$\forall_2$-axiomatizability

- A few more general facts
- A second use of the diagram technique
- A characterization result.

Two characterizations

Theorem *For each theory T the following are equivalent.*

(i) *T is $\forall_1$-axiomatizable*

(ii) *$Md(T)$ is closed under taking substructures.*

This was done earlier

Theorem *For each theory T the following are equivalent.*

(i) *T is $\forall_2$-axiomatizable.*

(ii) *$Md(T)$ is closed under unions of directed systems.*

(iii) *$Md(T)$ is closed under unions of ω-chains.*

This is stated on page 55 as Theorem 3.24
The proof of this will take some time.

Directed systems — Page 52, Section 3.4

A non-empty family $\mathcal{A}$ of structures is directed if for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{A}$, there is some $\mathfrak{C} \in \mathcal{A}$ with both $\mathfrak{A} \subseteq \mathfrak{C}$ and $\mathfrak{B} \subseteq \mathfrak{C}$.

Each chain is directed.

Given a directed family $\mathcal{A}$ we want to construct the union

$$\mathfrak{U} = \bigcup \mathcal{A}$$

a structure $\mathfrak{U}$ which extends each $\mathfrak{A} \in \mathcal{A}$ in a minimal way.

Thus we require the carrier U and an interpretation

$$\mathfrak{U}[\![K]\!] \quad \mathfrak{U}[\![R]\!] \quad \mathfrak{U}[\![O]\!]$$

for each symbol of the signature.

The construction — Page 53, Item 3.21

For U we set $$U = \bigcup\{A \mid A \text{ is the carrier of some } \mathfrak{A} \in \mathcal{A}\}$$

For K we set $$\mathfrak{U}[\![K]\!] = \mathfrak{A}[\![K]\!] \text{ for any } \mathfrak{A} \in \mathcal{A}$$

For R we set $$\mathfrak{U}[\![R]\!]a_1 \cdots a_n = \mathfrak{A}[\![R]\!]a_1, \cdots a_n$$ $$\text{for any } \mathfrak{A} \in \mathcal{A} \text{ with } a_1, \ldots, a_n \in \mathfrak{A}.$$

For O we set $$\mathfrak{U}[\![O]\!]a_1 \cdots a_n = \mathfrak{A}[\![O]\!]a_1, \cdots a_n$$ $$\text{for any } \mathfrak{A} \in \mathcal{A} \text{ with } a_1, \ldots, a_n \in \mathfrak{A}.$$

In each case we have to check that the constructed attribute is independent of the choice of $\mathfrak{A}$.

The interlacing trick — Exercise 3.11

Let $\mathcal{A}$ and $\mathcal{B}$ be two directed families which interlace in the sense that we have both

For each $\mathfrak{A} \in \mathcal{A}$, there is some $\mathfrak{B} \in \mathcal{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$

For each $\mathfrak{B} \in \mathcal{B}$, there is some $\mathfrak{A} \in \mathcal{A}$ with $\mathfrak{B} \subseteq \mathfrak{A}$

Under these circumstances

$$\bigcup \mathcal{A} = \bigcup \mathcal{B}$$

and, in fact, $\mathcal{A} \cup \mathcal{B}$ is a directed family.

$(i) \Rightarrow (ii)$ — Page 53, Lemma 3.22

Lemma *For each directed family $\mathcal{A}$ we have*

$$\mathcal{A} \models \sigma \Longrightarrow \bigcup \mathcal{A} \models \sigma$$

for each $\forall_2$-sentence σ.

Recall the σ is

$$(\forall u)\phi(u)$$

for some $\exists_1$-formula in a batch u.

Each point a of $\bigcup \mathcal{A}$ lives in some $\mathfrak{A} \in \mathcal{A}$, and

$$\mathfrak{A} \models \phi(a)$$

and $\exists_1$-formulas transfer upwards.

Kinds of embeddings — Page 42, Definition 3.3

$\mathfrak{A} \subseteq \mathfrak{B}$	$\mathfrak{A} \models \phi(a) \Rightarrow \mathfrak{B} \models \phi(a)$	For all $\phi \in \exists_1$
$\mathfrak{A} \prec_1 \mathfrak{B}$	$\mathfrak{A} \models \phi(a) \Rightarrow \mathfrak{B} \models \phi(a)$	For all $\phi \in \forall_1$ (hence $\exists_2$)
	$\vdots$	
$\mathfrak{A} \prec_n \mathfrak{B}$	$\mathfrak{A} \models \phi(a) \Rightarrow \mathfrak{B} \models \phi(a)$	For all $\phi \in \forall_n$ (hence $\exists_{n+1}$)
	$\vdots$	
$\mathfrak{A} \prec \mathfrak{B}$	$\mathfrak{A} \models \phi(a) \Rightarrow \mathfrak{B} \models \phi(a)$	For all $\phi \in \forall_\infty$

This gives a whole hierarchy of relations

$$\mathfrak{A} \prec_n \mathfrak{B}$$

between $\subseteq = \prec_0$ and $\prec = \prec_\omega$.

A preservation result — Lemma 3.23, Exercise 3.9

Lemma *Let $\mathcal{A}$ be a directed family of structures. If*

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_n \mathfrak{B}$$

for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{A}$. Then

$$\mathfrak{A} \prec_n \bigcup \mathcal{A}$$

for each $\mathfrak{A} \in \mathcal{A}$.

Corollary *If*

$$\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} \prec \mathfrak{B}$$

for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{A}$. Then

$$\mathfrak{A} \prec \bigcup \mathcal{A}$$

for each $\mathfrak{A} \in \mathcal{A}$.

Three similar results

$$\mathfrak{A} \models T \cap \forall_1 \text{ gives some } \mathfrak{B} \text{ with } \mathfrak{A} \subseteq \mathfrak{B} \models T$$

$$\mathfrak{A} \models T \cap \forall_2 \text{ gives some } \mathfrak{B} \text{ with } \mathfrak{A} \prec_1 \mathfrak{B} \models T$$

For the second show that the following is consistent.

$$T \cup \big(Th(\mathfrak{A}, \mathsf{a}) \cap \forall_1 \big)$$

$$\mathfrak{A} \prec_1 \mathfrak{B} \quad \text{gives some } \mathfrak{C} \text{ with} \quad \begin{array}{c} \mathfrak{A} \;\prec_1\; \mathfrak{B} \;\subseteq\; \mathfrak{C} \\ \underline{\qquad\quad}\prec\underline{\qquad\quad} \end{array}$$

Show that

$$Th(\mathfrak{A}, \mathsf{a}) \cup Diag(\mathfrak{B}, \mathsf{a}, \mathsf{b})$$

is consistent.

This is really a particular case of the first result.
Ex: Locate where these are done.

$(iii) \Rightarrow (i)$ Page 55

$\mathfrak{A} \models T \cap \forall_2$ gives $\mathfrak{A} \prec_1 \mathfrak{B} \models T$ which gives $\underbrace{\mathfrak{A} \prec_1 \mathfrak{B} \subseteq \mathfrak{A}'}_{\prec}$

Iterate to get

$$\left(\mathfrak{A}_i \mid i < \omega\right) \qquad \left(\mathfrak{B}_i \mid i < \omega\right)$$

with

$$\mathfrak{A}_0 = \mathfrak{A} \quad \mathfrak{A}_i \subseteq \mathfrak{B}_i \subseteq \mathfrak{A}_{i+1} \quad \mathfrak{B}_i \models T \quad \mathfrak{A}_i \prec \mathfrak{A}_{i+1}$$

There is a common union $\mathfrak{U}$

$$\bigcup \left(\mathfrak{A}_i \mid i < \omega\right) = \mathfrak{U} = \bigcup \left(\mathfrak{B}_i \mid i < \omega\right)$$

Since each $\mathfrak{B}_i \models T$ the hypothesis (iii) gives $\mathfrak{U} \models T$.
The presevation result gives $\mathfrak{A} \prec \mathfrak{U}$.
Hence $\mathfrak{A} \models T$.