The diagram technique

- First we sort out a few definitions and conventions.
- Then we look at a use of compactness and the diagram technique to obtain a characterization result.
- These two techniques are the cornerstone of basic model theory.

Informal satisfaction — Page 16

For a formula ϕ we often write

$$\phi(v_1, \ldots, v_n) \quad \text{to indicate} \quad \partial\phi = \{v_1, \ldots, v_n\}$$

then

$$\mathfrak{A} \models \phi\mathsf{x} \quad \text{depends only on} \quad a_1 = v_1\mathsf{x}, \ldots, a_n = v_n\mathsf{x}$$

so we write

$$\mathfrak{A} \models \phi(a_1, \ldots, a_n)$$

Often we even collapse

$$\begin{array}{lll} (v_1, \ldots, v_n) & \text{to a batch of variables} & v \\ (a_1, \ldots, a_n) & \text{to a point of } \mathfrak{A} & a \end{array}$$

and write

$$\mathfrak{A} \models \phi(a)$$

Substructure – Page 9, Definition 1.10

Let $\mathfrak{A}, \mathfrak{B}$ be two structures for the same language.
We have

$$\mathfrak{A} \subseteq \mathfrak{B}$$

precisely when

$$A \subseteq B$$

and the distinguished attributes of $\mathfrak{A}$ are the restrictions to A of those of $\mathfrak{B}$.

When this is so we have

$$\mathfrak{A} \models \delta(a) \Longleftrightarrow \mathfrak{B} \models \delta(a)$$

for each quantifier free formula $\delta(v)$ and point a of $\mathfrak{A}$.
(See Ex 1.16)

Elementary class .v. Theory — Pages 18 and 19

For a class $\mathcal{K}$ of structures and a set Σ of sentences we let

$$\mathcal{K} \models \Sigma \quad \text{mean} \quad (\text{For all } \mathfrak{A} \in \mathcal{K}, \sigma \in \Sigma)[\mathfrak{A} \models \sigma]$$

This sets up a galois correspondence.

$$\mathcal{K} \longmapsto Th(\mathcal{K}) \qquad \Sigma \longmapsto \mathcal{M}d(\Sigma)$$

$$\sigma \in Th(\mathcal{K}) \iff \mathcal{K} \models \sigma \qquad \mathfrak{A} \in \mathcal{M}d(\Sigma) \iff \mathfrak{A} \models \Sigma$$

A class $\mathcal{K}$ is elementary if $\mathcal{K} = \mathcal{M}d(\Sigma)$ for some Σ.

A set T is a theory if $T = Th(\mathcal{K})$ for some $\mathcal{K}$.

$$\Sigma \vdash \sigma \iff (\text{For all } \mathfrak{A})\left[\mathfrak{A} \models \Sigma \Rightarrow \mathfrak{A} \models \sigma\right] \iff \sigma \in Th(\mathcal{M}d(\Sigma))$$

Σ is a set of axioms for the theory T if $\qquad \Sigma \vdash \sigma \iff \sigma \in T$

Quantifier complexity — Pages 10 and 11

Each formula is equivalent to one in prenex normal form.
Taking note of the blocks of quantifiers we obtain a hierarchy.

QF	of the form	Quantifier Free
$\forall_1$	of the form	$(\forall \ldots)QF$
$\exists_1$	of the form	$(\exists \ldots)QF$
$\forall_2$	of the form	$(\forall \ldots)\exists_1$
$\exists_2$	of the form	$(\exists \ldots)\forall_1$
	$\vdots$	
$\forall_{n+1}$	of the form	$(\forall \ldots)\exists_n$
$\exists_{n+1}$	of the form	$(\exists \ldots)\forall_n$
	$\vdots$	

See picture on page 11.

A first example — Page 50, Theorem 3.17

Suppose the theory T is $\forall_1$-axiomatizable.
Then

$$(*) \quad \mathfrak{A} \subseteq \mathfrak{B} \models T \Longrightarrow \mathfrak{A} \models T$$

We convert this into a characterization.

Theorem *Suppose the theory T has (*). Then*

- $\mathfrak{A} \models T \cap \forall_1 \Longrightarrow \mathfrak{A} \models T$
- *T is $\forall_1$-axiomatizable.*

The trick: Given $\mathfrak{A} \models T \cap \forall_1$ we produce $\mathfrak{A} \subseteq \mathfrak{B} \models T$.

To do that we use the diagram technique.

Diagram technique — Page 434, Section 3.2

Let L be the parent language. Consider, and fix, $\mathfrak{A} \models T \cap \forall_1$. Enrich L, add a new constant

$$a \longmapsto K_a$$

for each element a of $\mathfrak{A}$. [In fact, we 'confuse' K_a with a.]

Let a be 'the' enumeration of $\mathfrak{A}$. The enriched language is $L(\mathsf{a})$ and

$$(\mathfrak{A}, \mathsf{a})$$

is its canonical structure.

Let

$$Diag(\mathfrak{A}, \mathsf{a})$$

be the set of QF sentences of $L(\mathsf{a})$ which hold in this structure.

What does this diagram look like

Each member is an $L(\mathsf{a})$-sentence

$$\delta(a_1, \dots, a_n) \qquad \delta(a)$$

where a

$$\delta(v_1, \dots, v_n) \qquad \delta(v)$$

is a QF formula of L instantiated by a

list $(a_1, \dots, a_n)$ point (a)

of $\mathfrak{A}$ matching the

list $(v_1, \dots, v_n)$ batch (v)

and where

$$\mathfrak{A} \models \delta(a)$$

We constructively confuse the elements of $\mathfrak{A}$ with the naming constants of $L(\mathsf{a})$.

The trick

Suppose

$$(\mathfrak{B}, \mathsf{b}) \models Diag(\mathfrak{A}, \mathsf{a})$$

where $\mathfrak{B}$ is an L-structure and b is a list of elements of $\mathfrak{B}$.

This sets up an embedding via the enriching constants.

$$\begin{array}{ccccc} \mathfrak{A} & & \text{Constants} & & \mathfrak{B} \\ a & \longmapsto & K_a & \longmapsto & \mathfrak{B}[\![K_a]\!] \end{array}$$

By replacing $\mathfrak{B}$ by a suitable isomorphic copy, we may suppose that $\mathfrak{A} \subseteq \mathfrak{B}$.

The body of the proof

Suppose T has (*) and $\mathfrak{A} \models T \cap \forall_1$.

Consider any

$$(\mathfrak{B}, \mathsf{b}) \models T \cup Diag(\mathfrak{A}, \mathsf{a})$$

Then (by taking an isomorphic copy of $\mathfrak{B}$) we have

$$\mathfrak{A} \subseteq \mathfrak{B} \models T$$

and hence

$$\mathfrak{A} \models T$$

as required.

(The missing soul) How do we know that $(\mathfrak{B}, \mathsf{b})$ exists?

The missing soul — Page 46, Lemma 3.9

We show the following is consistent.

$$T \cup Diag(\mathfrak{A}, \mathsf{a})$$

By compactness it suffices to show that each of

$$T \cup \{\delta(a_1, \ldots, a_n)\} \qquad \mathfrak{A} \models \delta(a) \qquad \delta(v) \text{ is QF}$$

is consistent.

If the set isn't consistent then

$$T \vdash \neg\delta(a_1, \ldots, a_n) \quad \text{that is} \quad T \vdash \neg\delta(K_1, \ldots, K_n)$$

so that ...

Moving between languages

Suppose

$$T \vdash \neg\delta(K_1, \ldots, K_n)$$

These constants don't occur in T, and so behave like variables. Thus we have

$$T \vdash \neg\delta(v_1, \ldots, v_n)$$

that is

$$T \vdash (\forall v_1, \ldots, v_n)\neg\delta(v_1, \ldots, v_n)$$

to give

$$(\forall v_1, \ldots, v_n)\neg\delta(v_1, \ldots, v_n) \in T \cap \forall_1$$

and hence

$$\mathfrak{A} \models (\forall v_1, \ldots, v_n)\neg\delta(v_1, \ldots, v_n)$$

which is not so (by instantiating the variables).