The large spectrum of an arbitrary lattice
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1 Preamble

2 Some background frame theory

As indicated in the preamble we fix a lattice Λ and work with that throughout. Recall that a congruence on Λ is an equivalence relation ≡ such that

\[
\begin{align*}
    a_1 \equiv a_2 & \implies \{ a_1 \lor b_1 \equiv a_2 \lor b_2 \} \\
    b_1 \equiv b_2 & \implies \{ a_1 \land b_1 \equiv a_2 \land b_2 \}
\end{align*}
\]

for all \( a_1, a_2, b_1, b_2 \in \Lambda \). Congruences are partially ordered by inclusion (with equality at the bottom and the total collapsing congruence at the top). The intersection of any family of congruences is a congruence, and hence we have a complete lattice.

3.1 DEFINITION. For an arbitrary lattice Λ let Cong(Λ) be the complete lattice of all congruences on Λ.

Infima in \( \text{Cong}(\Lambda) \) are just intersections, but suprema are more complicated. In fact, the join of two congruences can be quite difficult to describe. In this section we produce a different way of handling congruences which for our purposes is easier to work with. We then show that \( \text{Cong}(\Lambda) \) is a frame.
3.2 DEFINITION. Let Λ be an arbitrary lattice. For each $a, b \in \Lambda$ with $a \leq b$ we let

$$[a, b] = \{ x \in \Lambda \mid a \leq x \leq b \}$$

to obtain the interval between $a$ and $b$. In particular $[a, a]$ is the trivial interval $\{a\}$, and $[\bot, \top] = \Lambda$ is the improper interval.

A congruence $\equiv$ on $\Lambda$ collapses an interval $[a, b]$ of $\Lambda$ if $a \equiv b$. It is not too hard to see that a congruence is uniquely determined by the intervals it collapses. To analyse this we look at various families of sets of intervals.

3.3 DEFINITION. Let Λ be an arbitrary lattice Λ. For two intervals $I, J$ we write

$I \sim J$

and say $I$ and $J$ are similar if there are elements $l, r$ such that $I, J$ have the form

$$[l, l \vee r], [l \wedge r, r]$$

in some order. ■

By definition this relation is symmetric, and it is easy to see that it is reflexive. In general $\sim$ need not be transitive. In fact, in some lattices the transitive closure of $\sim$ can be quite weird.

3.4 DEFINITION. Let Λ be an arbitrary lattice.

(a) A set of intervals $\mathcal{A}$ is abstract if it is non-empty and closed under $\sim$, that is

$J \sim I \in \mathcal{A} \implies J \in \mathcal{A}$

for intervals $I, J$. Let $\mathcal{A}(\Lambda)$ be the collection of all abstract sets of intervals.

(b) A set of intervals $\mathcal{B}$ is basic if it is abstract and closed under taking subintervals, that is

$J \rightarrow I \in \mathcal{B} \implies J \in \mathcal{B}$

for intervals $I, J$. Let $\mathcal{B}(\Lambda)$ be the collection of all basic sets of intervals.

(c) A set of intervals $\mathcal{C}$ is a congruence set if it is basic and closed under abutting, that is

$[a, b], [b, c] \in \mathcal{C} \implies [a, c] \in \mathcal{C}$

for elements $a \leq b \leq c$. Let $\mathcal{C}(\Lambda)$ be the collection of all congruence sets of intervals. ■

Trivially, each of $\mathcal{A}(\Lambda) \supseteq \mathcal{B}(\Lambda) \supseteq \mathcal{C}(\Lambda)$ is a poset under inclusion. A few moment’s thought shows that each is closed under arbitrary intersections, and so each is a complete lattice. Furthermore, in $\mathcal{A}(\Lambda)$ and $\mathcal{B}(\Lambda)$ suprema are unions. However, suprema in $\mathcal{C}(\Lambda)$ are more complicated, but we do have the following.

3.5 LEMMA. For each lattice $\Lambda$ the poset $\mathcal{C}(\Lambda)$ is closed under arbitrary intersections and directed unions.
The lattice $A(\Lambda)$ is not too important. It is there just to get us going. Calculations in $B(\Lambda)$ are easy. Since it is closed under arbitrary intersections and unions it is a frame. (It’s actually a topology, but of a rather pathetic kind.) In due course we show that $C(\Lambda)$ is a quotient frame of $B(\Lambda)$. However, the first aim of this section is to observe that $C(\Lambda)$ is canonically isomorphic to $\text{Cong}(\Lambda)$. To help with that and later calculations we need a couple of preliminaries.

The following is a simple exercise.

3.6 \textbf{Lemma.} Let $\Lambda$ be a lattice and consider a non-empty set $B$ of intervals. Then $B$ is basic precisely when it is closed under translations, that is
\[ [a, b] \in B \implies [a \land x, b \land x], [a \lor x, b \lor x] \in B \]
for each interval $[a, b]$ and element $x$.

Often we find that this is a useful characterization of being basic. Congruence sets also have other useful properties.

3.7 \textbf{Lemma.} Let $\Lambda$ be an arbitrary lattice and consider $C \in C(\Lambda)$. Then $C$ is closed under pivoting, that is
\[ (\lor) \quad [a, x], [a, y] \in C \implies [a, x \lor y] \in C \]
\[ (\land) \quad [x, b], [y, b] \in C \implies [x \land y, b] \in C \]
for all $a, b, x, y \in A$.

As suggested above, congruences and congruence sets are different guises of the same thing. We now make that precise. The details of the proof of the following result is essentially the same as [Gratzer] (Lemma 8 on page 20, and Theorem 2 on page 172).

3.8 \textbf{Theorem.} Let $\Lambda$ be an arbitrary lattice. There is a bijective correspondence
\[ \begin{array}{rcl}
\text{Cong}(\Lambda) & \leftrightarrow & C(\Lambda) \\
\equiv & \leftrightarrow & B
\end{array} \]
between congruences on $\Lambda$ and congruence sets for $\Lambda$. This is given by
\[ (\equiv \leftrightarrow C) \quad a \leq b \text{ and } a \equiv b \iff [a, b] \in C \]
\[ (\equiv \leftrightarrow C) \quad a \equiv b \iff [a \land b, a \lor b] \in C \]
for $a, b \in A$.

The idea is to use $C(\Lambda)$ in place of $\text{Cong}(\Lambda)$. Many calculations that need can be done in $B(\Lambda)$ and then the result transferred to $C(\Lambda)$. We now look at how that transfer is done.

3.9 \textbf{Definition.} Let $\Lambda$ be an arbitrary lattice, and consider any basic set $B \in B(\Lambda)$. We let
\[ C\text{ng}(B) \]
be the set of all intervals $[a, b]$ which can be \textit{partitioned} into $B$. In other words there is a finite sequence
\[ a = x_0 \leq \cdots \leq x_i \leq \cdots \leq x_{m+1} = b \]
with $[x_i, x_{i+1}] \in A$ for each $0 \leq i \leq m$. ■
This is the crucial operation that takes us from $\mathbb{B}(\Lambda)$ to $\mathbb{C}(\Lambda)$.

3.10 LEMMA. Let $\Lambda$ be an arbitrary lattice.

(a) For each $B \in \mathbb{B}(\Lambda)$ the set of intervals $\mathbb{Cng}(B)$ is the least congruence set that includes $B$.

(b) The operation $\mathbb{Cng}$ is a closure operation on $\mathbb{B}(\Lambda)$, that is, it is inflationary, monotone, and idempotent.

(c) The operation $\mathbb{Cng}$ is a nucleus on $\mathbb{B}(\Lambda)$, that is

$$\mathbb{Cng}(A) \cap \mathbb{Cng}(B) \subseteq \mathbb{Cng}(A \cap B)$$

for each $A, B \in \mathbb{B}$.

(d) The fixed family of $\mathbb{Cng}$ is precisely the members of $\mathbb{C}(\Lambda)$, that is

$$\mathbb{Cng}(B) = B \iff B \in \mathbb{C}(\Lambda)$$

for each $B \in \mathbb{B}(\Lambda)$.

Proof. (a) Trivially, we have $B \subseteq \mathbb{Cng}(B)$ since each interval is a partition of itself. We check that $\mathbb{Cng}(B)$ is closed under translation. Suppose $[a, b] \in \mathbb{Cng}(B)$ and consider any $y \in \Lambda$. Suppose the partition $a = x_0 \leq \cdots \leq x_i \leq \cdots \leq x_{m+1} = b$ witnesses the membership of $\mathbb{Cng}(B)$, that is $[x_i, x_{i+1}] \in B$ for each part. Then we have $[x_i \wedge y, x_{i+1} \wedge y] \in B$, so that the partition

$$a \wedge y = x_0 \wedge y \leq \cdots \leq x_i \wedge y \leq \cdots \leq x_{m+1} \wedge y = b \wedge y$$

ensures that $[a \wedge y, b \wedge y] \in \mathbb{Cng}(B)$. A similar argument shows that $[a \vee y, b \vee y] \in \mathbb{Cng}(A)$.

This shows that $\mathbb{Cng}(B)$ is a basic set. By construction $\mathbb{Cng}(B)$ is closed under partitions, and hence is a congruence set. We have observed that $B \subseteq \mathbb{Cng}(B)$.

Consider any congruence set $C$ with $B \subseteq C$. Consider any interval $[a, b] \in \mathbb{Cng}(B)$. This interval can be partitioned into members of $B$, and each of these subintervals is in $C$, so that whole interval is in $C$ since $C \in \mathbb{C}(\Lambda)$. Thus $\mathbb{Cng}(B) \subseteq C$.

(b) We have $B \subseteq \mathbb{Cng}(B)$ for each $B \in \mathbb{B}(\Lambda)$. A trivial observation shows that $\mathbb{Cng}$ is monotone. To show that $\mathbb{Cng}$ is idempotent consider any $B \in \mathbb{B}(\Lambda)$ and any interval $[a, b] \in \mathbb{Cng}(\mathbb{Cng}(B))$. This interval can be partitioned into members of $\mathbb{Cng}(B)$. Each of these parts can be partitioned into members of $B$. This gives a partition of $[a, b]$ into members of $B$, and hence $[a, b] \in \mathbb{Cng}(B)$.

(c) Consider any $[a, b] \in \mathbb{Cng}(A) \cap \mathbb{Cng}(B)$. Since $[a, b] \in \mathbb{Cng}(A)$ there is a partition $a \leq \cdots \leq x \leq y \leq \cdots \leq b$ where $[x, y] \in A$ for each part. We have $[x, y] \subseteq [a, b] \in \mathbb{Cng}(B)$, so that $[x, y] \in \mathbb{Cng}(B)$ and this part $[x, y]$ itself has a partition $x \leq \cdots \leq u \leq v \leq \cdots \leq y$ where $[u, v] \in B$ for each part. We also have $[u, v] \subseteq [x, y] \in A$ so that $[u, v] \in A \cap B$. Thus, by combining all the partitions we see that $[a, b] \in \mathbb{Cng}(A \cap B)$, as required.
Consider any $B \in \mathbb{B}(\Lambda)$ with $\text{Cng}(B) = B$. Since $\text{Cng}(B) \in \mathbb{C}(\Lambda)$ this gives $B \in \mathbb{C}(\Lambda)$. Conversely, consider any $B \in \mathbb{C}(\Lambda)$ and any $[a, b] \in \text{Cng}(B)$. This interval can be partitioned into members of $\mathcal{B}$. But this $B$ is closed under abutting, so that whole interval is in $B$. Thus $\text{Cng}(B) = B$. ■

With this we have the main result of this section.

3.11 THEOREM. Let $\Lambda$ be an arbitrary lattice. Then $\mathbb{C}(\Lambda)$ is a quotient frame of $\mathbb{B}(\Lambda)$

$$
\begin{array}{ccc}
\mathbb{B}(\Lambda) & \longrightarrow & \mathbb{C}(\Lambda) \\
B & \longmapsto & \text{Cng}(B)
\end{array}
$$

given by the nucleus $\text{Cng}$.

Every frame has a point space. What is this space for $\mathbb{C}(\Lambda)$? That’s the topic of the next section.

[Hold in PUSSY/IDioms-../SPECTRUM/PAPER/004... Changed 23 September 2012]

4 The large spectrum

Each frame $\Omega$ has a point space $\text{pt}(\Omega)$ together with a surjective frame morphism

$$
\begin{array}{cc}
\Omega & \overset{U}{\longrightarrow} & \text{Opt}(\Omega)
\end{array}
$$

to the topology of the space. In general this need not be an isomorphism. This space can be constructed in several ways. Here is the method that is most convenient here. A point of $\Omega$ is a $\wedge$-irreducible element, and element $p \in \Omega$ with $p \neq \top$ and

$$
x \wedge y \leq p \implies x \leq p \text{ or } y \leq p
$$

for $x, y \in \Omega$. Let $\text{pt}(\Omega)$ be the set of all these. For each $a \in \Omega$

$$
p \in U(a) \iff a \neq p
$$

gives a subset $U(a)$ of $\text{pt}(\Omega)$. We easily check that

$$
\text{Opt}(\Omega) = \{U(a) \mid a \in \Omega\}
$$

is a topology on $\text{pt}(\Omega)$ and $U$ is a surjective frame morphism. In general this morphism need not be an isomorphism. There are some vary large frames with no points at all. To show that the morphism is an isomorphism we need enough separating points. That is, for each pair of elements $b \nleq a$ there must be a point $p$ (a $\wedge$-irreducible element) with $b \nleq p$ and $a \leq p$.

Each lattice $\Lambda$ has an associated frame $\mathbb{C}(\Lambda)$ which in turn has a point space. We now investigate that space.

4.1 DEFINITION. For each lattice $\Lambda$ we let

$$
\text{SPEC}(\Lambda) = \text{pt}(\mathbb{C}(\Lambda))
$$

to obtain the large spectrum of $\Lambda$. ■
This notation and terminology is perhaps a bit naughty since many lattices \( \Lambda \), particularly distributive lattices, already have a spectrum \( \text{spec}(\Lambda) \) (which is sometimes written \( \text{Spec}(\Lambda) \)). As we will see, in general these two are not the same but not unrelated. The upper case \( \text{SPEC}(\Lambda) \) should help to distinguish between the two.

Let

\[
C(\Lambda) \xrightarrow{U} \text{OSPEC}(\Lambda)
\]

be the canonical surjective morphism to the topology of the point space \( \text{SPEC}(\Lambda) \).

4.2 THEOREM. For each lattice \( \Lambda \) the congruence frame \( C(\Lambda) \) is spatial.

Proof. We must show that \( C(\Lambda) \) has enough separating points. Since \( C(\Lambda) \) is closed under directed unions we may invoke Zorn’s Lemma.

For each interval \( I \) of \( \Lambda \) the family

\[
\{ A \in C(\Lambda) \mid I \notin A \}
\]

is closed under directed unions and so each member of the family lies below a maximal member of the family. Let \( P \) be such a maximal member, and suppose

\[
P = \bigcap X
\]

for some family \( X \) of members \( X \in C(\Lambda) \). We show that \( P \in X \). By way of contradiction suppose \( P \notin X \). For each \( X \in X \) we certainly have \( P \subseteq X \), and then \( P \neq X \) (for otherwise \( P = X \in X \)). Thus \( I \in X \) by the maximality of \( P \). This shows that \( I \in X \) for each \( X \in X \), and hence \( I \in \bigcap X = P \), which is the contradiction. This shows that \( P \) is \( \cap \)-irreducible, and hence \( \cap \)-irreducible, that is

\[
A \cap B = P \implies A = P \text{ or } B = P
\]

for each \( A, B \in C(\Lambda) \). In particular, \( P \) is a point.

Now consider \( A, B \in C(\Lambda) \) with \( B \notin A \). There is some interval \( I \in B - A \). By the argument above there is some \( \cap \)-irreducible \( P \in C(\Lambda) \) with \( A \subseteq P \) and \( I \notin P \). In particular, we have

\[
B \notin P \quad A \subseteq P
\]

to give the required result.

You may recognize the content of this argument. It is essentially that used by Garret Birkhoff to obtain the result concerning sub-direct representation.

What are the properties of this space? Every point space is sober hence \( \text{SPEC}(\Lambda) \) is sober. (Sobriety is a separation property between \( T_0 \) and \( T_2 \) which is incompatible with \( T_1 \).) At the moment I know of only one other general property of \( \text{SPEC}(\Lambda) \), but this does ring a bell. To describe this property we need to look at \( C(\Lambda) \).

The frame \( C(\Lambda) \) is closed under arbitrary intersections. Thus for each interval \( [a, b] \) of \( \Lambda \) there is a least congruence set which contains \( [a, b] \). That is, there is a least quotient of \( \Lambda \) which collapses \( [a, b] \). Let’s give this congruence set a name.

4.3 DEFINITION. For each interval \( [a, b] \) of \( \Lambda \) let \( \langle a, b \rangle \) be the least congruence set \( A \) with \( [a, b] \in A \).
Observe that
\[ [a, b] \in \mathcal{A} \iff \langle a, b \rangle \subseteq \mathcal{A} \]
for each interval \([a, b]\) and \(\mathcal{A} \in \mathcal{C}(\Lambda)\). We also have
\[ \mathcal{A} = \bigvee \{ \langle a, b \rangle \mid [a, b] \in \mathcal{A} \} \]
but remember that this supremum need not be a union. In fact, we have
\[ \bigvee \mathcal{A} = Cng(\bigcup \mathcal{A}) \]
for each \(\mathcal{A} \subseteq \mathcal{C}(\Lambda)\).

4.4 **LEMMA.** For each lattice \(\Lambda\) and interval \([a, b]\) of \(\Lambda\), the associated congruence set \(\langle a, b \rangle\) is compact in \(\mathcal{C}(\Lambda)\).

**Proof.** Consider any family \(\mathcal{X}\) of congruence sets \(\mathcal{X}\) with
\[ \langle a, b \rangle \subseteq \bigvee \mathcal{X} \]
(where this supremum is computed in \(\mathcal{C}(\Lambda)\)). We must show that \(\langle a, b \rangle\) is included in some congruence set
\[ \mathcal{Y} = \mathcal{X}_1 \lor \cdots \lor \mathcal{X}_m \]
for some \(\mathcal{X}_1, \ldots, \mathcal{X}_m \in \mathcal{X}\). To this end let \(\mathcal{Y}\) be the family of all such \(\mathcal{Y}\) that can be formed from members of \(\mathcal{X}\). We have
\[ \bigvee \mathcal{X} = \bigvee \mathcal{Y} = \bigcup \mathcal{Y} \]
where the second equality holds since \(\mathcal{Y}\) is directed. We have
\[ \langle a, b \rangle \subseteq \bigcup \mathcal{Y} \]
so that
\[ [a, b] \in \bigcup \mathcal{Y} \]
and hence
\[ [a, b] \in \mathcal{Y} \]
that is
\[ \langle a, b \rangle \subseteq \mathcal{Y} \]
for some \(\mathcal{Y} \in \mathcal{Y}\), as required. \(\blacksquare\)

Since every congruence set is a supremum of these special congruence sets we have the following.

4.5 **COROLLARY.** For each lattice \(\Lambda\) the frame \(\mathcal{C}(\Lambda)\) is compactly generated (that is \(\bigvee\)-generated by its compact elements).

This is a different, and more ***, way of stating this result.

4.6 **THEOREM.** For each lattice \(\Lambda\) the large spectrum \(\text{SPEC}(\Lambda)\) is compact with a base of compact open sets.
Proof. Recall that the surjective frame morphism $U(\cdot)$, which in this case is an isomorphism, is given by

$$P \in U(A) \iff A \notin P$$

for $P \in \text{SPEC}(\Lambda)$, and $A \in \mathcal{C}(\Lambda)$. Since

$$A = \bigvee \{\langle a, b \rangle \mid [a, b] \in A\}$$

we have

$$U(A) = \bigcup \{U(a, b) \mid [a, b] \in A\}$$

where

$$U(a, b) = U(\langle a, b \rangle)$$

that is

$$P \in U(a, b) \iff [a, b] \notin P$$

for each interval $[a, b]$ of $A$ and $P \in \text{SPEC}(\Lambda)$. Corollary 4.5 now gives the required result.

This seems to show that $\text{SPEC}(\Lambda)$ is getting close to being a spectral space. What is missing is the intersection property, that the intersection of two compact open sets is itself compact open. We look at that in the next section.

5 The distributive case

Suppose the parent lattice $\Lambda$ is distributive. Then $\Lambda$ already has a spectrum $\text{spec}(\Lambda)$, its associated spectral space. For this case it is not unreasonable to expect there is some connection between $\text{SPEC}(\Lambda)$ and $\text{spec}(\Lambda)$. In this section we show there is connection, but it is not quite what we might expect.

Each interval $[a, b]$ of $\Lambda$ gives a congruence set $\langle a, b \rangle$, corresponding to the smallest congruence that collapses $[a, b]$. In general this set $\langle a, b \rangle$ is difficult to describe. For the distributive case it is easy. The following is essentially given in [Gratzer]Check (page 74, Theorem 3).

5.1 LEMMA. Let $\Lambda$ be a distributive lattice. Then

$$[x, y] \in \langle a, b \rangle \iff a \land y \leq x \text{ and } y \leq b \lor x$$

for all intervals $[a, b]$ and $[x, y]$ of $\Lambda$.

A few more simple calculations gives the following.

5.2 COROLLARY. For each distributive lattice $\Lambda$ we have

$$\langle a, b \rangle \cap \langle c, d \rangle = \langle a \lor c, b \land d \rangle$$

for each pair $[a, b]$ and $[c, d]$ of intervals.
Proof. Consider any member \([x, y]\) of the intersection. Then

\[
\begin{align*}
y & \leq b \lor x \quad y \leq d \lor x \\
 a \land y & \leq x \quad c \land y \leq x
\end{align*}
\]

to give

\[
\begin{align*}
y & \leq (b \lor x) \land (d \lor x) = (b \land d) \lor x \\
x & \geq (a \land y) \lor (c \land y) = (a \lor c) \land y
\end{align*}
\]

to give

\([x, y] \in \langle a \lor c, b \land d \rangle\)

for one of the required inclusions.

For the converse we are given

\[
x \geq (a \lor c) \land y \quad y \leq (b \land d) \lor x
\]

and then

\[
x \geq (a \lor c) \land y \geq a \land y \quad y \leq (b \land d) \lor x \leq b \lor x
\]

with two similar observations to give the required result. ■

For a distributive lattice \(\Lambda\) we wish to relate the two spaces

\[
\text{spec}(\Lambda) \quad \text{SPEC}(\Lambda) = \text{pt}(\mathbb{C}(\Lambda))
\]

the classical spectrum and the large spectrum. To do that we need to recall how \(\text{spec}(\Lambda)\) is constructed. There are three ways of producing the points. Here it is convenient to use the prime ideals \(P\) of \(\Lambda\).

5.3 Lemma. For each distributive lattice \(\Lambda\) there is a bijective correspondence

\[
\text{spec}(\Lambda) \longleftrightarrow \text{pt}(\mathbb{C}(\Lambda))
\]

given by

\[
(P \mapsto \mathcal{P}) \quad a \notin P \text{ or } b \in P \iff [a, b] \in \mathcal{P} \\
(P \mapsto \mathcal{P}) \quad x \in P \iff [\bot, x] \in \mathcal{P}
\]

for \(a, b, x \in \Lambda\).

Proof. We verify this in four phases. We first check that each of the two assignments does produce the required kind of gadget, and then we check that each 2-step trip gets back to the starting point.

For the first phase let \(P\) be a prime ideal of \(\Lambda\) and consider the indicated set \(\mathcal{P}\) of intervals, that is

\([a, b] \in \mathcal{P} \iff a \notin P \text{ or } b \in P\)

for elements \(a \leq b\) of \(\Lambda\). We show that \(\mathcal{P} \in \text{pt}(\mathbb{C}(\Lambda))\) by a series of steps.

We show that \(\mathcal{P}\) is closed under translations, and hence is basic. To this end consider \([a, b] \in \mathcal{P}\) so that either

\[a \notin P \text{ or } b \in P\]
holds. Consider also any \( x \in \Lambda \).

If \([a \land x, b \land x] \notin \mathcal{P}\), then we have both

\[
\begin{align*}
a \land x & \in P \\
b \land x & \notin P
\end{align*}
\]

and hence one of

\[
\begin{align*}
a & \in P \\
x & \in P
\end{align*}
\]

since \( P \) is prime. The first of these contradicts the given condition on \( a \), and the second contradicts the second given condition on \( x \), since \( P \) is a lower section of \( \Lambda \). This contradiction gives \([a \land x, b \land x] \in \mathcal{P}\).

If \([a \lor x, b \lor x] \notin \mathcal{P}\), then we have both

\[
\begin{align*}
a \lor x & \in P \\
b \lor x & \notin P
\end{align*}
\]

and hence

\[
a, x \in P
\]

since \( P \) is a lower section. Since \([a, b] \in \mathcal{P}\) this gives

\[
b, x \in P
\]

and hence \( b \lor x \in P \), since \( P \) is an ideal of \( \Lambda \). This contradiction gives \([a \lor x, b \lor x] \in \mathcal{P}\).

These two arguments show that \( \mathcal{P} \) is closed under translations, and hence is basic.

Now consider \([a, b], [b, c] \in \mathcal{P}\). Then

\[
\begin{align*}
a & \notin P \text{ or } b \in P \\
b & \notin P \text{ or } c \in P
\end{align*}
\]

so that

\[
a \notin P \text{ or } c \in P
\]

to show that \([a, c] \in \mathcal{P}\), and hence \( \mathcal{A} \) is closed under abutment. This shows that \( \mathcal{P} \in \mathbb{C}(\Lambda) \).

Trivially, we have \( \mathcal{P} \neq \mathcal{I}(\Lambda) \) (for otherwise \([\perp, \top] \in \mathcal{P}\) and hence either \( \perp \notin P \) or \( \top \in P \), neither of which is true since \( P \) is a proper ideal). It remains to show that \( \mathcal{P} \) is \( \land \)-irreducible in \( \mathbb{C}(\Lambda) \).

Consider any \( \mathcal{A}, \mathcal{B} \in \mathbb{C}(\Lambda) \) with \( \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{P} \) and, by way of contradiction, suppose that both

\[
\mathcal{A} \notin \mathcal{P} \quad \mathcal{B} \notin \mathcal{P}
\]

hold. This gives us intervals

\[
[a, b] \in \mathcal{A} \quad [c, d] \in \mathcal{B}
\]

where each of

\[
a \in P \quad b \notin P \quad c \in P \quad d \notin P
\]

holds. By translation we have

\[
[a \lor c, b \lor c] \in \mathcal{A} \quad [a \lor c, a \lor d] \in \mathcal{B}
\]

and hence

\[
[a \lor c, (b \lor c) \land (a \lor d)] \in \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{P}
\]

since this interval is a sub-interval of the two previous ones. But now one of

\[
a \lor c \notin P \quad (b \lor c) \land (a \lor d) \in P
\]
must hold. The first of these contradicts the known positions of \( a \) and \( c \). Thus we have
\[
b \lor c \in P \quad a \lor d \in P
\]
since \( P \) is prime. These contradict the known positions of \( b \) and \( d \).

This completes the proof that \( P \) is a point of \( C(\Lambda) \).

For the second phase consider any point \( P \) of \( C(\Lambda) \) and let \( P \) be the subset of \( \Lambda \) given by
\[
x \in P \iff [\bot, x] \in P
\]
(for \( x \in \Lambda \)). Since \( P \) is closed under subintervals, this \( P \) is a lower section of \( \Lambda \). A use of the translation property shows that \( P \) is an ideal of \( \Lambda \). Also \( P \) is proper since \( \Lambda = [\bot, \top] \notin P \).

For each \( a, b \in \Lambda \) a use of Corollary 5.2(a) gives
\[
a \land b \in P \implies [\bot, a \land b] \in P
\]
\[
\implies [\bot, a] \subseteq P
\]
\[
\implies [\bot, a] \cap [\bot, b] \subseteq P
\]
\[
\implies [\bot, a] \subseteq P \text{ or } [\bot, a] \subseteq P
\]
\[
\implies [\bot, a] \in P \text{ or } [\bot, a] \in P \implies a \in P \text{ or } b \in P
\]
to show that \( P \) is prime.

For the third phase consider a 2-step trip
\[
P \longleftarrow P \longleftarrow P'
\]
across the two assignments. Since \( \bot \in P \) we have
\[
x \in P' \iff [\bot, x] \in P \iff x \in P
\]
for each \( x \in \Lambda \), to show that \( P' = P \).

For the fourth and final phase consider a 2-step trip
\[
P \longleftarrow P \longleftarrow P'
\]
across the assignments. We observe that for each \( a \in \Lambda \) exactly one of
\[
[\bot, a] \in P \quad [a, \top] \in P
\]
must occur. They can not both occur, since otherwise \( \Lambda = [\bot, \top] \notin P \) by a use of abutment. Also, by Corollary 5.2(b) we have
\[
[\bot, a] \cap [a, \top] = \mathcal{O}(\Lambda) \in P
\]
so that at least one must occur.

Using this, for each interval \([a, b]\) we have
\[
[a, b] \in P' \implies a \notin P \text{ or } b \in P
\]
\[
\implies [\bot, a] \notin P \text{ or } [\bot, b] \in P
\]
\[
\implies [a, \top] \in P \text{ or } [\bot, b] \in P \implies [a, b] \in P
\]
so that \( \mathcal{P}' \subseteq \mathcal{P} \). Conversely, we have
\[
[\bot, a], [a, b] \in \mathcal{P} \implies [\bot, b] \in \mathcal{P}
\]
by abutment, so that
\[
[a, b] \in \mathcal{P} \implies [\bot, a] \notin \mathcal{P} \text{ or } [\bot, b] \in \mathcal{P}
\]
\[
\implies a \notin \mathcal{P} \text{ or } b \in \mathcal{P} \implies [a, b] \in \mathcal{P}'
\]
and hence \( \mathcal{P}' = \mathcal{P} \).

This completes the four phases.

This result is suggestive, but the suggestion is wrong. The bijection is not a homeomorphism between the two spaces, but between the patch space \( \text{spec}^*(\Lambda) \) of \( \text{spec}(\Lambda) \) and \( \text{SPEC}(\Lambda) \). Recall that for each \( a \in \Lambda \) we use
\[
P \in U(a) \iff a \notin \mathcal{P}
\]

to extract a subset of \( \text{spec}(\Lambda) \). These sets form the canonical basic open sets of \( \text{spec}(\Lambda) \).
The patch space \( \text{spec}^*(\Lambda) \) is the set \( \text{spec}(\Lambda) \) with the smallest topology for which each set \( U(a) \) is clopen. Thus, for \( a, b \in \Lambda \)
\[
U^*(a, b) = U(b) - U(a)
\]
is a typical basic open set of \( \text{spec}^*(\Lambda) \). A simple calculation shows that
\[
U^*(l, r) = U^*(l \land r, l \lor r)
\]
for all \( l, r \in \Lambda \). Thus the family \( U^*(a, b) \) for all intervals \([a, b]\) is a base for \( \text{spec}^*(\Lambda) \).

Now recall that for each interval \([a, b]\) gives a typical basic open set of \( \text{Spec}(\Lambda) \). Thus for each matching pair of points
\[
P \leftarrow \mathcal{P}
\]
and interval \([a, b]\), we have
\[
P \in U^*(a, b) \iff P \in U(b), P \notin U(a)
\]
\[
\iff b \notin P, a \in P
\]
\[
\iff [a, b] \notin \mathcal{P}
\]
\[
\iff P \in U(a, b)
\]
to show that the two canonical bases are match by the point correspondence. Thus we have the following.

5.4 THEOREM. For each distributive lattice \( \Lambda \) the bijective correspondence

\[
\text{spec}^*\Lambda \longleftrightarrow \text{pt}(\mathbb{C}(\Lambda))
\]

\[
P \longleftrightarrow \mathcal{P}
\]
is a homeomorphism.

I suspect there is more to be dug out of this.
6 Further remarks and open problems

In this final section I make a few random observations and suggest a couple of topics for further investigation.

Let’s begin with a possible extension of the distributive case. We know that for a distributive lattice \( \Lambda \) the spectrum \( \text{spec}(\Lambda) \) determines \( \Lambda \). This is just the Stone representation result. We also know that the patch space \( \text{spec}^*(\Lambda) \) determines \( \Lambda \) provided we have access to the specialization order of \( \text{spec}(\Lambda) \). This is the Priestley representation result. By [**refer to earlier**] we know that the patch space can be generalized to an arbitrary lattice, namely the large spectrum \( \text{SPEC}(\Lambda) \). Can the representation results be extended.

6.1 QUESTION. For an arbitrary lattice \( \Lambda \) is there a sheaf-like representation of \( \Lambda \) over the space \( \text{SPEC}(\Lambda) \).

I haven’t look at this in any detail but I suspect the Birkhoff result mentioned earlier may be relevant.

Some intro chat
Mention the papers later

Let \( \Lambda \) be an arbitrary lattice. Theorem 4.6 gives us two properties that \( \text{SPEC}(\Lambda) \) has. I do not know if there are other properties that this space always has. For certain lattices this space will have other properties. Theorem 5.4 gives an example of that. What about various other properties? Let’s look at the crude end. When is the space discrete? The answer to this has been known for many years, but not usually stated in this form. I think it is worth looking at some of the details of this result.

Since the space is sober it is discrete exactly when its topology is boolean. Thus we wish to find out when \( \mathbb{C}(\Lambda) \) is boolean. To describe that we need a bit of background information. In particular, we need to consider how basic sets can be generated.

6.2 DEFINITION. Let \( \Lambda \) be an arbitrary lattice. For each interval \( I \) let

\[
\text{Bsc}(I)
\]

be the least basic set that contains \( I \).

This set \( \text{Bsc}(I) \) is formed by starting from \( I \) and forming chains of intervals by repeatedly taking sub-intervals and similar intervals. An interval is in \( \text{Bsc}(I) \) if it occurs in such a chain. In general the length of the witnessing chain can be arbitrarily long, but for a modular lattice it is short.

6.3 LEMMA. Let \( \Lambda \) be a modular lattice. For each pair of intervals \( I, J \) we have \( J \in \text{Bsc}(I) \) precisely when \( J \sim K \subseteq I \) for some interval \( K \).

We don’t need a proof of this for we are going to consider a more general situation. The notion of weak modularity is defined at the bottom of page 175 of [Gratzer]. This can be rephrased as follows. Recall that \( \mathcal{O} = \mathcal{O}(\Lambda) \) is the set of trivial intervals, the singletons, and \( \mathcal{I} = \mathcal{I}(\Lambda) \) is the set of all intervals.  

\(^{1}\text{Use } \mathcal{O} \text{ and } \mathcal{I} \text{ earlier} \)
6.4 DEFINITION. Let \( \Lambda \) be an arbitrary lattice. We say \( \Lambda \) is weakly modular if for each pair of intervals \( I, J \) with \( J \in \mathbb{B}_{sc}(I) - \mathcal{O} \) there is an interval \( K \subseteq I \) with \( K \in \mathbb{B}_{sc}(J) - \mathcal{O} \). ■

The following is extracted from the first part of the proof of Theorem 9 on page 204 of [Gratzer].

6.5 LEMMA. Let \( \Lambda \) be an arbitrary lattice, and suppose \( C(\Lambda) \) is boolean. Then \( \Lambda \) is weakly modular.

Proof. Consider any pair of intervals \( I, J \) with \( J \in \mathbb{B}_{sc}(I) - \mathcal{O} \). We must produce an interval \( K \subseteq I \) with \( K \in \mathbb{B}_{sc}(J) - \mathcal{O} \). To do that let

\[
\mathcal{B} = \mathbb{B}_{sc}(J) \quad \mathcal{C} = \mathbb{C}_{ng}(\mathcal{B})
\]

to produce a basic set and its congruence closure. Since \( C(\Lambda) \) is complemented we have

\[
\mathcal{C} \cap \mathcal{D} = \mathcal{O} \quad \mathcal{C} \cup \mathcal{D} = \mathbb{C}_{ng}(\mathcal{C} \cup \mathcal{D}) = \mathcal{I}
\]

for some \( \mathcal{D} \in \mathbb{C}(\Lambda) \). From the second of these we have \( I \in \mathbb{C}_{ng}(\mathcal{C} \cup \mathcal{D}) \), and hence \( I \) decomposes into finitely many non-trivial sub-intervals \( L \subseteq I \) with \( L \in \mathcal{C} \cup \mathcal{D} \). We show that \( L \in \mathcal{C} \) for at least one such component \( L \). By way of contradiction suppose \( L \in \mathcal{D} \) for each component \( L \). Then \( I \in \mathcal{D} \), to give

\[
J \in \mathbb{B}_{sc}(I) \cap \mathcal{D} \subseteq \mathcal{C} \cap \mathcal{D} = \mathcal{O}
\]

which is not so. We thus have

\[
L \in \mathcal{C} = \mathbb{C}_{ng}(\mathcal{B})
\]

for at least one non-trivial sub-interval \( L \subseteq I \). In turn this decomposes into finitely many non-trivial sub-intervals \( K \) with

\[
K \subseteq L \subseteq I \quad K \in \mathcal{B} = \mathbb{B}_{sc}(J)
\]

for each one. Any such component \( K \) will do. ■

We know that \( C(\Lambda) \) is a frame. In particular, each \( \mathcal{B} \in \mathbb{B}(\Lambda) \) has a negation on \( \mathbb{B}(\Lambda) \), there is a unique largest basic set \( \mathcal{A} \) such that

\[
\mathcal{B} \cap \mathcal{A} = \mathcal{O}
\]

holds. It can be shown that for an arbitrary lattice \( \Lambda \) this negation \( \mathcal{A} \) consists of all those intervals \( I \) such that

\[
(\forall K \in \mathbb{B}_{sc}(I))[K \in \mathcal{B} \implies K \in \mathcal{O}]
\]

holds. Weak modularity enables this to be simplified.

6.6 DEFINITION. Let \( \Lambda \) be an arbitrary weakly modular lattice, and consider any basic set \( \mathcal{B} \in \mathbb{B}(\Lambda) \). We let

\[
\text{Neg}(\mathcal{B})
\]

be the set of intervals \( I \) for which

\[
(\forall K \subseteq I)[K \in \mathcal{B} \implies K \in \mathcal{O}]
\]

holds.
This notation indicates what the next result is.

6.7 Lemma. Let $\Lambda$ be an arbitrary weakly modular lattice, and consider any basic set $B \in \mathbb{B}(\Lambda)$. Then $\text{Neg}(B)$ is basic and is the negation of $B$ in $\mathbb{B}(\Lambda)$.

Proof. We first check that $\text{Neg}(B)$ is abstract. To this end consider any intervals

$$L \sim I \in \text{Neg}(B)$$

and by way of contradiction suppose $L \notin \text{Neg}(B)$. Thus there is an interval $J$ such that

$$J \subseteq L \quad J \in B \quad J \notin \mathcal{O}$$

hold. In particular

$$J \subseteq L \sim I \quad J \notin \mathcal{O}$$

and hence $J \in \mathbb{Bsc}(I)$ with $J \notin \mathcal{O}$. A use of weak modularity now gives an interval $K$ such that

$$K \in \mathbb{Bsc}(J) \subseteq B \quad K \subseteq I \quad K \notin \mathcal{O}$$

hold. This contradicts the assumption that $I \in \text{Neg}(B)$.

This shows that $\text{Neg}(B)$ is abstract, and trivially it is closed under taking sub-intervals. Thus $\text{Neg}(B)$ is basic.

Next we observe that

$$B \cap \text{Neg}(B) = \mathcal{O}$$

hold. For consider any member $I$ of this intersection. With $K = I$ we have

$$I \in \text{Neg}(B) \quad K \subseteq I \quad K \in B$$

so that $I = K \in \mathcal{O}$, as required.

Finally, consider any basic set $A$ such that

$$B \cap A = \mathcal{O}$$

holds. We require $A \subseteq \text{Neg}(B)$. But for each intervals

$$I \in A \quad K \subseteq I \quad K \in B$$

we have

$$K \in B \cap A = \mathcal{O}$$

which gives the required result. ■

There does seem to be a bit of slack in this proof, so perhaps the result can be generalized. In particular, you might like to show that

$$C \in \mathbb{C}(\Lambda) \implies \text{Neg}(C) \in \mathbb{C}(\Lambda)$$

provided $\Lambda$ is weakly modular. We use this observation on the proof of the next result.

For that we look at the notion of separability as defined at the top of page 204 of [Gratzer]. This translates as follows.
6.8 DEFINITION. Let \( \Lambda \) be an arbitrary lattice. A congruence set \( \mathcal{C} \in \mathbb{C}(\Lambda) \) is **separable** if for each non-trivial interval \([a, b]\) there is a finite decomposition

\[
a = x_0 \leq x_1 \leq \cdots \leq x_i \leq \cdots \leq x_{n+1} = b
\]

such that for each component \([y, z]\) one of

\[
(+) \quad [y, z] \in \mathcal{C} \quad \quad (-) \quad (\forall y \leq u \leq v \leq z)[[u, v] \in \mathcal{C} \to u = v]
\]

holds.

Take another look at clauses \((-\))\). This says that for each component \(I\) of the decomposition

\[
(\forall K \subseteq I)[K \in \mathcal{C} \implies K \in \mathcal{O}]
\]

holds. When \(\Lambda\) is weakly modular this says that \(I \in \text{Neg}(\mathcal{C}_{\text{cal}})\), which helps with the proof of the following.

6.9 LEMMA. Suppose \(\Lambda\) is weakly modular. Then a congruence set \(\mathcal{C} \in \mathbb{C}(\Lambda)\) is separable precisely when it is complemented in \(\mathbb{C}(\Lambda)\).

Proof. Suppose first that the congruence set \(\mathcal{C} \in \mathbb{C}(\Lambda)\) is separable. Each interval has a finite decomposition into sub-intervals such that

\[
I \in \mathcal{C} \cup \text{Neg}(\mathcal{C})
\]

for each component \(I\). Thus the original interval is in

\[
\mathbb{Cng}(\mathcal{C} \cup \text{Neg}(\mathcal{C}))
\]

to show that

\[
\mathcal{C} \cup \text{Neg}(\mathcal{C}) = \mathcal{I}(\Lambda)
\]

where this join is taken in \(\mathbb{C}(\Lambda)\). Since we always have

\[
\mathcal{C} \cap \text{Neg}(\mathcal{C}) = \mathcal{O}(\Lambda)
\]

this shows that \(\mathcal{C}\) is complemented.

Conversely, suppose that \(\mathcal{C}\) is complemented. Each interval is a member of

\[
\mathcal{I}(\Lambda) = \mathbb{Cng}(\mathcal{C} \cup \text{Neg}(\mathcal{C}))
\]

and so has a finite decomposition into intervals where each component is in \(\mathcal{C}\) or \(\text{Neg}(\mathcal{C})\). By the remarks above, this ensures that \(\mathcal{C}\) is separable. \(\blacksquare\)

There is an obvious weakening of this notion of separability obtained by replacing the \((-\)) clause by

\[
(\forall K \in \mathbb{B}_{\text{sc}}(I))[K \in \mathcal{C} \implies K \in \mathcal{O}]
\]

for each component \(I\). This may be worth investigating.

We are now ready to prove the result we have been working towards. If you have been following the citations it should be no suprise that this is the following result due to Grätzer and Schmidt in 1958.
6.10 THEOREM. Let $\Lambda$ be an arbitrary lattice. Then $\text{SPEC}(\Lambda)$ is discrete, that is $\mathbb{C}(\Lambda)$ is boolean, precisely when $\Lambda$ is weakly modular and each $C \in \mathbb{C}(\Lambda)$ is separable.

Proof. Suppose first that $\mathbb{C}(\Lambda)$ is boolean. Then $\Lambda$ is weakly modular by Lemma 6.5, and each $C \in \mathbb{C}(\Lambda)$ is separable by Lemma 6.9. The converse is an immediate consequence of Lemma 6.9.

For later I haven’t been able to work out which prime ideals the $\cap$-irreducible members of $\mathbb{C}(\Lambda)$ correspond to. My instinct is that they correspond to the principal prime filters.

Some remarks

The book [?] has an appendix of 12 pages, written by Grätzer, concerning the congruence lattice $\mathbb{C}(\Lambda)$ of a lattice. In that I can find no recognition that $\mathbb{C}(\Lambda)$ is a spatial frame (a topology).

There are two fairly recent surveys of congruence lattices, [?, ?]. The first of these, [?], is entirely about finite lattices. The second, [?], has a short section on [?, ?] which, at first sight, seems to be about Theorem 5.2 I have not seen [?, ?] but from reading [?] it seems that the space considered is not $\text{Spec}(\Lambda)$ based on $\cap$-irreducible members of $\mathbb{C}(\Lambda)$, but the subspace based on the $\cap$-irreducible members, those points $P$ such that

$$P = \bigcap X \iff P \in X$$

for each $X \subseteq \mathbb{C}(\Lambda)$. Notice that the proof of Theorem 5.2 shows that there are enough of these points to separate $\mathbb{C}(\Lambda)$. However, that doesn’t mean it is the correct space to consider. I think they have missed the point (pun intended).

Warning: The specialization order on $\text{Spec}(\Lambda)$ is the reverse of the inclusion comparison, so the closed sets are lower section. Here it is more convenient to use the reverse ordering.