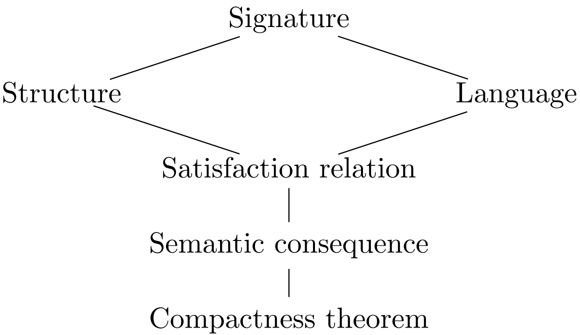
Basic notions — Syntax and semantics



Signature — Page 4, Definition 1.1

A signature is an indexed family of symbols where each is either

- a constant symbol K
- a relation symbol R with a nominated arity
- an operation symbol O with a nominated arity

respectively.

Each arity is a non-zero natural number. We speak of an n-placed relation symbol or an n-placed operation symbol to indicate the the arity is n.

Eventually we use the signature to generate a first order language.

Structure — Page 4, Definition 1.2

A structure $\mathfrak{A}$ for a given signature consists of the following.

- A non-empty set A, the carrier of $\mathfrak{A}$.
- For each constant symbol K, a nominated element $\mathfrak{A}[\![K]\!]$ of A.
- For each n-placed relation symbol R, a nominated n-placed relation $\mathfrak{A}[\![R]\!]$ on A.
- For each n-placed operation symbol O, a nominated n-placed operation $\mathfrak{A}[\![O]\!]$ on A.

These nominated gadgets are the distinguished attributes of $\mathfrak{A}$.

Primitive symbols — Page 4, Definition 1.4

For a given signature, the associated primitive symbols are:

- The symbols of the signature.
- The equality symbol $\doteq$.
- An unlimited stock of variables v.
- The connectives $\neg, \wedge, \vee, \rightarrow$
- The quantifiers $\forall$ and $\exists$.
- The constant sentences which are true and false.
- The punctuation symbols (and).

A string is a finite list of primitive symbols.

Term — Page 5, Definition 1.5

The terms of a signature are generated as follows.

- Each variable v is a term, and $\partial v = \{v\}$.
- Each constant symbol K is a term, and $\partial K = \emptyset$.
- For each n-placed operation symbol O and each list $t_1, \ldots, t_n$ of terms, the compound

  $$(Ot_1 \cdots t_n)$$

  is a term and

  $$\partial(Ot_1, \cdots t_n) = \partial t_1 \cup \cdots \cup \partial t_n$$

  is its set of free variables.

Atomic formula —- Page 6, Definition 1.6

The atomic formulas for a signature are

$$\mathsf{true} \qquad \mathsf{false} \qquad (t_1 \doteq t_2) \qquad (Rt_1 \cdots t_n)$$

where $t_1, t_2, \ldots t_n$ are terms and R is an n-placed relation symbol. Each such atomic formula θ has a set $\partial\theta$ of free variables given by

$$\emptyset \qquad \emptyset \qquad \partial t_1 \cup \partial t_2 \qquad \partial t_1 \cup \cdots \cup \partial t_n$$

respectively.

Formula — Page 6, Definition 1.7

The formulas for a signature are generated as follows.

Each atomic formula is a formula with free variables, as given

$$\frac{\psi}{\neg\psi} \qquad \partial\neg\psi = \partial\psi$$

$$\frac{\theta \quad \psi}{(\theta \wedge \psi)} \qquad \frac{\theta \quad \psi}{(\theta \vee \psi)} \qquad \frac{\theta \quad \psi}{(\theta \to \psi)} \qquad \frac{\partial\theta \quad \partial\psi}{\partial\theta \cup \partial\psi}$$

$$\frac{\psi}{(\forall v)\psi} \qquad \frac{\psi}{(\exists v)\psi} \qquad \frac{\partial\psi}{\partial\psi - \{v\}}$$

A sentence of a signature is a formula σ with $\partial\sigma = \emptyset$.

Satisfaction — Page 13, Section 1.3

Given a structure $\mathfrak{A}$ and a sentence σ of the same signature, we write

$$\mathfrak{A} \models \sigma$$

and say

the structure $\mathfrak{A}$ satisfies σ

(or some such phrase) if 'σ is true in $\mathfrak{A}$'.
It is perfectly clear what this means, but we do need a formal definition. This uses a little trick.

An $\mathfrak{A}$-assignment is a function x which attaches to each variable v an element $v\mathsf{x}$ of $\mathfrak{A}$.

Interpretation of a term - Page 13, Defn 1.13

For each structure $\mathfrak{A}$, each $\mathfrak{A}$-assignment x, and each term t the element

$$\mathfrak{A}[\![t]\!]\mathsf{x}$$

of $\mathfrak{A}$ is generated as follows.

- $\mathfrak{A}[\![v]\!]\mathsf{x} = v\mathsf{x}$
- $\mathfrak{A}[\![K]\!]\mathsf{x} = \mathfrak{A}[\![K]\!]$
- $\mathfrak{A}[\![(Ot_1 \cdots t_n)]\!]\mathsf{x} = \mathfrak{A}[\![O]\!]a_1 \cdots a_n \quad$ where $a_i = \mathfrak{A}[\![t_i]\!]\mathsf{x}$

Look at Lemma 1.14.

The satisfaction relation - Page 14, Defn 1.15

$\mathfrak{A} \models (\mathsf{true})\mathsf{x}$ always $\quad \mathfrak{A} \models (\mathsf{false})\mathsf{x}$ never

$\mathfrak{A} \models (t_1 \doteq t_2)\mathsf{x} \iff \mathfrak{A}[\![t_1]\!]\mathsf{x} = \mathfrak{A}[\![t_2]\!]\mathsf{x}$

$\mathfrak{A} \models (Rt_1 \cdots t_n)\mathsf{x} \iff \mathfrak{A}[\![R]\!]a_1 \cdots a_n$ where $a_i = \mathfrak{A}[\![t_i]\!]\mathsf{x}$

$\mathfrak{A} \models (\neg\psi)\mathsf{x} \iff \text{not}(\mathfrak{A} \models \psi\mathsf{x})$

$\mathfrak{A} \models (\theta \land \psi)\mathsf{x} \iff \mathfrak{A} \models \theta\mathsf{x}$ and $\mathfrak{A} \models \psi\mathsf{x}$

$\mathfrak{A} \models (\theta \lor \psi)\mathsf{x} \iff \mathfrak{A} \models \theta\mathsf{x}$ or $\mathfrak{A} \models \psi\mathsf{x}$

$\mathfrak{A} \models (\theta \to \psi)\mathsf{x} \iff \text{not}(\mathfrak{A} \models \theta\mathsf{x})$ or $\mathfrak{A} \models \psi\mathsf{x}$

$\mathfrak{A} \models ((\forall v)\psi)\mathsf{x} \iff \mathfrak{A} \models \psi\mathsf{y}$ for each $\mathfrak{A}$-assignment y which agrees with x except possibly in the v-position

$\mathfrak{A} \models ((\exists v)\psi)\mathsf{x} \iff \mathfrak{A} \models \psi\mathsf{y}$ for some $\mathfrak{A}$-assignment y which agrees with x except possibly in the v-position

Crude compactness – Page 23, Theorem 1.25

Consider a fixed signature with associated language L.
Let Σ be a set of sentences of this language.
If Σ is finitely satisfiable, then it has a model.

A set is finitely satisfiable if each of its finite subsets has a model (is satisfiable).

A first example — Page 24, Theorem 1.26

Let L be any language.

(a) The class of all infinite structures is elementary but not strictly elementary.
(b) The class of all finite structures is not elementary.
(c) A sentence holds in all infinite structures if and only if it holds in all sufficiently large finite structures.
(d) The theory of the class of finite structures has an infinite model.
(e) The theory of the class of infinite structures has no finite model.