Quantifier Elimination

Strictly speaking this is not model theory, but it had a big influence on the in initial development of model theory.

We look at a couple of the more simple examples.

Basics — Definition 2.1, Theorem 2.2

Let T be a theory in some language.

Two formulas ϕ and ψ are T-equivalent if

$$T \vdash (\forall v)[\phi \leftrightarrow \psi]$$

where v is a batch of variables which includes $\partial\phi \cup \partial\psi$.

The theory T has *EQ* (elimination of quantifiers) if each formula is T-equivalent to some quantifier-free formula.

Theorem *To eliminate quantifiers for a theory T it is sufficient to find a quantifier-free equivalent of each formula*

$$(\exists w)\delta(w, v_1, \ldots, v_k)$$

where δ is a conjunction of literals and where the single quantified variable w occurs in each such literal.

The theory of lines — Section 2.2

A line is a structure

$$\mathfrak{A} = (A, \leq)$$

which is a dense linear order without end points. Thus it is a linearly ordered set which has no first point, no last point, and between any two distinct points there is a third.

Both $\mathbb{Q}$ and $\mathbb{R}$ (with their natural orderings) are lines.

The theory T of lines is $\forall_2$-axiomatizable, but can not be $\forall_1$-axiomatized.

We show that T has EQ.

The modified signature for linear orders

Working just with $\leq$ and $\doteq$ can tedious. It is useful to have the strict comparison $<$ as well. The theory of linear orders ensures the following equivalences.

$$(u < v) \leftrightarrow (v \not\leq u) \qquad (u \neq v) \leftrightarrow (u < v) \vee (v < u)$$

$$(u \doteq v) \leftrightarrow (u \leq v) \wedge (v \leq u)$$

This enables us to get rid of negation and equality.

Lemma *Relative to the theory of linearly ordered sets, each quantifier-free formula is equivalent to a* $\{\wedge, \vee\}$*-combination of formulas*

$$(u \leq v) \qquad (u < v)$$

for appropriate variables u, v.

Eliminating a quantifier

$$(\exists w)[L_1 \wedge \cdots \wedge L_n]$$

This conjunction of positive literals says that the other variables must relate to w in a certain way.

Certain variables must occur to the left of w, and some of these must occur strictly to the left.

Certain variables must occur to the right of w, and some of these must occur strictly to the right.

In a line this is possible if and only if the left and right don't overlap. Thus the formula is equivalent to a conjunction of conditions

left variable $\leq$ right variable   left variable $<$ right variable

The completeness of T

There are only two QF-sentences, so for each sentences σ one of

$$T \vdash \sigma \leftrightarrow \mathsf{true} \qquad T \vdash \sigma \leftrightarrow \mathsf{false}$$

holds, and hence we have one of

$$T \vdash \sigma \qquad T \vdash \neg\sigma$$

to show that T is complete.

This means that, as lines,

$$\mathbb{Q} \equiv \mathbb{R}$$

which is a bit surprising.

Let $\mathfrak{N} = (\mathbb{N}, S, 0)$. We show that $Th(\mathfrak{N})$ has EQ.

The Dedekind axioms which characterize $\mathfrak{N}$ are

(0) $(\forall v)[(Sv \not\doteq 0)]$

(1) $(\forall u, v)[(Su \doteq Sv) \rightarrow (u \doteq v)]$

(Ind) Each subset X of $\mathbb{N}$ which contains 0 and is closed under S must be the whole of $\mathbb{N}$

The axiom (Ind) is not first order, but we can convert it into a first order scheme.

In fact, we can get rid of all the induction.

Two theories T^+ and T

Let T^+ be the theory axiomatized by

(0) $(\forall v)[(Sv \not\doteq 0)]$

(1) $(\forall u, v)[(Su \doteq Sv) \rightarrow (u \doteq v)]$

$$\text{(Ind)} \quad (\forall u_1, \ldots u_n) \left[\begin{bmatrix} \phi(u, 0) \\ \wedge \\ (\forall v)[\phi(u, v) \rightarrow \phi(u, Sv)] \end{bmatrix} \rightarrow (\forall v)\phi(u, v) \right]$$

for each formula $\phi(u_1, \ldots, u_n, v)$.

Let T be the theory axiomatized by (0,1) and

(2) $(\forall v)[(v \doteq \ulcorner 0 \urcorner) \vee (\exists w)[Sw \doteq v]]$
(3) $(\forall v)[S^{k+1}v \not\doteq v]$

for each $k \in \mathbb{N}$.

Two simple results — Lemmas 2.9 and 2.11

$$T \subseteq T^+ \subseteq Th(\mathfrak{N})$$

For each $k \in \mathbb{N}$

$$T \vdash (\forall v)\left[(\exists w)[S^{k+1}w \doteq v] \leftrightarrow (v \neq \ulcorner 0 \urcorner) \wedge \cdots \wedge (v \neq \ulcorner k \urcorner)\right]$$

We show that T has EQ, hence so does T^+ and $Th(\mathfrak{N})$.

In fact, since T decides all quantifier-free sentences, we have

$$T = T^+ = Th(\mathfrak{N})$$

so that T is a complete characterization of $\mathfrak{N}$, and this is achieved without the use of induction.

Eliminating the quantifiers

Is done using a tedious analysis of linear equations over $\mathbb{N}$.

Model theoretic consequences

We have two quite different examples of theories for which the quantifiers can be eliminated. There are also other more complicated examples.

There doesn't seem to be much structure or common features in these algorithms.

However, model theory does isolate the conditions which ensure that quantifiers can be eliminated.

We work towards the appropriate characterization.