(T) \Longleftrightarrow T[\mathfrak{A}] \text{ is consistent}$$

When is the theory (axiomatized by) $T[\mathfrak{A}]$ complete for certain $\mathfrak{A} \in \mathcal{S}(T)$?

Model complete and submodel complete theories

A theory T is model complete if the enriched theory (axiomatized by) $T[\mathfrak{A}]$ is complete for each $\mathfrak{A} \in \mathcal{M}d(T)$.

A theory T is submodel complete if the enriched theory (axiomatized by) $T[\mathfrak{A}]$ is complete for each $\mathfrak{A} \in \mathcal{S}(T)$.

That was the original definition of model completeness, but there is a much neater characterization.

Theorem *A theory T is model complete if and only if*

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec \mathfrak{B}$$

for all models $\mathfrak{A}, \mathfrak{B}$ of T.

Exercise: Prove this.

More characterizations of model completeness

Theorem *For each theory T the following are equivalent.*

(i) *Each formula is T-equivalent to an $\forall_1$-formula.*

(ii) *Each formula is T-equivalent to an $\exists_1$-formula.*

(iii) *Each $\forall_1$-formula is T-equivalent to an $\exists_1$-formula.*

(iv) *We have*

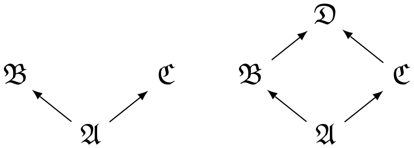
$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

for all models $\mathfrak{A}, \mathfrak{B}$ of T.

(v) *T is model complete.*

The amalgamation property

A theory T has AP if for each wedge of embedding between models of T, there is a model $\mathfrak{D}$



of T and a commuting square of embeddings.

There is a syntactic characterization of having AP which we will look at later.

The syntactic characterization of AP

Theorem *For each theory T the following are equivalent.*

(i) *T has AP.*

(ii) *For each pair ψ, ϕ of $\forall_1$-formulas with*

$$T \vdash \psi \vee \phi$$

there are $\exists_1$-formulas λ, ρ such that

$$T \vdash \lambda \vee \rho \qquad T \vdash \lambda \to \psi \qquad T \vdash \rho \to \phi$$

hold.

A characterization of submodel completeness

Theorem *For each theory T, the following are equivalent.*

(i) T *is submodel complete.*

(ii) T *is model complete and* $T \cap \forall_1$ *has AP.*

(iii) *The implication* $(\mathfrak{A}, a) \equiv_0 (\mathfrak{B}, b) \Longrightarrow (\mathfrak{A}, a) \equiv (\mathfrak{B}, b)$ *holds for all* $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(T)$ *and points* a *from* $\mathfrak{A}$ *and* b *from* $\mathfrak{B}$.

(iv) *The implication* $(\mathfrak{A}, a) \equiv_0 (\mathfrak{B}, b) \Longrightarrow (\mathfrak{A}, a) \equiv_1 (\mathfrak{B}, b)$ *holds for all* $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(T)$ *and points* a *from* $\mathfrak{A}$ *and* b *from* $\mathfrak{B}$.

(v) *For each* $\forall_1$*-formula* ϕ, *there is a quantifier-free formula* δ *such that* $T \vdash \phi \leftrightarrow \delta$

(vi) T *has EQ.*