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The fundamental triangle of a space

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This rather curious title doesn't really explain what this set of notes is about. So perhaps I should let you into the secret.

From [10] we know that each frame A has a point space

$$S = \mathsf{pt}(A)$$

together with a surjective frame morphism

$$A \xrightarrow{U_A} \mathcal{O}S$$

indexing the carried topology. The construction

$$A \longmapsto S = \mathsf{pt}(A)$$

is a contravariant functor from ***Frm*** to ***Top***, and the composite construction

$$A \longmapsto \mathcal{O}S$$

is an endofunctor on ***Frm*** for which the indexing morphism $U_\bullet$ is natural.

From [9] we know that each frame A has an assembly

$$NA$$

of all nuclei on A, and this itself is a frame. The construction

$$A \longmapsto NA$$

is an endofunctor on ***Frm***, and there is a natural epic embedding

$$A \xrightarrow{n_A} NA$$

of a frame into its assembly.

What happens when we compose these two endofunctors? We are going to look at the two constuctions

$$A \longmapsto \mathcal{O}S \longmapsto N\mathcal{O}S$$

$$A \longmapsto NA \longmapsto \mathcal{O}T$$

where

$$S = \mathsf{pt}(S) \qquad T = \mathsf{pt}(NA)$$

are the two spaces. We will find that T depends only on S (and not on the parent frame A), and there is a commuting triangle

$$\begin{array}{ccc} \mathcal{O}S & \xrightarrow{n_{\mathcal{O}S}} & N\mathcal{O}S \\ & \searrow^{\iota_S} & \downarrow{\scriptstyle \sigma_S} \\ & & \mathcal{O}T \end{array}$$

of frame morphisms. This is the fundamental triangle of the title.

Many of the results in this set of notes have been known, to some people, since the middle 1980s or earlier. Many are taken from [3] or [6], and some were stated without proof in [5]. However, the full well rounded picture has not been seen. Indeed, some of the later published results have obscured this picture.

Contents

1 The point space of an assembly

We know that the points of a frame can be viewed in three different ways.

($\uparrow$) As completely prime filters

($\leftrightarrow$) As ***Frm***-characters

($\downarrow$) As $\wedge$-irreducible elements

The character view is always worth remembering, for it can make certain functorial aspects almost trivial. Certain people preach the obligatory use of completely prime filters. However, almost always $\wedge$-irreducible elements are easier to work with. Some of that came through in [10], but now we have a really good example of the benefits of $\wedge$-irreducible elements.

Let us recall from [10] and [9] some of the general functorial properties.

Let A be an arbitrary frame. We fix this throughout this section. Let NA be its assembly and, as above, let

$$S = \mathsf{pt}(A) \qquad T = \mathsf{pt}(NA)$$

be the two points spaces each with its canonical topology. (Eventually we will furnish S with a second topology.) We have the natural embedding

$$A \underset{e_A}{\overset{n_A}{\rightleftarrows}} NA$$

where e_A is the right adjoint of n_A. By Corollary 3.3 of [9] we have

$$e_A(j) = j(\bot)$$

for each $j \in NA$. By Theorem 1.12 and just before of [10] this induces a continuous map

$$S \xleftarrow{\ \varepsilon_A\ } T$$

where

$$\varepsilon_A = e_A|T$$

that is

$$\varepsilon_A(\ell) = \ell(\bot)$$

for each $\ell \in T$. In particular, this shows that $p = \ell(\bot)$ is $\wedge$-ireducible in A, and gives us a clue as to what ℓ might be.

What are the points of NA? We are looking for those $\ell \in NA$ which are $\wedge$-irreducible in NA, that is $\ell(\bot) \neq \top$ (so that $\ell \neq \top_N$) and

$$j \wedge k \leq \ell \Longrightarrow j \leq \ell \text{ or } k \leq \ell$$

for all $j, k \in NA$. There is a nice characterization of these nuclei.

We need to remember that

$$j \leq \mathsf{w}_a \Longleftrightarrow j(a) \leq a$$

for all $a \in A$ and $j \in NA$. We use this several times in this section.

The following characterization appears as Lemma 3.2 of [3]. I give a slightly different proof.

1.1 LEMMA. *For each frame A and $\ell \in NA$, the following are equivalent.*

(i) ℓ is $\wedge$-irreducible in NA.

(ii) ℓ is 2-valued, that is $\ell(\bot) \neq \top$ and these are the only values of ℓ.

(iii) $\ell = \mathsf{w}_p$ where $p = \ell(\bot)$ and this is $\wedge$-irreducible in A.

Proof. (i)⇒(ii). Assuming (i) we show that

$$\ell(x) = \begin{cases} \top \text{ if } x \not\leq p \\ p \text{ if } x \leq p \end{cases}$$

for each $x \in A$, where $p = \ell(\bot)$. We know that $p \neq \top$, for otherwise $\ell = \top_N$.

For later, note that this is the only form a 2-valued nucleus can take.

Given $x \in A$ we have

$$\mathsf{u}_x \wedge \mathsf{v}_x = \bot_N \le \ell$$

so that either

$$\mathsf{u}_x \le \ell \quad \text{or} \quad \mathsf{v}_x \le \ell$$

by (i). By evaluation at $\bot$ and x these give

$$x \le \ell(\bot) = p \quad \text{or} \quad \top \le \ell(x)$$

which leads to the suggested description of ℓ.

(ii)⇒(iii). Assuming (ii) we first show that $p = \ell(\bot)$ is $\wedge$-irreducible.

We have $p \ne \top$, otherwise $\ell = \top_N$. For the $\wedge$-irreducibility suppose

$$x \wedge y \le p$$

for some $x, y \in A$. Then

$$\ell(x) \wedge \ell(y) = \ell(x \wedge y) \le \ell(p) = p$$

and

$$\ell(x) \in \{p, \top\} \quad \text{and} \quad \ell(y) \in \{p, \top\}$$

since ℓ is 2-valued. If

$$\ell(x) = \top = \ell(y)$$

then

$$\top = \ell(x) \wedge \ell(y) \le p$$

which is not so. Thus, one of

$$x \le \ell(x) = p \qquad y \le \ell(y) = p$$

must hold, as required.

Next, knowing that p is $\wedge$-irreducible, we show that

$$(x \supset p) = \begin{cases} p \text{ if } x \not\le p \\ \top \text{ if } x \le p \end{cases}$$

for each $x \in A$. To this end let

$$y = (x \supset p)$$

for arbitrary $x \in A$. Then

$$x \wedge y \le p$$

so that

$$x \le p \quad \text{or} \quad y \le p$$

to give

$$y = \top \quad \text{or} \quad p \le y \le p$$

for the required result.

Finally, two uses of this formula gives

$$\mathsf{w}_p(x) = ((x \supset p) \supset p) = \left\{ \begin{matrix} (p \supset p) \text{ if } x \not\le p \\ (\top \supset p) \text{ if } x \le p \end{matrix} \right\} = \left\{ \begin{matrix} (p \supset p) \text{ if } x \not\le p \\ (\top \supset p) \text{ if } x \le p \end{matrix} \right\} = \ell(x)$$

to show that $\ell = \mathsf{w}_p$. Here the last step uses the observation above concerning the shape of a 2-valued nucleus.

(iii)⇒(i). let $p \in A$ be $\wedge$-irreducible. We show that w_p is $\wedge$-irreducible in NA.

We have $\mathsf{w}_p)\bot) = p \neq \top$, so that $\mathsf{w}_p \neq \top_N$. For $j, k \in NA$ we have

$$j \wedge k \leq \mathsf{w}_p \Longrightarrow j(p) \wedge k(p) \leq \mathsf{w}_p(p) = p \Longrightarrow j(p) \leq p \text{ or } k(p) \leq p \Longrightarrow j \leq \mathsf{w}_p \text{ or } k \leq \mathsf{w}_p$$

as required. ■

This result can be rephrased as follows.

1.2 THEOREM. *For a frame A let*

$$S = \mathsf{pt}(A) \qquad T = \mathsf{pt}(NA)$$

with the canonical topologies. Then the two assignments

$$\begin{array}{ccc} \ell(\bot) & \longleftarrow\!\!\!\dashv & \ell \\ S & \xleftarrow{\ \varepsilon_A\ } & T \\ & \xrightarrow{\ \varpi_A\ } & \\ p & \longmapsto & \mathsf{w}_p \end{array}$$

form an inverse pair of bijections. Furthermore, ε_A is continuous.

Proof. For convenience we let

$$\varepsilon = \varepsilon_A \qquad \varpi = \varpi_A$$

to avoid a bit of clutter.

By Lemma 1.1 we know that each $p \in S$ gives a point

$$\varpi(p) = \mathsf{w}_p \in T$$

and each $\ell \in T$ has the form

$$\ell = \mathsf{w}_p = \varpi(p)$$

for some $p \in S$. In particular, for $p \in S$ we have

$$(\varepsilon \circ \varpi)(p) = \varepsilon(\mathsf{w}_p) = \mathsf{w}_p(\bot) = p$$

to show that $\varepsilon \circ \varpi = \boldsymbol{id}_S$. Similarly, for each $\ell = \mathsf{w}_p \in T$ we have

$$(\varpi \circ \varepsilon)(\ell) = \varpi(l(\bot)) = \varpi(p) = \mathsf{w}_p = \ell$$

to show that $\varpi \circ \varepsilon = \boldsymbol{id}_T$.

We observed above that the general functorial properties ensure that ε is continuous. A direct proof is worth seeing.

For each $a \in A$ and $j \in NA$ the two sets $U_A(a), U_{NA}(j)$ given by

$$p \in U_A(a) \Longleftrightarrow a \nleq p \qquad j \nleq \mathsf{w}_p \Longleftrightarrow \mathsf{w}_p \in U_{NA}(j)$$

are respective typical open sets of S and T. We show that

$$\varepsilon^{\leftarrow}(U_A(a)) = U_{NA}(\mathsf{u}_a)$$

for each $a \in A$.

For each $p \in S$ we have

$$\begin{aligned} \mathsf{w}_p \in U_{NA}(\mathsf{u}_a) &\iff \mathsf{u}_a \not\leq \mathsf{w}_p \\ &\iff a \vee p \not\leq p \\ &\iff a \not\leq p \qquad \iff \varepsilon(\mathsf{w}_p) = p \in U_A(a) \end{aligned}$$

to give the required result. ■

This is looking good isn't it. But now comes the stumble. In general the assignment ϖ is not continuous. We can easily see what the problem is, and so conjure up a solution.

A typical open set of $T = \mathsf{pt}(NA)$ has the form

$$U_{NA}(j)$$

for some $j \in NA$. Since

$$j = \bigvee\{\mathsf{u}_{j(a)} \wedge \mathsf{v}_a \mid a \in A\}$$

we have

$$U_{NA}(j) = \bigvee\{U_{NA}(\mathsf{u}_{j(a)}) \wedge U_{NA}(\mathsf{v}_a) \mid a \in A\}$$

and hence the sets

$$U_{NA}(\mathsf{u}_a) \qquad U_{NA}(\mathsf{v}_a)$$

(for varying $a \in A$) form a subbase of $\mathcal{O}T$. Observe that these two sets are complementary in T, so both are clopen.

The calculation in the proof of Theorem 1.2 gives

$$\varpi^{\leftarrow}(U_{NA}(\mathsf{u}_a)) = U_A(a)$$

and hence

$$\varpi^{\leftarrow}(U_{NA}(\mathsf{v}_a)) = U_A(a)'$$

which need not be open in S.

There is an obvious, rather crude, way to solve this problem. We furnish S with a new topology by declaring that each member of the old topology $\mathcal{O}S$ should become clopen. Surprisingly, this works.

1.3 DEFINITION. For an arbitrary space S, with topology $\mathcal{O}S$, the front topology $\mathcal{O}^f S$ is the smallest topology on S for which each $U \in \mathcal{O}S$ is clopen.

The front space ${}^f S$ is the set S furnished with $\mathcal{O}^f S$. ■

If you think that this construction looks a bit stupid (because, surely, ${}^f S$ is discrete) don't worry. You are not the first to make that mistake, and you won't be the last. We take a closer look at the construction in Section 2. For now, let's see what we get with it.

1.4 THEOREM. *For a frame A let*

$$S = \mathsf{pt}(A) \qquad T = \mathsf{pt}(NA)$$

where each carries the canonical topology. Then the two assignments

$$\begin{array}{ccc} \ell(\bot) & \longleftarrow\!\!\!\shortmid & \ell \\ {}^f S & \xleftarrow{\ \varepsilon_A\ } & T \\ & \xrightarrow{\ \varpi_A\ } & \\ p & \longmapsto & \mathsf{w}_p \end{array}$$

form an inverse pair of homeomorphisms, where now ${}^f S$ carries the front topology.

Proof. By Theorem 1.2 the two assignments form an inverse pair of bijections, so it suffices to show that each is continuous.

As shown at the end of Section 1 we know that the sets

$$U_{NA}(\mathsf{u}_a) \qquad U_{NA}(\mathsf{v}_a)$$

for $a \in A$ form a subbase of $\mathcal{O}T$. Also from there we have

$$\varpi^{\leftarrow}(U_{NA}(\mathsf{u}_a)) = U_A(a) \in \mathcal{O}^f S \qquad \varpi^{\leftarrow}(U_{NA}(\mathsf{v}_a)) = U_A(a)' \in \mathcal{O}^f S$$

to show that ϖ is continuous. More or less the same calculation shows that

$$\varepsilon^{\leftarrow}(U_A(a)) = U_{NA}(\mathsf{u}_a) \qquad \varepsilon^{\leftarrow}(U_A(a)') = U_{NA}(\mathsf{v}_a)$$

and hence ε is continuous. ■

This result shows that once we know the point space S of a frame A, then we also know the point space of its assembly NA. We may take it to be ${}^fS$. If S is T_1 then ${}^fS$ is discrete (since each singleton becomes clopen). For certain weaker separated space ${}^fS$ need not be discrete.

Observe that we also know the point space of the second assembly N^2A. It is ${}^{ff}S$, the front space of the front space of S. This is always discrete. There after the higher level assemblies $N^3A, N^4A, \ldots$ all have the same point space, namely S with the discrete topology.

What do you think this means?

2 The front space

Each space S carries its nominated topology $\mathcal{O}S$, but the set S may still carry other topologies. In particular, we can enlarge $\mathcal{O}S$ by declaring that we want certain subsets of S to become open, and then generating a new topology. The front topology is a rather crude examples of this idea. This section is a self contained account of the front construction and its various ramifications.

We begin with a repeat of Definition 1.3.

2.1 DEFINITION. For an arbitrary space S, with topology $\mathcal{O}S$, the front topology $\mathcal{O}^fS$ is the smallest topology on S for which each $U \in \mathcal{O}S$ is clopen.

The front space ${}^fS$ is the set S furnished with $\mathcal{O}^fS$. ■

The front topology is sometime called the Skula topology. We will explain the name 'front' after Lemma 2.3.

By construction we have an insertion

$$\mathcal{O}S \hookrightarrow \mathcal{O}^f S$$

for which each $U \in \mathcal{O}S$ becomes complemented (that is, clopen) in $\mathcal{O}^fS$. Thus the insertion solves the complementation problem for $\mathcal{O}S$ (but perhaps not universally). This indicates that in some way the front topology $\mathcal{O}^fS$ is connected with the assembly $N\mathcal{O}S$. We return to this observation in Section 3.

But first, we need some information about $\mathcal{O}^f S$.

The set S carries two topologies

$$\mathcal{O}S \qquad\qquad \mathcal{O}^f S$$

the parent topology on the left and the produced topology on the right. To distinguish between these we speak of

$$\text{open} \quad \text{closed} \qquad\qquad f\text{-open} \quad f\text{-closed}$$

subsets of S. We write

$$(\cdot)^\circ \quad (\cdot)^- \qquad\qquad (\cdot)^\square \quad (\cdot)^=$$

for the corresponding

$$\text{interior} \quad \text{closure} \qquad\qquad f\text{-interior} \quad f\text{-closure}$$

operations on $\mathcal{P}S$.

By construction the family of sets

$$U \cap X \quad (U \in \mathcal{O}S, X \in \mathcal{C}S)$$

forms a base for the topology $\mathcal{O}^f S$. There is a variant of this which is sometimes useful.

2.2 LEMMA. *For each $p \in S$ the family*

$$U \cap p^- \quad (p \in U \in \mathcal{O}S)$$

forms a base for the f-open neighbourhoods of p. In other words, for each

$$p \in E \in \mathcal{O}^f S$$

we have

$$p \in U \cap p^- \subseteq E$$

for some $U \in \mathcal{O}S$.

Proof. For $U \in \mathcal{O}S$ and $X \in \mathcal{C}S$ we have

$$U \cap X = \bigcup \{U \cap p^- \mid p \in U \cap X\}$$

which leads to the required result. ■

A similar idea gives the following.

2.3 LEMMA. *For each subset $E \subseteq S$ we have*

$$p \in E^\square \iff p \in (E \cup p^{-\prime})^\circ \qquad p \in E^= \iff p \in (E \cap p^-)^-$$

for each point $p \in S$.

Proof. Consider any point $p \in E^{\Box}$. By Lemma 2.2 we have

$$p \in U \cap p^{-} \subseteq E$$

for some $U \in \mathcal{O}S$. But now $U \subseteq E \cup p^{-\prime}$ so that

$$p \in U \subseteq (E \cup p^{-\prime})^{\circ}$$

for one implication.

Conversely, if

$$p \in U = (E \cup p^{-\prime})^{\circ}$$

then

$$p \in U \subseteq E \cup p^{-\prime}$$

so that

$$p \in U \cap p^{-} \subseteq E$$

which leads to the converse implication.

The equivalence for $(\cdot)^{=}$ follows by a manipulation of complements. ■

The right hand equivalence is the reason for the name 'front', since to determine whether or not a point p is in a front closure only the points in front of p (that is in p^{-}) need be considered.

At first sight the front space ${}^{f}S$ looks a rather silly idea. Surely, it is just a discrete space. Not quite. Here is an example to indicate that the front construction can have some hidden complexities.

2.4 EXAMPLE. Consider the set $\mathbb{R}$ of reals.

Let $\mathcal{O}_l\mathbb{R}$ be the family of all intervals

$$(-\infty, a)$$

for $a \in \mathbb{R}$ together with $\emptyset$ and $\mathbb{R}$. A few moments thought shows that $\mathcal{O}_l\mathbb{R}$ is a topology on $\mathbb{R}$. Note that this space is T_0 but is not T_1.

The front topology

$$\mathcal{O}_r\mathbb{R} = \mathcal{O}_l{}^{f}\mathbb{R}$$

is generated by all the intervals

$$[a, b)$$

for $a \leq b$ from $\mathbb{R}$. This is T_3, but it is certainly not discrete.

Let $\mathcal{O}_m\mathbb{R}$ be the metric topology on $\mathbb{R}$. This is generated by all he intervals

$$(a, b)$$

for $a \leq b$ from $\mathbb{R}$. Observe that

$$\mathcal{O}_l\mathbb{R} \subseteq \mathcal{O}_m\mathbb{R} \subseteq \mathcal{O}_r\mathbb{R}$$

to suggest that the front topology might not be as simple as it first seems. ■

Given a space S we may form

$$S \quad {}^{f}S \quad {}^{ff}S \quad {}^{fff}S \quad \cdots$$

by repeated use of the front construction. How different can these be?

Suppose S is T_1, that is 'points are closed'. By Lemma 2.2 we see that ${}^{f}S$ is discrete. In fact, there is a slightly weaker separation property which ensures that ${}^{f}S$ is discrete. We look at this in a moment.

By Example 2.4 there is a space S where $S \neq {}^{f}S$ and ${}^{f}S$ is not discrete. For that example we see that ${}^{ff}S$ is discrete, and so $S, {}^{f}S, {}^{ff}S$ can be distinct. We are going to show that this is as far as we can go. For every space S the topology $\mathcal{O}{}^{ff}S$ is boolean and ${}^{fff}S = {}^{ff}S$. In fact, if S is T_0 then ${}^{ff}S$ is discrete.

2.5 DEFINITION. A space S is T_D if for each $p \in S$ we have

$$U \cap p^{-} = \{p\}$$

for some $U \in \mathcal{O}S$. ■

In other words, by Lemma 2.2, a space is T_D precisely when ${}^{f}S$ is discrete. A few moment's thought gives

$$T_1 \Longrightarrow T_D \Longrightarrow T_0$$

and Example 2.4 shows that the right hand implication is not an equivalence. An even simpler example shows that the left hand implication is not an equivalence.

2.6 EXAMPLE. Let S be a partially ordered set (with at least two elements, one above the other). Let

$$\mathcal{O}S = \Upsilon S$$

be the family of all upper section of S. This is a topology, the Alexandroff topology, on S. For each $p \in S$ the principal sections

$$\uparrow p \qquad \downarrow p$$

are, respectively,

open closed

subsets of S. In particular, we have $p^{-} = \downarrow p$ so that S is not T_1 since it has at least one non-closed point. Also

$$\uparrow p \cap p^{-} = \{p\}$$

to show that S is T_D. ■

Notice that siepinski space $\mathsf{2}$ is an example of a T_D space that is not T_1.

To show that ${}^{fff}S = {}^{ff}S$ we need a bit of preparation.

2.7 DEFINITION. For each $p \in S$ let

$$p^{-} = \bigcap\{X \in \mathcal{C}S \mid p \in X\} \qquad p^{\circ} = \bigcap\{U \in \mathcal{O}S \mid p \in U\} \qquad p^{\ominus} = p^{\circ} \cap p^{-}$$

to obtain three sets containing p. Thus p^{-} is just the closure of $\{p\}$, but p° need not be open. We call p° the hull of p, and we call $p^{\ominus}$ the monad of p. ■

Almost trivially we have

$$p \in q^- \iff q \in p^\circ$$

for $p, q \in S$, and a simple exercise shows that the four conditions

$$p^- = q^- \qquad p^\circ = q^\circ \qquad p^\ominus = q^\ominus \qquad p \in q^\ominus$$

are equivalent. Furthermore, we find that

$$\begin{aligned} S \text{ is } T_1 &\iff (\forall p \in S)[p^- = \{p\}] \\ S \text{ is } T_0 &\iff (\forall p \in S)[p^\ominus = \{p\}] \end{aligned}$$

where the first is the standard 'points are closed' characterization of T_1. Later we will use these sets to generalize the notion of an isolated point.

These gadgets are concerned with the parent topology on S. There are similar gadgets for the f-topology. Fortunately, things don't get too complicated.

2.8 LEMMA. *For each point $p \in S$, the monad $p^\ominus$ is the f-closure, the f-hull, and the f-monad of p.*

Proof. For the given point p we first show that $p^\square \subseteq p^=$. To this end consider any $q \in p^\square$ and, by way of contradiction, suppose $q \notin p^=$. We have $q \in p^{='}$ and the set $p^{='}$ is f-open, so that Lemma 2.2 gives

$$q \in U \cap q^- \subseteq p^{='}$$

for some $q \in U \in \mathcal{O}S$. But now

$$p \in p^= \subseteq U' \cup q^{-'}$$

and $U \cup q^{-'}$ is f-clopen, to give

$$q \in p^\square \subseteq U \cup q^{-'}$$

which leads to the required contradiction.

This shows that, in fact, the f-hull $p^\square$ is also the f-monad of p.

Since

$$p^\ominus = p^\circ \cap p^-$$

is an intersection of f-closed sets we have

$$p^\square \subseteq p^= \subseteq p^\ominus$$

for the given point p. It remains to tighten these inclusions.

Using Lemma 2.2 we have

$$p^\square = \bigcap \{U \cap p^- \mid p \in U \in \mathcal{O}S\} = \bigcap \{U \mid p \in U \in \mathcal{O}S\} \cap p^- = p^\circ \cap p^- = p^\ominus$$

for the required result. ■

This result shows that

$$p^= = p^\ominus = p^\square$$

for each point p of a space. Furthermore, $p^=$ is the f-monad of p. This has a consequence for the second front space.

2.9 LEMMA. *Suppose $p^\circ = p^-$ for each point p of a space S. Then ${}^{ff}S = {}^{f}S$.*

Proof. By Lemma 2.8 we have

$$p^= = p^\circ \cap p^- = p^-$$

for each $p \in S$.

Consider any $F \in \mathcal{O}^{ff}S$. We show that F is f-open. To this end consider any $p \in F$. We show that $p \in F^\Box$.

By Lemma 2.2 applied to ${}^fS$ we have

$$p \in E \cap p^= \subseteq F$$

for some $E \in \mathcal{O}^fS$. But now by Lemma 2.2 applied to S we have

$$p \in U \cap p^- \subseteq E$$

for some $U \in \mathcal{O}S$. Since $p^= = p^-$ this gives

$$p \in U \cap p^- \subseteq E \cap p^= \subseteq F$$

and hence $p \in F^\Box$, as required. ■

We now apply this result to ${}^fS$. By Lemma 2.8 we know that the f-hull and the f-closure of a point are the same. Thus we have the following.

2.10 COROLLARY. *For each space S we have ${}^{fff}S = {}^{ff}S$.*

In the remainder of this section we use these ideas to make precise the informal notion of a nearly scattered space. You might think that this has nothing to do with frames and assemblies, but you will see the relevance in Section 5.

Recall that a space is scattered if each of its non-empty closed sets has at least one isolated point. In some ways we want to think of such a space as rather pathological; we can't creep up on an isolated point. However, this official version doesn't quite capture this informal idea.

Consider a large indiscrete space (where 'large' can mean 'having at least two points'). Surely this space is pathological, but it doesn't have any isolated points. Every non-empty open set contains every point.

The problem is that the official version of 'isolated point' tacitly assumes a modicum of separation. Here we re-work the idea to get round this problem.

Why are we doing this here? Because in Section 5 we show that for a space S we have

$$N\mathcal{O}S \text{ is spatial} \iff S \text{ is weakly scattered}$$

where being weakly scattered is an appropriate refinement of being scattered that comes out of our analysis.

2.11 DEFINITION. Let S be a space, let $X \in \mathcal{C}S$ be a closed set of S, and let $p \in X$. We say p is

(i) an isolated point (ii) a detached point (iii) a loose point of X if there is some $U \in \mathcal{O}S$ such that

$$\text{(i)} \quad p \in X \cap U \subseteq \{p\} \qquad \text{(ii)} \quad p \in X \cap U \subseteq p^{\ominus} \qquad \text{(iii)} \quad p \in X \cap U \subseteq p^{-}$$

holds, respectively. Let

$$I(X) \qquad D(X) \qquad L(X)$$

be these respectively sets of points. ■

Trivially we have

$$I(X) \subseteq D(X) \subseteq L(X) \subseteq X$$

and $I(X)$ is just the usual set of isolated points of X.

The first part of the next result is essentially Lemma 1.4 of [6].

2.12 LEMMA. *For each space S and closed set $X \in \mathcal{C}S$ we have*

$$\begin{aligned} p \in I(X) &\iff p \in D(X) \text{ and } p^{\ominus} = \{p\} \\ p \in D(X) &\iff p \in L(X) \text{ and } p^{\ominus} \in O^{f}S \end{aligned}$$

for each point $p \in S$.

Furthermore, if S is T_0 then $L(X)$ is a discrete subspace of X (and S).

Proof. The two equivalences are almost trivial. Let's show that $L = L(X)$ is a discrete subspace when S is T_0.

Consider any $p \in L$. Thus

$$p \in L \cap U \subseteq X \cap U \subseteq p^{-}$$

for some open set U. We show that $L \cap U = \{p\}$. To this end consider any $q \in L \cap U$. Then

$$q \in X \cap V \subseteq q^{-}$$

for some open set V. Since $q \in V \cap p^{-}$ we have $p \in V$, and hence $p \in q^{-}$. This gives $p^{-} = q^{-}$, and hence $p = q$ since S is T_0. ■

Notice that if S is T_0 (so that $p^{\ominus} = \{p\}$) then $I(X) = D(X)$. The slightly stronger separation property T_D (which is still weaker than T_1) ensures that $I(X) = L(X)$. On the whole it does no harm to restrict our attention to T_0 spaces, but we don't want to assume anything stronger.

The next result is an improved version of Lemma 5.1 of [7].

2.13 LEMMA. *For each space S we have*

$$L(X) \text{ is } f\text{-closed} \qquad D(X) \text{ is } f\text{-clopen} \qquad D(X) = L(X)^{\Box}$$

for each closed set $X \in \mathcal{C}S$.

Proof. Fix a closed set $X \in \mathcal{CS}$ and for convenience let $L = L(X)$ and $D = D(X)$.

Consider any $p \in L^{=}$. Thus $p \in (L \cap p^{-})^{-}$, and hence both

$$\text{(i)} \quad p \in L^{-} \subseteq X \qquad \text{(ii)} \quad L \cap p^{-} \neq \emptyset$$

hold. From (ii) there is some $q \in L \cap p^{-}$, and hence

$$\text{(iii)} \quad q \in p^{-} \qquad \text{(iv)} \quad q \in X \cap U \subseteq q^{-}$$

for some open set U. From (iv) we have $q \in U$, and hence (iii) gives $p \in U$, so that (i) gives $p \in X \cap U$. But (iii) gives $q^{-} \subseteq p^{-}$, so that

$$p \in X \cap U \subseteq q^{-} \subseteq p^{-}$$

by (iv), to give $p \in L$. Thus $L^{=} \subseteq L$, as required.

A similar proof shows that D is f-closed. We use $p^{\ominus}$ in place of p^{-}, and (iii) becomes $q \in p^{\ominus}$, to give $q^{\ominus} = p^{\ominus}$.

To complete the proof we show that $D = L^{\square}$.

Consider any $p \in D$. We have

$$p \in X \cap U \subseteq p^{\ominus}$$

for some open set U. For each $q \in X \cap U$ we have $q \in p^{\ominus}$, and hence $q^{\ominus} = p^{\ominus}$, so that

$$q \in X \cap U \subseteq q^{\ominus}$$

and hence $q \in D$. In fact, this shows that

$$p \in X \cap U \subseteq D \subseteq L$$

and hence $p \in L^{\square}$. Since this holds for each $p \in D$, we have $D \subseteq L^{\square}$.

Conversely, consider any $p \in L^{\square}$. We have

$$p \in U \cap p^{-} \subseteq L$$

for some open U, and so

$$p \in p^{\ominus} \subseteq U \cap p^{-} \subseteq L$$

holds. Consider any $q \in U \cap p^{-}$. We have $q \in L$ and hence

$$q \in X \cap V \subseteq q^{-}$$

for some open V. But now $q \in V$ and $q \in p^{-}$, so that $p \in X \cap V \subseteq q^{-}$, to give $q^{-} = p^{-}$, and hence $q^{\ominus} = p^{\ominus}$. In fact, this shows that $p^{\ominus} = U \cap p^{-}$. Finally, since $p \in L$ we have

$$p \in X \cap W \subseteq p^{-}$$

for some open W, and hence

$$p \in X \cap U \cap W \subseteq U \cap p^{-} = p^{\ominus}$$

to show $p \in D$, as required. ■

Using these ideas we can refine the notion of a scattered space. Here we require just one such refinement, but it is worth looking at the general method.

2.14 DEFINITION. Let S be a space, and let

$$p \longmapsto p^{\boxdot}$$

be any operation from points to subsets such that

$$p \in p^{\boxdot} \subseteq p^{-}$$

for each $p \in S$. We say such an operation is suitable

(a) For each closed set $X \in \mathcal{C}S$ we use

$$p \in \boxdot(X) \Longleftrightarrow (\exists U \in \mathcal{O}S)[p \in X \cap U \subseteq p^{\boxdot}]$$

to extract a subset $\boxdot(X)$ of X.

(b) We say S is $\boxdot$-scattered if

$$\boxdot(X) = \emptyset \Longrightarrow X = \emptyset$$

holds for each $X \in \mathcal{C}S$. ■

Notice the restriction

$$p \in p^{\boxdot} \subseteq p^{-}$$

on a suitable operation $(\cdot)^{\boxdot}$. We use this several times.

We have three particular examples

$$p^{\boxdot} \qquad \{p\} \qquad p^{\ominus} \qquad p^{-}$$

of suitable operations, and these give the subsets

$$\boxdot(X) \qquad I(X) \qquad D(X) \qquad L(X)$$

of X respectively. In particular, a space is $\{\cdot\}$-scattered if it is scattered in the usual sense. In [6] the

$$(\cdot)^{\ominus}\text{-scattered} \qquad (\cdot)^{=}\text{-scattered}$$

spaces were termed

dispersed corrupt

respectively. Here we will use the terminology of [3].

2.15 DEFINITION. A space is weakly scattered precisely when it is $(\cdot)^{=}$-scattered, that is when each non-empty closed set has at least one loose point. ■

There is another characterizations of being nearly scattered. To obtain that we use the following observation.

2.16 LEMMA. *Let $(\cdot)^{\boxdot}$ be a suitable operation on the space S. We have*

$$p \in \boxdot(X) \Longleftrightarrow p \in X \text{ and } (\exists U \in \mathcal{O}S)[\emptyset \neq X \cap U \subseteq p^{\boxdot}]$$

for each $X \in \mathcal{C}S$ and $p \in S$.

Proof. The implication $\Rightarrow$ is trivial, so it suffices to show the converse.

Consider $p \in X \in CS$ and suppose

$$q \in X \cap U \subseteq p^{\boxdot}$$

for some $U \in \mathcal{O}S$ and $q \in S$. Since

$$q \in U \cap p^{\boxdot} \subseteq U \cap p^{-}$$

we have $p \in U$. [1] Thus

$$p \in X \cap U \subseteq p^{\boxdot}$$

as required. ■

With this we can obtain a useful characterization of being $\boxdot$-scattered.

2.17 LEMMA. *A space S is $\boxdot$-scattered precisely when*

$$X = \boxdot(X)^{-}$$

for each $X \in \mathcal{C}S$.

Proof. If $X = \boxdot(X)^{-}$ then

$$\boxdot(X) = \emptyset \Longrightarrow X = \emptyset$$

to give us one of the required implications.

For the other suppose S is $\boxdot$-scattered, and consider

$$Y = (X - \boxdot(X)^{-})^{-} = (X \cap \boxdot(X)^{-\prime})^{-}$$

for some $X \in \mathcal{C}S$. We require $Y = \emptyset$.

By way of contradiction, suppose $Y \neq \emptyset$. Since S is $\boxdot$-scattered we have $\boxdot(Y) \neq \emptyset$, to give

$$p \in Y \cap V \subseteq p^{\boxdot}$$

for some $V \in \mathcal{O}S$ and $p \in S$. Remembering that Y is a closure this gives some

$$q \in X \cap \boxdot(X)^{-\prime} \cap V \subseteq p^{\boxdot}$$

where, of course,

$$U = \boxdot(X)^{-\prime} \cap V$$

is open. By Lemma 2.16 this gives $p \in \boxdot(X)$, and hence

$$q \in p^{\boxdot} \subseteq p^{-} \subseteq \boxdot(X)^{-}$$

so that

$$q \in \boxdot(X)^{-} \cap \boxdot(X)^{-\prime}$$

which is the contradiction. ■

Remembering Lemma 2.12 we have the following instance of Lemma 2.17.

[1] The following argument should be done earlier.
If $p \notin U$ then $p \in U'$ so that $p^{-} \subseteq U'$ to give $U \subseteq p^{-\prime}$, and hence $U \cap p^{-} \subseteq p^{-\prime} \cap p^{-} = \emptyset$.

2.18 COROLLARY. *A space S is weakly scattered precisely when $L(X)^- = X$ for each close set X.*

A T_0 space is weakly scattered precisely when each closed set has a discrete dense subset.

I said earlier that the official notion of isolated point tacitly assumes a modicum of separation. I can now make this more precise. It is convenient to compare the notions of scattered and weakly scattered. Thus for a space S these are defined by

$$\begin{array}{ll}(\mathrm{Sct}) & (\forall X \in \mathcal{C}S)[I(X) = \emptyset \Longrightarrow X = \emptyset] \\ (\mathrm{WSc}) & (\forall X \in \mathcal{C}S)[L(X) = \emptyset \Longrightarrow X = \emptyset]\end{array}$$

respectively. Recall also that

$$T_D \quad (\forall p \in S)(\exists U \in \mathcal{O}S)[U \cap p^- = \{p\}]$$

gives the separation property between T_0 and T_1.

We use these in the proof of the following.

2.19 LEMMA. *A space S is scattered precisely when it is T_D and weakly scattered.*

Proof. Suppose first that S is scattered. Since

$$I(X) \subseteq L(X)$$

for each $x \in \mathcal{C}S$, we see that S is weakly scattered. To show that S is T_D consider any $p \in S$ and let $X = p^-$. Since this is non-empty, it has an isolated point, so that

$$U \cap p^- = \{q\}$$

for some $U \in \mathcal{O}S$ and $q \in S$. But now $p \in U$ (for otherwise $p \in U'$ so that $q \in p^- \subseteq U'$, which is not so) and hence

$$p \in U \cap p^- = \{q\}$$

to give $p = q$ for the required result.

Conversely suppose that S is T_D and weakly scattered, and consider any non-empty $X \in \mathcal{C}S$. We must find an isolated point of X. Since S is weakly scattered it certainly has a loose point, so that

$$p \in X \cap U \subseteq p^-$$

for some $p \in X$ and $U \in \mathcal{O}S$. Since S is T_D we have

$$V \cap p^- = \{p\}$$

for some $V \in \mathcal{O}S$. But now

$$p \in X \cap U \cap v \subseteq V \cap p^- = \{p\}$$

to show that p is isolated in X. ■

It is now time to get back to the central topic of these notes.

3 The fundamental triangle

By construction, for each space S the insertion

$$\mathcal{O}S \hookrightarrow \mathcal{O}^f S$$

solves the complementation problem for $\mathcal{O}S$ (but perhaps not universally). Hence, by Theorem 5.4 of [9], there is a commuting triangle

$$\begin{array}{ccc} \mathcal{O}S & \longrightarrow & N\mathcal{O}S \\ & \searrow & \downarrow \sigma_S \\ & & \mathcal{O}^f S \end{array}$$

for some unique frame morphism σ_S. This triangle first appeared in [5]. In this section we describe and analyse the morphism σ_S. Later we will determine when σ_S is an isomorphism.

In due course we show that the following is the morphism we want.

3.1 DEFINITION. Let S be an arbitrary space, For a nucleus $j \in N\mathcal{O}S$ let $\sigma_S(j)$ be the subset of S given by

$$p \in \sigma(j) \Longleftrightarrow p \in j(p^{-\prime})$$

(for $p \in S$). ■

As usual we often write σ for σ_S when there is only one space around. For instance we do that now.

Almost trivially

$$\sigma(\bot_{N\mathcal{O}S}) = \emptyset \qquad \sigma(\top_{N\mathcal{O}S}) = S$$

and

$$\sigma(j \wedge k) = \sigma(j) \wedge \sigma(k)$$

for $j, k \in N\mathcal{O}S$. Thus we see that σ is a $\wedge$-semilattice morphism. To show that it is a frame morphism we exhibit its right adjoint. For this we use the spatially induced nuclei. These are discussed in Section 4 of [8], but we also review the pertinent facts here.

3.2 DEFINITION. For each space S and subset $E \subseteq S$ we set

$$[E](U) = (E \cup U)^{\circ}$$

for each $U \in \mathcal{O}S$, to obtain an nucleus $[E]$ on $\mathcal{O}S$.

The spatially induced nuclei on $\mathcal{O}S$ are those of the form $[E]$ for some $E \subseteq S$. ■

Of course, we need to check that $[E]$ is a nucleus, but that is straight forward and is done in [8]. Remember also why these nuclei are spatially induced.

Each continuous map

$$T \xrightarrow{\ \phi\ } S$$

to the space S induces a frame morphism and its adjoint

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T$$

where

$$\phi^*(U) = \phi^{\leftarrow}(U) \qquad \phi_*(W) = \phi^{\rightarrow}(W')^{-\prime}$$

for each $U \in \mathcal{O}S$ and $V \in \mathcal{O}T$. This morphism has a kernel

$$\phi_* \circ \phi^* = [E]$$

where $E = T - \phi^{\rightarrow}(S)$, the complement of the range of ϕ.

The complements in the formulas above have led to some confusion in the literature, and some rather silly terminology.

The topology $\mathcal{O}S$ always carries its spatially induced nuclei. In general, it can carry many more nuclei, which is one reason why an analysis of the assembly $N\mathcal{O}S$ is useful. There are some spaces S for which every nucleus on $\mathcal{O}S$ is spatially induced. Later, in Section 5 we will give a characterization of these spaces, and we will see that they have a modicum of pathology.

Each subset $E \subseteq S$ gives a spatially induced nucleus, but different subsets can give the same nucleus. This is where the front topology is useful.

3.3 LEMMA. *For each space S we have*

$$[E] \le [F] \Longleftrightarrow E^{\Box} \subseteq F^{\Box}$$

for all $E, F \subseteq S$.

Proof. Suppose $[E] \le [F]$ and consider any $p \in E^{\Box}$. We have

$$p \in (E \cup p^{-\prime})^{\circ} = [E](p^{-\prime}) \subseteq [F](p^{-\prime}) = (F \cup p^{-\prime})^{\circ}$$

and hence $p \in F^{\Box}$, as required.

Conversely, suppose $E^{\Box} \subseteq F^{\Box}$ and consider

$$V = [E](U) = (E \cup U)^{\circ} \subseteq E \cup U$$

so that we required $V \subseteq F \cup U$. But

$$V \cap U' \subseteq E$$

and $V \cap U'$ is f-open to give

$$V \cap U' \subseteq E^{\Box} \subseteq F^{\Box} \subseteq F$$

for the required result. ■

This result has a couple of simple consequences which go back to Lemma 14 of [1].

3.4 COROLLARY. *For each space S we have*

$$[E] = [F] \Longleftrightarrow E^{\Box} = F^{\Box} \qquad [E] = [E^{\Box}]$$

for all $E, F \subseteq S$.

Proof. Two uses of Lemma 3.3 gives the left hand equivalence, and then the observation $E^{\Box} = E^{\Box\Box}$ gives the right hand equality. ■

This last result show that we have an injection

$$\begin{array}{ccc} \mathcal{O}^f S & \longrightarrow & N\mathcal{O}S \\ E & \longmapsto & [E] \end{array}$$

from the front topology of S to the assembly $N\mathcal{O}S$. A simple calculation verifies that

$$[E] \wedge [F] = [E \cap F]$$

for $E, F \in \mathcal{O}^f S$, and trivially we have

$$[S] = \top_{N\mathcal{O}S}$$

to give the following.

3.5 LEMMA. *For each space S the assignment*

$$\mathcal{O}^f S \xrightarrow{\;[\,\cdot\,]\;} N\mathcal{O}S$$

is a $\{\wedge, \top\}$-embedding.

Perhaps you have already guessed that this assignment $[\,\cdot\,]$ is the right adjoint of the morphism σ_S. To prove that we need a bit of preparation.

3.6 LEMMA. *For each space S we have*

$$\sigma_S([E]) = E^{\Box}$$

for each subset $E \subseteq S$.

Proof. Consider any point $p \in \sigma([E])$. We have

$$p \in [E](p^{-\prime})$$

and hence

$$p \in U \subseteq E \cup p^{-\prime}$$

for some $U \in \mathcal{O}S$. This gives

$$p \in U \cap p^{-} \subseteq E$$

and hence $p \in E^{\Box}$.

Conversely, if $p \in E^{\Box}$ then

$$p \in U \cap p^{-} \subseteq E$$

for some $U \in OS$, which leads to $p \in [E](p^{-\prime})$, and hence $p \in \sigma([E])$. ■

With this we can complete the proof that σ_S is a frame morphism, and get a bit more information.

3.7 THEOREM. *For each space S the assignments*

$$NOS \underset{[\,\cdot\,]}{\overset{\sigma_S}{\rightleftarrows}} \mathcal{O}^f S$$

form a frame morphism and its right adjoint, $\sigma_S \dashv [\cdot]$. Furthermore, we have $\sigma_S \circ [\cdot] = \boldsymbol{id}$ and the right adjoint is injective.

Proof. We know that both σ and $[\,\cdot\,]$ are $\wedge$-semilattice morphisms, with $[\,\cdot\,]$ injective, and Lemma 3.6 gives $\sigma \circ [\,\cdot\,] = \boldsymbol{id}$. Thus it remains to show that

$$\sigma(j) \subseteq E \Longleftrightarrow j \leq [E]$$

for each $j \in N\mathcal{O}S$ and $E \in \mathcal{O}^f S$.

Suppose first that $\sigma(j) \subseteq E$. Then, for each $U \in OS$ and $p \in S$

$$\begin{aligned} p \in j(U) &\Longrightarrow p \in j(U) - U \text{ or } p \in U \\ &\Longrightarrow p \in \sigma(j) \subseteq E \text{ or } p \in U \quad \Longrightarrow p \in E \cup U \end{aligned}$$

so that

$$j(U) \subseteq E \cup U$$

and hence

$$j(U) \subseteq [E](U)$$

as required.

Conversely, if $j \leq [E]$, then

$$\sigma(j) \subseteq \sigma([E]) = E^{\square} = E$$

using Lemma 3.6. ■

By Lemma 3.6 we see that the morphism σ_S is surjective but, as we will see, it need not be injective. We characterize when this is the case (and hence σ_S is an isomorphism) in Section 5.

4 The assembly diagram

We have produced various functorial and natural facets. Let's gather these together in one place.

Each frame A induces a diagram in ***Frm*** as shown in Table 1. Here, as usual, we have

$$S = \mathsf{pt}(A) \qquad T = \mathsf{pt}(NA)$$

and all the arrows have been constructed earlier.

We know that cell (1) commutes. This is the functoriality of $N(\cdot)$ and the naturality of $n_\bullet$.

Cell (2) commutes, for this is just the fundamental triangle of S.

We will show that cell (3) commutes. To do that we remember an earlier observation and that n_A is epic,

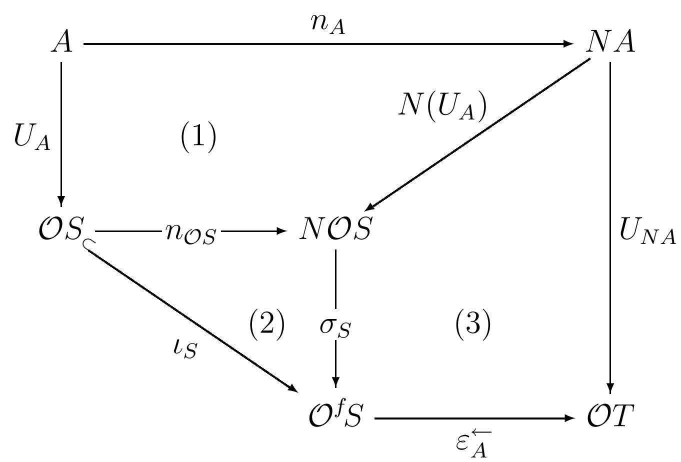
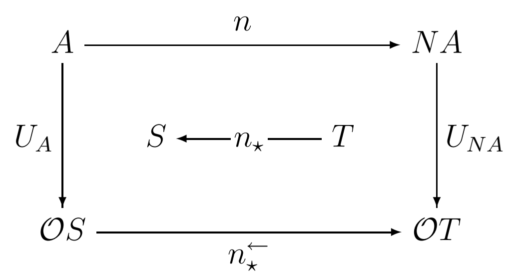


Table 1: The assembly diagram of a frame

Let $n = n_A$. We know that this morphism induces a commuting square



where $n_\star = n_*|T$ with $n \dashv n_*$. What is $n_\star^{\leftarrow}$? Here we must be a bit careful with our current notation. We do *not* have

$$n_\star^{\leftarrow} = \varepsilon_A$$

for ε_A has silently changed its identity! Perhaps you can locate where this happened.

We know that

$$^fS \xleftarrow{\ \varepsilon_A\ } T$$

is a homeomorphism, and so ε_A is also continuous to target S. In other words the continuous map $n_\star$ is

$$S \xleftarrow{\ \varrho_S\ } {^fS} \xleftarrow{\ \varepsilon_A\ } T$$

where ϱ_S is the identity function on S viewed as a continuous map. From this we see that

$$n_\star^{\leftarrow} = \varepsilon_A^{\leftarrow} \circ \varrho_S^{\leftarrow} = \varepsilon_A^{\leftarrow} \circ \iota_S$$

since a few moment's thought gives $\varrho_S^{\leftarrow} = \iota_S$.

This shows that the outer cell of the diagram of of Table 1 commutes.

Using this information we have

$$U_{NA} \circ n_A = \varepsilon_A^{\leftarrow} \circ \iota_S \circ U_A = \varepsilon_A^{\leftarrow} \circ \sigma_S \circ n_{\mathcal{O}S} \circ U_A = \varepsilon_A^{\leftarrow} \circ \sigma_S \circ N(U_A) \circ n_A$$

and hence

$$U_{NA} = \varepsilon_A^{\leftarrow} \circ \sigma_S \circ N(U_A)$$

since n_A is epic.

It can be checked that this whole diagram is natural form variation of A along a frame morphism. Further information about this can be found in Section 4 of [4]

5 Totally spatial frames

Each frame A has a point space $S = \mathsf{pt}(A)$ together with a surjective frame morphism

$$A \xrightarrow{U_A} \mathcal{O}S$$

indexing the topology. As with any frame morphisms, this morphism has a kernel $\mathfrak{s} \in NA$ which, by Lemma 1.9 of [10] in this case, is given by [2]

$$\mathfrak{s}(a) = \bigwedge\{p \in S \mid a \le p\}$$

for each $\in A$. This assumes that we view the points of S as the $\wedge$-irreducible elements of A, and the infimum is computed in A. Consider any $j \in NA$. This gives us a pair of quotients

$$A \xrightarrow{j^*} A_j \xrightarrow{U_j} \mathcal{O}S_j$$

where $S_j = \mathsf{pt}(A_j)$ and U_j is the associated indexing morphism. We wish to describe the kernel $\mathfrak{s}_j \in NA$ of this composite morphism.

To do that we must first locate S_j, the set of points of A_j. Remember that each such point is determined by a character

$$A_j \longrightarrow \mathsf{2}$$

of A_j, and this gives a character

$$A \xrightarrow{j^*} A_j \longrightarrow \mathsf{2}$$

of A. Thus $S_j \subseteq S$. This gives us some information about A_j, but, as usual, it is easier to deal with $\wedge$-irreducible elements.

5.1 LEMMA. *For each nucleus $j \in NA$ on a frame A, we have*

$$S_j = S \cap A_j$$

where $S = \mathsf{pt}(A)$ and $S_j = \mathsf{pt}(A_j)$.

[2]The earlier notation of [10] should be changed to match this better one.

Proof. Consider $p \in S_j$. Thus $p \in A_j$ and is $\wedge$-irreducible in A_j. But now, for $x, y \in A$, we have

$$x \wedge y \leq p \Longrightarrow j(x) \wedge j(y) \leq j(p) = p \Longrightarrow x \leq j(x) \leq p \text{ or } y \leq j(y) \leq p$$

to show that p is $\wedge$-irreducible in A. Thus

$$S_j \subseteq S \cap A_j$$

and the converse inclusion is even easier. ■

With this we can describe the kernel of the composite map.

5.2 LEMMA. *For each frame A and $j \in NA$, the kernel $\mathfrak{s}_j$ of the morphism*

$$A \xrightarrow{\;j^*\;} A_j \xrightarrow{\;U_j\;} \mathcal{O}S_j$$

(where $S_j = \mathsf{pt}(A_j)$) is given by

$$\mathfrak{s}_j(a) = \bigwedge\{p \mid p \in S \cap A_l \text{ and } a \leq p\}$$

for each $a \in A$.

Proof. The composite morphism is

$$\begin{array}{ccccc} A & \xrightarrow{\;j^*\;} & A_j & \xrightarrow{\;U_j\;} & \mathcal{O}S_j \\ a & \longmapsto & j(a) & \longmapsto & U_j(j(a)) \end{array}$$

where U_j is the indexing map of A_j. By Corollary 3.13 of [8] we have

$$x \leq \mathfrak{s}_j(a) \Longleftrightarrow U_j(j(x)) \subseteq U_j(j(a))$$

for $a, x \in A$. This right hand side unravels to

$$(\forall p \in S_j)[j(x) \not\leq p \Longrightarrow j(a) \not\leq p]$$

that is

$$(\forall p \in S \cap A_j)[j(a) \leq p \Longrightarrow j(x) \leq p]$$

using Lemma 5.1. For $p \in A_j$ we have

$$x \leq p \Longrightarrow j(x) \leq j(p) = p$$

so that

$$(\forall p \in S \cap A_j)[a \leq p \Longrightarrow x \leq p]$$

is a further unravelling of the right hand side. This latest version is just

$$x \leq \bigwedge\{p \mid p \in S \cap A_j \text{ and } a \leq p\}$$

which gives the required result. ■

Notice that when $j = \boldsymbol{id}_A$ we have $\mathfrak{s}_{\boldsymbol{id}_A} = \mathfrak{s}$ where $\mathfrak{s}$ is the nucleus described at the beginning of this section.

5.3 DEFINITION. A frame A is **totally spatial** if A_j is spatial for each $j \in NA$. ■

By considering the particular case $j = \boldsymbol{id}_A$ we see that each totally spatial frame is spatial. What kind of spaces arise in this way. In due course we will see that they are precisely the weakly scattered spaces of Definition 2.15. These spaces also have a more fundamental property.

Each frame A has an assembly NA which has a point space

$$NA \longrightarrow \mathcal{O}\mathsf{pt}(NA)$$

which has a kernel. This is some second level nucleus $\mathfrak{S} \in N^2A$. We know that

$$\mathsf{pt}(NA) = \{\mathsf{w}_p \mid p \in S\}$$

where $S = \mathsf{pt}(A)$. Thus, using the description of $\mathfrak{s}$ applied to NA, we have

$$\mathfrak{S}(j) = \bigwedge\{\mathsf{w}_p \mid p \in S \text{ and } j \leq \mathsf{w}_p\}$$

for each $j \in NA$. This infimum is computed in NA. In other words, it is computed pointwise.

We carry these various notations into the next result.

5.4 THEOREM. *For each frame A and nucleus $j \in NA$ we have*

$$\mathfrak{S}(j) = \mathfrak{s}_j$$

where $\mathfrak{S} \in N^2A$ and $\mathfrak{s}_j \in NA$ are as given above.

Proof. For each $j \in NA$ and $a \in A$ we have

$$\begin{aligned} \mathfrak{S}(j)(a) &= \bigwedge\{\mathsf{w}_p(a) \mid p \in S \qquad \text{and } j \leq \mathsf{w}_p\} \\ \mathfrak{s}_j(a) &= \bigwedge\{\quad p \quad \mid p \in S \cap A_j \text{ and } a \leq p\} \end{aligned}$$

so it suffices to compare the two right hand sides. We simplify the right hand side for $\mathfrak{S}(j)(a)$.

First of all we have

$$j \leq \mathsf{w}_p \Longleftrightarrow j(p) = p \Longleftrightarrow p \in A_j$$

so that

$$\mathfrak{S}(j)(a) = \bigwedge\{\mathsf{w}_p(a) \mid p \in S \cap A_j\}$$

for the first step of the simplification.

Secondly, as in the proof of Lemma 1.1, we have

$$\mathsf{w}_p(a) = \begin{cases} \top \text{ if } a \nleq p \\ p \text{ if } a \leq p \end{cases}$$

so we may omit those $p \in S \cap A_j$ where $a \nleq p$. Thus

$$\mathfrak{S}(j)(a) = \bigwedge\{p \mid p \in S \cap A_j \text{ and } a \leq p\}$$

as required. ■

With this we can obtain the equivalence (1) ⇔ (2) of Theorem 3.4 of [3]. We look at the other equivalents of that result in Section 6.

5.5 THEOREM. *A frame has a spatial assembly precisely when it is totally spatial.*

Proof. A frame A has a spatial assembly precisely when the associated second level nucleus $\mathfrak{S}$ is $\boldsymbol{id}_{NA}$. By Theorem 5.4 this occurs precisely when

$$\mathfrak{s}_j = \mathfrak{S}(j) = j$$

for each $j \in NA$. In other words, when each quotient A_j is spatial. ■

In Theorem 5.8 below we look at a different proof of this result which gives us different information about the situation.

Consider an arbitrary space S. The quotient morphism

$$\mathcal{O}S \xrightarrow{\quad \sigma_S \quad} \mathcal{O}^f S$$

is canonically equivalent to the spatial reflection

$$N\mathcal{O}S \xrightarrow{U_{N\mathcal{O}S}} \mathcal{O}(\mathsf{pt}(N\mathcal{O}S))$$

of the assembly $N\mathcal{O}S$! This can be seen by considering the particular case $A = \mathcal{O}S$ in the assembly diagram Table 1. For this case the cell (1) collapses since U_A is just equality. You have to be a bit careful with this. The S there is not the S here. There it is the point space of $\mathcal{O}S$ for our S, in other words the *sober reflection* of our space S.

This shows that the assembly $N\mathcal{O}S$ is spatial precisely when the morphism σ_S is injective, and hence is an isomorphism. How can we test for that condition? For any frame A each nucleus is an infimum of w_a-nuclei. That is true for $A = \mathcal{O}S$, so perhaps we should see what σ_S does to these particular nuclei. This is where the loose points of a closed set come into play.

The next result first appeared as Theorem 6.1 of [7].

5.6 LEMMA. *For each space S and closed set $X \in \mathcal{C}S$ we have*

$$\sigma_S(\mathsf{w}_E) = L(X)'$$

where $E = X'$.

Proof. Consider any point $p \in S$. We show that

$$(i)\quad p \in L(X) \qquad\qquad (ii)\quad p \in X \cap U(p) \qquad\qquad (iii)\quad p \notin \sigma_S(\mathsf{w}_E)$$

are equivalent where

$$U(p) = (E \cup p^-)^\circ$$

is the open set attached to p.

(i)⇒(ii). Assuming (i) we have

$$p \in X \cap V \subseteq p^-$$

for some $V \in \mathcal{O}S$. The right hand inclusion gives

$$V \subseteq E \cup p^-$$

so that $V \subseteq U(p)$, and hence

$$p \in X \cap V \subseteq X \cap U(p)$$

as required.

(ii)⇒(iii). By the definition of σ_S we have

$$p \in \sigma_S(\mathsf{w}_E) \Longleftrightarrow p \in \mathsf{w}_E(p^{-\prime})$$

so it suffices to unravel the right hand side.

We have

$$(p^{-\prime} \supset E) = (p^{-} \cup)^{\circ} = U(p)$$

so that

$$\mathsf{w}_E(p^{-\prime})' = (U(p) \supset E)' = (U(p)' \cup E)^{\circ\prime} = (X \cap U(p))^{-}$$

from which we obtain the required implication.

(iii)⇒(i). Assuming (iii) we have

$$p \in (X \cap U(p))^{-}$$

by the calculation of the previous part. In particular

$$X \cap U(p) \neq \emptyset$$

so that

$$\emptyset \neq X \cap U(p) = X \cap (E \cup p^{-}) = X \cap p^{-} \subseteq p^{-}$$

to give $x \in L(X)$ by Lemma 2.16 applied to the particular case $(\cdot)^{\boxdot} = (\cdot)^{-}$. ■

With this we can prove the crucial result, which is an improved version of Lemma 4.1 of [6].

5.7 LEMMA. *For each space S and closed set $X \in \mathcal{C}S$, the three conditions*

(i) w_E *is spatially induced* *(ii)* $X = L^{-}$ *(iii)* $\mathsf{w}_E = [L']$

are equivalent, where $E = X'$ and $L = L(X)$.

Furthermore, if the base space S is T_0 then these hold precisely when X has a discrete dense subspace.

Proof. We first prove the three implications (i)⇒(ii)⇒(iii)⇒(i) in turn.

(i)⇒(ii). Assuming (i) we have

$$\mathsf{w}_E = [F]$$

for some $F \in \mathcal{O}^f S$. This gives

$$E = \mathsf{w}_E(\emptyset) = [F](\emptyset) = F^{\circ}$$

and Lemma 5.6 gives

$$L' = \sigma_S(\mathsf{w}_E) = \sigma_S([F]) = F$$

so that

$$X = F^{\circ\prime} = L^{\prime\circ\prime} = L^{-}$$

as required.

(ii)⇒(iii). On general grounds we have

$$\mathsf{w}_E \leq ([\,\cdot\,] \circ \sigma_S)(\mathsf{w}_E) = [L']$$

using Lemma 5.6. We have

$$[L'](E) = (L' \cup E)^{\circ} = (L \cap X)^{-\prime} = L^{-\prime}$$

since $L \subseteq X$, so that (ii) gives

$$[L'](E) = X' = E$$

and hence

$$[L'] \leq \mathsf{w}_E$$

by a use of Lemma 3.10 of [9].

(iii)⇒(i). This is trivial.

Now suppose that S is T_0.

By Lemma 2.12 we know that L is a discrete subspace of X. Thus if $X = L^{-}$ then L is the discrete dense subspace we are after.

Conversely, suppose M is a discrete dense subspace of X. Since $M^{-} = X$, it suffices to show that $M \subseteq L$. Consider any $p \in M$. Since M is discrete we have

$$M \cap U = \{p\}$$

for some open set U. We show that $X \cap U \subseteq p^{-}$. Thus, consider any $q \in X \cap U$, and consider any open set V with $q \in V$. Since $q \in M^{-} \cap U \cap V$ we have some point

$$r \in M \cap U \cap V \subseteq \{p\}$$

and hence $p \in V$ (since $r = p$). This show that $q \in p^{-}$, as required. ■

With these preliminaries we can prove the main result, which first appeared as Theorem 4.4 of [6].

5.8 THEOREM. *For each space S the three conditions*

(i) The canonical morphism σ_S is monic (and hence is an isomorphism).

(ii) Each nucleus on $\mathcal{O}S$ is spatially induced.

(iii) Each closed set of S is the closure of its set of loose points, that is S is weakly scattered.

are equivalent.

Furthermore, when S is T_0 these occur precisely when each closed set has a discrete dense subset.

Proof. We first prove the three implications (i)⇒(ii)⇒(iii)⇒(i) in turn.

(i)⇒(ii). Assuming (i) consider any nucleus $j \in N\mathcal{O}S$ and let $E = \sigma_S(j)$. Then $E \in \mathcal{O}^f S$ and Lemma 3.6 gives

$$\sigma_S([E]) = E = \sigma_S(j)$$

and hence $j = [E]$, by (i).

(ii)⇒(iii). This is an immediate consequence of Lemma 5.7.

(iii)⇒(i). Assuming (iii) consider a distinct pair j, k of nuclei. By symmetry we may suppose $j \not\leq k$. By a standard representation thus gives some $A \in \mathcal{O}S$ with

$$j \leq w_A \qquad k \not\leq w_A$$

and hence, by (iii) and Lemma 5.7, we have

$$j \leq [E] \qquad k \not\leq [E]$$

for some $E \in \mathcal{O}^f S$. The adjunction properties (of Theorem 3.7) now give

$$\sigma(j) \subseteq E \qquad \sigma(k) \not\subseteq E$$

and hence $\sigma(j) \neq \sigma(k)$, as required.

The remainder of the required result is a direct consequence of Lemma 5.7. ■

Some ten years after the appearance of [6] and [7] the characterization

$$S \text{ is sober and } NOS \text{ is spatial} \iff \text{each closed set has a discrete dense subset}$$

for T_0 spaces S was produced by Isbell (as Theorem 7 of [2]). The refinements of that results have been incorporated into Theorem 5.8.

What is needed now is a collection of examples of various spaces and their assemblies. These can be found in [13].

6 Essential points

As said in the Preamble, many of the results in this set of notes are taken from [3] or [6]. There is some overlap between those two papers, but that is no bad thing for they look at the topic from different points of view. However, there is one aspect of [3] that is missing from [6]. That is the relationship with spectra of commutative rings.

A decent account of this topic would take us too far away from the main theme of this set of notes. There has been quite a lot of work done in this area since [3], and at the time of writing I am not familiar with all the relevant material. But, just as an appetizer, let's look at the results in [3].

Before we get to the pertinent aspect we need a bit of notation.

6.1 DEFINITION. Let A be a frame with point space $S = \mathsf{pt}(A)$ viewed as the set of $\wedge$-irreducible elements.

For each $a \in A$ we let

$$\mathcal{P}(a) = \{p \in S \mid a \leq p\}$$

be the set of points above a, and we let

$$\mathcal{M}(a) = \min \mathcal{P}(a)$$

be the set of minimal members of $\mathcal{P}(a)$. ■

The construction of $\mathcal{M}(a)$ needs a bit more explanation.

We have $\mathcal{P}(a) \subseteq A$, and so $\mathcal{P}(a)$ can be viewed as a poset with the comparison inherited from A. We extract the minimal members of that poset. Thus $p \in \mathcal{M}(a)$ if $p \in \mathcal{P}(a)$ and

$$q \leq p \Longrightarrow q = p$$

for all $q \in \mathcal{P}(a)$.

Note that $p \in \mathcal{M}(a)$ need not be minimal in A, only in the smaller poset $\mathcal{P}(a)$. Note also that we may view $\mathcal{P}(a)$ as a subset of S. It then carries a second comparison, the restriction of the specialization order on S. This is just the reverse of the comparison inherited from A.

We know that the kernel $\mathfrak{s}$ of the canonical morphism

$$A \xrightarrow{U_A} \mathcal{O}S$$

is given by

$$\mathfrak{s}(a) = \bigwedge \mathcal{P}(a)$$

for each $a \in A$. This infimum is computed in A. Is it necessary to use the whole of $\mathcal{P}(a)$ to obtain $\mathfrak{s}(a)$? Clearly not, for if we have $p, q \in \mathcal{P}(a)$ with $q \leq p$, then we can forget about p. In fact, as we are going to show, we can forget about every point not in $\mathcal{M}(a)$. For this we need a preliminary observation.

6.2 LEMMA. *For a frame A let $\mathcal{X}$ be a downwards directed subset of the point space $S = \mathsf{pt}(A)$. Then $\bigwedge \mathcal{X} \in S$.*

Proof. Recall that $\mathcal{X}$ is downwards directed if it is non-empty and if for each $p, q \in \mathcal{X}$ there is some $r \in \mathcal{X}$ with $r \leq p, q$.

In particular $\bigwedge \mathcal{X} \neq \top$ (since $\mathcal{X}$ is non-empty). By way of contradiction suppose

$$x \wedge y \leq \bigwedge \mathcal{X} \qquad x \nleq \bigwedge \mathcal{X} \qquad y \nleq \bigwedge \mathcal{X}$$

for some $x, y \in A$. The second and third of these give

$$x \nleq p \qquad y \nleq q$$

for some $p, q \in \mathcal{X}$. But now there is some $r \in \mathcal{X}$ with $r \leq p, q$, so that

$$x \wedge y \leq r$$

and hence either

$$x \leq r \leq p \quad \text{or} \quad y \leq r \leq q$$

which is the contradiction. ■

Recall that ZL (Zorn's Lemma) says that if P is a poset for which each upwards directed subset has an upper bound, then each member of P lies below a maximal member. By turning $\mathcal{P}(a)$ upside-down we obtain the following.

6.3 COROLLARY. *For each frame A, element $a \in A$, and point $p \in \mathcal{P}(a)$, there is some $q \in \mathcal{M}(a)$ with $q \leq p$.*

This has an immediate consequence about the nature of the kernel $\mathfrak{s}$.

6.4 COROLLARY. *For each frame A we have*

$$\mathfrak{s}(a) = \bigwedge \mathcal{M}(a)$$

for each $a \in A$.

This shows that when computing $\mathfrak{s}(a)$ the points in $\mathcal{P}(a) - \mathcal{M}(a)$ are not needed. But are all the points in $\mathcal{M}(a)$ needed? We are going to analyse this question.

6.5 DEFINITION. For a frame a, element $a \in A$, and point $p \in \mathcal{P}(a)$, we say p is essential over a if

$$\mathfrak{s}(a) \neq \bigwedge(\mathcal{P}(a) = \{p\})$$

holds. We let $\mathcal{E}(a)$ be the set of points that are essential over a. ■

By Corollary 6.3 we have

$$\mathcal{E}(a) \subseteq \mathcal{M}(a) \subseteq \mathcal{P}(a)$$

for each $a \in A$. Is it the case that $\mathcal{E}(a) = \mathcal{M}(a)$? We will show that $\mathcal{E}(a) = \emptyset \neq \mathcal{M}(a)$ can happen.

You should compare the notion of Definition 6.5 with that of an essential prime of [3]. Here every element $a \in A$ has an associated set $\mathcal{E}(a)$, whereas there an element $a \in A$ has an associated set $\mathrm{Ess}(a)$ *only* if $\mathfrak{s}(a) = a$. Of course, we always have

$$\mathcal{E}(a) = \mathcal{E}(\mathfrak{s}(a)) = \mathrm{Ess}(\mathfrak{s}(a))$$

and $\mathcal{E}(a) = \mathrm{Ess}(a)$ when $\mathfrak{s}(a) = a$. In this account I have made explicit this hidden spatiality condition.

Keeping this in mind we see that the following is a slight generalization of Proposition 2.3 of [3].

6.6 LEMMA. *For each frame A, element $a \in A$, and point $p \in \mathcal{P}(a)$ the four conditions*

(i) The point p is essential over a, that is $p \in \mathcal{E}(a)$.

(ii) We have

$$z \wedge p \leq \mathfrak{s}(a) \qquad z \not\leq \mathfrak{s}(a)$$

for some $z \in A$.

(iii) We have

$$j(\mathfrak{s}(a)) = \mathfrak{s}(a) \Longrightarrow j(p) = p$$

for all $j \in NA$.

(iv) The point p is a maximal element of the boolean quotient $A_{\mathsf{w}_{s(a)}}$.

are equivalent.

Proof. We first show the equivalence

$$(i) \iff (ii)$$

and then deal with (iii) and (iv).

(i)⇒(ii). Assuming (i) let

$$z = \bigwedge\{\mathcal{P}(a) - \{p\}\}$$

so that

$$z \wedge p = \bigwedge\mathcal{P}(a) = \mathfrak{s}(a)$$

and $z \not\leq \mathfrak{s}(a)$ since $p \in \mathcal{E}(a)$.

(ii)⇒(i). Assuming (i) let

$$y = \bigwedge\{\mathcal{M}(a) - \{p\}\}$$

so that $\mathfrak{s}(a) \leq y$ and we require $y \neq \mathfrak{s}(a)$. By way of contradiction suppose $y = \mathfrak{s}(a)$. For each

$$q \in \mathcal{M}(a) - \{p\}$$

using the separating element from (ii) we have

$$z \wedge p \leq \mathfrak{s}(a) = y \leq q$$

so that one of

$$z \leq q \qquad p \leq q$$

holds. The second of these with the minimality of q gives $q = p$, and this has been excluded. Thus we have $z \leq q$ for all $q \in \mathcal{M}(a) - \{p\}$. This gives

$$z \leq \bigwedge\{\mathcal{M}(a) - \{p\}\} = y = \mathfrak{s}(a)$$

which is the contradiction.

(ii)⇒(iii). Assuming (ii) consider any $j \in NA$ with $j(\mathfrak{s}(a)) = \mathfrak{s}(a)$. Then, using the separating condition, we have

$$j(z) \wedge j(p) = j(z \wedge p) \leq j(\mathfrak{s}(a)) = \mathfrak{s}(a) \leq p$$

so that one of

$$z \leq j(z) \leq p \qquad j(p \leq p$$

holds. The first gives

$$z = z \wedge p \leq \mathfrak{s}(a)$$

which is not so. Thus $j(p) = p$, as required.

(iii)⇒(iv). Assuming (iii), consider $j = \mathsf{w}_{\mathfrak{s}(a)}$. Then

$$j(\mathfrak{s}(a) = \mathfrak{s}(a)$$

so that $j(p) = p$, which leads to the required result.

(iv)⇒(ii). Assuming (iv) let

$$z = (p \supset \mathfrak{s}(a))$$

so that

$$z \wedge p \le \mathfrak{s}(a)$$

which is one half of the required separation. For the other half observe that if $z \le \mathfrak{s}(a)$ then (iv) gives

$$p = \mathsf{w}_{\mathfrak{s}(a)} = (z \supset \mathfrak{s}(a)) = \top$$

which is not so. ■

The important part of this result is the equivalence

$$(i) \Longleftrightarrow (ii)$$

which gives a characterization of essentiality in terms of a separation property. However, as we will see, the equivalent (iv) is not without interest.

We now move towards a version of Theorem 3.4 of [3]. As indicated above, throughout that result there is a hidden assumption of spatiality. Here we make that explicit. To do that we make a preliminary observation.

The non-standard numbering of the clauses in the following result will help in the proof of Theorem 6.8 later.

6.7 LEMMA. *For each frame A the three conditions*

(ii) $(\forall a \in A)[A_{\mathsf{w}_{\mathfrak{s}(a)}} \text{ is atomless} \Longrightarrow \mathfrak{s}(a) = \top]$

(iii) $(\forall a \in A)[\mathcal{E}(a) = \emptyset \Longrightarrow \mathfrak{s}(a) = \top]$

(iv) $(\forall a \in A)[\mathfrak{s}(a) = \bigwedge \mathcal{E}(a)]$

are equivalent.

Proof. (ii)⇔(iii). The atoms of a boolean algebra are in bijective correspondence with its maximal elements, hence the equivalence $(i) \Leftrightarrow (iv)$ of Lemma 6.6 gives

$$\mathcal{E}(a) = \emptyset \Longleftrightarrow A_{\mathsf{w}_{\mathfrak{s}(a)}} \text{ is atomless}$$

for each $a \in A$. From this we see that conditions (ii) and (iii) are little more than rephrasings of each other.

(iii)⇒(iv). Assuming (iii), consider an $a \in A$, let

$$b = \bigwedge \mathcal{E}(a) \qquad c = (b \supset \mathfrak{s}(a))$$

so that $\mathfrak{s}(a) \le b$ and we require $c = \top$.

Observe that

$$c \wedge b \le \mathfrak{s}(a)$$

so that

$$\mathfrak{s}(c) \wedge b \le \mathfrak{s}(c \wedge b) \le \mathfrak{s}^2(a) = \mathfrak{s}(a)$$

and hence

$$\mathfrak{s}(c) \le (b \supset \mathfrak{s}(a)) = c$$

that is

$$\mathfrak{s}(c) = c$$

and we require $\mathfrak{s}(c) = \top$.

By way of contradiction suppose $\mathfrak{s}(c) \neq \top$. Then (ii) gives $\mathcal{E}(c) \neq \emptyset$, and so provides some $p \in \mathcal{E}(c)$, and hence a separation

$$z \wedge p \leq \mathfrak{s}(c) = c \qquad z \not\leq \mathfrak{s}(c) = c$$

by Lemma 6.6. This gives

$$z \wedge b \wedge p \leq \mathfrak{s}(a) \qquad z \wedge b \not\leq \mathfrak{s}(a)$$

to witness that $p \in \mathcal{E}(a)$, again by Lemma 6.6. But now $b \leq p$, so that

$$z \wedge b = z \wedge b \wedge p \leq \mathfrak{s}(a)$$

which is the contradiction.

(iv)⇒(iii). Assuming (iv), for each $a \in A$ we have

$$\mathcal{E}(a) = \emptyset \Longrightarrow \mathfrak{s}(a) = \bigwedge \mathcal{E}(a) = \bigwedge \emptyset = \top$$

for the required result. ■

The equivalent conditions of this result do not ensure spatiality. To see that consider any non-trivial frame A with an empty point space. Then $\mathcal{E}(a) = \emptyset$ for each element a, and

$$\mathfrak{s}(a) = \bigwedge \emptyset = \top$$

to show that (ii,ii,v) hold.

We can now obtain a version of Theorem 3.4 of [3]. In this I have made explicit the spatiality requirements.

6.8 THEOREM. *For each frame A the four conditions*

(i) The frame A is totally spatial.

(ii) The frame A is spatial and

$$A_{\mathsf{w}_a} \text{ is atomless} \Longrightarrow a = \top$$

for each $a \in A$.

(iii) The frame A is spatial and

$$\mathcal{E}(a) = \emptyset \Longrightarrow a = \top$$

for each $a \in A$.

(iv) We have

$$a = \bigwedge \mathcal{E}(a)$$

for each $a \in A$.

(v) The assembly NA is spatial.

are equivalent.

Proof. (i)⇒(ii). Assuming (i), the frame is spatial, by considering the quotient $A = A_{\boldsymbol{id}}$, so it suffices to verify the implication. In fact, we verify the contrapositive.

Thus consider any $a \in A$ with $a \neq \top$. Then $\mathsf{w}_a \neq \top_N$, and hence the quotient A_{w_a} is non-trivial. By (i) this gives a point of A_{w_a}, and hence also an atom.

(ii)⇔(iii). Going in either direction we have $\mathfrak{s} = \boldsymbol{id}$ since the frame is spatial. Thus, be the equivalence (i)⇔(iv) of Lemma 6.6 we have

$$\mathcal{E}(a) = \emptyset \Longleftrightarrow A_{\mathsf{w}_a} \text{ is atomless}$$

for each $a \in A$. From this we see that conditions (ii) and (iii) are little more than rephrasings of each other.

(iii)⇒(iv). Assuming (iii) we have $\mathfrak{s} = \boldsymbol{id}$ since the frame is spatial. The rest of (ii) gives

$$(\forall a \in A)[\mathcal{E}(a) = \emptyset \Longrightarrow \mathfrak{s}(a) = a = \top]$$

and hence

$$(\forall a \in A)[a = \mathfrak{s}(a) = \bigwedge \mathcal{E}(a)]$$

by Lemma 6.7.

(iv)⇒(v). Assuming (iv), for each $a \in A$ we have

$$\mathfrak{s}(a) = \bigwedge \mathcal{M}(a) \leq \bigwedge \mathcal{M}(a) = a$$

to show that A is spatial.

To show that NA is spatial consider any $j \in NA$ and let

$$k = \mathfrak{S}(j) = \bigwedge \{\mathsf{w}_p \,|\, p \in S, j(p) = p\}$$

where $S = \mathsf{pt}(A)$. We have $j \leq k$, and we require $j = k$.

By way of contradiction suppose $k \not\leq j$, This gives some $a \in A$ with

$$k(A) \not\leq j(a)$$

and, by replacing a be $j(a)$, we may suppose $j(a) = a$. By (iv) we have

$$k(a) \not\leq \bigwedge \mathcal{E}(a)$$

to give

$$k(a) \not\leq p$$

for some $p \in \mathcal{E}(a)$. By the separation property of Lemma 6.6 we have

$$z \wedge p \leq \mathfrak{s}(a) = a \qquad x \not\leq \mathfrak{s}(a) = a$$

for some $z \in A$. The first of these gives

$$j(z) \wedge j(p) \leq j(a) = a \leq p$$

so that one of

$$z \le j(z) \le p \qquad j(p) = p$$

holds. This left hand alternative can not hold, for otherwise

$$z = z \wedge p \le a$$

which is not so.

We thus have some $p \in S$ with $a \le p = j(p)$, and hence

$$k(a) \le k(p) \le \mathsf{w}_p(p) = p$$

which is the contradiction.

(v)⇒(i). This follows as in Theorem 5.5. ■

The implication of part (iii) of this result is reminiscent of the characterization

$$(\forall x \in \mathcal{C}S)[L(X) = \emptyset \Longrightarrow X = \emptyset]$$

of weakly scattered spaces. This is not a coincident, as Lemma 3.6 of [3] shows.

6.9 LEMMA. *For each space S we have*

$$p \in L(X) \Longleftrightarrow p^{-\prime} \in \mathcal{E}(X')$$

for each $X \in \mathcal{C}S$ and $p \in S$.

Proof We are concerned with the space S and its associated frame $\mathcal{O}S$. Each point $p \in S$ gives a point $p^{-\prime}$ of $\mathcal{O}S$. Consider any $X \in \mathcal{C}S$. We look at the corresponding member X' of $\mathcal{O}S$.

Note that since $\mathcal{O}S$ is spatial, we have $\mathfrak{s}(X') = X'$.

By separation characterization of Lemma 6.6 we have

$$p^{-\prime} \in \mathcal{E}(X')$$

precisely

$$X' \subseteq p^{-\prime} \quad \text{and} \quad (\exists W \in \mathcal{O}S)[W \cap p^{-\prime} \subseteq X' \text{ and } W \nsubseteq X']$$

that is

$$p^{-} \subseteq X \quad \text{and} \quad (\exists W \in \mathcal{O}S)[X \cap W \cap p^{-\prime} = \emptyset \text{ and } X \cap W \neq \emptyset]$$

which is

$$p \in X \quad \text{and} \quad (\exists W \in \mathcal{O}S)[\emptyset \neq X \cap W \subseteq p^{-}]$$

and this is equivalent to

$$p \in L(X)$$

by the appropriate case of Lemma 2.16. ■

The final section of [3] gives us a glimpse of the connection between frames and rings. It is just a glimpse, for there is far more than could be said there or here. For instance, the analysis of ranking techniques and decomposition theories for torsion theories. But that's for another day.

7 Regular algebras

The results of this final section are derived from various observations in [6]. I'm not sure where these fit in the general scheme of things, but here seems to be as good a place as any.

We begin with a general observation and then look at a particular case. [3]

Consider an arbitrary frame morphism

$$A \xrightarrow{\ f\ } B$$

together with nuclei $j \in NAS$ and $k \in NB$ on the source and target. This gives us a 3-sided diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle j^*}\big\downarrow & & \big\downarrow{\scriptstyle k^*} \\ A_j & & B_k \end{array}$$

which cries out for a fill-in to produce a commuting square. Since j^* is surjective there can be at most one such fill-in. By Theorems 3.14 and 3.20 of [8], such a fill-in exists precisely when

$$j \leq \ker(k^* \circ f)$$

and when it does exist it is just the restriction of f to A_j. As in Section 5 of [9] (just before Theorem 5.13), this fill-in exists precisely when

$$f \circ j \leq k \circ f$$

holds.

Now consider the case where both j and k are double negation. Thus we have a commuting square

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \big\downarrow & & \big\downarrow \\ A_{\neg\neg} & & B_{\neg\neg} \\ a & \longmapsto & \neg\neg f(a) \end{array}$$

precisely when

$$f(\neg\neg a) \leq \neg\neg f(a)$$

for all $a \in A$. When it does exist it is given by the indicated assignment.

We look at a particular case of this set up.

Consider an arbitrary space S and the frame morphism

$$NOS \underset{[\,\cdot\,]}{\overset{\sigma_S}{\rightleftarrows}} \mathcal{O}^f S$$

[3]This general observation should have been made explicit in Section 5 of [9].

given by Theorem 3.7. Here

$$\sigma_S \dashv [\,\cdot\,]$$

that is $[\,\cdot\,]$ is the right adjoint of σ_S. Recall also that

$$\sigma_S \circ [\,\cdot\,] = \boldsymbol{id}_{\mathcal{O}^f S}$$

to show that σ_S is surjective.

We have a 3-sided diagram

$$\begin{array}{ccc} N\mathcal{O}S & \xrightarrow{\sigma_S} & \mathcal{O}^f S \\ \downarrow & \sqcap & \downarrow \\ (N\mathcal{O}S)_{\neg\neg} & & (\mathcal{O}^f S)_{\neg\neg} \end{array}$$

where $(N\mathcal{O}S)_{\neg\neg}$ is a certain complete boolean algebra and $(\mathcal{O}^f S)_{\neg\neg}$ is the particular complete boolean algebra of front-regular open subsets of S. We will show that for this case there is a fill-in, and it is an isomorphism.

We fix S throughout the discussion, so we may write σ for σ_S when this is convenient.

We need to relate negation on $N\mathcal{O}S$ with negation on $\mathcal{O}^f S$. In fact, we can relate certain implications.

7.1 LEMMA. *For each space S, nucleus $j \in N\mathcal{O}S$, and f-open set $F \in \mathcal{O}^f S$, we have*

$$(j \supset [F]) = [\sigma_S(j) \supset F]$$

where the implication on the left is computed in $N\mathcal{O}S$ and that on the right is computed in $\mathcal{O}^f S$.

Proof. We make use of the adjunction $\sigma \dashv [\,\cdot\,]$. For $k \in N\mathcal{O}S$ we have

$$\begin{aligned} k \le (j \supset [F]) &\iff k \wedge j \le [F] \\ &\iff \sigma(k \wedge j) \subseteq F \\ &\iff \sigma(k) \wedge \sigma(j) \subseteq F \\ &\iff \sigma(k) \subseteq (\sigma(j) \supset F) \quad \iff k \le [(\sigma(j) \supset F)] \end{aligned}$$

to give the required result. ■

Recall that in $\mathcal{O}^f S$ negation and double negation are given by

$$\neg E = E^{=\prime} \qquad \neg\neg E = E^{=\square}$$

for $E \in \mathcal{O}^f S$. Thus by taking $F = \emptyset$ in Lemma 7.1 we obtain the following.

7.2 COROLLARY. *For each space S and $j \in N\mathcal{O}S$ we have*

$$\neg j = [E^{=\prime}] \qquad \neg\neg j = [E^{=\square}]$$

where $E = \sigma_S(j)$.

Proof. The left hand equality is a particular case of Lemma 7.1. This gives

$$\sigma(\neg j) = \sigma[E^{=\prime}] = E^{=\prime}$$

so that another use of the left hand equality gives

$$\neg\neg j = \neg[E^{=\prime}] = [E^{=\prime=\prime}] = [E^{=\Box}]$$

as required. ■

We also have a characterization of $(N\mathcal{O}S)_{\neg\neg}$.

7.3 COROLLARY. *For each space S and $j \in \mathcal{O}S$, we have $j \in (N\mathcal{O}S)_{\neg\neg}$ precisely when*

$$j = [E]$$

for some $E \in (\mathcal{O}^f S)_{\neg\neg}$, and then $\sigma_S(j) = E$.

Proof. Suppose $j \in (N\mathcal{O}S)_{\neg\neg}$ and let $E = \sigma(j)$. Then

$$j = \neg\neg j = [E^{=\Box}]$$

and

$$E = \sigma(j) = \sigma([E^{=\Box}]) = E^{=\Box}$$

to show that $j = [E]$ with $E \in (\mathcal{O}^f S)_{\neg\neg}$.

Conversely, suppose that $j = [E]$ with $E \in (\mathcal{O}^f S)_{\neg\neg}$. Then $\sigma(j) = E$ and so

$$\neg\neg j = [E^{=\Box}] = [E] = j$$

to show $j \in (N\mathcal{O}S)_{\neg\neg}$. ■

We now return to the 3-sided diagram ($\sqcap$).

From our preliminary observations we know there is a fill-in precisely when

$$\sigma(\neg\neg j) \subseteq \neg\neg(\sigma(j))$$

for each $j \in N\mathcal{O}S$. With $E = \sigma(j)$ this condition is

$$\sigma([E^{=\Box}]) \subseteq E^{=\Box}$$

which, since $\sigma \circ [\,\cdot\,] = \boldsymbol{id}$, always holds.

Thus we do get a fill-in. And just for being good, we get an extra treat.

7.4 THEOREM. *For each space S there is a commuting square of frame morphism*

$$\begin{array}{ccc} N\mathcal{O}S & \xrightarrow{\sigma_S} & \mathcal{O}^f S \\ \downarrow & & \downarrow \\ (N\mathcal{O}S)_{\neg\neg} & \xrightarrow[\rho_S]{} & (\mathcal{O}^f S)_{\neg\neg} \end{array}$$

where ρ_S is the restriction of σ_S to $(N\mathcal{O}S)_{\neg\neg}$. Furthermore, ρ_S is an isomorphism.

Proof. From the calculations above we know that

$$\rho_S = \sigma_S|_{(N\mathcal{O}S)_{\neg\neg}}$$

is a frame morphism which makes the square commute. Thus it suffices to show that this ρ is injective and surjective.

To show that ρ is injective suppose

$$\rho(j) = \rho(k)$$

for $j, k \in (N\mathcal{O}S)_{\neg\neg}$. Then

$$\sigma(j) = \rho(j) = \rho(k) = \sigma(k)$$

so that Corollary 7.3 gives

$$j = [\sigma(j)] = [\sigma(k)] = k$$

as required.

To show that ρ is surjective consider any $E \in (\mathcal{O}^f S)_{\neg\neg}$ and let $j = [E]$. Then, again by Corollary 7.3, we have $j \in (N\mathcal{O}S)_{\neg\neg}$ and

$$\rho(j) = \sigma(j) = \sigma([E]) = E$$

for the required result. ■

As an application of these ideas let's have a look at the result which initially prompted this analysis. The proof we give of the following is not the best one, for it hides a lot of the relevant facts. A much better proof, which exposes these facts, is given in [12].

7.5 THEOREM. *For a T_0 space S, the assembly $N\mathcal{O}S$ is boolean precisely when S is scattered.*

Proof. Suppose first that $N\mathcal{O}S$ is boolean. Then each nucleus $j \in N\mathcal{O}S$ is regular and hence is spatially induced by Corollary 7.3. Thus, by Theorem 5.8, we know that $N\mathcal{O}S$ and $\mathcal{O}^f S$ are isomorphic and S is weakly scattered. But now $\mathcal{O}^f S$ is a boolean algebra and hence, since S is T_0, it is just the power set of S. This shows that S is T_D and weakly scattered, and hence is scattered by Lemma 2.19.

Conversely, suppose S is scattered. Then it is weakly scattered and T_D by Lemma 2.19. By by Theorem 5.8, we know that $N\mathcal{O}S$ and $\mathcal{O}^f S$ are isomorphic. Since S is T_D the front space ${}^f S$ is discrete, and hence $\mathcal{O}^f S$ is a boolean algebra. ■

As I said at the beginning of this section, I am not quite sure of the relevance of these ideas, but no doubt they will be useful somewhere.

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The whole collection of notes can be found on my web pages with

/FRAMES/frames.html

holding the relevant documents. Here are the first few parts.

[8] The basics of frame theory.

[9] The assembly of a frame.

[10] The point space of a frame.

[11] The fundamental triangle of a space.

[12] The higher level CB properties of frames.

[13] Examples of higher level assemblies.

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