This version was compiled on July 22, 2004

The Vietoris modifications of a frame

Harold Simmons

Department of Computer Science, The University, Manchester, England

hsimmons@cs.man.ac.uk

Leopold Vietoris, 4 June 1891 – 9 April 2002

The many different point-sensitive Vietoris modifications of a topological space have been around for many years. These constructions produce what are often called hyper-spaces of a space. The point-free version, which is rather more modern, was introduced in [3] and reworked in [4]. Since then nothing much worth reading has been written about the construction, although [6] is an exception to this. Much of what has been written seems to have been from the rather narrow perspective of squeezing a bit more out of some power domain construction or other. I take the wider view that the constructions are bits of topological gadgetry which are interesting in their own right.

Here I set down some of the basic material together with a few observation I have made over the years. As will become clear, these notes are based in part on [3, 4] with the benefit of some 20 years of further development in the use of frames as a tool for studying certain aspects of point set topology. I have also gathered together the contents of various scraps of paper I have had hanging around for a while. I'm not sure there is much new here, but you might find them useful.

Contents

1 Point-sensitive background

In this section I will set down the required point-sensitive background material, that is the material we need from point set topology. Some of this is entirely standard, but some is a little specialized. Some is quite new but has already proved its worth.

We begin with the notation and terminology.

1.1 DEFINITION. Let S be a topological space. This has associated sets

$$\mathcal{O}S \qquad \mathcal{C}S \qquad \mathcal{U}S \qquad \mathcal{Q}S$$

of

open closed saturated compact saturated

subsets of S, respectively.

For a subset E of S we write

$$E^{\circ} \qquad E^{-} \qquad E^{\uparrow}$$

for the

interior closure saturation

of E, respectively.

The specilization order on S is the comparison $\sqsubseteq$ of points given by

$$p \sqsubseteq q \Longleftrightarrow p \in q^{-}$$

for $p, q \in S$. This is a pre-order, is a partial order precisely when the space is T_0, and is equality precisely when the space is T_1.

A saturated set is an upper section of this comparison. Every open set is saturated, and the saturation $E^{\uparrow}$ of a set E is the intersection of all its open supersets. ∎

For those of you who haven't yet got used to the 21st century or are afraid that the bourbakiman might get you, I do not assume that every space is T_2, nor is it necessary to quasify non-hausdorff notions. However, it will do you no harm if you assume that each space is T_0.

I am not quite sure why Vietoris needed his construction. It is something to do with attempting to measure the 'distance' between subsets of a space. He certainly wasn't the first to do this, and there are several other constructions still in use. They often go by the name of hyperspaces. However, without prejudice to other possibilities, let's set down the construction we are interested in here.

1.2 CONSTRUCTION. Let S be any topological space and let $\mathcal{K}S$ be any set of compact subsets K. For $U \in \mathcal{O}S$ let $\Diamond(U)$ and $\Box(U)$ be the subsets of $\mathcal{K}S$ given by

$$K \in \Diamond(U) \Longleftrightarrow K \text{ meets } U \qquad K \in \Box(U) \Longleftrightarrow K \subseteq U$$

(for $K \in \mathcal{K}S$). We use these to topologize $\mathcal{K}S$. That is we take the smallest topology in which each of these sets is open. Thus we use the given topology $\mathcal{O}S$ to doubly indexed a subbase of the topology constructed on $\mathcal{K}S$.

This is the V-modification of S on $\mathcal{K}S$. ∎

In the original Vietoris construction the space S is compact T_2 and $\mathcal{K}S = \mathcal{C}S$ (the family of closed sets). We will look at some other possibilities shortly.

You may wonder what the boxes and diamonds are doing here. They are taken from modal logic (which rarely has anything worth nicking). Essentially a diamond, $\Diamond$, encodes a use of an existential quantifier, $\exists$, and a box, $\Box$, encodes a use of a universal quantifier, $\forall$. Write out the definition of $\Diamond(U)$ and $\Box(U)$ in symbols and you will get the idea.

You may also wonder why we use compact sets (and not arbitrary sets). That will become clear in the Section 4.

1.3 EXAMPLES. (a) The standard V-modification of a space S uses the set $\mathcal{Q}S$ of all compact saturated sets.

(b) For any space S if we let $\mathcal{K}S$ be the set of singletons that we retrieve the parent topology.

(c) A more interesting case is to use the set $\mathcal{L}S$ of all compact lenses of S. ■

This last example needs some explanation, begining with the definition of a lens.

1.4 DEFINITION. A lens of a space S is a subset of the form

$$L = R \cap X$$

where $R \in \mathcal{U}S$ and $X \in \mathcal{C}S$.

A lens is compact if it is a compact subset of the parent space. ■

In the specialization order a lens is a convex set since it is the intersection of an upper set and a lower set. However, not every convex set is a lens since the the lower set must be closed.

Lenses are easy to generate.

1.5 LEMMA. *For each subset E of a space S*

$$E^\ell = E^\uparrow \cap E^-$$

is the smallest lens that includes E.

Proof. Trivially, this set E^ℓ is a lens and includes E. Let

$$L = R \cap X$$

be any lens that includes E. Then

$$E \subseteq R \in \mathcal{U}S \qquad E \subseteq X \in \mathcal{C}S$$

to give

$$E^\uparrow \subseteq R^\uparrow = R \qquad E^- \subseteq X^- = X$$

so that

$$E^\ell \subseteq R \cap X = L$$

as required. ■

Notice that $(\cdot)^\ell$ is a closure operation on subsets of S (but, in general, not a topological closure operation).

By definition, each lens can be represented as the intersection of a closed set and a saturated set. Although there may be many such representations, there is a canonical one. For each lens L we have

$$L = L^{\uparrow} \cap L^{-}$$

(by a particular case of the Lemma).

The family of all lenses has its uses, but here we are concerned with those that are compact (that is, compact as a subset of the parent space).

1.6 LEMMA. *For each lens L of a space S the three conditions*

(i) L is compact

(ii) $L^{\uparrow} \in \mathcal{Q}S$

(iii) L has the form $Q \cap X$ for some $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$.

are equivalent.

Proof. $(i) \Rightarrow (ii)$. Since each open set is saturated, it is easy to check that the saturation of a compact set is compact.

$(ii) \Rightarrow (iii)$. This follows since $L = L^{\uparrow} \cap L^{-}$.

$(iii) \Rightarrow (i)$. Suppose

$$L = Q \cap X$$

where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$, and consider any family $\mathcal{U}$ of open sets with $L \subseteq \bigcup \mathcal{U}$. In the usual way we may assume that $\mathcal{U}$ is directed. We have

$$Q \subseteq X' \cup \bigcup \mathcal{U}$$

and X' is open, so that

$$Q \subseteq X' \cup U$$

for some $U \in \mathcal{U}$ (since Q is compact). Thus $L \subseteq U$, to give the required result. ■

These compact lenses are the sets used in Example 1.3(c). We will see that they arise also in a quite different way.

Each $Q \in \mathcal{Q}S$ is a compact lens (since $Q^{\uparrow} = Q \subseteq Q^{-}$). Thus we have an insertion

$$\mathcal{Q}S \hookrightarrow \mathcal{L}S$$

at the set level. Similarly, for each $p \in S$ the set $p^{\uparrow} \cap p^{-}$ is a compact lens. When S is T_0 this is just the singleton $\{p\}$. Thus, with a bit of not very poetic licence, we have an insertion

$$S \hookrightarrow \mathcal{L}S$$

again at the set level. The three sets $S, \mathcal{Q}S, \mathcal{L}S$ carry topologies, the given topology on S and the V-modifications on the other two. A couple of simple calculations gives the following.

1.7 LEMMA. *For each T_0 space S the two insertions*

$$S \hookrightarrow \mathcal{L}S \qquad\qquad \mathcal{Q}S \hookrightarrow \mathcal{L}S$$

are topological embeddings.

At first sight it looks as though this result can be improved. For each $p \in S$ the saturation $p^{\uparrow}$ is compact, so we have an insertion

$$S \hookrightarrow \mathcal{Q}S$$

(when S is T_0). However, in general, this map is not continuous. Each $U \in \mathcal{O}S$ gives two subbasic open sets $\Diamond(U), \Box(U)$ of $\mathcal{Q}S$. The inverse image of the second is just U, but the inverse image of the first is $U^{\downarrow}$, the downward closure of U. This need not be open in S.

Every space has a specialization order. In particular, both the spaces $\mathcal{Q}S$ and $\mathcal{L}S$ carry a comparison which, in general, is not inclusion, as might be expected

1.8 LEMMA. *For each space S the specialization order on $\mathcal{Q}S$ and $\mathcal{L}S$ are given by, respectively,*

$$Q \sqsubseteq R \Longleftrightarrow R \subseteq Q \subseteq R^{-} \qquad L \sqsubseteq M \Longleftrightarrow M \subseteq L^{\uparrow} \text{ and } L \subseteq M^{-}$$

for $Q, R \in \mathcal{Q}S$ and $L, M \in \mathcal{L}S$, where $(\cdot)^{-}, (\cdot)^{\uparrow}$ are the gadgets of the parent space S.

Proof. Let $(\cdot)^{\sim}$ be the closure operation on $\mathcal{Q}S$ (so as not to confuse it with the closure operation $(\cdot)^{-}$ of S). Then for $Q, R \in \mathcal{Q}S$ we have

$$\begin{aligned} Q \sqsubseteq R &\Longleftrightarrow Q \in R^{\sim} \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S) \begin{bmatrix} Q \in \Diamond(U) \Longrightarrow R \in \Diamond(U) \\ Q \in \Box(U) \Longrightarrow R \in \Box(U) \end{bmatrix} \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S) \begin{bmatrix} Q \text{ meets } U \Longrightarrow R \text{ meets } U \\ Q \subseteq U \Longrightarrow R \subseteq U \end{bmatrix} \quad \Longleftrightarrow \begin{bmatrix} Q \subseteq R^{-} \\ R \subseteq Q \end{bmatrix} \end{aligned}$$

to give the required result. At the last step we remember that the saturation of a set is the intersection of all its open supersets.

For the second part let $(\cdot)^{\sim}$ be the closure operation on $\mathcal{L}S$. For $L, M \in \mathcal{L}S$ we have

$$\begin{aligned} L \sqsubseteq M &\Longleftrightarrow Q \in R^{\sim} \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S) \begin{bmatrix} L \in \Diamond(U) \Longrightarrow M \in \Diamond(U) \\ L \in \Box(U) \Longrightarrow M \in \Box(U) \end{bmatrix} \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S) \begin{bmatrix} L \text{ meets } U \Longrightarrow M \text{ meets } U \\ L \subseteq U \Longrightarrow M \subseteq U \end{bmatrix} \quad \Longleftrightarrow \begin{bmatrix} L \subseteq M^{-} \\ M \subseteq L^{\uparrow} \end{bmatrix} \end{aligned}$$

to give the required result. ■

Later on we will generalize this result.

So far most of this section has been standard material, or at least material that doesn't look too strange. However, when first seen the next part seems a bit pointless. In fact, we will make good use of it in Sections 6 and 7 where we will see that it is not pointless but is the point-sensitive view of a point-free construction.

We fixed a space S which we may assume to be T_0, and fix some $Q \in \mathcal{Q}S$. We use this to produce an operation on $\mathcal{C}S$. (In fact, we could use any subset Q of S but that generality is not needed here.)

1.9 DEFINITION. Let S be a space and let $Q \in \mathcal{Q}S$. For each $X \in \mathcal{C}S$ we set

$$\partial_Q(X) = \bigcap \big\{ (X \cap U)^- \,|\, Q \subseteq U \in \mathcal{O}S \big\}$$

to produce the Q-derivative on $\mathcal{C}S$. ■

Why does anyone want to use this? Be patient and all will be explained. For the time being let's look at some of its properties. Trivially we have

$$\partial_Q : \mathcal{C}S \longrightarrow \mathcal{C}S$$

and the operation is deflationary and monotone, that is

$$\partial_Q(X) \subseteq X \qquad \partial_Q(Y) \subseteq \partial_Q(X)$$

for all $X, Y \in \mathcal{C}S$ with $Y \subseteq X$. The operation has a more important property which partly explains the terminology 'derivative'.

1.10 LEMMA. *We have*

$$\partial_Q(X \cup Y) = \partial_Q(X) \cup \partial_Q(Y)$$

for all $X, Y \in \mathcal{C}S$.

Proof. Since the operation is monotone is suffices to show that

$$\partial_Q(X \cup Y) \subseteq \partial_Q(X) \cup \partial_Q(Y)$$

holds. In fact, we prove the contrapositive inclusion. Thus consider any point

$$p \notin \partial_Q(X) \cup \partial_Q(Y)$$

(so we want $p \notin \partial_Q(X \cup Y)$). Since we have both

$$p \notin \partial_Q(X) \qquad p \notin \partial_Q(Y)$$

we have

$$p \notin (X \cap U_X)^- \qquad p \notin (Y \cap U_Y)^-$$

for some open sets $U_X, U_Y \in \mathcal{O}S$ with $Q \subseteq U_X \cap U_Y$. This gives some $Q \subseteq U \in \mathcal{O}S$ with

$$p \notin (X \cap U)^- \qquad p \notin (Y \cap U)^-$$

(for we may take $U = U_X \cap U_Y$). But now we have

$$X \cap U \cap V_X = \emptyset = Y \cap U \cap V_Y$$

for some open sets $V_X, U_V \in \mathcal{O}S$ with $p \in V_X \cap V_Y$. This gives some $p \in V \in \mathcal{O}S$ with

$$X \cap U \cap V = \emptyset = Y \cap U \cap V$$

(for we may take $V = V_X \cap V_Y$). Finally, since

$$(X \cup Y) \cap U \cap V = (X \cap U \cap V) \cup (Y \cap U \cap V) = \emptyset$$

we have

$$p \notin ((X \cup Y) \cap U)^-$$

to give the required result. ■

You may not recognize these properties, but you have seen them before. Let lim be the Cantor-Bendixson derivative on the space S. Thus

$$\mathsf{lim} : \mathcal{C}S \longrightarrow \mathcal{C}S$$

is the operation which extracts the set $\mathsf{lim}(X)$ of limit points of each closed set X. This operation is deflationary and monotone and, more importantly, we have

$$\mathsf{lim}(X \cup Y) = \mathsf{lim}(X) \cup \mathsf{lim}(Y)$$

for all $X, Y \in \mathcal{C}S$. This derivative is used to attach an ordinal valued rank, the CB-rank, to a space. This is an important indicator of some of the pathological properties of a space.

In the same way the Q-derivative can be used to attach a rank to S. A full discussion of this can be found in [7]. We won't need this here but we will need the analogue of the perfect part of a space.

1.11 DEFINITION. Let S be a space and let $Q \in \mathcal{Q}S$. For each ordinal α let

$$Q(\alpha) = \partial_Q^\alpha(S)$$

that is set

$$Q(0) = S \qquad Q(\alpha+1) = \partial_Q(Q(\alpha)) \qquad Q(\lambda) = \bigcap\{Q(\alpha) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. ■

This produces a descending chain of closed sets with

$$Q \subseteq Q(\alpha)$$

for each ordinal α. On cardinality grounds the chain eventually stabilizes at some ordinal ∞. It is know that this Q-rank can be arbitrarily large. Furthermore, we have

$$Q^- \subseteq Q(\infty)$$

and these can be far apart. In Section 9 we describe a large space S which is T_1 and sober and where Q^- is a singleton but $Q(\infty)$ is the whole space.

We will see that the behaviour of these Q-chains has a significant impact of the nature of the various V-modifications of a space.

1.12 DEFINITION. Let S be a space and let $Q \in \mathcal{Q}S$. For each $X \in \mathcal{C}S$ we write

$$Q \ltimes X$$

if $\partial_Q(X) = X$, that is if

$$X \subseteq (X \cap U)^-$$

for each $Q \subseteq U \in \mathcal{O}S$. ■

The sets $X \in \mathcal{C}S$ with $Q \ltimes X$ are the analogues of the perfect subsets. We will see that the pairs (Q, X) where $Q \ltimes X$ and X is large enough form the points of a V-modification of the parent space which, in some ways, is better than those given by Construction 1.2.

2 Point-free background

Much of what we do takes place within the category ***Frm*** of frames and uses is battery of gadgets, so a short refresher won't go amiss.

A frame is a complete lattice

$$(A,\ \leq,\ \top,\ \wedge,\ \bot,\ \bigvee)$$

for which the distributive law

$$a \wedge \bigvee X = \bigvee\{a \wedge x \mid \in X\}$$

holds for each element $a \in A$ and subset $X \subseteq A$. In other words, a frame A is a complete heyting algebra, and so carries an implication $(\cdot \supset \cdot)$ given by

$$x \leq (a \supset b) \Longleftrightarrow a \wedge x \leq b$$

for $a, b, x \in A$.

A frame morphism

$$A \xrightarrow{\ f = f^*\ } B$$

is a monotone function between the frames which preserves the distinguished attributes $\leq, \top, \wedge, \bot, \bigvee$. It need not preserve other operations (such as the infimum $\bigwedge$ or the implication $\supset$ carried by the frames.)

Each frame morphism $f = f^*$, as above, has a right adjoint

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

satisfying

$$f^*(x) \leq y \Longleftrightarrow x \leq f_*(y)$$

for $x \in A$ and $y \in B$. This adjoint f_* need not be a frame morphism, but it is a useful tool at times. When it is important we write f^* for the associated morphism, otherwise we write f (which accounts for the dual naming above).

This sets up the category ***Frm*** of frames. It is related to the category ***Top*** of topological spaces via a schizophrenic adjunction.

For each space S the topology $\mathcal{O}S$ is a frame (under the obvious set theoretic operations). For each continuous map

$$T \xrightarrow{\ \phi\ } S$$

the inverse image function

$$\mathcal{O}S \xrightarrow{\ \phi^{\leftarrow}\ } \mathcal{O}T$$

is a frame morphism. You may like to work out the right adjoint of $\phi^{\leftarrow}$.

This sets up one half of a contravariant adjunction

$$\textit{\textbf{Top}} \underset{\mathsf{pt}}{\overset{\mathcal{O}}{\rightleftarrows}} \textit{\textbf{Frm}}$$

where the other functor pt is more interesting.

2.1 LEMMA. *For a frame A the following gadgets are in bijective correspondence.*

- *Completely prime filters on A, that is filters P such that*

$$\bigvee X \in P \Longrightarrow X \cap P \neq \emptyset$$

  for each subset X.

- *Characters of A, that is from morphisms*

$$A \xrightarrow{\ \pi\ } \mathsf{2}$$

  to the 2-element frame.

- *Meet-irreducible elemements of A, that is elements $p \neq \top$ such that*

$$x \wedge y \leq p \Longrightarrow x \leq p \text{ or } y \leq p$$

  for each $x, y \in A$.

The correspondences are given by

$$x \leq p \Longleftrightarrow \pi(x) = 0 \qquad \pi(x) = 1 \Longleftrightarrow x \in P$$

for $x \in A$.

These are the points of A. We may view them in any of the three ways, but it is usually the $\wedge$-irreducible elements that provide the clearer picture.

Let $S = \mathsf{pt}(A)$ be the set of points of A. Viewing these points as $\wedge$-irredicible, for each $x \in A$ we extract $\circledcirc^*(x) \subseteq \mathsf{pt}(A)$ by

$$p \in \circledcirc^*(x) \Longleftrightarrow x \nleq p$$

(for $p \in S$). We find that

$$\mathcal{O}S = \{\circledcirc^*(x) \mid x \in A\}$$

is a topology on S, and the assignment $x \longmapsto \circledcirc^*(x)$ is a surjective frame morphism.

We call $\mathsf{pt}(A)$ the point space of A, and the morphism $\circledcirc^*$ is the spatial reflation of A. As a morphism this spatial reflectyion has a right adjoint

$$A \underset{\circledcirc_*}{\overset{\circledcirc^*}{\rightleftarrows}} \mathcal{O}S$$

and we find that

$$\circledcirc_*(X') = \bigwedge X \qquad p \in X \Longleftrightarrow p \leq \bigwedge X$$

for $X \in \mathcal{C}S$ and $p \in S$. We will return to this shortly.

In the usual way of universal algebra, each surjective frame morphism $A \longrightarrow B$ is determined up to canonical isomorphism by a congruence on A. For frames there is a neater way of getting at these.

2.2 DEFINITION. Let A be a frame.

A pre-nucleus on A is a function $f : A \longrightarrow A$ which is inflationary, monotone, and preserves meets, that is

$$x \leq f(x) \qquad x \leq y \Longrightarrow f(x) \leq f(y) \qquad f(x \wedge y) = f(x) \wedge f(y)$$

for each $x, y \in A$.

A nucleus on A is a pre-nucleus j that is idempotent, that is $j^2 = j$. ■

We have already met several examples of pre-nuclei (in disguised form). Consider a space S, some $Q \in QS$, and let ∂_Q be the asosciated Q-derivative. The dual complement f_Q of ∂_Q is a pre-nucleus on $\mathcal{O}S$. This is given by

$$f_Q(U) = \partial_Q(U')'$$

for each $U \in \mathcal{O}S$. Lemma 1.10 converts into the crucial property.

Each frame A carries families of nuclei

$$u_a \qquad v_a \qquad w_a$$

indexed by $a \in A$, and given by

$$u_a(x) = a \vee x \qquad v_a(x) = (a \supset x) \qquad w_a(x) = ((x \supset a) \supset a)$$

for $x \in A$. These are the componenents out of which many nuclei are built.

By setting

$$j = f_* \circ f^*$$

each morphism from a frame

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

gives a nucleus on that frame. This is the kernel of the morphism.

Conversely, each nucleus j on a frame A is the kernel of a morphism frame A. We take

$$A_j = j[A] = \{x \in A \mid j(x) = x\}$$

with a fairly obvious structure, to show that

$$\begin{array}{ccc} A & \xrightarrow{\ j^*\ } & A_j \\ x & \longmapsto & j(x) \end{array}$$

is a frame morphism with kernel j. You might like to work out the right adjoint of j^*.

Let NA be the poset of nuclei on A ordered pointwise. This is a complete lattice with infima cimputed pointwise. However, suprema are not computed pointwise, and often involve a hidden ordinal iteration. This lattice NA is itself a frame, and is the assembly of A.

Let A be a frame with points space $S = \mathsf{pt}(A)$. The spatial reflection

$$A \underset{\circledcirc_*}{\overset{\circledcirc^*}{\rightleftarrows}} \mathcal{O}S$$

can be used to generate other spatial quotients of A. Let K be any subset of S and view K as a subspace of S. The embedding of K in S is continuous and so gives a frame morphism $\mathcal{O}S \longrightarrow \mathcal{O}K$ which we may adjoin to the spatial reflection

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{O}S & \longrightarrow & \mathcal{O}K \\ a & \longmapsto & & & \circledcirc(a) \cap K \end{array}$$

to obtain a quotient of A. This gives us an adjoint pair

$$A \underset{k_*}{\overset{k^*}{\rightleftarrows}} \mathcal{O}K$$

and a nucleus k on A, where

$$k^*(a) = \circledcirc(a) \cap K \qquad k = k_* \circ k^*$$

(for $a \in A$). We describe the right adjoint k_* and the kernel k.

Each $E \subseteq K$ is a subset of S, and hence is a set of elements of A (the $\wedge$-irreducible elements that belong to K). In particular we can form the infimum $\bigwedge E$ in A.

Each open set of K has the form $K \cap U$ for some $U \in \mathcal{O}S$. This gives a closed subset $K - U$ of K.

2.3 LEMMA. *Let A be a frame with points space $S = \mathsf{pt}(A)$, let K be a subset of S, and consider the morphism*

$$A \longrightarrow \mathcal{O}K$$

as described above. For each open set $U \in \mathcal{O}S$ and $a \in A$ we have

$$k_*(K \cap U) = \bigwedge(K - U) \qquad k(a) = \bigwedge\{p \in K \mid a \leq p\}$$

where each infimum is taken in A.

Proof. For each element $a \in A$ we have

$$\begin{aligned} a \leq k_*(K \cap U) &\iff k^*(a) \subseteq K \cap U \\ &\iff (\forall p \in S)[p \in \circledcirc(a) \cap K \Longrightarrow p \in K \cap U] \\ &\iff (\forall p \in K)[p \in \circledcirc(a) \Longrightarrow p \in U] \\ &\iff (\forall p \in K)[p \notin U \Longrightarrow a \leq p] \\ &\iff (\forall p \in S)[p \in (K - U) \Longrightarrow a \leq p] \qquad \iff a \leq \bigwedge(K - U) \end{aligned}$$

as required.

Using this for each $a \in A$ we have

$$k(a) = k_*(k^*(a)) = \bigwedge(K - \circledcirc(a)) = \bigwedge\{p \in K \mid a \leq p\}$$

to give the required result. ■

We will need some information about the block structure of a frame. This ought to be better known but isn't.

2.4 DEFINITION. Let A be a frame.

(1) An element $x \in A$ is admitted by a nucleus j on A if $j(x) = \top$.

(2) For a nucleus j on A the set $\nabla(j)$ of all elements admitted by j is the admitting filter of j.

(3) A filter on A is admissible if it has the form $\nabla(j)$ for at least one nucleus j. ■

There are two observations which we will expand on in this section. Firstly, there may be filters on a frame that are not admissible. Secondly, although each nucleus j determines an admissible filter $\nabla(j)$, the filter need not determine the parent nucleus. Although the validity of these observations may not be clear just yet, they will be by the end of this section.

2.5 EXAMPLES. Let A be any frame.

(a) Each completely prime filter P on A is related to a $\wedge$-irreducible element p and a character $\pi : A \longrightarrow 2$ of A. In fact

$$x \in P \Longleftrightarrow \pi(x) = q \Longleftrightarrow x \not\leq p$$

which, more or less, shows that P is admissible. It can be checked that

$$w_p(x) = \begin{cases} \top \text{ if } x \in P \\ p \text{ if } x \leq p \end{cases}$$

to show that $\nabla(w_p) = P$.

(b) Each element $a \in A$ gives a nucleus v_a, and for this we have

$$x \in \nabla(v_a) \Longleftrightarrow (a \supset x) = \top \Longleftrightarrow a \leq x$$

so that $\nabla(v_a)$ is the principal filter above a.

This shows that in any frame each principal filter is admissible.

(c) Now suppose that the frame A is boolean. Consider any admissible filter $\nabla(j)$ on A. Since A is boolean the parent nucleus j has the form $j = v_a$ for some $a \in A$ (namely $\neg(j\bot)$). Thus the filter is principal.

In other words, for a boolean frame the admissible filters are precisely the principal filters. ■

In the next section we will see a larger class of admissible filters that includes all the completely prime filters.

It is not hard to see that many different nuclei can give the same admissible filter. We take a closer look at this feature.

2.6 DEFINITION. Let A be a frame. For nuclei j, k on A we say j, k are companions if $\nabla(j) = \nabla(k)$.

This puts an equivalence relation on NA, and a block is an equivalence class of this relation.

A nucleus is alone if it is the only member of its block. ■

The examples of Subsections 9.2 and 9.3 show that a block can have a quite complicated structure. However, no matter how complicated it is a block always has a special member.

Consider any family $\mathcal{J}$ of nuclei all in the same block with the common admitting filter ∇. Then for each $x \in \nabla$ we have

$$(\bigwedge \mathcal{J})x = \bigwedge\{jx \mid j \in \mathcal{J}\} = \bigwedge\{\top\} = \top$$

so that $\bigwedge \mathcal{J}$ is also in the same block. This proves the following.

2.7 LEMMA. *Each block of a frame is closed under arbitrary infima. In particular, each block has a least member.*

These least nuclei play a special role in the analysis of frames, so they are given a special name.

2.8 DEFINITION. A nucleus is fitted if it is the least nucleus in its block. ■

We are going to spend a little while obtaining various properties of fitted nuclei and their impact on frames. We begin with a simple example. The proof of the following result is a routine exercise.

2.9 LEMMA. *For a frame A we have*

$$v_a \leq j \iff a \in \nabla(j)$$

for each $a \in A$ and nucleus j on A.

In particular, each nucleus v_a is fitted.

Not every fitted nucleus is a v-nucleus, but every fitted nucleus is built out of v-nuclei in a uniform way.

Let ∇ be any filter on the frame A. For $a, b \in \nabla$ we have

$$v_a \vee v_b = v_{a \wedge b}$$

so that

$$V_\nabla = \{v_a \mid a \in \nabla\}$$

is a directed family of nuclei. Let $f = f_F$ be the pointwise supremum of this family V_∇. Thus

$$f(x) = \bigvee\{(a \supset x) \mid a \in \nabla\}$$

for each $x \in A$, and f is a pre-nucleus. Consider the family

$$(f^\alpha \mid \alpha \text{ an ordinal})$$

of ordinal iterates of f generated by

$$f^0(x) = x \qquad f^{\alpha+1} = f(f^\alpha(x)) \qquad f^\lambda(x) = \bigvee\{f^\alpha(x) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ (and $x \in A$). It is easy to check that this is an ascending chain of pre-nuclei. On cardinality grounds the chain stabilizes at some ordinal ∞, and then

$$v_\nabla = f^\infty$$

is a nucleus. In fact,

$$v_\nabla = \bigvee V_\nabla$$

is the supremum of V_∇ in NA.

We continue with this notation in the proof of the following.

2.10 LEMMA. *Let ∇ be a filter on the frame A.*

(1) The nucleus v_∇ is the least nucleus which admits each member of ∇.

(2) The filter $\nabla(v_\nabla)$ is the least admissible filter which includes ∇.

(3) The nucleus v_∇ is fitted.

(4) If $\nabla = \nabla(j)$ (for some nucleus j) then v_∇ is the least companion of j.

Proof. We use the pre-nucleus f constructed above.

(1) For each $a \in \nabla$ we have $v_\nabla(a) = \top$, and hence $f(a) = \top$. Thus $v_\nabla(a) = \top$, that is v_∇ admits a. Conversely, suppose j admits each member of ∇, that is $j(a) = \top$ for each $a \in \nabla$. Consider any $a \in \nabla$ and $x \in A$. Let $y = v_a(x) = (a \supset x)$. Then $y \wedge a \leq x$ so that

$$y \leq j(y) = j(y) \wedge j(a) = j(y \wedge a) \leq j(x)$$

to show that $v_a \leq j$. Thus $f \leq j$, and hence $v_\nabla = f^\infty \leq j$, as required.

(2) Consider any admissible filter $\nabla(j)$ with $\nabla \subseteq \nabla(j)$. As indicated, this second filter arises from some nucleus j. But this j admits each member of $\nabla(j)$, and hence each member of ∇, so that $v_\nabla \leq j$ by part (1). Thus $\nabla(v_\nabla) \subseteq \nabla(j)$, as required.

(3) Suppose j is a companion of v_∇. Then j admits each member of ∇, and hence $v_\nabla \leq j$ by part (1).

(4) This is another immediate consequence of (1). ■

This result gives us a better handle on fitted nuclei.

2.11 COROLLARY. *A nucleus j on a frame A is fitted precisely when*

$$j = \bigvee V$$

for some set V of v-nuclei.

Proof. Part (4) of the Lemma shows that every fitted nucleus can be decomposed in this way. Conversely, suppose $j = \bigvee V$ where

$$V = \{v_y \mid y \in Y\}$$

for some subset Y of A. Let ∇ be the filter generated by Y. Thus ∇ is the set of all elements $a \in A$ such that

$$y_1 \wedge \cdots \wedge y_n \leq a$$

for some $y_1, \ldots, y_n \in Y$. For each such selection we have

$$v_a \leq v_{y_1 \wedge \cdots \wedge y_n} = v_{y_1} \vee \cdots \vee v_{y_n} \leq j$$

so that

$$v_\nabla \leq \bigvee V = j$$

and the converse comparison is trivial (since $Y \subseteq \nabla$. Thus j is fitted. ■

Each filter ∇ on a frame A determines a block within NA. This block has a least member v_∇ which (because of the ordinal iteration involved) can be quite difficult to calculate. Furthermore, the structure of the block can be quite intricate. Beginning in Section 3 we will look at some special kinds of blocks, when the following result will at least motivate the notation used. You may like to amuse yourself by finding a proof.

2.12 LEMMA. *For each frame A and $a \in A$ the nucleus w_a is the greatest member of its block.*

Each filter ∇ on a frame A gives a fitted nucleus v_∇ on A, and so determines a quotient

$$A_\nabla = A_{v_\nabla}$$

of A. When ∇ is admissible it can be retrieved from v_∇ (for in that case $\nabla = \nabla(v_\nabla)$, if this isn't too nablous). Each of A and A_∇ has a point space and, on general grounds, there is a continuous map

$$\mathsf{pt}(A_\nabla) \longrightarrow \mathsf{pt}(A)$$

induced by the quotient morphism $A \longrightarrow A_\nabla$. We locate $\mathsf{pt}(A_\nabla)$ as a subspace of $\mathsf{pt}(A)$.

2.13 LEMMA. *Let A be a frame with $S = \mathsf{pt}(A)$. Let ∇ be a filter of A with associated pre-nucleus $f = f_\nabla$. Then*

$$f(p) = \begin{cases} \top & \textit{if } p \in \nabla \\ p & \textit{if } p \notin \nabla \end{cases}$$

for each $p \in S$.

Proof. We have

$$f(p) = \bigvee \{(x \supset p) \mid x \in \nabla\}$$

so that $p \leq f(p)$ and $f(p) = \top$ if $p \in \nabla$. Suppose $p \notin \nabla$, consider any $x \in \nabla$, and let $y = (x \supset p)$. We have $x \wedge y \leq p$ and $x \nleq p$ (for otherwise $p \in \nabla$) so that $y \leq p$. Letting x vary through ∇ gives $f(p) \leq p$, as required. ■

Remember that we can view a point of A_∇ as a character

$$A_\nabla \longrightarrow 2$$

and this gives a character

$$A \longrightarrow A_\nabla \longrightarrow 2$$

of A. This is how the map $\mathsf{pt}(A_\nabla) \longrightarrow \mathsf{pt}(A)$ comes about. Which points of A arise in this way? By Lemma 2.13 we have

$$v_\nabla(p) = \begin{cases} \top & \text{if } p \in \nabla \\ p & \text{if } p \notin \nabla \end{cases}$$

for each $p \in S$. Thus each point p of A is either destroyed by sending it to $\top$ or preserved as a point of A_∇. This proves the following.

2.14 LEMMA. *Let A be a frame with $S = \mathsf{pt}(A)$, and let ∇ be a filter of A. Then*

$$\mathsf{pt}(A_\nabla) = S - \nabla$$

where this difference is viewed as a subspace of S.

We are going to develop this theme quite a bit further.

3 The HM result done properly

We have seen that for a space S the family $\mathcal{Q}S$ of compact saturated subsets is quite heavily involved in the various point-sensitive V-modifications of S. In this section we obtain a point-free analogue of $\mathcal{Q}S$ as well as some other useful gadgets. The original Hofmann-Mislove results shows that for a sober space S there is a bijective correspondence between $\mathcal{Q}S$ and the open filters on $\mathcal{O}S$ (where these will be defined shortly). More generally, for a frame A with point space S, there is a bijective correspondence between $\mathcal{Q}S$ and the open filters on A. Each such filter is admissible and so determines a block within NA. As usual, this block has a least member, but unusually such a block also has a greatest member. The structure of these blocks is intimately connected with the various V-modifications we are going to look at.

Before we get to the details of this correspondence it is useful to set down a bit of general background material.

Let A be a frame with point space S and consider

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{O}S \\ a & \longmapsto & \circledcirc(a) \end{array}$$

the canonically associated spatial reflection frame morphism. Each subset K of S produces a surjective morphism

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{O}S & \longrightarrow & \mathcal{O}K \\ a & \longmapsto & & & \circledcirc(a) \cap K \end{array}$$

by viewing K as a subspace of S. This situation is discussed in Lemma 2.3. In particular, the kernel k of this morphism is given by

$$k(a) = \bigwedge \{p \in K \mid a \leq p\}$$

(for $a \in A$) where this infimum is taken in A. The admitting filter $\nabla(K)$ of k is given by

$$a \in \nabla(K) \iff \circledcirc(a) \cap K = K \iff K \subseteq \circledcirc(a)$$

(for $a \in A$). In other words $\nabla(K)$ is an indexing of the open neighbourhood filter of K.

Notice that although the set K determines the filter $\nabla(A)$, the filter does not determine the set. Let $K^\uparrow$ be the saturation of the set K in S. Since open sets are saturated we have

$$a \in \nabla(K^\uparrow) \iff \uparrow K \subseteq \circledcirc(a) \iff K \subseteq \circledcirc(a) \iff a \in \nabla(K)$$

to show that $\nabla(K^\uparrow) = \nabla(K)$.

3.1 LEMMA. *For a subset K of the point space S of a frame A we have*

$$p \in K^{\uparrow} \iff p^{-\prime} \notin \nabla(K)$$

for each $p \in S$.

Proof. Remembering that the closure p^{-} of a point $p \in S$ is just the downward closure of p in S, we have

$$p \notin K^{\uparrow} \iff p^{-} \cap K = \emptyset \iff K \subseteq p^{-\prime} \iff p^{-\prime} \in \nabla(K)$$

to give the required result. ■

This result shows that each $Q \in \mathcal{Q}S$ is determined by the filter $\nabla(Q)$, and this is the filter $\nabla(K)$ for any $K \subseteq Q$ with $K^{\uparrow} = Q$. What kind of filter is $\nabla(Q)$, and is there any 'canonical' generating subset $K \subseteq Q$? We can answer both these questions in a nice way.

A filter ∇ on A is a non-empty upper section which is closed under binary meets. The complement $A - \nabla$ in A is a lower section but need not be closed under binary joins. We impose a restriction on the complement and so obtain a special class of filters.

3.2 DEFINITION. A filter ∇ on a frame A is (Scott) open if the complement $A - \nabla$ is closed under directed suprema. ■

In other words the filter ∇ is open precisely when

$$\bigvee X \in \nabla \Longrightarrow X \text{ meets } \nabla$$

holds for each directed subset X of A. Intuitively this is some kind of compactness property, and for once our intuition is correct.

3.3 LEMMA. *For a set K of points of a frame A, the filter $\nabla(K)$ is open precisely when the set K is compact (in the point space).*

Proof. Suppose first that $\nabla(K)$ is open, and consider any directed subset X of A which gives a covering

$$\mathcal{U} = \{\circledcirc(x) \mid x \in X\}$$

of K. Thus

$$K \subseteq \bigcup \mathcal{U} = \circledcirc(\bigvee X)$$

so that $\bigvee X \in \nabla(K)$, and hence X meets $\nabla(K)$ at x say, to give $K \subseteq \circledcirc(x)$.

Conversely, suppose K is compact, and consider any directed subset X of A with $\bigvee X \in \nabla(K)$. Then

$$\mathcal{U} = \{\circledcirc(x) \mid x \in X\}$$

is a directed covering of K, which leads to some $x \in X \cap \nabla(K)$. ■

This shows that each $Q \in \mathcal{Q}S$ determines and is determined by an open filter $\nabla(Q)$. But does every open filter arise in this way? Yes it does.

We need to fix a bit of notation.

Let ∇ be an open filter on the frame A. We look at the complement $A - \nabla$ in A. Let M be the set of maximal elements of this complement, that is the set of elements $m \in A - \nabla$ such that

$$m \leq x \in \nabla \Longrightarrow m = x$$

holds for each $x \in A$. In general there is no reason to suppose that M is non-empty, but since ∇ is open we know that $A - \nabla$ is closed under directed suprema, and hence a application of Zorn's Lemma give these following.

3.4 LEMMA. *Let ∇ be an open filter in the frame A and let M be the maximal non-members of ∇, as above. Then for each $a \in A - \nabla$ there is some $m \in M$ with $a \leq m$.*

This is the crucial result from which all the other non-trivial results follow.

Recall that we view the points of A as the $\wedge$-irreducible members of A. In this way we convert the open filter ∇ into a set of points.

3.5 LEMMA. *For the situation above, each member m of M is a point of A.*

Proof. Since $\top \in \nabla$ we have $m \neq \top$. Consider elements $x, y \in A$ with $x \wedge y \leq m$ and, by way of contradiction, suppose both $x \not\leq m$ and $y \not\leq m$. Then $m < m \vee x$ and $m < m \vee y$ so that $m \vee x, m \vee y \in \nabla$ (by the maximality of m). This gives

$$m = m \vee (x \wedge y) = (m \vee x) \wedge (m \vee y) \in \nabla$$

which is the contradiction. ■

This associates with the open filter ∇ a set of points M, and hence gives us a second filter $\nabla(M)$. Or does it?

3.6 LEMMA. *For each open filter ∇ on the frame A we have $\nabla = \nabla(M)$ where M is the set of maximal non-members of ∇.*

Proof. Consider any $a \in A$. Remembering how the open set $\circledcirc(a)$ is constructed we have

$$a \in \nabla(M) \Longleftrightarrow M \subseteq \circledcirc(a) \Longleftrightarrow (\forall m \in M)[a \not\leq m]$$

so that taking the contrapositive gives

$$a \notin \nabla(M) \Longleftrightarrow (\exists m \in M)[a \leq m] \Longleftrightarrow a \in A - \nabla$$

where the last equivalence follows by Lemma 3.4. Thus

$$A - \nabla(M) = A - \nabla$$

and hence $\nabla = \nabla(M)$, as required. ■

By Lemma 3.3 the set M is compact (in S) and hence so is its saturation $Q = \uparrow M$, and we have $\nabla = \nabla(Q)$. At this stage we must remember that the specialization order $\sqsubseteq$ on S is the reverse of the comparison induced by that on A. Thus

$$p \sqsubseteq q \Longleftrightarrow q \leq p$$

for $p, q \in S$, that is $\wedge$-irreducible elements of A. The saturation of M is taken with respect to $\sqsubseteq$, and hence in terms of $\leq$ we have

$$q \in Q \Longleftrightarrow (\exists m \in M)[q \leq m]$$

for each $q \in S$.

This more or less finishes the proof of the following variant of the HM result.

3.7 THEOREM. *For each frame A there is a bijective correspondence*

$$\nabla \longleftrightarrow Q$$

between the open filters ∇ on A and the compact saturated subsets Q of the point space S of A. This is given by

$$\nabla = \nabla(Q) \qquad Q = S - \nabla$$

(where S is viewed as the set of $\wedge$-irreducible elements of A).

Proof. There are still one or two details to sort out. The main one is the equality $Q = S - \nabla$. Thus starting from the open filter ∇ let M be the set of maximal members of $A - \nabla$ and let Q be the saturation of M. The characterization of Q in terms of $\leq$ and Lemma 3.4 show that $Q = S - \nabla$. ■

Of course, each sober space S determines and is determined by a frame, its topology. The original HM result is the corresponding spatial case of Theorem 3.7.

3.8 COROLLARY. *For each sober space S there is a bijective correspondence between the open filters ∇ on $\mathcal{O}S$ and the members $Q \in \mathcal{Q}S$. This is given by*

$$U \in \nabla \Longleftrightarrow Q \subseteq U \qquad Q = \bigcap \nabla$$

(for $U \in \mathcal{O}S$).

We will use the general correspondence $\nabla \longleftrightarrow Q$ for a frame A many times. We will also use the set M of maximal elements of $A - \nabla$. Since each $m \in M$ is in S (by Lemma 3.5) we have $M \subseteq Q$. Futhermore, we have $Q = M^{\uparrow}$ and relative to the specialization order M is the set of minimal members of Q. We refer to M as the minimal generating set of Q.

This construction has another important consequence which was originally proved by a quite different method. For the open filter ∇ on A, let K be any subset of S with $M \subseteq K \subseteq Q$. The admitting filter of the kernel of the surjective frame morphism

$$\begin{array}{rcl} A & \longrightarrow & \mathcal{O}K \\ a & \longmapsto & \circledcirc(a) \cap K \end{array}$$

is $\nabla(K) = \nabla$, to give the following.

3.9 COROLLARY. *Each open filter on a frame is admissible.*

As mentioned, this result was first proved by a quite different method. That method can be refined to extract a lot more information about the open filters on a frame. We don't need that refinement here, but details can be found in [7].

The open filter ∇ on A is admissible and, as explained in Section 2, determines a block in NA with least member v_∇. This gives a two-step quotient

$$A \longrightarrow A_\nabla \longrightarrow \mathcal{O}\mathsf{pt}(A_\nabla)$$

of A, and Lemma 2.14 tells us what the point space is.

3.10 LEMMA. *Let A be a frame with point space S. Let ∇ be an open filter on A, and let Q be the corresponding member of $\mathcal{Q}S$. The points space of A_∇ is Q viewed as a subspace of S.*

Proof. By Lemma 2.14 and Theorem 3.7 we have

$$\mathsf{pt}(A_\nabla) = S - \nabla = Q$$

as required. ■

In general for an admissible filter we can not say much more about the corresponding block, but open filters are rather special.

Using Lemma 3.10 each open filter ∇ on the frame A determines a chain

$$A \longrightarrow A_\nabla \longrightarrow \mathcal{O}Q \longrightarrow \mathcal{O}M$$

of three quotients of A where the new step arises from the subspace M of Q. Thus the three-step quotient is

$$\begin{array}{rcl} A & \longrightarrow & \mathcal{O}M \\ a & \longmapsto & U(a) \cap M \end{array}$$

and its admitting filter is

$$\nabla(M) = \nabla$$

the original open filter. This suggest that the corresponding nucleus might be worth looking at, and we are not disappointed.

3.11 DEFINITION. Let ∇ be an open filter ∇ on a frame A, and let M be the set of maximal non-member of ∇. For each $a \in A$ let

$$w_\nabla(a) = \bigwedge\{m \in M \mid a \leq m\}$$

to obtain a nucleus w_∇ on A. ■

Because of the special position of M we can expect w_∇ to be in a special position in its block, but do we expect the following?

3.12 LEMMA. *For each open filter ∇ on a frame the associated nucleus w_∇ is the greatest member of its block.*

Proof. Consider any companion j of w_∇, that is any nucleus j on A with $\nabla(j) = \nabla$. Consider any $m \in M$. If $jm < m$ then $jm \in \nabla$ (by the maximality of m), so that $jm = j(jm) = \top$, and hence $m \in \nabla$, which is false. Thus $jm = m$.

Now consider any $a \in A$ and $m \in M$ with $a \leq m$. Then $ja \leq jm = m$, so that $ja \leq ka$ (since m is arbitrary), as required. ■

For a frame A each open filter ∇ on A gives us a bounded block

$$[v_\nabla, w_\nabla]$$

in NA. We will see from the examples in Section 9 that such a block can be quite complicated, and its structure is related to the properties of the Vietoris construction.

Notice that the diagram above gives us another member of the block, namely the nucleus of the quotient to $\mathcal{O}Q$. This is best seen for $\mathcal{O}S$. We have nuclei

$$v_Q \leq [Q'] \leq w_Q = [M']$$

where v_Q and w_Q are the extremes of the corresponding block in $N\mathcal{O}S$. The natural instinct is to think that $v_Q = [Q']$, but this need not be the case. In fact, $[Q']$ can float about between the two extremes, and we will see examples of this later. What we can say is that if S is T_2 then our instinct is correct, we do have $v_Q = [Q']$. This will be discussed in Section 7.

4 The point-free V-modification

We can now begin the main topic of these notes, a description of the point-free approach to producing a V-modification of a space. In fact, we produce a V-modification of a frame, and then apply the construction to a topology.

We know as much as we need to about the category ***Frm*** of frames. We now modifying that category by splitting its arrows in two so as to obtain a new category.

4.1 DEFINITION. Let ***Vrm*** be the category of V-frames. The objects of this are frames. However, each arrow is a pair of functions

$$A \underset{f_\Box}{\overset{f^\Diamond}{\rightrightarrows}} B$$

satisfying

$$(\wedge) \qquad f_\Box(x \wedge y) = f_\Box(x) \wedge f_\Box(y) \qquad f_\Box(\top) = \top$$

$$(\mathrm{c}) \qquad f_\Box(\bigvee X) = \bigvee\{f_\Box(x) \mid x \in X\}$$

$$(\bigvee) \qquad f^\Diamond(\bigvee Y) = \bigvee\{f^\Diamond(y) \mid y \in Y\} \qquad f^\Diamond(\bot) = \bot$$

$$(\mathrm{m}) \qquad (q)\ \ f_\Box(x) \wedge f^\Diamond(y) \leq f^\Diamond(x \wedge y) \qquad (s)\ \ f_\Box(x \vee y) \leq f_\Box(x) \vee f^\Diamond(y)$$

for each pair x, y of elements, each directed subset X, and each subset Y of A. ■

Notice that by taking $Y = \emptyset$ the equality $f^\Diamond(\bot) = \bot$ is a particular case of the general clause ($\bigvee$). However, it does no harm to state this explicitly.

We don't want to confuse the arrows of ***Vrm*** with those of ***Frm***. Thus we use a double shaft to indicate arrows of ***Vrm***. We write

$$A \overset{}{=\!(f^\Diamond, f_\Box)\!\Longrightarrow} B \qquad \text{or} \qquad A =\!=\!f\!=\!\Longrightarrow B$$

to indicate such a ***Vrm***-arrow. On the left we have indicated both components in full. On the right we have abbreviated this pair.

The 'right' component $f_\Box$ is a $\wedge$-semilattice morphism which is (Scott) continuous (by condition (c)). The 'left' component $f^\Diamond$ is a $\bigvee$-semilattice morphism. These two components must interact as in the two mixing conditions (q) and (s). It is routine to check that these objects and arrows do form a category, that is these arrows are closed under the obvious composition.

You might question why anyone should be interested in this category. In fact, when it is describe in this way there does not seem to be a good answer to this question. The only thing I can say is wait and see what happens. (Some of you might want to prattle on about power domains, but that is even less convincing.)

There is an obvious forgetful functor

$$\textbf{\textit{Vrm}} \xleftarrow[\;¿\;]{} \textbf{\textit{Frm}} \quad \text{where} \quad (¿f)^\Diamond = f \qquad (¿f)_\Box = f$$

which simply uses the given frame morphism f as both components of the ***Vrm***-arrow $¿f$. When any forgetful functor arises we should always ponder the existence of an adjoint on one side or the other. Let's see what we can make of this.

We pose a problem.

4.2 Problem. Let A be a frame. Take two copies of the elements

$$\Diamond(x) \qquad \Box(x)$$

for $x \in A$. Is there a frame generated by these elements satisfying

($\wedge$)	$\Box(x \wedge y) = \Box(x) \wedge \Box(y)$	$\Box(\top) = \top$
(c)	$\Box(\bigvee X) = \bigvee\{\Box(x) \mid x \in X\}$	
($\bigvee$)	$\Diamond(\bigvee Y) = \bigvee\{\Diamond(y) \mid y \in Y\}$	$\Diamond(\bot) = \bot$
(m)	$(q)\ \Box(x) \wedge \Diamond(y) \le \Diamond(x \wedge y)$	$(s)\ \Box(x \vee y) \le \Box(x) \vee \Diamond(y)$

for each pair x, y of elements, each directed subset X, and each subset Y of A? ■

There is a silly answer to this, for we can always take the trivial, 1-element, frame. There are other, more interesting, solutions some of which we have seen in Section 1.

Let S be any space and let $\mathcal{K}S$ be any set of compact subsets of S. Consider the topology $\mathcal{O}\mathcal{K}S$ on $\mathcal{K}S$ generated by Construction 1.2. This is given by a pair of assignments

$$\mathcal{O}S \overset{\Diamond}{\underset{\Box}{\rightrightarrows}} \mathcal{O}\mathcal{K}S$$

and it doesn't take to long to see the following.

4.3 LEMMA. *For each space S the Construction 1.2 solves Problem 4.2 for the spatial frame $\mathcal{O}S$.*

You should go through the proof of this. It is not very difficult, but you will still learn something. You will find that all except the continuity condition (c) translate into simple properties of quantifiers, with $\Diamond$ behaving as $\exists$ and $\Box$ behaving as $\forall$. In particular, the rather odd looking mixed conditions (m) are just the two obvious ways that the quantifiers interact.

The continuity condition (c) holds since each $K \in \mathcal{K}S$ is compact. This is why we restrict $\mathcal{K}S$ to compact sets.

(The similarity

$$\Diamond \sim \exists \qquad \Box \sim \forall$$

is why we write the $\Diamond$-component of a ***Vrm***-arrow on top with the $\Box$-component on the bottom, and refer to these as the 'left' component and 'right' component, respectively. Often existential quantification can be seen as the left adjoint of something or other with universal quantification as the right adjoint.)

As a particular case of this result we have a ***Vrm***-arrow

$$\mathcal{O}S \Longrightarrow \mathcal{O}\mathcal{L}S$$

from the parent topology to the constructed topology on the space of compact lenses. We also have an embedding $\mathcal{Q}S \hookrightarrow \mathcal{L}S$ which induces a frame morphism that we can attach to the ***Vrm***-arrow

$$\mathcal{O}S \Longrightarrow \mathcal{O}\mathcal{L}S \longrightarrow \mathcal{O}\mathcal{Q}S$$

to produce another ***Vrm***-arrow. This is nothing more than the ***Vrm***-arrow given by the Construction using $\mathcal{Q}S$.

These examples show that the Problem does have some interesting solutions. However, what we want is a *universal* solution. If possible, we wish to attach to a frame A a frame VA together with a ***Vrm***-arrow

$$A \Longrightarrow VA$$

which is universal among such. That is, for each ***Vrm***-arrow

$$A \overset{(f^\Diamond, f_\Box)}{\Longrightarrow} B$$

there is a unique frame morphism

$$VA \xrightarrow{\ f\ } B$$

such that the following diagram commutes.

$$A \xRightarrow{(f^\Diamond, f_\Box)} B$$

$$VA \xrightarrow{\ f\ }$$

This can always be done.

4.4 THEOREM. *For each frame A the Problem 4.2 has a universal solution. In particular, the forgetful functor ¿ has a left adjoint.*

We won't prove this in full, for we don't need all the details. The point-free construction $A \longmapsto VA$ proceed in two steps. The first converts A into a certain $\wedge$-semilattice, and the second step is a nice example of the coverage technique. In fact, it is one of the first uses of this technique applied to frames. An account of that technique can be found in [9] but this particular example is not discussed there.

The end product of the two-step construction is a ***Vrm***-arrow

$$\begin{array}{c} x \longmapsto \Diamond(x) \\ A \Longrightarrow VA \\ x \longmapsto \Box(x) \end{array}$$

from the given frame A to the constructed frame VA. These two assignments form a double indexing of a set of generating elements of VA. It turns out that each element of VA is a supremum of elements of the form

$$\Box(x) \wedge \Diamond(y_1) \wedge \cdots \wedge \Diamond(y_n)$$

for $x, y_1, \ldots, y_n \in A$.

The conditions in Definition 4.1 and in Problem 4.2 can take a bit of getting used to. The use of the coverage technique helps with this, for several small manipulations have to be carried out. We don't do much of that here, but Subsection 9.1 illustrates the kind of thing that can happen.

As with almost any universal construction, there are some functorial and naturality aspects hanging around. Some of these are discussed in Section 8.

What has this got to do with V-modifications of a space?

Each frame A has a point space $\mathsf{pt}(A)$. It also has a universal V-modification VA which in turn has a point space $\mathsf{pt}(VA)$. In particular, we may apply the construction to the topology $A = \mathcal{O}S$ of a given space. How are these various spaces related? That is what we are going to analyse. To do that we do not need the details of VA. The universality of the construction will do.

5 The V-space of a frame

We have attached to each frame A a second frame VA via a ***Vrm***-arrow

$$A \Longrightarrow VA$$

which is a double indexing of the generators. We know also that each frame A has a point space $S = \mathsf{pt}(A)$ where the topology is connected to A

$$A \longrightarrow \mathcal{O}S$$

via a surjective frame morphism. What is the connection between the V-modification VA and the spatial reflection $\mathcal{O}S$ of A? Since VA is a frame it too has a point space $\mathsf{pt}(VA)$ and we can expect this to have some relationship with $S = \mathsf{pt}(A)$. In this section we begin the analysis of this connection.

Recall that in general we view the points of a frame A as its $\wedge$-irreducible elements. These are in bijective correspondence with the characters

$$A \longrightarrow \mathsf{2}$$

of A, the frame morphisms from A to the two-element frame $\mathsf{2} = \{0,1\}$. (They are also in bijective correspondence with with the completely prime filters on A.) Thus the V-points of A, that is the points of VA, are in bijective correspondence with the frame morphisms $VA \longrightarrow \mathsf{2}$. From the universality of the construction these V-points are in bijective correspondence with the V-characters of A

$$A \underset{[\cdot]}{\overset{\langle\cdot\rangle}{\Longrightarrow}} \mathsf{2}$$

the ***Vrm***-arrows from A to to $\mathsf{2}$. For each such V-character, the two component functions $\langle\cdot\rangle$ and $[\cdot]$ must satisfy

($\wedge$) $\quad [\cdot]\,(x \wedge y) = [\cdot]\,(x) \wedge [\cdot]\,(y) \qquad [\cdot]\,(\top) = 1$

(c) $\quad [\cdot]\,(\bigvee X) = \bigvee\{[\cdot]\,(x) \mid x \in X\}$

($\bigvee$) $\quad \langle\cdot\rangle\,(\bigvee Y) = \bigvee\{\langle\cdot\rangle\,(y) \mid y \in Y\} \qquad \langle\cdot\rangle\,(\bot) = 0$

(m) $\quad (q)\;\; [\cdot]\,(x) \wedge \langle\cdot\rangle\,(y) \le \langle\cdot\rangle\,(x \wedge y) \qquad (s)\;\; [\cdot]\,(x \vee y) \le [\cdot]\,(x) \vee \langle\cdot\rangle\,(y)$

for each pair x, y of elements, each directed subset X, and each subset Y of A. Since each value of $[\cdot]$ and $\langle\cdot\rangle$ is either 0 or 1, we may use ($\wedge$) and ($\bigvee$) to obtain a filter ∇ and element a where

$$x \in \nabla \Longleftrightarrow [\cdot]\,(x) = 1 \qquad x \le a \Longleftrightarrow \langle\cdot\rangle\,(x) = 0$$

for each $x \in A$. This pair (∇, a) determines the ***Vrm***-arrow $A \Longrightarrow \mathsf{2}$. In other words, the V-points are certain pairs (∇, a). The continuity (c) merely says that ∇ is open (that is, Scott open), but what do the two mixed conditions (q,s) say?

5.1 LEMMA. *Consider a pair (∇, a) where ∇ is an open filter and a is an element of the frame A. Then the two mixed conditions correspond to*

(∇-*closed*) $\quad (\forall x)[x \in \nabla \Longrightarrow (x \supset a) = a] \qquad$ (∇-*small*) $\quad (\forall x)[a \vee x \in \nabla \Longrightarrow x \in \nabla]$

respectively.

Proof. We show first that (q) is equivalent to a being ∇-closed.

Suppose that the mixed condition (q) holds, and consider any $x \in \nabla$. Thus we have $[\cdot]\,(x) = 1$ and hence

$$\langle\cdot\rangle\,(y) \le \langle\cdot\rangle\,(x \wedge y)$$

for each $y \in A$. Consider $y = (x \supset a)$. Then $x \wedge y \le a$ so that

$$\langle\cdot\rangle\,(y) \le \langle\cdot\rangle\,(x \wedge y) \le \langle\cdot\rangle\,(a) = 0$$

and hence $y \le a$, as required to show that a is ∇-closed.

Conversely suppose that a is ∇-closed, and consider any $x, y \in A$. If either $[\cdot](x) = 0$ or $\langle\cdot\rangle(x \wedge y) = 1$ then we are done. Thus we may suppose $[\cdot](x) = 1$ and $\langle\cdot\rangle(x \wedge y) = 0$, that is $x \in \nabla$ and $x \wedge y \leq a$. This gives $y \leq (x \supset a) = a$ since a is ∇-closed, and hence $\langle\cdot\rangle(y) = 0$, which leads to the required result.

Next we show that (s) is equivalent to a being ∇-small.

Suppose that the mixed condition (s) holds, and consider any $x \in A$ with $a \vee x \in \nabla$. Then

$$[\cdot](x) = [\cdot](x) \vee \langle\cdot\rangle(a) \geq [\cdot](x \vee a) = 1$$

to give $x \in \nabla$, as required.

Conversely suppose that a is ∇-small, and consider any $x, y \in A$. If either $[\cdot](x \vee y) = 0$ or $\langle\cdot\rangle(y) = 1$ then we are done. Thus we may suppose $[\cdot](x \vee y) = 1$ and $\langle\cdot\rangle(y) = 0$, that is $x \vee y \in \nabla$ and $y \leq a$. Since ∇ is a filter these give $x \vee a \in \nabla$, so that $x \in \nabla$, and hence $[\cdot](x) = 1$, which leads to the required result. ■

When is an element ∇-closed or ∇-small? To answer this we use the information about the block structure of A we set down in Sections 2 and 3.

The filter ∇ is open and hence admissible with a smallest nucleus v_∇ and a largest nucleus w_∇ in its block. This smallest nucleus gives a quotient A_∇ of A. The nucleus v_∇ is the closure of the pre-nucleus f where

$$f(y) = \bigvee\{(x \supset y) \mid x \in \nabla\}$$

for each $y \in A$. The elements of A_∇ are precisely those elements of A that are fixed by f. Thus we have the following (which explains the terminology).

5.2 LEMMA. *Let ∇ be an open filter of the frame A. Then an element $a \in A$ is ∇-closed precisely when $a \in A_\nabla$.*

Proof. Suppose a is ∇-closed. Then, using the pre-nucleus f, as above, we have

$$f(a) = \bigvee\{(x \supset a) \mid x \in \nabla\} = \bigvee\{a \mid x \in \nabla\} = a$$

to show that $a \in A_\nabla$.

Conversely, suppose $a \in A_\nabla$. Then for each $x \in \nabla$ we have

$$(x \supset a) \leq f(a) = a$$

to show that a is ∇-closed. ■

This result shows that the bottom value $v_\nabla(\bot)$ is ∇-closed, in fact it is the least ∇-closed element. And there's a bonus.

5.3 LEMMA. *Let ∇ be an open filter of the frame A. Then the element $v_\nabla(\bot)$ is the least ∇-closed element, and is ∇-small.*

Proof. As observed above, $a = v_\nabla(\bot)$ is the least ∇-closed element. To show it is ∇-small consider any $x \in A$ with $a \vee x \in \nabla$. Since both

$$a = v_\nabla(\bot) \leq v_\nabla(x) \qquad x \leq v_\nabla(x)$$

we have $v_\nabla(x) \in \nabla$, to give $x \in \nabla$, as required. ■

Next we look at the ∇-small elements and eventually arrive at a companion result to Lemma 5.3.

As usual, an element a of a frame A is dense if $\neg a = \bot$. In particular an open set $U \in \mathcal{O}S$ in a topology is dense (in this sense) exactly when

$$U^{-\prime} = \neg U = \emptyset$$

which is when U is topologically dense. The general notion unravels as

$$(\forall x)[a \wedge x = \bot \Longrightarrow x = \bot]$$

which is a form we may dualize. An element is dense if it is large, in a certain sense. Thus, let us say let us say an element $a \in A$ is small if

$$(\forall x)[a \vee x = \top \Longrightarrow x = \top]$$

holds. The ∇-small condition is a relativized version smallness.

5.4 LEMMA. *For an open filter ∇ of a frame A, an element $a \in A$ is ∇-small if and only if its image in A_∇ is small (in that quotient).*

Proof. Suppose that a is ∇-small. Consider any $x \in A_\nabla$ for which the join in A_∇ with $v_\nabla(a)$ is $\top$. Thus

$$v_\nabla(v_\nabla(a) \vee x) = \top$$

to give

$$v_\nabla(a) \vee x \in \nabla$$

so that

$$v_\nabla(a \vee x) \in \nabla$$

(since this is a larger element), to give

$$a \vee x \in \nabla$$

so that

$$x \in \nabla$$

(since a is ∇-small) and hence (since $x \in A_\nabla$) we have

$$x = v_\nabla(x) = \top$$

as required.

Conversely suppose $v_\nabla(a)$ is small in A_∇ and consider any $x \in A$ with $a \vee x \in \nabla$. Then

$$v_\nabla(v_\nabla(a) \vee v_\nabla(x)) = \top$$

and this is the join of $v_\nabla(a)$ and $v_\nabla(x)$ in A_∇. The smallness in A_∇ gives $v_\nabla(a) = \top$, so that $x \in \nabla$, as required. ■

Almost trivially, the set of ∇-small elements of A is a (non-empty) lower section. If a, b are both ∇-small then

$$a \vee b \vee x \in \nabla \Longrightarrow b \vee x \in \nabla \Longrightarrow x \in \nabla$$

to show that the ∇-small elements form an ideal. Finally, since ∇ is open, we see that this ideal is principal.

5.5 LEMMA. *For an open filter ∇ of a frame A, the set of ∇-small elements form a principal ideal, and its generator is ∇-closed.*

Proof. Only the last part is not immediate. Consider any element a which is ∇-small. We show that $v_\nabla(a)$ is also ∇-small.

Consider $x \in A$ with $v_\nabla(a)\vee x \in \nabla$. Since $v_\nabla(a)\vee x \le v_\nabla(a\vee x)$ we have $v_\nabla(a\vee x) \in \nabla$, and hence

$$v_\nabla(a \vee x) = v_\nabla^2(a \vee x) = \top$$

to give $a \vee x \in \nabla$. But a is ∇-small, so that $x \in \nabla$, as required. ■

These results show that for each open filter ∇ of A, the elements $a \in A$ for which the pair (∇, a) is a V-point form a principal lower section of A_∇. There is always a smallest possible such a, namely $v_\nabla(\bot)$, and there is a largest possible such a. This suggest several questions. What is this largest possible a? When is there just one possible element a? Are there any 'special' associated elements a?

Let $S = \mathsf{pt}(A)$ be the point space of A. The open filter ∇ is controlled by some $Q \in \mathcal{Q}S$. Within the specialization order of S each point $q \in Q$ sits above a minimal member $m \in Q$. Let M be this set of minimal members, so that

$$w_\nabla(x) = \bigwedge\{m \in M \mid x \le m\}$$

gives the largest member of the block associated with ∇. Of course, this infimum is taken in A.

As promised, we can now give the companion to Lemma 5.3.

5.6 LEMMA. *Let ∇ be an open filter of the frame A. Then the element $w_\nabla(\bot) = \bigwedge M$ is the largest ∇-small element, and is ∇-closed.*

Proof. Since $M \subseteq Q = \mathsf{pt}(A_\nabla)$, we see that the infimum $\bigwedge M$ is a member of A_∇ (since an infimum taken in a quotient agrees with that taken in the parent). Thus $w_\nabla(\bot)$ is ∇-closed.

To show that $w_\nabla(\bot)$ is ∇-small, we verify

$$x \notin \nabla \Longrightarrow w_\nabla(\bot) \vee x \notin \nabla$$

(the contrapositive). If $x \notin \nabla$ then, by Lemma 3.4, we have some $x \le m \in M$. But now

$$w_\nabla(\bot) \vee x \le m \notin \nabla$$

to give the required result.

Finally, suppose $a \in A$ is ∇-small. Consider any $m \in M$. If $a \not\le m$, then $m < a \vee m$ and hence $a\vee m \in \nabla$ (by the maximality of m). Thus (since a is ∇-small) we have $m \in \nabla$, which is not so. Thus $a \le m$, and hence $a \le w_\nabla(\bot)$. ■

For an open filter ∇ on a frame A the elements which are both ∇-closed and ∇-small form a principal lower section of A_∇. The least such element is $v_\nabla(\bot)$ and the greatest such element is $w_\nabla(\bot)$, the bottom elements of the least and greatest nuclei of the ∇-block. Lets gather together this information in one place.

5.7 THEOREM. *For each frame A the V-points are the pairs (∇, a) where ∇ is an open filter on A and a is a member of the principal lower section*

$$[v_\nabla(\bot), w_\nabla(\bot)]$$

of the quotient A_∇.

For a fitted frame A we have $v_\nabla = w_\nabla$ and hence the V-points of A are essentially the open filters on A. Eventually we will obtain a vast improvement of this observation. We will also give examples to show that this interval can be very complicated.

We are building up quite a nice picture of the space $\mathsf{pt}(VA)$. It's time we had a look at the topology.

We know that each V-point corresponds to a pair (∇, a) which determines a character

$$VA \xrightarrow{(\nabla, a)(\cdot)} 2$$

of VA. The opens of the space $\mathsf{pt}(VA)$ are the sets given by

$$(\nabla, a) \in \circledcirc(\alpha) \Longleftrightarrow (\nabla, a)(\alpha) = 1$$

where α varies through VA. We don't know all of these elements α, but we do know a set of generators, and these will index a subbase of the topology. The generators are

$$\Box(x) \qquad \Diamond(x)$$

as x varies through A. Thus setting

$$[\circ]\,(x) = \circledcirc(\Box(x)) \qquad \langle\circ\rangle\,(x) = \circledcirc(\Diamond(x))$$

gives the subbase. For later let's state this as a result.

5.8 LEMMA. *For each frame A the subbasic open sets of $\mathsf{pt}(VA)$ are given by*

$$(\nabla, a) \in [\circ]\,(x) \Longleftrightarrow x \in \nabla \qquad (\nabla, a) \in \langle\circ\rangle\,(x) \Longleftrightarrow x \not\leq a$$

for $x \in A$ and V-points (∇, a).

Proof. Each side of the equivalence unravels to

$$(\nabla, a)(\Box(x)) = 1 \qquad (\nabla, a)(\Diamond(x)) = 1$$

respectively. ■

As always, whenever we construct a space it is worth looking at the specialization order.

5.9 LEMMA. *For each frame A we have*

$$(\nabla, a) \sqsubseteq (\Pi, b) \Longleftrightarrow \nabla \subseteq \Pi \text{ and } b \leq a$$

for all $(\nabla, a), (\Pi, b) \in \mathsf{pt}(VA)$.

Proof. Unravelling the various definitions we have

$$\begin{aligned}(\nabla, a) \sqsubseteq (\Pi, b) &\iff (\nabla, a) \in (\Pi, b)^{-}\\ &\iff (\forall x \in A)\begin{bmatrix}(\nabla, a) \in [\circ]\,(x) \Longrightarrow (\Pi, b) \in [\circ]\,(x)\\ (\nabla, a) \in \langle\circ\rangle\,(x) \Longrightarrow (\Pi, b) \in \langle\circ\rangle\,(x)\end{bmatrix}\\ &\iff (\forall x \in A)\begin{bmatrix}x \in \nabla \Longrightarrow (x \in \Pi\\ x \not\leq a \Longrightarrow x \not\leq b\end{bmatrix} \qquad \iff \qquad \begin{bmatrix}\nabla \subseteq \Pi\\ b \leq a\end{bmatrix}\end{aligned}$$

as required. ■

We haven't had many examples yet, mainly because we are saving them for Section 9. However, it is worth giving a simple one here, for it clears up what is sometimes a source of confusion.

5.10 EXAMPLE. For any frame A the whole set $\nabla = A$ is an open filter. This corresponds to the empty subset of the point space S. Since $\perp \in \nabla$ the corresponding inflator on A send everything to $\top$. Thus $(A, \top)$ is the only V-point associated with A as a filter.

The bottom $\perp$ gives an open set $[\circ]\,(\perp)$ of $\mathsf{pt}(VA)$. But, for each V-point $\nabla, a)$ we have

$$(\nabla, a) \in [\circ]\,(\perp) \iff \perp \in \nabla \iff \nabla = A$$

to show that

$$[\circ]\,(\perp) = (A, \top)$$

and hence $(A, \top)$ is isolated in $\mathsf{pt}(VA)$.

For this reason A is often discounted as an open filter, and $\emptyset$ is often discounted as a compact saturated set. ■

There is a direct comparison between $\mathsf{pt}(A)$ and $\mathsf{pt}(VA)$.

Consider any $p \in \mathsf{pt}(A)$ (viewed as a $\wedge$-irreducible). The associated filter $\nabla(p)$ given by

$$x \in \nabla(p) \iff x \not\leq p \iff p \in \circledcirc(x)$$

is open (in fact, completely prime). The corresponding subsets of $S = \mathsf{pt}(A)$ are

$$Q = \uparrow p = \{p\}^{\uparrow} \qquad M = \{p\}$$

where $\uparrow p$ is the saturation in S, that is the set of all $q \in S$ with $q \leq p$ in A. Remember that the specialization order of S is the reverse of the comparison inherited from A. In particular, we have

$$\bigwedge M = p$$

to give the following.

5.11 LEMMA. *For each point p of the frame A the pair $(\nabla(p), p)$ is a V-point.*

This sets up an injection

$$\begin{array}{rcl}\mathsf{pt}(A) & \xrightarrow{\theta} & \mathsf{pt}(VA)\\ p & \longmapsto & (\nabla(p), p)\end{array}$$

and so gives a possible second topology on $\mathsf{pt}A$ (the canonical topology and the one induced by the canonical topology on $\mathsf{pt}(VA)$).

5.12 LEMMA. *The function θ is an embedding, that is the canonical topology and the induced topology are the same.*

Proof. As x varies through A the sets

$$\theta^{\leftarrow}([\circ](x)) \qquad \theta^{\leftarrow}(\langle\circ\rangle(x))$$

give a subbase of the induced topology. But, for each point p we have

$$\begin{aligned} p \in \theta^{\leftarrow}([\circ](x)) &\iff (\nabla(p), p) \in [\circ](x) \iff x \in \nabla(p) \iff p \in \circledcirc(x) \\ p \in \theta^{\leftarrow}(\langle\circ\rangle(x)) &\iff (\nabla(p), p) \in \langle\circ\rangle(x) \iff x \not\leq p \qquad \iff p \in \circledcirc(x) \end{aligned}$$

so that

$$\theta^{\leftarrow}([\circ](x)) = \circledcirc(x) = \theta^{\leftarrow}(\langle\circ\rangle(x))$$

which gives the required result. ■

It is tempting to think of $\mathsf{pt}(VA)$ as just a space. However, it probably has a more sophisticated structure, or is the external view of an object in a non-standard universe. In the last part of this section I will give you just a glimpse of this. (In fact, I haven't seen much more myself).

In Section 2 we saw that each frame A has several associated filter spaces. The largest is the space of all filters and then, as subspaces, all admissible filters, all (Scott) open filters, and all completely prime filters. This last one, of course, is just the point space $S = \mathsf{pt}(A)$ of A looked at from a different angle. Let's take a look at the next larger one.

5.13 DEFINITION. For a frame A let $\mathsf{dp}(A)$ be the space of all (Scott) open filters with the canonical topology. ■

Each $x \in A$ gives a subset $D(x)$ of $\mathsf{dp}(A)$ were

$$\nabla \in D(x) \iff x \in \nabla$$

for each open filter ∇. We easily check that

$$D(x) \cap D(y) = D(x \wedge y)$$

(for $x, y \in A$) and so, by definition, these sets form the canonical base for the topology on $\mathsf{dp}(A)$.

Each open filter is a component of at least one V-point of A, and we have the expected result.

5.14 LEMMA. *For each frame A the assignment*

$$\begin{array}{ccc} \mathsf{pt}(VA) & \longrightarrow & \mathsf{dp}(A) \\ (\nabla, a) & \longmapsto & \nabla \end{array}$$

is a continuous surjection.

Proof. We know the assignment is surjective, so it suffices to check that it is continuous. To this end consider a typical basic open set $D(x)$ of $\mathsf{dp}(A)$. A trivial calculations shows that the inverse image of this is $[\circ]\,(x)$, which is open in $\mathsf{pt}(VA)$. ∎

This displays $\mathsf{pt}(VA)$ as a bundle over $\mathsf{dp}(A)$ to produce a structure which ought to be investigated.

What are the fibres of this bundle? For each open filter ∇ let S_∇ be the set of all V-points of A of the form (∇, a). In other words, this is a certain principal lower section of the quotient A_∇ of A. This is the fibre over ∇. Of course, S_∇ inherits a comparison $\leq$ from A_∇, and we can expect this to be related to the specialization order on S_∇.

5.15 LEMMA. *For each open filter ∇ the topology on S_∇ is the smallest for which each principal $\leq$-upper section is closed. In particular, the specialization order is the opposite of the comparison inherited from A.*

The canonical subbase of S_∇ is doubly indexed by the elements of A. Thus for $x \in A$ we have

$$a \in [\triangledown]\,(x) \iff (\nabla, a) \in [\circ]\,(x) \iff x \in \nabla$$
$$a \in \langle\triangledown\rangle\,(x) \iff (\nabla, a) \in \langle\circ\rangle\,(x) \iff x \not\leq a$$

for each $a \in S_\nabla$. In particular, we have

$$[\triangledown]\,(x) = \begin{cases} S_\nabla & \text{if } x \in \nabla \\ \emptyset & \text{if } x \notin \nabla \end{cases}$$

so we can discount these sets $[\triangledown]\,(x)$. For the other sets $\langle\triangledown\rangle\,(x)$ we remember that each $a \in S_\nabla$ is v_∇-closed. Thus

$$a \notin \langle\triangledown\rangle\,(x) \iff x \leq a \iff v_\nabla(x) \leq a$$

to show that

$$\{(\uparrow x)' \mid x \in S_\nabla\}$$

is a subbase for the topology on S_∇. ∎

Let's conclude this section with a simple example which illustrates that the difference between $\mathsf{pt}(VA)$ and $\mathsf{pt}(VOS)$ can be quite dramatic.

5.16 EXAMPLE. Suppose A is a compact frame, that is the top $\top$ is a compact element, and so $\{\top\}$ is an open filter. The point space S is compact and is the corresponding member of $\mathcal{Q}S$. The generating subset M of this $Q = S$ is the set of maximal element of A. There may be many of these or few. The infimum $m = \bigwedge M$ (taken in A) may be quite large or small. Thus there may be many or few V-points $(\{\top\}, a)$.

As a particular example consider a frame B with no points. Form A be adding a new top to B. Thus A is compact and has just one point, namely the top p of the original frame B. (This construction is the analogue of the 1-point compactification of a space.) In this case $M = \{p\}$, and the points associated with $\{\top\}$ are in bijective correspondence with the elements of B. ∎

Clearly, there is quite a bit more to be sorted out here, but before we can do that we need some information about the V-space of a topology.

6 The V-modifications of a space

In Construction 1.2 we produce various point-sensitive V-modifications of a space S. To do this we select some set $\mathcal{K}S$ of compact subsets of S and then use the parent topology $\mathcal{O}S$ to doubly index a subbase for a topology on $\mathcal{K}S$. Instinctively the canonical choice for $\mathcal{K}S$ is $\mathcal{Q}S$, the set of compact saturated sets of S. However, there are other possibilities, and perhaps the set $\mathcal{L}S$ of compact lenses is a better canonical choice. In this section we will see that the point-free approach can lead to other spatial V-modifications.

In Sections 4 and 5 we have seen how each frame A has a V-modification VA (as a frame), and this produces a space $\mathsf{pt}(VA)$ which has some connection with the point space $\mathsf{pt}(A)$ of the original frame. We might expect that $\mathsf{pt}(VA)$ is some kind of V-modification of the space $\mathsf{pt}(A)$. It is, but not quite of the point-sensitive kind.

In this section we begin to investigate the nature of the space $\mathsf{pt}(VA)$. However, initially we do not look at the most general situation. For the time being we assume the parent frame A is spatial. Only later do we look at the general situation.

Put another way, we assume given a space S and compare the space $\mathsf{pt}(V\mathcal{O}S)$ with the various direct V-modifications of S. Throughout this section we assume that S is sober, but there are one or two places where this is not strictly necessary.

Our first job is to describe the points of $V\mathcal{O}S$, or rather to knock them into a point-sensitive shape.

The V-points of $\mathcal{O}S$ are the pairs (∇, W) where ∇ is an open filter on $\mathcal{O}S$ and W is a certain associated open set. By Corollary 3.8 (and since S is sober) we may replace ∇ by its controlling set $Q \in \mathcal{Q}S$. To preserve the symmetry we also replace the associated open set by its complement. Thus we view the V-points of $\mathcal{O}S$ as pairs

$$(Q, X)$$

where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. Of course, X must satisfy a couple of conditions relating to Q. The complement X' must be ∇-closed and ∇-small. Let's rephrase the conditions in terms of Q.

The matching pair (∇, Q) induces a quotient $(\mathcal{O}S)_\nabla$ of $\mathcal{O}S$, and the complement X' must be a member of this quotient. Let us say X must be Q-closed. We have seen this notion before.

Recall that in Definition 1.9 we attached to each $Q \in \mathcal{Q}S$ a derivative ∂_Q on $\mathcal{C}S$. In Definition 1.12 we introduced a relation $Q \ltimes X$ between $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. These are connected by

$$Q \ltimes X \iff \partial_Q(X) = X$$

for each $X \in \mathcal{C}S$. In other words $Q \ltimes \cdot$ picks out the fixed sets of ∂_Q. The matching pair (∇, Q) also gives a pre-nucleus $f = f_\nabla$ on $\mathcal{O}S$.

6.1 LEMMA. *For each sober space S and matching pair (∇, Q), the pre-nucleus f_∇ and derivative ∂_Q are dual complements, that is*

$$\partial_Q(X) = f_\nabla(X')'$$

for each $X \in \mathcal{C}S$.

Proof. We have

$$\partial_Q(X) = \bigcap\{(X\cap U)^- \mid Q\subseteq U\in\mathcal{O}S\} \qquad f_\nabla(W) = \bigcup\{(W\supset U)\mid Q\subseteq U\in\mathcal{O}S\}$$

for each $X\in\mathcal{C}S$ and $W\in\mathcal{O}S$. Passing complements gives the required result. ■

A couple more complements gives the following.

6.2 COROLLARY. *For each sober space S and $Q\in\mathcal{Q}S$, a set $X\in\mathcal{C}S$ is Q-closed precisely when $Q\ltimes X$, that is when*

$$Q\subseteq U \Longrightarrow X\subseteq (X\cap U)^-$$

holds for each $U\in\mathcal{O}S$.

Continuing with these pleasantries let us say $X\in\mathcal{C}S$ is Q-small if X' is ∇-small. To unravel this notion recall that Q is generated from its set M of minimal members (that is, minimal in the specialization order). Of course, these need not be minimal members of the whole space.

6.3 LEMMA. *For each sober space S and $Q\in\mathcal{Q}S$, a set $X\in\mathcal{C}S$ is Q-small precisely when $M\subseteq X$.*

Proof. By a direct translation we find that X is Q-small precisely when

$$(\forall U\in\mathcal{O}S)[X\cap Q\subseteq U\Longrightarrow Q\subseteq U]$$

holds. We show that is equivalent to $M\subseteq X$.

Suppose that X is Q-small and consider any point $m\in M$. We have $m\in M\subseteq Q$, and hence $Q\not\subseteq m^{-\prime}$, to give $X\cap Q\not\subseteq m^{-\prime}$. This produces some point $q\in Q\cap X$ with $q\in m^-$, that is with $q\in Q$ and $q\sqsubseteq m$. But m is a minimal member of Q, so that $m=q\in X$, as required.

Conversely, consider any $U\in\mathcal{O}S$ with $X\cap Q\subseteq U$. We have

$$M\subseteq X\cap Q\subseteq U\in\mathcal{U}S$$

so that $Q=M^{\uparrow}\subseteq U$, as required. ■

These results combine to give us the characterization we want.

6.4 THEOREM. *For each sober space S the V-points of OS are the pairs (Q,X) where $Q\in\mathcal{Q}S$ and $X\in\mathcal{C}S$ and where both*

$$Q\ltimes X \qquad M^-\subseteq X\subseteq Q(\infty)$$

hold.

Proof. The only aspect not covered by Lemmas 6.2 and 6.3 is the inclusion $X \subseteq Q(\infty)$. But recall that

$$Q(\alpha) = \partial_Q^\alpha(S)$$

for each ordinal α, so that $Q(\infty)$ is the largest Q-closed set. In other words, we have

$$Q \ltimes X \Longrightarrow X \subseteq Q(\infty)$$

and the inclusion $X \subseteq Q(\infty)$ could be omitted. ■

This enables us to exhibit a few of the V-points of $\mathcal{O}S$.

6.5 COROLLARY. *For each sober space S and each $Q \in \mathcal{Q}S$, each of*

$$(Q, M^-) \quad (Q, Q^-) \quad (Q, Q(\infty))$$

is a V-point of $\mathcal{O}S$.

The two closed sets

$$M^- \qquad Q(\infty)$$

are nothing more than the complements of the bottom values

$$w_\nabla(\emptyset) \qquad v_\nabla(\emptyset)$$

of the largest and smallest nuclei associated with ∇. We always have

$$w_\nabla \geq [Q'] \geq v_\nabla(\emptyset)$$

and Q^- is just the complement of the bottom value of this intermediate nucleus. Of course, these three V-points need not be distinct. For many spaces S we find that $M^- = Q(\infty)$ for each $Q \in \mathcal{Q}S$, and so the set of points of $V\mathcal{O}S$ is essentially $\mathcal{Q}S$. This is the case of original interest, but we will see that it is not the most interesting case.

Theorem 6.4 gives us the points of $V\mathcal{O}S$, so we know what we should do next, don't we? We must first show that the frame is spatial, and then characterize the associated topology. Unfortunately, or to put it better, fortunately the frame $V\mathcal{O}S$ need not be spatial. An example of such a space S is referred to in Subsection 9.1.

Even though $V\mathcal{O}S$ need not be spatial, it still has a point space $\mathsf{pt}(V\mathcal{O}S)$ with its canonical topology. In Lemma 5.8 we described the topology on $\mathsf{pt}(VA)$ for a general frame A. This is given by a subbase which is doubly indexed by the elements of A. For $A = \mathcal{O}S$ each $U \in OS$ give a pair of subbasic open set of $\mathsf{pt}(VOS)$. Following through the various rephrasings we obtain the following.

6.6 LEMMA. *For each sober space S the doubly indexed families given by*

$$(Q, X) \in \langle \circ \rangle (U) \Longleftrightarrow X \textit{ meets } U \qquad (Q, X) \in [\circ] (U) \Longleftrightarrow Q \subseteq U$$

(for V-points (Q, X) and $U \in \mathcal{O}S$) form a subbase of the topology on $\mathsf{pt}(V\mathcal{O}S)$.

You should compare this description with that of Construction 1.2 for the particular case $\mathcal{K}S = \mathcal{Q}S$. The box condition is the same, but there is a difference in the diamond condition. In fact, this difference arises because we have produced a larger space.

As always, whenever we set up a space it is worth writing done the specialization order. We have done this in Lemma 5.9 for the general case $\mathsf{pt}(VA)$. More or less the same proof gives the following.

6.7 LEMMA. *For each sober space S we have*

$$(Q, X) \sqsubseteq (R, Y) \iff R \subseteq Q \text{ and } X \subseteq Y$$

for all $(Q, X), (R, Y) \in \mathsf{pt}(V\mathcal{O}S)$.

For each $Q \in \mathcal{Q}S$ the pair (Q, Q^-) is a V-point, and so we have an insertion, at the set level, of $\mathcal{Q}S$ into $\mathsf{pt}(V\mathcal{O}S)$. Guess what.

6.8 THEOREM. *For each sober space S the insertion*

$$\begin{array}{rcl} \mathcal{Q}S & \longrightarrow & \mathsf{pt}(V\mathcal{O}S) \\ Q & \longmapsto & (Q, Q^-) \end{array}$$

exhibits the V-modification $\mathcal{Q}S$ as a subspace of $\mathsf{pt}(V\mathcal{O}S)$

Proof. Using Lemma 6.6 we have

$$(Q, Q^-) \in \langle\circ\rangle(U) \iff Q^- \text{ meets } U \iff Q \text{ meets } U \iff \Diamond(U)$$

and

$$(Q, Q^-) \in \langle\circ\rangle(U) \iff X \text{ meets } U \iff Q \in \Box(U)$$

(for each $Q \in \mathcal{Q}S$ and $U \in \mathcal{O}S$) to give the required result. ■

On seeing this result anyone who still believes in fairies might be tempted to think that the embedding is a homeomorphism. It isn't, and we will look at some appropriate examples in Section 8. What we will do here it to enlarge the space $\mathcal{Q}S$ to one which is a much more plausible copy of $\mathsf{pt}(V\mathcal{O}S)$.

Remember, from Section 1, the set $\mathcal{L}S$ of compact lenses carries a V-modification topology and includes $\mathcal{Q}S$ as a subspace.

(Although this result is stated for sober spaces it can be generalized slightly. We will look at a non-sober variant in Subsection 9.4.)

6.9 LEMMA. *For each compact lens L of a sober space S the pair $(L^\uparrow, L^-)$ is a V-point.*

Proof. Consider a compact lens $L = Q \cap L^-$ where $Q = L^\uparrow \in \mathcal{Q}S$. We show that $X = L^-$ is Q-closed and Q-small.

Consider any $Q \subseteq U \in \mathcal{O}S$. Then

$$L \subseteq X \cap Q \subseteq X \cap U$$

to give $X = L^- \subseteq (X \cap U)^-$, as required.

Consider any $X \cap Q \subseteq U \in \mathcal{O}S$. Then

$$L \subseteq X \cap Q \subseteq U$$

to give $Q = L^\uparrow \subseteq U$, as required. ■

With this we have a more interesting version of Theorem 6.8.

6.10 THEOREM. *For each sober space S the insertion*

$$\begin{array}{rcl} \mathcal{L}S & \xrightarrow{\lambda} & \mathsf{pt}(V\mathcal{O}S) \\ L & \longmapsto & (L^{\uparrow}, L^{-}) \end{array}$$

exhibits the V-modification $\mathcal{L}S$ as a subspace of $\mathsf{pt}(V\mathcal{O}S)$

Proof. We show that the inverse image function $\lambda^{\leftarrow}$ matches the canonical subbasic sets of $\mathsf{pt}(V\mathcal{O}S)$ with those of $\mathcal{L}S$.

For each $U \in \mathcal{O}S$ and $L \in \mathcal{L}S$ we have

$$L \in (\lambda^{\leftarrow} \circ [\circ])(U) \Longleftrightarrow (L^{\uparrow}, L^{-}) \in [\circ])(U) \Longleftrightarrow L^{\uparrow} \subseteq U \Longleftrightarrow L \subseteq U \Longleftrightarrow L \in \Box(U)$$

to give

$$\lambda^{\leftarrow} \circ [\circ] = \Box$$

as required.

For each $U \in \mathcal{O}S$ and $L \in \mathcal{L}S$ we have

$$\begin{aligned} L \in (\lambda^{\leftarrow} \circ \langle\circ\rangle)(U) &\Longleftrightarrow (L^{\uparrow}, L^{-}) \in \langle\circ\rangle)(U) \\ &\Longleftrightarrow L^{-} \text{ meets } U \\ &\Longleftrightarrow L \text{ meets } U \qquad \Longleftrightarrow L \in \Diamond(U) \end{aligned}$$

to give

$$\lambda^{\leftarrow} \circ \langle\circ\rangle = \Diamond$$

as required. ■

From Lemma 1.7 and Theorem 6.10 we have two composite embeddings

$$\begin{array}{lcr} S \hookrightarrow \mathcal{L}S \hookrightarrow \mathsf{pt}(V\mathcal{O}S) & \qquad & \mathcal{Q}S \hookrightarrow \mathcal{L}S \hookrightarrow \mathsf{pt}(V\mathcal{O}S) \\ p \longmapsto (p^{\uparrow}, p^{-}) & \qquad & Q \longmapsto (Q, Q^{-}) \end{array}$$

where a couple of simple calculations verify these descriptions. In particular, the right-hand one is that of Theorem 6.8. It is, I suppose, a reasonable to hope that the embedding λ (of Theorem 6.10) may be a homeomorphism. Unfortunately, as we will see, not even Tinkerbell exist.

Let us now turn to the more general, non-spatial case.

We have attached to each frame A a collection of spaces. Firstly there is the point space $S = \mathsf{pt}(A)$ of A, and we have added to this the point space $\mathsf{pt}(VA)$ of the point-free V-modification of A. The point space space S has two point-sensitive V-modifications, $\mathcal{Q}S$ and $\mathcal{L}S$. Finally, as we have just seen, the topology $\mathcal{O}S$ has a point-free V-modification which gives yet another space $\mathsf{pt}(V\mathcal{O}S)$. The last four of these spaces form a three-step hierarchy

$$\mathcal{Q}S \hookrightarrow \mathcal{L}S \hookrightarrow \mathsf{pt}(V\mathcal{O}S) \hookrightarrow \mathsf{pt}(VA)$$

where the first two steps are discussed above. We now consider the last, right-most, step.

The first thing we have to do is set up the assignment. There is an 'obvious' way to do this. Since we have a morphism $A \longrightarrow \mathcal{O}S$ the functorial properties of V give a morphism $VA \longrightarrow V\mathcal{O}S$ which we may hit with $\mathsf{pt}(\cdot)$ to obtain a continuous map

$\mathsf{pt}(V\mathcal{O}S) \longrightarrow \mathsf{pt}(VA)$. Although this is the map we want, in this section we construct it in a more direct fashion. The functorial construction will be analysed in Section 8.

Consider any V-point (Q, X) of S. Here $Q \in \mathcal{Q}S$ and, in the usual way, is related to an open filter ∇ on A by

$$x \in \nabla \iff Q \subseteq \circledcirc(x)$$

for $x \in A$. The second component of the V-point is a certain closed set $X \in \mathcal{C}S$. In particular, it is a set of $\wedge$-irreducible elements of A, and so we may form the infimum

$$a = \bigwedge X$$

(where this is taken in A). On general grounds we know that

$$p \in X \iff p \leq a$$

for $p \in S$.

6.11 LEMMA. *Let A be a frame with $S = \mathsf{pt}(A)$. For each V-point (Q, X) of $\mathcal{O}S$ the pair $(\nabla, \bigwedge X)$ is a V-point of A (where ∇ is the open filter on A associated with Q).*

Proof. As above let $a = \bigwedge X$ so we must show that a is ∇-closed and ∇-small. We know that $M \subseteq X \subseteq Q(\infty)$ so that

$$a \leq \bigwedge M$$

and hence a is ∇-small.

To show that a is ∇-closed we use the relationship $Q \ltimes X$.

Consider any $x \in \nabla$, let $y = (x \supset a)$, so we required $y \leq a$. To this end consider any $p \in X$, so we want $y \leq p$. By way of contradiction suppose this is not the case, that is $p \in \circledcirc(y)$. Since $x \in \nabla$, we have $Q \subseteq \circledcirc(x)$ and hence

$$p \in X \subseteq (X \cap \circledcirc(x))^-$$

(since $Q \ltimes X$). This gives

$$X \cap \circledcirc(x) \cap \circledcirc(y) \neq \emptyset$$

and hence there is a point q such that

$$q \in X \qquad x \not\leq q \qquad y \not\leq q$$

holds. In particular, we have

$$x \wedge y \leq a \leq q$$

which leads to the required contradiction. ■

This gives us an assignment $(Q, X) \longmapsto (\nabla, a)$ from $\mathsf{pt}(V\mathcal{O}S)$ to $\mathsf{pt}(VA)$. From the way the ∇ determines Q and a determines X we see this is an insertion at the set level. We improve this along the lines of Theorems 6.8 and 6.10.

6.12 THEOREM. *For each frame A with point space $S = \mathsf{pt}(A)$, the insertion*

$$\begin{array}{rcl} \mathsf{pt}(V\mathcal{O}S) & \xrightarrow{\varphi} & \mathsf{pt}(VA) \\ (Q,X) & \longmapsto & (\nabla, \bigwedge X) \end{array}$$

exhibits $\mathsf{pt}(V\mathcal{O}S)$ *as a subspace of* $\mathsf{pt}(VA)$.

Proof. We must compare the two topologies. By Lemma 6.6 and 5.8 these are given by

$$\begin{array}{ll} (Q,X) \in \langle\circ\rangle(U) \iff X \text{ meets } U & (\nabla, a) \in \Diamond(x) \iff x \nleq a \\ (Q,X) \in [\circ](U) \iff Q \subseteq U & (\nabla, a) \in \Box(x) \iff x \in \nabla \end{array}$$

for each $(Q,X) \in \mathsf{pt}(V\mathcal{O}S), (\nabla, a) \in \mathsf{pt}(VA), U \in \mathcal{O}S, x \in A$.

Each U has the form $\circledcirc(x)$ for some $x \in A$. Thus, if $(Q,X) \longmapsto (\nabla, a)$ then

$$\begin{array}{l} (Q,X) \in \langle\circ\rangle(\circledcirc(x)) \iff \circledcirc(x) \nsubseteq X' \iff x \nleq a \iff (\nabla, a) \in \Diamond(x) \\ (Q,X) \in [\circ](\circledcirc(x)) \iff Q \subseteq \circledcirc(x) \iff x \in \nabla \iff (\nabla, a) \in \Box(x) \end{array}$$

to give the required result. ■

As far as I am aware not much seems to be known about the relationship between these two spaces $\mathsf{pt}(V\mathcal{O}S)$ and $\mathsf{pt}(VA)$, though no doubt relevant results can be found in in odd places.

7 The clans of a space

For each sober space S the V-points of $\mathcal{O}S$ have the form (Q,X) where $Q \in \mathcal{Q}S$ and X is a related closed set. There is always at least one such V-point for each $Q \in \mathcal{Q}S$, and there can be many. In this section we fix some $Q \in \mathcal{Q}S$ and consider what the family of all associated V-points might look like. We do this via the related closed sets X. Let's introduce some terminology.

7.1 DEFINITION. For a sober space S and $Q \in \mathcal{Q}S$ the clan $\dagger Q$ of Q is the collection of closed sets $X \in \mathcal{C}S$ such that (Q,X) is a V-point of $\mathcal{O}S$. ■

In other words the clan $\dagger Q$ is the interval

$$M^- \subseteq X \subseteq Q(\infty)$$

of those closed sets $X \in \mathcal{C}S$ which satisfy $Q \ltimes X$. The two extremes M^- and $Q(\infty)$ are in $\dagger Q$, and we know that Q^- is also a member. This member Q^- seems to have some special status, but does not always occur in the same position. We illustrate later that as Q changes it can float about between the two extremes. There is something to be investigated here, and this section may throw a little light on the problem.

The clan $\dagger Q$ is a subspace of $\mathsf{pt}(V\mathcal{O}S)$. By Lemma 6.7 the specialization order on $\dagger Q$ is just inclusion. However, by Lemma 5.15 the clan $\dagger Q$ is not very interesting as a space, for the topology is the smallest such that for each $X \in \dagger Q$ the lower set

$$\{Y \in \dagger Q \mid Y \subseteq X\}$$

is closed. Nevertheless, as some kind of algebra, the clan can be quite complex.

For the whole space we have two embeddings

$$
\begin{array}{ccccc}
\mathcal{Q}S & \hookrightarrow & \mathcal{L}S & \overset{\lambda}{\hookrightarrow} & \mathsf{pt}(V\mathcal{O}S) \\
 & & L & \longmapsto & (L^{\uparrow}, L^{-}) \\
Q & \longmapsto & & & (Q, Q^{-})
\end{array}
$$

which select certain V-points of the topology. Here we fix Q and consider which points are selected, that is we concentrate on the second component X of the selected pair (Q, X). The two-step embedding merely selects the spatial member Q^{-}, but the lens embedding can select many more.

7.2 DEFINITION. For a sober space S and $Q \in \mathcal{Q}S$ a focal point of $\dagger Q$ is one that arises from a lens. That is, it has the form L^{-} for some lens L with $L^{\uparrow} = Q$.

The focal clan $\ddagger Q$ of Q is the collection of focal points in $\dagger Q$. ■

Recall that each $X \in \dagger Q$ satisfies $Q \ltimes X$, that is

$$X = (X \cap U)^{-}$$

for each $Q \subseteq U \in \mathcal{O}S$. The focal points satisfy a similar, but stronger property.

7.3 LEMMA. *Let S be a sober space with $Q \in \mathcal{Q}S$. A set $X \in \dagger Q$ is a focal point if and only if $X = (Q \cap X)^{-}$.*

Proof. Suppose first that X is a focal point of of $\dagger Q$. Thus there is some $L \in \mathcal{L}S$ with $Q = L^{\uparrow}, X = L^{-}$. But now, since L is a lens, we have

$$X = L^{-} = (L^{-} \cap L^{\uparrow})^{-} = (Q \cap X)^{-}$$

as required.

Conversely, suppose $X = (Q \cap X)^{-}$. We show that $L = Q \cap X$ is a lens with $L^{\uparrow} = Q$ and $L^{-} = X$. The equality $L^{-} = X$ holds by hypothesis, and the inclusion $L \subseteq Q$ gives $L^{\uparrow} \subseteq Q$. The Q-smallness of X gives

$$L \subseteq U \Longrightarrow Q \subseteq U$$

for each $U \in \mathcal{O}S$. For each $q \in Q$ we have $Q \not\subseteq q^{-\prime}$, so that $L \not\subseteq q^{-\prime}$, which provides some $p \in L$ with $p \sqsubseteq q$, and hence $q \in L^{\uparrow}$. Thus $Q \subseteq L^{\uparrow}$ and hence $L^{\uparrow} = Q$. Finally

$$L \subseteq L^{-} \cap L^{\uparrow} = X \cap Q = L$$

to give $L = L^{-} \cap L^{\uparrow}$, and hence L is a lens. ■

This characterization helps us to locate some of the focal points of a space.

7.4 LEMMA. *Let S be a sober space, and let $Q \in \mathcal{Q}S$.*

Both M^{-} and Q^{-} are in $\ddagger Q$.

If $X \in \ddagger Q$ then $X \subseteq Q^{-}$.

Proof. We know that $M^-, Q^- \in \dagger Q$. We have

$$M \subseteq Q \cap M^- \subseteq M^-$$

and hence $M^- = (Q \cap M^-)^-$, to show that $M^- \in \ddagger Q$. Trivially we have $(Q \cap Q^-)^- = Q^-$, to give $Q^- \in \ddagger Q$,

Finally, if $X \in \ddagger Q$ then

$$X = (Q \cap X)^- \subseteq Q^-$$

as required. ■

This result suggests that the focal points on $\dagger Q$ lie towards the lower end. I do not know the full story, but we do have the following.

7.5 LEMMA. *Let S be a sober space, and let $Q \in \mathcal{Q}S$. For each $X \in \dagger Q$ we have $(Q \cap X)^- \in \ddagger Q$, and this is the largest focal point below X.*

Proof. Consider $X \in \dagger Q$ and set $L = Q \cap X$. By Lemma 1.6 this is a lens. Since $M \subseteq X$ we have $M \subseteq L \subseteq Q$ and hence $L^\uparrow = M^\uparrow = Q$, to show that

$$(Q \cap X)^- = L^- \in \ddagger Q$$

as required for the first part.

For the second part consider any $Y \in \ddagger Q$ with $Y \subseteq X$. By Lemma 7.3 we have

$$Y = (Q \cap Y)^- \subseteq (Q \cap X)^-$$

as required. ■

At the end of Section 6 we suggested that the embedding λ just might be a homeomorphism. In general, it is not. In terms of $Q \in \mathcal{Q}S$ this is concerned with the difference between $\ddagger Q$ and $\dagger Q$, and this can be extreme.

At this stage it it useful to look at the salient features of a couple of examples. The full details of these are given in Section 9.

7.6 EXAMPLES. (a) The example of Subsection 9.2 illustrates one extreme of the many possibilities. We can describe some of the features here, but the full details are dealt with in that Subsection.

Thus, let A be the real interval $[0, 1]$ viewed as a frame (with bottom 0 and top 1). This is spatial with point space $S = [0, 1)$. Here we will confuse A with $\mathcal{O}S$ (for that will help to give a better impression of the actual situation). Each point $m \in S$ gives an open filter $\nabla = (m, 1]$ on A, and every open filter has this form. The corresponding $Q \in \mathcal{Q}S$ is the lower section $[0, m]$ (for the specialization order on S is the reverse of the given comparison). The singleton $M = \{m\}$ is the minimal generating set of Q. We find that $v_\nabla = [Q']$ so here $[Q']$ is as small as possible. In the terminology of Definition 7.7 (below) the space is strongly stacked, and so by Lemma 7.10 every V-point is a focal point.

The quotient A_∇ is the subset $[0, m] \cup \{1\}$ viewed as a frame, whereas the quotient by w_∇ is the two-point frame $\{m, 1\}$. This suggests that the block $[v_\nabla, w_\nabla]$ is quite large and, in fact, as in the Example 9.2 we can exhibit a vast array of its members.

The V-points associated with ∇ are essentially the reals a with $0 \leq a \leq m$.

(b) The example of Subsection 9.3 illustrates another extreme of the many possibilities. Again we can describe some of the features here, but the full details are dealt with in that Subsection.

Consider the set of all words on an uncountable alphabet. These form a tree $\mathbb{S}$ of height ω. We adjoint to that a new point and impose a certain topology on $S = \{\star\} \cup \mathbb{S}$. This gives a space which is T_1 and sober. (In particular, the comparison on the tree $\mathbb{S}$ is not related to the specialization order of S, for that is just equality.) We use the singleton $Q = \{\star\} = M$ as the selected member of $\mathcal{Q}S$ with its minimal generating set.

We have

$$w_\nabla = [M'] = [Q']$$

so here $[Q']$ is as large as possible. Note that

$$w_\nabla(\emptyset) = Q' = \mathbb{S}$$

the original tree. For this Q there is just one focal point, namely that given by Q itself.

The lower end v_∇ of the block is harder to describe, but we do have

$$v_\nabla(\emptyset) = \emptyset$$

and we can exhibit many members of $(\mathcal{O}S)_\nabla$ in the form of closed set X with $Q \ltimes X$.

Consider any subtree $\mathbb{T}$ of $\mathbb{S}$ such that for each node $x \in \mathbb{T}$ there is a lot of splitting. (This is made precise in the Example.) There are many such subtrees $\mathbb{T}$, and for each one $X = \{\star\} \cup \mathbb{T}$ satisfies $Q \ltimes X$.

This shows that the clan $\dagger Q$ is extremely complicated and large. In particular, the whole space S is in $\ddagger Q$. Even when sober a clan can be pretty nasty. ■

In the remainder of this section we discuss a couple of properties which ensure the clan structure is quite simple. This is also the point where, as mentioned in the preamble, we see a connection with another piece of point-free gadgetry, the patch assembly of a frame. We need not go into the details here, they can be found in [7], especially Chapter 8. Let us simple accept that the following two notions come out of that work.

7.7 DEFINITION. Let S be a (sober) space and let $Q \in \mathcal{Q}S$. We say S is

Q-stacked $\qquad$ Q-strongly stacked

if, respectively,

$$Q \ltimes X \Longrightarrow X \subseteq Q^- \qquad Q \ltimes X \Longrightarrow X \subseteq (Q \cap X)^-$$

holds for each $X \in \mathcal{C}S$.

We say S is

stacked $\qquad$ strongly stacked

if, respectively, it is

Q-stacked $\qquad$ Q-strongly stacked

for each $Q \in \mathcal{Q}S$. ■

Remember that we always have

$$Q \ltimes X \Longrightarrow X \subseteq Q(\infty)$$

so that being Q-stacked is a strengthening of this universal property, and being strongly stacked is a further strengthening.

These properties are defined in this way to emphasize their common aspects. There is also another way to look at them. Recall that each $Q \in \mathcal{Q}S$ gives us an open filter ∇ on $\mathcal{O}S$ and a block of nuclei

$$v_\nabla \leq [Q'] \leq [M'] = w_\nabla$$

with indicated least and greatest members. As we have seen in Examples 7.6 the special member $[Q']$ of the block can occur at either extreme. Each of the three sets

$$Q(\infty) \supseteq Q^- \supseteq M^-$$

is the complement of the bottom value of the corresponding nucleus.

Being stacked or strongly stacked improves a certain comparison to an equality.

7.8 LEMMA. *Let S be a (sober) space and let $Q \in \mathcal{Q}S$. Then S is*

Q-stacked *Q-strongly stacked*

if and only if

$$Q^- = Q(\infty) \qquad v_\nabla = [Q']$$

respectively.

Proof. Suppose S is Q-stacked. Since $Q \ltimes Q(\infty)$, we have $Q(\infty) \subseteq Q^-$, and hence $Q^- = Q(\infty)$ (since the other inclusion always holds). Conversely, if $Q^- = Q(\infty)$ then

$$Q \ltimes X \Longrightarrow X \subseteq Q(\infty) = Q^-$$

and hence S is Q-stacked.

We always have

$$v_\nabla \leq [Q']$$

and these two nuclei will agree when they fix the same open sets. For a close set $X \in \mathcal{C}S$ the open set X' is fixed by either nucleus precisely when

$$Q \ltimes X \qquad X = (Q \cap X)^-$$

respectively. This leads to the required second equivalence. ■

Clearly, these stacking properties have some impact on the nature of the V-points of a space. The precise connections are not known, but we do know something.

Recall that in a T_2 space every compact subset is closed. By extracting a bit more out of the proof of that we obtain the following.

7.9 LEMMA. *Let S be a T_2 space. Then*

$$Q \ltimes X \Longrightarrow X \subseteq Q$$

for each $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. In particular, the space is strongly stacked.

Proof. Consider $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$ with $Q \ltimes X$, consider any $p \in X$, and by way of contradiction suppose $p \notin Q$. Since S is T_2, by the standard covering argument (as used to show that Q is closed) we have

$$Q \subseteq U \qquad p \in V \qquad U \cap V = \emptyset$$

for some $U, V \in \mathcal{O}S$. But now $U^- \subseteq V'$ and hence, since $Q \ltimes X$, we have

$$p \in X \subseteq (X \cap U)^- \subseteq U^- \subseteq V'$$

which is the contradiction. ■

This shows that in Bourbaki's world nothing very interesting happens as far as V-points are concerned. In a T_2 space we have $v_\nabla = w_\nabla$ for each open filter ∇ on the topology, and each clan is a singleton. (In fact, for this result we can replace T_2 be something much weaker. The details are not needed here, but can be found in [8].)

As often happens, non-T_2 spaces are more interesting.

For a space S and $Q \in \mathcal{Q}S$ we say S is Q-**focused** if $\ddagger Q = \dagger Q$.

7.10 THEOREM. *Let S be a sober space and let $Q \in \mathcal{Q}S$.*
If S is Q-strongly stacked then it is Q-focused,
If S is Q-focused then it is Q-stacked.

Proof. Suppose S is Q-strongly stacked and consider any $X \in \dagger Q$. Then $Q \ltimes X$, and hence $X \subseteq (Q \cap X)^-$ (since S is Q-strongly stacked) which, by Lemma 7.3, gives $X \in \ddagger Q$.

Suppose S is Q-focused. Then $Q(\infty) \in \dagger Q = \ddagger Q$ so that $Q(\infty) \subseteq Q^-$, by Lemma 7.4, and hence S is Q-stacked by Lemma 7.8. ■

I am not sure if either of these implications can be turned into an equivalence. My resident expert tells me she beleives there is at least one space that is stacked and not strongly stacked, but she hasn't written down the details.

We have seen that in a T_2 space each clan is a singleton, In fact, for most everyday spaces the clans and focal clans are quite small.

7.11 THEOREM. *Let S be a sober space.*
The space S is T_1 precisely when $\ddagger Q$ is a singleton for each $Q \in \mathcal{Q}S$.
The space S is T_1 and stacked precisely when $\dagger Q$ is a singleton for each $Q \in \mathcal{Q}S$.

Proof. Suppose that S is T_1, and consider any $X \in \ddagger Q$ for $Q \in \mathcal{Q}S$. We have $M^- \subseteq X \subseteq Q^-$, but S is T_1, and hence $M = Q$, to give $M^- = X = Q^-$.

Conversely, suppose each $Q \in \mathcal{Q}S$ is uniquely focused, and consider any pair of points $q \sqsubseteq p$ in S. We show that $q = p$, and hence S is T_1. Let $Q = \uparrow q$ and $Y = q^-, X = p^-$. A simple calculation shows that both $X, Y \in \ddagger Q$, so that $X = Y$, to give $p = q$, as required.

Suppose that S is T_1 and stacked, and consider any $X \in \dagger Q$. We have $M^- \subseteq X \subseteq Q(\infty)$. But the T_1 property and Lemma 7.8 give $M^- = Q^- = Q(\infty)$, and hence X is this set.

Conversely, suppose $Q \in \mathcal{Q}S$ carries a unique V-point. Then $M^- = Q(\infty)$, which leads to the required result. ■

There is much that is not known here, and these notions deserve a thorough going over. The Thesis [7] makes a start at this.

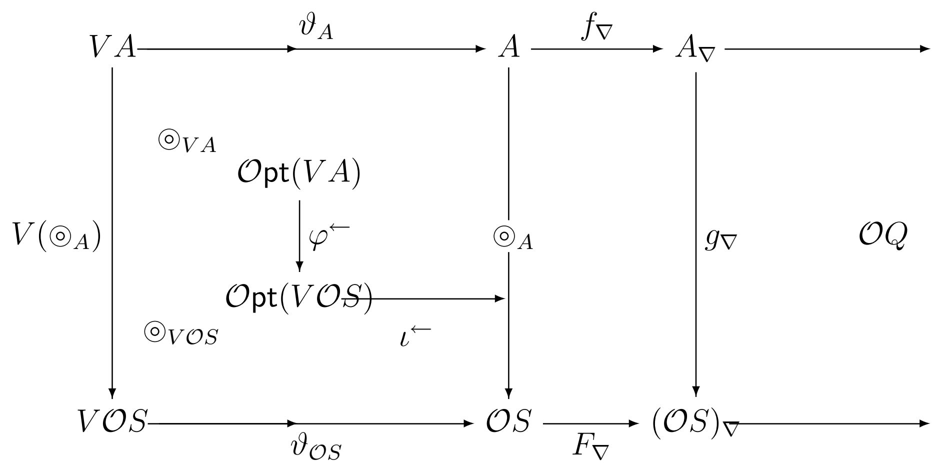


Table 1: Various arrows induced by the spatial reflection and an open filter

8 Functorial matters

In this section we collect together various functorial and other properties of the V-construction. On the whole these are not directly related to the main topic of these notes and this section could be omitted entirely. However, the results described might be useful in further work.

Consider the diagram of Table 1 on page 45. Most of this is generated from the spatial reflection

$$A \xrightarrow{\circledcirc_A} \mathcal{O}S$$

of a frame A to the topology $\mathcal{O}S$ of its point space $S = \mathsf{pt}(A)$. This arrow appears in a downwards direction somewhere near the centre, and so splits the diagram in two. Everything to the left is induced by $\circledcirc_A$. The arrows $f_\nabla, F_\nabla, g_\nabla$ to the right are induced by an open filter ∇ on A. Of course Q is the member of $\mathcal{Q}S$ associated with ∇, and the two unnamed arrows are the spatial reflections. Our job here is to construct all these arrows and to show that all the cells commute.

To begin we set up and analyse the left hand part of the diagram, that part which does not depend on ∇. We look at the ∇-part later.

We have already met several of the arrows in this diagram; now we will see how these fit into the general scheme of things.

We have a pair of categories and a forgetful functor

$$\boldsymbol{Vrm} \xleftarrow[\text{¿}]{} \boldsymbol{Frm}$$

via which we may think of ***Frm*** as a subcategory of ***Vrm***. Both categories have the same objects, frames, but ***Vrm*** has more arrows. By definition, the object construction $A \mapsto VA$ (on frames) provides the universal solution to Problem 4.2. On general grounds this is the object assignment of the left adjoint to the forgetful functor, that is it provides a reflection from ***Vrm*** to ***Frm***.

Each frame A (thought of as a $\boldsymbol{Vrm}$-object) is furnished with a $\boldsymbol{Vrm}$-arrow

$$A \Longrightarrow VA$$

to its V-modification. This is the unit of the adjunction, and with the universal property it determines all the other components.

The adjunction is an inverse pair of arrow transposition

$$\boldsymbol{Vrm}[A, ¿B] \underset{(\cdot)^{\flat}}{\overset{(\cdot)^{\sharp}}{\rightleftarrows}} \boldsymbol{Frm}[VA, B]$$

for each pair A, B of frames. For each $\boldsymbol{Vrm}$-arrow

$$A \overset{f^{\Diamond}}{\underset{f_{\Box}}{\Longrightarrow}} B$$

the transpose

$$VA \xrightarrow{(f^{\Diamond}, f_{\Box})^{\sharp}} B$$

is the frame morphism

$$VA \xrightarrow{\ f\ } B$$

given by the universal solution property. Because of the uniqueness of this f the unit has an epic-like property. Namely, for each parallel pair

$$VA \underset{h}{\overset{g}{\rightrightarrows}} B$$

of frame morphisms, if the two composites

$$A \Longrightarrow VA \xrightarrow{\ g\ } B \qquad\qquad A \Longrightarrow VA \xrightarrow{\ h\ } B$$

(as $\boldsymbol{Vrm}$-arrows) agree, then $g = h$. This is useful when showing that certain squares commute.

We have the object assignment $A \mapsto VA$ of the functor. The arrow assignment is obtained from the universal property. For each $\boldsymbol{Vrm}$-arrow

$$A \overset{f}{\Longrightarrow} B$$

the $\boldsymbol{Frm}$-arrow

$$VA \xrightarrow{V(f)} B$$

is the unique frame morphism such that the $\boldsymbol{Vrm}$-square

$$\begin{array}{ccc} A & \Longrightarrow & VA \\ {\scriptstyle (f^{\Diamond}, f_{\Box})} \Downarrow & & \downarrow {\scriptstyle V(f)} \\ B & \Longrightarrow & VB \end{array}$$

commutes. Notice that we can apply $V(\cdot)$ to any frame morphism (since each is a ***Vrm***-arrow).

We have the unit, but what about the co-unit

$$VA \longrightarrow A$$

of the adjunction? By general principles this is the $(\cdot)^\sharp$-transpose of the identity ***Vrm***-arrow on A which, in this case, is the pair

$$A \overset{id}{\underset{id}{\Longrightarrow}} A$$

of identity frame morphisms. Thus we have the following.

8.1 LEMMA. *For each frame A the co-unit*

$$VA \xrightarrow{\ \vartheta_A\ } A$$

is the unique frame morphism such that the composite

$$A \Longrightarrow VA \xrightarrow{\ \vartheta_A\ } A$$

*(as a **Vrm**-arrow) is the identity **Vrm**-arrow on A.*

In other words ϑ_A is the unique frame morphism such that

$$(\vartheta \circ \Diamond)(x) = x \qquad (\vartheta \circ \Box)(x) = x$$

for each $x \in A$.

There are a couple of useful observations concerning this morphism ϑ. The first is nothing more than the naturality of the co-unit in any adjunction.

8.2 LEMMA. *For each **Vrm**-arrow*

$$A =\!(f^\Diamond, f_\Box)\!\Longrightarrow B$$

*the **Vrm**-square*

$$\begin{array}{ccc} VA & \xrightarrow{\ \vartheta_A\ } & A \\ {\scriptstyle V(f)}\downarrow & & \Downarrow{\scriptstyle (f^\Diamond, f_\Box)} \\ VB & \xrightarrow[\ \vartheta_B\ ]{} & B \end{array}$$

commutes.

Remember also that we can apply this to any frame morphism f, in which case we obtain a commuting ***Frm***-square. As a particular case of this we may start from the spatial reflection $\circledcirc_A$ of a frame A. In that case we obtain the outer square on the left of the diagram of Table 1. We will insert into this square several more arrows to obtain four commuting cells.

The second observation is slightly more interesting since it is not just a general fact. Remember, by Lemma 5.12, for each frame A we have an embedding

$$\begin{array}{rcl} S & \xrightarrow{\ \theta\ } & \mathsf{pt}(VA) \\ p & \longmapsto & (\nabla(p), p) \end{array}$$

of the point space $S = \mathsf{pt}(A)$ into the point space of the V-modification of A. (In fact, we have located larger subspaces of $\mathsf{pt}(VA)$ but we don't need those here.)

8.3 LEMMA. *For each frame A with point space $S = \mathsf{pt}(A)$, the **Frm**-square*

$$\begin{array}{ccc} VA & \xrightarrow{\ \vartheta\ } & A \\ {\scriptstyle \circledcirc_{VA}}\downarrow & & \downarrow{\scriptstyle \circledcirc_A} \\ \mathcal{O}\mathsf{pt}(VA) & \xrightarrow[\ \theta^{\leftarrow}\ ]{} & \mathcal{O}S \end{array}$$

commutes.

Proof. Because of the epic-like property of the unit $A \Longrightarrow VA$, it suffices to show that the two paths from VA to $\mathcal{O}S$ agree on the generators $\Diamond(x)$ and $\Box(x)$ of VA (for $x \in A$.

The defining property of ϑ gives

$$(\circledcirc_A \circ \vartheta \circ \Diamond)(x) = \circledcirc_A(x) = (\circledcirc_A \circ \vartheta \circ \Box)(x)$$

for each $x \in A$. Also, by definition of $\langle\circ\rangle$ and $[\circ]$, we have

$$\circledcirc_{VA} \circ \Diamond = \langle\circ\rangle \qquad \circledcirc_{VA} \circ\Box = [\circ]$$

(as in Section 5). Thus for each $x \in A$ and $p \in S$ we have

$$\begin{aligned} p \in (\theta^{\leftarrow} \circ \circledcirc_{VA} \circ \Diamond)(x) &\iff (\nabla(p), p) \in \langle\circ\rangle\,(x) \iff x \nleq p \iff p \in \circledcirc_A(x) \\ p \in (\theta^{\leftarrow} \circ \circledcirc_{VA} \circ \Box)(x) &\iff (\nabla(p), p) \in [\circ]\,(x) \iff x \nleq p \iff p \in \circledcirc_A(x) \end{aligned}$$

where

$$(\nabla(p), p) \in [\circ]\,(x) \iff x \in \nabla(p) \iff x \nleq p$$

is used in the lower sequence of equivalences. This leads to the required result. ■

As observed above, each ***Vrm***-arrow

$$A \overset{f}{\Longrightarrow} B$$

induces a commuting ***Vrm***-square

$$\begin{array}{ccc} A & \Longrightarrow & VA \\ \Big\Downarrow (f^{\Diamond}, f_{\Box}) & & \Big\downarrow V(f) \\ B & \Longrightarrow & VB \end{array} \qquad \begin{array}{c} \mathsf{pt}(VA) \\ \Big\uparrow \Phi(f) \\ \mathsf{pt}(VB) \end{array}$$

as on the left. Shortly we will need the two components $(f^{\Diamond}, f_{\Box})$ of the given ***Vrm***-arrow f. Since $V(f)$ is a frame morphism, this induces a continuous map $\Phi(f)$, as on the right. Let's get inside this map.

Consider a typical V-point (Π, b) of B, Thus Π is a certain filter on B and b is a certain element. This pair encodes a character

$$VB \longrightarrow \mathsf{2}$$

of VB, which gives a character

$$VA \xrightarrow{V(f)} VB \longrightarrow \mathsf{2}$$

of VA, and this is encoded by a V-point (∇, a) of A. It is this pair

$$(\nabla, a) = \Phi(f)(\Pi, b)$$

we wish to describe.

The character of VB is determined by a V-character

$$\begin{array}{ccccc} B & \Longrightarrow & VB & \longrightarrow & \mathsf{2} \\ y & \longmapsto & \Diamond(y) & \longmapsto & \begin{cases} 1 \text{ if } y \not\leq b \\ 0 \text{ if } y \leq b \end{cases} \\ y & \longmapsto & \Box(y) & \longmapsto & \begin{cases} 1 \text{ if } y \in \Pi \\ 0 \text{ if } y \notin \Pi \end{cases} \end{array}$$

of B. The two components of this are displayed below the arrows.

The required character of VA is determined by the V-character given by the top compound below.

$$\begin{array}{ccccccc} A & \Longrightarrow & VA & \xrightarrow{V(f)} & VB & \longrightarrow & \mathsf{2} \\ \\ A & \overset{(f^{\Diamond}, f_{\Box})}{\Longrightarrow} & B & \Longrightarrow & VB & \longrightarrow & \mathsf{2} \\ x & \longmapsto & f^{\Diamond}x & \longmapsto & \Diamond(f^{\Diamond}x) & \longmapsto & \begin{cases} 1 \text{ if } f^{\Diamond}x \not\leq b \\ 0 \text{ if } f^{\Diamond}x \leq b \end{cases} \\ x & \longmapsto & f_{\Box}x & \longmapsto & \Box(f_{\Box}x) & \longmapsto & \begin{cases} 1 \text{ if } f_{\Box}x \in \Pi \\ 0 \text{ if } f_{\Box}x \notin \Pi \end{cases} \end{array}$$

By the commuting square above, this is the same as the second compound. The two components of this V-character are displayed below the arrows. These components determined the required V-point (∇, a) of A, where ∇ comes from the $\Box$-assignment and a from $\Diamond$-assignment.

Next remember that $f_\Box$ is a $\wedge$-semilattice morphism, so the inverse image function $f_\Box^\leftarrow$ takes filters into filters. The other component $f^\Diamond$ is a $\bigvee$-morphism, so has a right adjoint

$$A \underset{f_*^\Diamond}{\overset{f^\Diamond}{\rightleftarrows}} B$$

where

$$f^\Diamond x \le y \Longleftrightarrow x \le f_\star^\Diamond y$$

for $x \in A$ and $y \in B$.

8.4 LEMMA. *For each* ***Vrm****-arrow*

$$A \overset{f = (f^\Diamond, f_\Box)}{\Longrightarrow} B$$

the induced continuous map

$$\mathsf{pt}(VB) \xrightarrow{\Phi(f)} \mathsf{pt}(VA)$$

is given by

$$\Phi(f) = (f_\Box^\leftarrow, f_*^\Diamond)$$

(where $f_\Box^\leftarrow$ acts on open filters and $f_*^\Diamond$ acts on elements).

Proof. Consider any V-point (Π, b) of B. From the calculations above we see that the V-point

$$(\nabla, a) = \Phi(f)(\Pi, b)$$

of A is given by

$$x \in \nabla \Longleftrightarrow f_\Box \in \Pi \qquad x \le a \Longleftrightarrow f^\Diamond x \le b$$

which leads to the required result. ■

Notice that we did not show that the pair (∇, a) is a V-point of A. In fact, this is automatic. However, you will find it instructive to check this directly using the properties of the pair $(f^\Diamond, f_\Box)$.

Let's look at a two particular cases of this result. In each case the parent ***Vrm***-arrow f is a frame morphism.

For the first particular case we start from a continuous map ϕ between sober spaces

$$\begin{array}{ccc} S & \mathcal{O}S & \mathsf{pt}(V\mathcal{O}S) \\ \phi \uparrow & \downarrow \phi^\leftarrow & \uparrow \Phi(\phi^\leftarrow) \\ T & \mathcal{O}T & \mathsf{pt}(V\mathcal{O}T) \end{array}$$

as on the left. This gives a frame morphism $\phi^\leftarrow$, as in the centre, which in turn gives a continuous map, as on the right. We view V-points of $\mathcal{O}S$ as pairs (Q, X), and V-points of $\mathcal{O}T$ as pairs (R, Y).

8.5 LEMMA. *For each continuous map*

$$T \xrightarrow{\ \phi\ } S$$

between sober spaces, the induced map

$$\begin{array}{rcl} \mathsf{pt}(V\mathcal{O}T) & \longrightarrow & \mathsf{pt}(V\mathcal{O}S) \\ (R, Y) & \longmapsto & (\phi[R]^{\uparrow}, \phi[Y']^{-\prime}) \end{array}$$

is obtained by taking modified direct images, as indicated.

Proof. Consider any V-point (R, Y) of $\mathcal{O}T$, and let (Q, X) be the resulting V-point (Q, X) of $\mathcal{O}S$, as given by Lemma 8.4. This correspondence goes via the filters $\nabla(Q)$ on $\mathcal{O}S$ and $\nabla(R)$ on $\mathcal{O}T$ given by Q and R, respectively, and the opens $X' \in \mathcal{O}S$ and $Y' \in \mathcal{O}T$. Thus, using Lemma 8.4, for each $U \in \mathcal{O}S$ we have

$$U \in \nabla(Q) \Longleftrightarrow \phi^{\leftarrow}(U) \in \nabla(R) \Longleftrightarrow R \subseteq \phi^{\leftarrow}(U) \Longleftrightarrow \phi[R] \subseteq U$$

and hence

$$Q \subseteq U \Longleftrightarrow U \in \nabla(Q) \Longleftrightarrow \phi[R] \subseteq U \Longleftrightarrow \phi[R]^{\uparrow} \subseteq U$$

to give $Q = \phi[U]^{\uparrow}$. Similarly, for each $U \in \mathcal{O}S$ we have

$$U \subseteq X' \Longleftrightarrow \phi^{\leftarrow}(U) \subseteq Y' \Longleftrightarrow U \subseteq \phi[Y'] \Longleftrightarrow U \subseteq \phi[Y']^{\circ}$$

so that

$$X = \phi[Y']^{\circ\prime} = \phi[Y']^{-\prime}$$

to give the required result. ■

Notice that again we did not show that the result $(\phi[R]^{\uparrow}, \phi[Y']^{-\prime})$ is a V-point of $\mathcal{O}S$. However, here this is not automatic, and it is instructive to fill in the missing details.

For the second particular case start from the spatial reflection

$$A \xrightarrow{\ \circledcirc^{*} = \circledcirc_A\ } \mathcal{O}S$$

of a frame A. Here, as usual $S = \mathsf{pt}(A)$, and the parent frame morphism f is $\circledcirc_A$. We view each V-point of $\mathcal{O}S$ as a certain pair (Q, X) where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. This will mean a bit of manipulation to bring this in line with the filter-element view.

By a particular case of Lemma 2.3 the right adjoint of the spatial reflection morphism

$$A \underset{\circledcirc_{*}}{\overset{\circledcirc^{*}}{\rightleftarrows}} \mathcal{O}S$$

is given by

$$\circledcirc_{*}(X') = \bigwedge X$$

for each $X \in \mathcal{C}S$. We have seen the induced continuous map before.

8.6 LEMMA. *For each frame A, the spatial reflection morphism*

$$A \xrightarrow{\circledcirc} \mathcal{O}S$$

induces the continuous map

$$\begin{array}{ccc} \mathsf{pt}(V\mathcal{O}S) & \xrightarrow{\varphi} & \mathsf{pt}(VA) \\ (Q, X) & \longmapsto & (\nabla, \bigwedge X) \end{array}$$

as described in Theorem 6.12.

Proof. Consider any point (Q, X) of $V\mathcal{O}S$. To use Lemma 8.4 we view this as a pair $(\nabla(Q), X')$ where $\nabla(Q)$ is the open filter on $\mathcal{O}S$ given by Q. We have

$$\varphi(Q, X) = \Phi(\circledcirc)(\nabla(Q), X') = \big(\circledcirc^{\leftarrow}(\nabla(Q)), \circledcirc_*(X')\big)$$

so it suffices to unravel the right hand side.

The filter $\nabla(Q)$ on $\mathcal{O}S$ is given by

$$U \in \nabla(Q) \iff Q \subseteq U$$

for each $U \in \mathcal{O}S$, and the associated filter on A is given by

$$x \in \nabla \iff Q \subseteq \circledcirc(x)$$

for each $x \in A$. Thus, for $x \in A$, we have

$$x \in \circledcirc^{\leftarrow}(\nabla(Q)) \iff \circledcirc(x) \in \nabla(Q) \iff Q \subseteq \circledcirc(x) \iff x \in \nabla$$

which gives the required result for the right hand component.

The result for the left hand component is an immediate consequence of the formula $\circledcirc_*(X') = \bigwedge X$ given above. ■

We are now in a position to explain the components of the left hand part of the diagram in Table 1.

We start from the spatial reflection $\circledcirc_A$ of A. The outer cell is an instance of Lemma 8.2 where in this case the parent ***Vrm***-arrow f is a frame morphism. Within that square the left hand cell is an instance of the functorality and naturality properties of the spatial reflection construction. Here the continuous map

$$\mathsf{pt}(V\mathcal{O}S) \xrightarrow{\varphi} \mathsf{pt}(VA)$$

is that set up by Lemma 8.6. We combine this with the embeddings

$$\begin{array}{ccccccccc} S & \hookrightarrow & \mathcal{Q}S & \hookrightarrow & \mathcal{L}S & \longrightarrow & \mathsf{pt}(V\mathcal{O}S) & \xrightarrow{\varphi} & \mathsf{pt}(VA) \\ p & \longmapsto & & & & & (p^{\uparrow}, p^{-}) & & \end{array}$$

and let ι be the composite embedding from S to $\mathsf{pt}(V\mathcal{O}S)$, as indicated.

8.7 LEMMA. *In the situation described above we have $\varphi \circ \iota = \theta$.*

Proof. Consider any $p \in S$. By Lemma 8.6 we have

$$(\varphi \circ \iota)(p) = (\nabla, \bigwedge p^-)$$

where ∇ is the open filter on A given by $p^\uparrow \in \mathcal{Q}S$, and the infimum $\bigwedge p^-$ is taken in A.

This filter is nothing more than $\nabla(p)$.

To compute $\bigwedge p^-$ remember that p^- is the set of $\wedge$-irreducible elements of A *above* p. Thus $\bigwedge p^- = p$. ■

As a consequence of this we see that in the Table the pentagon at the top right of the square is just the square of Lemma 8.3. Thus the pentagon commutes.

There is one cell within the square left to look at.

8.8 LEMMA. *In the diagram of Table 1 the bottom triangle commutes, that is*

$$\iota^\leftarrow \circ \circledcirc_{V\mathcal{O}S} = \vartheta_{\mathcal{O}S}$$

holds.

Proof. By the epic-like property of the unit it suffices to show that both the composites (as ***Vrm***-arrows)

$$\mathcal{O}S \Longrightarrow V\mathcal{O}S \xrightarrow{\iota^\leftarrow \circ \circledcirc_{V\mathcal{O}S}} \mathcal{O}S \qquad\qquad \mathcal{O}S \Longrightarrow V\mathcal{O}S \xrightarrow{\vartheta_{\mathcal{O}S}} \mathcal{O}S$$

agree. In other words, we must show that both the function composites from the left hand side are the identity on $\mathcal{O}S$.

To this end consider any $U \in \mathcal{O}S$, making use of Lemma 6.6, for each $p \in S$ we have both

$$\begin{aligned} p \in (\iota^\leftarrow \circ \circledcirc_{V\mathcal{O}S} \circ \Diamond)(U) &\iff (p^\uparrow, p^-) \in \langle\circ\rangle\,(U) \iff p^- \text{ meets } U \iff p \in U \\ p \in (\iota^\leftarrow \circ \circledcirc_{V\mathcal{O}S} \circ \Box)(U) &\iff (p^\uparrow, p^-) \in [\circ]\,(U) \iff \quad p^\uparrow \subseteq U \quad \iff p \in U \end{aligned}$$

to give the required result. ■

This deals with the left hand part of the diagram. We now turn to the right hand part, which is induced by an arbitrary open filter ∇ on A.

The open filter ∇ determines a fitted nucleus on A and an associated quotient

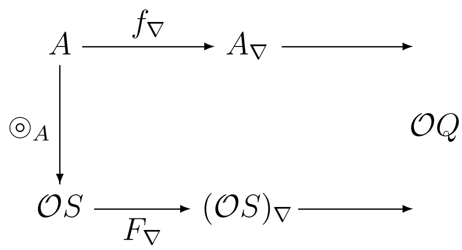
$$A \xrightarrow{f_\nabla} A_\nabla$$

which gives the top arrow of the right hand square. The filter also determines an open filter on $\mathcal{O}S$, namely the open neighbourhood filter of the associated $Q \in \mathcal{Q}S$. In turn this determines a fitted nucleus on $\mathcal{O}S$ and an associated quotient

$$\mathcal{O}S \xrightarrow{F_\nabla} (\mathcal{O}S)_\nabla$$

which gives the bottom arrow of the right hand square.

Notice the slight abuse of notation here, since we write $(\mathcal{O}S)_\nabla$ for the quotient of $\mathcal{O}S$ (to remind us of the parent open filter ∇). By Lemma 3.10 the set Q is the point space of A_∇. As a particular case of this we see that Q is also the point space of $(\mathcal{O}S)_\nabla$. Thus we obtain a diagram



using the two constructed arrows f_∇, F_∇. The two unnamed arrows are the spatial reflections. Both the arrows from A to $\mathcal{O}Q$ (via A_∇ and $(\mathcal{O}S)_\nabla$) are

$$a \longmapsto \circledcirc(a) \cap Q$$

(for $a \in A$), and hence the diagram commutes.

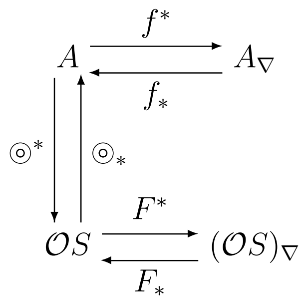
In the remainder of this section we construct a morphism

$$A_\nabla \xrightarrow{\ g_\nabla\ } (\mathcal{O}S)_\nabla$$

to make the left hand square (in the small diagram) commute. Since f_∇ is surjective this will also make the right hand triangle commute.

To produce g_∇ we need to look at the right adjoint of each of the morphisms involved. Thus before we start the construction proper let's clean up the notation.

The three known sides of the square give a diagram



where

$$\circledcirc^* = \circledcirc_A \qquad f^* = f_\nabla \qquad F^* = F_\nabla$$

and each has the indicated right adjoint. We have dropped the subscripts 'A' and '∇' since these are fixed throughout. We have to find a fill-in morphism

$$A_\nabla \xrightarrow{\ g^*\ } (\mathcal{O}S)_\nabla$$

which makes the morphism square commute. Since each of the three given morphisms is surjective, there can be at most one fill-in morphism. In fact, there isn't much choice, but let's not rush.

We need some information about the various components of this diagram. For each $y \in A$ and $V \in \mathcal{O}S$ we set

$$f(y) = \bigvee\{(x \supset y) \mid x \in \nabla\} \qquad F(V) = \bigcup\{(U \supset V) \mid Q \subseteq U\}$$

to produce inflators f on A and F on $\mathcal{O}S$. The closures f^∞ and F^∞ are the nuclei on A and $\mathcal{O}S$ which produce the quotients A_∇ and $(\mathcal{O}S)_\nabla$. Thus

$$f_* \circ f^* = f^\infty \qquad F_* \circ F^* = F^\infty$$

where the right components are just insertions and

$$f^*(y) = f^\infty(y) \qquad F^*(V) = F^\infty(V)$$

for $y \in A, V \in \mathcal{O}S$.

The connection between ∇ and Q suggests there is a connection between f and F.

8.9 LEMMA. *We have* $\circledcirc^* \circ f \le F \circ \circledcirc^*$.

Proof. Observe that

$$\circledcirc^*(x \supset y) \subseteq (\circledcirc^*(x) \supset \circledcirc^*(y))$$

for each $x, y \in A$. In particular, if $x \in \nabla$ then $Q \subseteq \circledcirc^*(x)$, so that

$$\circledcirc^*(x \supset y) \subseteq (F \circ \circledcirc^*)(y)$$

holds. But now, since $\circledcirc^*$ is a frame morphism, we have

$$(\circledcirc^* \circ f)(y) = \circledcirc^* \left(\bigvee\{(x \supset y) \mid x \in \nabla\}\right) = \bigvee\{\circledcirc^*(x \supset y) \mid x \in \nabla\}) \le (F \circ \circledcirc^*)(y)$$

to give the required result. ■

This comparison between the associated inflators extends to the corresponding nuclei.

8.10 COROLLARY. *We have* $\circledcirc^* \circ f^\infty \le F^\infty \circ \circledcirc^*$.

Proof. We show that

$$\circledcirc^* \circ f^\alpha \le F^\alpha \circ \circledcirc^*$$

for all ordinals α. Of course, we proceed by induction.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$\circledcirc^* \circ f^{\alpha+1} = \circledcirc^* \circ f \circ f^\alpha \le F \circ \circledcirc^* \circ f^\alpha \le F \circ F^\alpha \circ \circledcirc^* = F^{\alpha+1} \circ \circledcirc^*$$

using first Lemma 8.9 and then the induction hypothesis.

The induction leap to a limit ordinal is almost immediate. ■

What can the required morphism g^* be? We know that

$$G^* = F^* \circ \circledcirc^*$$

is a frame morphism, and we require some morphism g^* such that

$$f^* \circ g^* = G^*$$

holds. Since both the right adjoints f_*, F_* are insertion (at the set level) we see that

$$\begin{array}{rcl} A_\nabla & \xrightarrow{\;g^*\;} & (\mathcal{O}S)_\nabla \\ y & \longmapsto & G^*(y) \end{array}$$

is the assignment that will work. In other words, we must show that $g^* = G^*|_{A_\nabla}$ is the required morphism.

Certainly this makes the morphism square commute, and it is a $\wedge$-semilattice morphism, so it suffices to check that it is a $\bigvee$-morphism. The crunch, of course, is that suprema on A_∇ and $(\mathcal{O}S)_\nabla$ are not computed in the same was as on A and $\mathcal{O}S$; the appropriate nucleus gets involved. For convenience let us write

$$\widehat{\bigvee} \qquad \widetilde{\bigvee}$$

for, respectively, the supremum on A_∇ and $(\mathcal{O}S)_\nabla$. Thus

$$\widehat{\bigvee} = f^\infty \circ \bigvee \qquad \widetilde{\bigvee} = F^\infty \circ \bigcup$$

that is

$$\widehat{\bigvee} Y = f^\infty \left(\bigvee Y\right) \qquad \widetilde{\bigvee} \mathcal{U} = F^\infty \left(\bigcup \mathcal{U}\right)$$

for $Y \subseteq A_\nabla$ and $\mathcal{U} \subseteq (\mathcal{O}S)_\nabla$. Since G^* is a frame morphism we have

$$G^* \circ \bigvee = \widetilde{\bigvee} \circ G^*$$

(at the subset level). We need a corresponding result for g^*.

8.11 LEMMA. *We have* $g^* \circ \widehat{\bigvee} = \widetilde{\bigvee} \circ g^*$.

Proof. In the usual way a comparison

$$g^* \circ \widehat{\bigvee} \leq \widetilde{\bigvee} \circ g^*$$

will give the required result. But, remembering that we supply inputs from A_∇, we have

$$g^* \circ \widehat{\bigvee} = G^* \circ f^\infty \circ \bigvee = F^* \circ \circledcirc^* \circ f^\infty \circ \bigvee \leq F^* \circ F^\infty \circ \circledcirc^* \circ \bigvee$$

where the first two steps unravel the definition of $\widehat{\bigvee}$ and G^*, and the third step uses Corollary 8.10. Since

$$F^* \circ F^\infty = F^* \circ F_* \circ F^* = F^*$$

we have

$$g^* \circ \widehat{\bigvee} \leq F^* \circ \circledcirc^* \circ \bigvee = G^* \circ \bigvee = \widetilde{\bigvee} \circ G^* = \widetilde{\bigvee} \circ g^*$$

using the given property of G^*, as required. ■

This shows that there is a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{f^*} & A_\nabla \\ {\scriptstyle ⊚^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{O}S & \xrightarrow[F^*]{} & (\mathcal{O}S)_\nabla \end{array}$$

of surjective frame morphisms associated with each open filter ∇ on a frame A, and so completes the construction of the diagram of Table 1.

As far as I am aware, there is not much known about the relationship between A_∇ and $(\mathcal{O}S)_\nabla$ (not because it is difficult, but because it hasn't been looked at great detail). Let's finish this subsection with another use of Example 5.16.

8.12 EXAMPLE. Let A be a compact frame, thus $\nabla = \{\top\}$ is an open filter, and we know that A need not be spatial. The inflator f associated with ∇ is

$$f(y) = (\top \supset y) = y$$

for each $y \in A$, so that

$$A \xrightarrow{f^*} A_\nabla$$

is an isomorphism.

Similarly, $Q = S - \{\top\}$, and we find that

$$\mathcal{O}S \xrightarrow{F^*} (\mathcal{O}S)_\nabla$$

is an isomorphism. Thus g^* is agrees with $⊚^*$, and this need not be an isomorphism. ■

We know that the pair (∇, Q) watches over a family of V-points of A and of $\mathcal{O}S$. The fact that g^* need not be an isomorphism suggests that these two families need not be the same. The precise relationship between these two spaces ought to be exposed.

I suspect there is much more to be uncovered here.

9 A selection of examples

In this section I describe a few of the more substantial examples. I have not chosen these in any systematic fashion, there are merely the ones I had notes on. Thus there may be some aspects of the general material that are not illustrated here.

9.1 A computational example

The arrows of ***Vrm*** look at bit strange, and it is not clear what will happen when we start to compute with them. This first example uses quite a few of such calculations and, in the end, confirms our suspicion that strange things can happen. The example is a modified version of one used in [3, 4] but I'm not going to tell you which one because that might spoil the surprise.

Let A, B be a pair of frames. The product

$$A \times B$$

in ***Frm*** is produced in the obvious way. It is the set of all pairs

$$(a, b)$$

where $a \in A$ and $b \in B$. The operations are calculated pointwise. Let's have a look at the supremum. Thus suppose Z is a subset of $A \times B$ and consider the two sets $X \subseteq A$ and $Y \subseteq B$ given by

$$a \in X \iff (\exists b)[(a, b) \in Z] \qquad b \in Y \iff (\exists a)[(a, b) \in Z]$$

(for $a \in A$ and $b \in B$). We can use his to form a superset

$$X \times Y = \{(a, b) \mid a \in X, b \in Y\}$$

of Z. However, for each $(a, b) \in X \times Y$ there are $x \in X, y \in Y$ with

$$(a, b) \leq (a \vee x, b \vee y)$$

and hence

$$\bigvee Z = \bigvee X \times \bigvee Y$$

to show that even suprema in $A \times B$ are easy to handle.

There is an obvious pair of projections

$$A \times B \longrightarrow$$

$$A \longleftarrow \qquad B$$

which are frame morphisms, and it is easy to check that this gives a product wedge in ***Frm***. We won't name these projections for we are going to concentrate on ***Vrm***-arrows in the opposite direction (and we need the letters for these).

9.1 LEMMA. *For each pair A, B of frames the assignments*

$$\begin{array}{ccc} a \longmapsto (a, \bot) & \qquad & (\bot, b) \longleftarrow\!\dashv b \\ A \overset{p^\Diamond}{\underset{p_\Box}{\rightrightarrows}} A \times B & & A \times B \overset{q^\Diamond}{\underset{q_\Box}{\leftleftarrows}} B \\ a \longmapsto (a, \top) & & (\top, b) \longleftarrow\!\dashv b \end{array}$$

*form a pair of **Vrm**-arrows*

$$A =\!(p^\Diamond, p_\Box)\!\Longrightarrow A \times B \qquad A \times B \Longleftarrow\!(q^\Diamond, q_\Box)\!= B$$

both to the product.

Proof. We ought to check that the various conditions of Definition 4.1 hold. These are straight forward, but it is worth looking at the mixed conditions. We do the p-case. Thus for $a, x \in A$ we have

$$\begin{aligned} p_{\Box}(a) \wedge p^{\Diamond}(x) &= (a, \top) \wedge (x, \bot) = (a \wedge x, \bot) = p^{\Diamond}(a \wedge x) \\ p_{\Box}(a) \vee p^{\Diamond}(x) &= (a, \top) \vee (x, \bot) = (a \vee x, \top) = p_{\Box}(a \vee x) \end{aligned}$$

so that here we have equalities, not just comparisons. ■

Each of these ***Vrm***-arrows can be composed with the canonical ***Vrm*** arrow

$$A \times B \longrightarrow V(A \times B)$$

and then the associated universal solution property gives the following.

9.2 LEMMA. *For each pair A, B of frames there are **Frm**-arrows*

$$V(A) \xrightarrow{\;p\;} V(A \times B) \xleftarrow{\;q\;} V(B)$$

*such that the **Vrm**-diagram*

$$\begin{array}{ccc} A & \Longrightarrow & V(A) \\ \Big\| & & \\ (p^{\Diamond}, p_{\Box}) & & \Big\downarrow p \\ \Downarrow & & \\ A \times B & \Longrightarrow & V(A \times B) \\ \Uparrow & & \\ (q^{\Diamond}, q_{\Box}) & & \Big\uparrow q \\ \Big\| & & \\ B & \Longrightarrow & V(B) \end{array}$$

commutes.

Of course, we don't have a full description of p and q, but what we can say is that

$$p(\Diamond(a)) = \Diamond(a, \bot) \quad p(\Box(a)) = \Box(a, \bot) \qquad p(\Diamond(b)) = \Diamond(\bot, b) \quad q(\Box(b)) = \Box(\bot, b)$$

for each $a \in A$ and $b \in B$.

Notice that the arrows p, q seem to go in the wrong direction. Because of the product involved we might expect arrows in the opposite direction. Perhaps you can guess what is going to happen.

In the next three results we look at a wedge

$$\begin{array}{ccc} V(A) & \xrightarrow[\;f\;]{} & \\ (\triangleright) & & C \\ V(B) & \xrightarrow{\;g\;} & \end{array}$$

of ***Frm***-arrows where A, B are the frames under consideration and C is an arbitrary frame.

9.3 LEMMA. *For each pair A, B of frames and wedge $(\triangleright)$ in* ***Frm***, *the pair of assignments*

$$A \times B \underset{h_\Box}{\overset{h^\Diamond}{\rightrightarrows}} C$$

given by

$$h^\Diamond(a,b) = f(\Diamond(a)) \vee g(\Diamond(b)) \qquad h_\Box(a,b) = f(\Box(a)) \wedge g(\Box(b))$$

(for $a \in A$ and $b \in B$) form a ***Vrm****-arrow.*

Proof. We must check the conditions of Definition 4.1. Of these only the mixed conditions cause any trouble, so let's sort those out.

(q) For each $a, x \in A$ and $b, y \in B$ we have

$$\begin{aligned}
h_\Box(a,b) \wedge h^\Diamond(x,y) &= f(\Box(a)) \wedge g(\Box(b)) \wedge \big(f(\Diamond(x)) \vee g(\Diamond(y))\big) \\
&= \begin{cases} f(\Box(a)) \wedge g(\Box(b)) \wedge f(\Diamond(x)) \\ \qquad\qquad \vee \\ f(\Box(a)) \wedge g(\Box(b)) \wedge g(\Diamond(y)) \end{cases} \\
&= \begin{cases} f(\Box(a) \wedge \Diamond(x)) \wedge g(\Box(b)) \\ \qquad\qquad \vee \\ g(\Box(b) \wedge \Diamond(y)) \wedge f(\Box(a)) \end{cases} \\
&\le \begin{cases} f(\Diamond(a \wedge x)) \wedge g(\Box(b)) \\ \qquad\qquad \vee \\ g(\Diamond(b \wedge y)) \wedge f(\Box(a)) \end{cases} \\
&\le \begin{cases} f(\Diamond(a \wedge x)) \\ \qquad \vee \\ g(\Diamond(b \wedge y)) \end{cases} \\
&= h^\Diamond((a \wedge x, b \wedge y)) \qquad\qquad = h^\Diamond((a,b) \wedge (x,y))
\end{aligned}$$

as required. The fourth step (the first comparison) uses the condition (q) required by The Problem 4.2.

(s) For each $a, x \in A$ and $b, y \in B$ we have

$$\begin{aligned}
h_\Box(a,b) \vee h^\Diamond(x,y) &= \big(f(\Box(a)) \wedge g(\Box(b))\big) \vee f(\Diamond(x)) \vee g(\Diamond(y)) \\
&= \begin{cases} f(\Box(a)) \vee f(\Diamond(x)) \vee g(\Diamond(y)) \\ \qquad\qquad \wedge \\ g(\Box(b)) \vee f(\Diamond(x)) \vee g(\Diamond(y)) \end{cases} \\
&= \begin{cases} f(\Box(a) \vee \Diamond(x)) \vee g(\Diamond(y)) \\ \qquad\qquad \wedge \\ g(\Box(b) \vee \Diamond(y)) \vee f(\Box(a)) \end{cases} \\
&\ge \begin{cases} f(\Box(a \vee x)) \vee g(\Diamond(y)) \\ \qquad\qquad \wedge \\ g(\Box(b \vee y)) \vee f(\Box(a)) \end{cases} \\
&\ge \begin{cases} f(\Box(a \vee x)) \\ \qquad \wedge \\ g(\Box(b \vee y)) \end{cases} \\
&= h_\Box((a \wedge x, b \wedge y)) \qquad\qquad = h_\Box((a,b) \wedge (x,y))
\end{aligned}$$

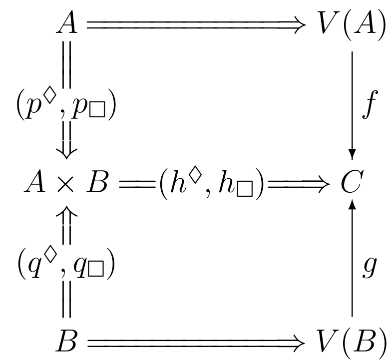
as required. The fourth step (the first comparison) uses condition (s) required by The Problem 4.2. ■

When stated and proved like this Lemma 9.3 looks a bit of a trick. However, it can be set up in a more categorical fashion. You should compare this result with Lemma 9.2.

9.4 LEMMA. *For each pair A, B of frames and wedge ($\triangleright$) in* ***Frm****, there is a unique* ***Vrm****-arrow*

$$A \times B = (h^{\Diamond}, h_{\Box}) \Longrightarrow C$$

such that the ***Vrm****-diagram*



commutes. Furthermore, the arrow $(h^{\Diamond}, h_{\Box})$ is that given by Lemma 9.3.

Proof. Suppose first that we have such an arrow $(h^{\Diamond}, h_{\Box})$. Then

$$\begin{array}{ll} h^{\Diamond} \circ p^{\Diamond} = f \circ \Diamond & h_{\Box} \circ p_{\Box} = f \circ \Box \\ h^{\Diamond} \circ q^{\Diamond} = g \circ \Diamond & h_{\Box} \circ q_{\Box} = g \circ \Box \end{array}$$

and hence

$$\begin{array}{ll} h^{\Diamond}(a, \bot) = (h^{\Diamond} \circ p^{\Diamond})(a) = f(\Diamond(a)) & h_{\Box}(a, \top) = (h_{\Box} \circ p_{\Box})(a) = f(\Box(a)) \\ h^{\Diamond}(\bot, b) = (h^{\Diamond} \circ q^{\Diamond})(b) = f(\Diamond(b)) & h_{\Box}(\top, b) = (h_{\Box} \circ q_{\Box})(b) = f(\Box(b)) \end{array}$$

for each $a \in A, b \in B$. But now

$$\begin{aligned} h^{\Diamond}(a, b) &= h^{\Diamond}((a, \bot) \vee (\bot, b)) = h^{\Diamond}(a, \bot) \vee h^{\Diamond}(\bot, b) = f(\Diamond(a)) \vee g(\Diamond(b)) \\ h_{\Box}(a, b) &= h_{\Box}((a, \top) \wedge (\top, b)) = h_{\Box}(a, \bot) \wedge h_{\Box}(\bot, b) = f(\Box(a)) \wedge g(\Box(b)) \end{aligned}$$

to show that $h^{\Diamond}, h_{\Box}$ must be the pair given by Lemma 9.3.

Conversely, we must show that this pair makes the various square commute. If we remember that

$$\Diamond(\bot) = \bot \qquad \Box(\top) = \top$$

the the required calculations are straight forward. ■

Have you spotted what is going to happen yet? Somehow the V-modification gets confused between products and coproducts.

9.5 THEOREM. *For each pair A, B of frames the induced morphisms*

$$V(A) \xrightarrow{p} V(A \times B) \xleftarrow{q} V(B)$$

(of Lemma 9.2) is a coproduct wedge in $\boldsymbol{Frm}$

Proof. Consider any wedge ($\triangleright$) in $\boldsymbol{Frm}$. We must obtain a mediating arrow

$$\begin{array}{ccc} V(A) & \longrightarrow & \\ p\big\downarrow & f & \\ V(A\times B) & \!\!-h\!\!\rightarrow & C \\ q\big\uparrow & g & \\ V(B) & \longrightarrow & \end{array}$$

and show that there is only one such.

By Lemma 9.3 the pair (f, g) of morphisms produce a $\boldsymbol{Vrm}$-arrow

$$A \times B = (h^{\Diamond}, h_{\Box}) \Longrightarrow C$$

which, by the universal property, factorizes as

$$A \times B \Longrightarrow V(A \times B) \xrightarrow{h} C$$

for some frame morphism h. We show this is the required morphism.

Consider the following $\boldsymbol{Vrm}$-diagram.

$$\begin{array}{ccccc} A & \Longrightarrow & V(A) & \longrightarrow & \\ \| & & & & \\ (p^{\Diamond}, p_{\Box}) & & p\big\downarrow & f & \\ \Downarrow & & & & \\ A \times B & \Longrightarrow & V(A\times B) & \!\!-h\!\!\rightarrow & C \\ \Uparrow & & & & \\ (q^{\Diamond}, q_{\Box}) & & q\big\uparrow & g & \\ \| & & & & \\ B & \Longrightarrow & V(B) & \longrightarrow & \end{array}$$

By Lemma 9.2 the two left hand square commute. By Lemma 9.4 the two outer cells commute (the ones from A to C and B to C). The two $\boldsymbol{Vrm}$-arrows

$$A \Longrightarrow V(A) \qquad\qquad B \Longrightarrow V(B)$$

index the generators of $V(A)$ and $V(B)$, respectively. Thus we find that the two right hand triangles commute, as required.

This shows that we have found one mediating arrow, and a similar argument shows that it is the only possible mediating arrow. ■

In [3, 4] this construction is used to show that for a sober space S the V-modification $V\mathcal{O}S$ need not be spatial. We don't go into the details of that here.

9.2 A linear example

Consider the real interval $A = [0, 1]$ as a linearly ordered frame (with top 1 and bottom 0). Much of what we do with this will generalize in some form or other to an arbitrarily linear frame, although there are a few places where we have to be careful (since there may be gaps in the linear order).

Because A is linear the implication is easy to handle. A simple calculation gives

$$(y \supset z) = \begin{cases} 1 \text{ if } y \leq z \\ z \text{ if } z < y \end{cases}$$

for each $y, z \in A$. Using this we find that

$$u_m(x) = \begin{cases} x \text{ if } a \leq x \\ a \text{ if } x < m \end{cases} \qquad v_m(x) = \begin{cases} 1 \text{ if } m \leq x \\ x \text{ if } x < m \end{cases} \qquad w_m(x) = \begin{cases} 1 \text{ if } m < x \\ m \text{ if } x \leq m \end{cases}$$

$$\nabla(u_m) = \begin{cases} A \text{ if } m = 1 \\ \{1\} \text{ if } a \neq 1 \end{cases} \qquad \nabla(v_m) = [m, 1] \qquad \nabla(w_m) = \begin{cases} A \text{ if } m = 1 \\ (m, 1] \text{ if } a \neq 1 \end{cases}$$

for each $m, x \in A$. In particular, for $m \neq 1$ the nuclei u_m all have the same admitting filter, namely $\{1\}$.

To help later we need a large stock of nuclei. Consider any interval $I = [b, a]$ of A and, to avoid some silliness, suppose $a \neq 1$. Let

$$k_I = v_b \wedge u_a$$

to obtain a nucleus with

$$k_I(x) = \begin{cases} x \text{ if } x \notin I \\ a \text{ if } x \in I \end{cases}$$

for each $x \in A$. In particular $\nabla(k_I) = \{1\}$, and we have a large number of nuclei with $\{1\}$ as admitting filter. In fact, the corresponding block is even larger.

Let $\mathcal{I}$ be any set of pairwise disjoint intervals (none of which contain 1), and for each $x \in A$ set

$$k_{\mathcal{I}}(x) = \bigvee \{k_I(x) \mid I \in \mathcal{I}\} = \begin{cases} x \text{ if } x \notin \bigcup \mathcal{I} \\ a_I \text{ if } x \in I \in \mathcal{I} \end{cases}$$

where in the lower clause a_I is the top end of the interval I. It is easy to check that $k_{\mathcal{I}}$ is a nucleus, and is the supremum

$$\bigvee \{k_I \mid I \in \mathcal{I}\}$$

of the component nuclei. (In particular, in this case the supremum of the nuclei is just the pointwise supremum, which is not usual.)

For each such family $\mathcal{I}$ we have $\nabla(k_{\mathcal{I}}) = \{1\}$, which illustrates just how complicated a block can be. Shortly we will use this complexity in a slightly different way.

What about the V-properties? Consider any filter ∇ on A. This has one of the forms

$$\text{(Open)} \quad \nabla = (m, 1] \qquad \text{(Closed)} \quad \nabla = [m, 1]$$

for some $m \in A$, namely $m = \bigwedge \nabla$. But now one of

$$\nabla(w_m) = \nabla \qquad \nabla(v_m) = \nabla$$

holds, to show that every filter is admissible.

Observe that the filter (Open) is (Scott) open, but the filter (Closed) is not. (For an arbitrary linearly ordered frame things are not so straight forward. There may be gaps in the order, and then the difference between open and not open filters is more delicate. The Hausdorff analysis of the order becomes involved, but that need to worry us here.)

The frame is spatial. The set of points is $S = \mathsf{pt}(A) = [0,1)$, the set of non-top elements. The spatial reflection morphism

$$\begin{array}{ccc} A & \xrightarrow{\circledcirc} & \mathcal{O}S \\ a & \longmapsto & [0,a) \end{array}$$

sends each element to the set of points strictly below it. In particular, the specialization order on S is the reverse on the given comparison on A. This is a bit of a nuisance, but in the end there is no way round it. We could work entirely in terms of the space S, or entirely in terms of A, but is is more convenient to have the freedom to flit between the two, and this does illustrate a few things.

Consider an arbitrary open filter ∇ on A. Thus

$$\nabla = (m, 1]$$

for some $0 \le m < 1$. This corresponds to some $Q \in \mathcal{Q}S$, and it isn't hard to check that $Q = [0, m]$. In particular $M = \{m\}$ is the minimal generating set of Q.

The set $\mathcal{Q}S$ is in bijective correspondence with S itself, and it doesn't take too long to check that, as spaces, these are homeomorphic. In other words the standard V-modification of S is itself.

What about $\mathcal{L}S$? A compact lens has the form

$$L = Q \cap X = [0, m] \cap [a, 1] = [a, m]$$

where $0 \le a \le m < 1$. (Actually, the empty set is also a compact lens.) The open sets $\mathcal{O}S$ are indexed by the $x \in A$, and it is easy to check that

$$L \in \Diamond(x) \Longleftrightarrow a < x \qquad L \in \Box(x) \Longleftrightarrow m < x$$

give the subbasic opens of $\mathcal{L}S$. Thus here $\mathcal{L}S$ is much richer that $\mathcal{Q}S$.

What about the point-free modification of A? Consider a connected connected pair (∇, Q). As above we assume that $\nabla = (m, 1]$ and $Q = [0, m]$ for some $0 \le m < 1$. The block has a least and greatest associated nucleus. Almost trivially w_m is the greatest nucleus, but the least one is a bit more interesting. Consider the inflator $f = f_\nabla$ of A given by

$$f(x) = \bigvee\{(y \supset x) \mid y \in \nabla\}$$

for $x \in A$. Using the description of $\cdot \supset \cdot$ we find that

$$f(x) = \begin{cases} 1 \text{ if } m < x \\ x \text{ if } x \le m \end{cases}$$

for each $x \in A$, and then

$$l_m = f^\infty$$

is the least nucleus v_∇. However, in this case we find that $f^2 = f$ so this is already a nucleus, and $l_m = f$. In fact, we have

$$l_m = v_m \wedge w_m$$

(by an easy calculation).

Since the frame A is spatial we could do all this in terms of the topology $\mathcal{O}S$. We then have

$$v_\nabla \leq [Q'] \leq w_\nabla = [M']$$

where each of these is a nucleus on $\mathcal{O}S$. These correspond to nuclei on A which we can locate by passing to and fro across the spatial reflection isomorphism. Of course

$$v_\nabla \circ \circledcirc = \circledcirc \circ l_m \qquad w_\nabla \circ \circledcirc = \circledcirc \circ w_m$$

for these are selecting the least and greatest members of the corresponding block. The interesting question is where $[Q']$ fits into the algebraic side.

9.6 LEMMA. *We have $[Q'] \circ \circledcirc = \circledcirc \circ l_m$, that is $v_\nabla = [Q']$.*

Proof. For each $x \in A$ we have

$$[Q'](\circledcirc(x)) = (Q' \cup [0,x])^\circ = ([0,x] \cup (m,1])^\circ = \left\{ \begin{array}{ll} S & \text{if } m < x \\ {[0,x]} & \text{if } x \leq m \end{array} \right\} = \circledcirc(l_m(x))$$

to give the required result. ■

In the terminology of Definition 7.7 this shows that the space S is strongly stacked, and hence by Theorem 7.10 each V-point is a focal point. In other words $\mathsf{pt}(VA)$ is essentially the space $\mathcal{L}S$ of compact lenses of S which as we have seen, is much bigger then the space $\mathcal{Q}S$. We will verify this directly.

What are the V-points of A? We may represent each open filter ∇ by its 'open' generator, so we are looking for pairs (m, a) of elements where

$$l_m(a) = a \leq w_m(0) = m$$

holds. Remembering the description of l_m gives the following.

9.7 LEMMA. *The V-points of the frame A are the pairs (m, a) where $0 \leq a \leq m < 1$*

A description of the topology is just as easy. From Lemma 5.8 we have the following.

9.8 LEMMA. *For the frame A the subbasic open sets of $\mathsf{pt}(VA)$ are given by*

$$(m,a) \in \langle\circ\rangle(x) \Longleftrightarrow a < x \qquad (m,a) \in [\circ](x) \Longleftrightarrow m < x$$

for $x \in A$ and V-points (m, a).

Finally we can demonstrate just how complicated a block can be. As before consider a connected pair (∇, Q) given by some $0 \leq m < 1$. Let $\mathcal{I}$ be a family of pairwise disjoint intervals I where

$$I = [b, a] \subseteq [0, m]$$

for some $0 \leq b \leq a \leq m < 1$. Using the nucleus $k_{\mathcal{I}}$, constructed above, let

$$j_{\mathcal{I}}(x) = l_m(x) \vee k_{\mathcal{I}}(x) = \bigvee\{k_I(x) \mid I \in \mathcal{I}\} = \begin{cases} 1 \text{ if } m < x \\ x \text{ if } x \leq m, x \notin \bigcup \mathcal{I} \\ a_I \text{ if } x \leq m, x \in I \in \mathcal{I} \end{cases}$$

for each $x \in A$. As above, in the lower clause a_I is the top end of the interval I. We may check that $j_{\mathcal{I}}$ is a nucleus and $\nabla(j_{\mathcal{I}}) = \nabla$, so we have many nuclei in this block.

By choosing $\mathcal{I}$ in a suitable fashion it is easy to produce a continuum running through the block, and a countably infinite family of pairwise disjoint members of the block. However, I do not know if there is a continuum family of pairwise disjoint members of the block.

9.3 A tree example

In this subsection we produce a space S which is sober and T_1 and in which each compact set if finite. This space has a special point $\star$ and the clan associated with $Q = \{\star\}$ has a multitude of members, including the whole space S. Since the space is T_1 the singleton $Q = \{\star\}$ is closed and is the only member of $\ddagger\star$. Thus the example shows there are many non-focal points associated with Q. Notice also that since $S \in \dagger Q$ we have $Q(\infty) = S$, so the space is far from being stacked.

The example is taken from [7] (in Chapter 11) where it is used for a slightly different purpose. We won't describe all the details of its properties here, but we will certainly give the immediately relevant ones.

The space has the form

$$S = \{\star\} \cup \mathbb{S}$$

where $\mathbb{S}$ is a certain tree (of which $\star$ is not a member). In fact many different trees can be used, but here we will concentrate on what seems to be the canonical example.

9.9 DEFINITION. Let I be an uncountable alphabet, and let $\mathbb{S}$ be the set of all words on I, including the empty word $\perp$. Thus each $x \in \mathbb{S}$ is a list (finite sequence)

$$x = \perp i_1 \ldots i_l$$

of letters $i_1, \ldots, i_l \in I$. Here l is the length of x and $l = 0$ is allowed. These words are partially ordered by extension, thus $x \leq y$ (for words x, y) precisely when

$$y = xi_1 \ldots i_l$$

for some sequence $i_1, \ldots, i_l$ of letters (and again $l = 0$ is allowed). This makes $\mathbb{S}$ a well founded tree.

For each word x

$$I(x) = \{xi \mid i \in I\}$$

is the set of immediate successors of x. Each set $I(x)$ has the same cardinality as I. ■

You may be wondering why we need I to be uncountable. That will be made clear at the appropriate point.

9.10 DEFINITION. For $x \in \mathbb{S}$ we say a subset of $I(x)$ is small if it is countable, otherwise it is large. ■

Notice that the whole set $I(x)$ is large. In fact, $I(x)$ can not be covered by countably many small sets.

It is possible to modify the tree $\mathbb{S}$ and the notion of smallness, and still get the same kind of example, but we do not need those refinements here. However, it is worth emphasizing that taking I to be countable and small to mean finite will not work.

The tree $\mathbb{S}$ carries its partial ordering $\leq$, and so we have the usual notion of a lower section of $\mathbb{S}$. Such a section is itself a tree, but may not have very interesting splitting properties.

9.11 DEFINITION. A subtree of $\mathbb{S}$ is a lower section $\mathbb{T}$. Such a subtree is rampant if for each $x \in \mathbb{T}$ the set $\mathbb{T} \cap I(x)$ is large. ■

We can generate many different rampant subtrees. Start from the root $\perp$, and select a large subset of $I(\perp)$. For each x in this large selected set select a large subset of $I(x)$. Then for each y in each of these large sets select a large subset of $I(y)$. Repeat this process through all levels of $\mathbb{S}$. One simple way of producing a rampant tree is to nominate a large subset $J \subseteq I$, and then consider those members of $\mathbb{S}$ built up using only the letters on J. This gives an indication of just how big the collection of rampant trees is.

Eventually we will see that for each rampant tree $\mathbb{T}$ the set $\{\star\} \cup \mathbb{T}$ is a member of $\dagger\star$.

We furnish the tagged set

$$S = \{\star\} \cup \mathbb{S}$$

with a topology.

9.12 DEFINITION. Let $\mathcal{O}S$ be the family of subsets $U \subseteq S$ such that both

$$\star \in U \Longrightarrow (\forall x \in \mathbb{S})[I(x) - U \text{ is small}] \qquad (\forall x \in \mathbb{S})[x \in U \Longrightarrow I(x) - U \text{ is small}]$$

hold. ■

It doesn't take too long to show that this is a topology on S. Note also that a subset $X \subseteq S$ is closed precisely when both

$$\star \notin X \Longrightarrow (\forall x \in \mathbb{S})[I(x) \cap X \text{ is small}] \qquad (\forall x \in \mathbb{S})[x \notin X \Longrightarrow I(x) \cap X \text{ is small}]$$

hold, or equivalently when both

$$(\exists x \in \mathbb{S})[I(x) \cap X \text{ is large}] \Longrightarrow \star \in X \qquad (\forall x \in \mathbb{S})[I(x) \cap X \text{ is large} \Longrightarrow x \in X]$$

hold.

We need a couple of examples of closed sets.

9.13 EXAMPLES. (a) For each $x \in \mathbb{S}$ the upper section

$$U = \uparrow x = \{y \in \mathbb{S} \mid x \leq y\}$$

of $\mathbb{S}$ is open. By construction $\star \notin U$, so it suffices to consider those $y \in U$. For such a y we have $I(y) \subseteq U$ so that $I(y) - U = \emptyset$ which is certainly small.

(b) Each countable subset $X \subseteq \mathbb{S}$ is closed in S, since there is no $x \in X$ for which $I(x) \cap X$ is large. This is a crucial use of the uncountability of I.

(c) For each subtree $\mathbb{T}$ of $\mathbb{S}$, the set $X = \{\star\} \cup \mathbb{T}$ is closed. For each $x \in \mathbb{S}$, if $I(x) \cap X$ is large, then it is non-empty, so that $xi \in \mathbb{T}$ for some letter i, and hence $x \in \mathbb{T}$ (since $\mathbb{T}$ is a lower section). ■

By fiddling about with this notion you will begin to see that there are some quite intricate closed sets.

9.14 LEMMA. *The space S is T_1 and sober.*

Proof. By Examples 9.13(b) each countable subset is closed. In particular, each singleton is closed, and hence S is T_1.

The proof of sobriety is a bit more delicate. A full proof can be found in [7]. We don't need the details here. ■

Since the space is T_1, each subset is saturated and $\mathcal{Q}S$ is just the family of compact subsets. We will still make use of the comparison $\leq$ on $\mathbb{S}$, but this should not be confused with the specialization order of S (which is just equality).

In any space each finite subset is compact. In this space the converse holds. This is the crucial place where the uncountability of I is used.

9.15 LEMMA. *Each compact subset Q of S is finite.*

Proof. Let Q be a fixed compact subset. To show that Q is finite we first show something weaker.

Let L be any antichain of $\mathbb{S}$. We show that $Q \cap L$ is finite.

By way of contradiction suppose $Q \cap L$ is infinite. Thus there is a countably infinite subset $X \subseteq Q \cap L$. This set X is closed (by Examples 9.13(b)). Thus, using Examples 9.13(a) we see that

$$X' \cup \{\uparrow x \mid x \in X\}$$

is an open cover of S (since $\star \in X'$). The compactness of Q gives

$$X \subseteq Q \subseteq X' \cup \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

for some $x_1, \ldots, x_m \in X$, and hence

$$X \subseteq L \cap (\uparrow x_1 \cup \cdots \cup \uparrow x_m)$$

holds. Consider any $y \in X$. Then $y \in L$ and $x_i \leq y$ for some $1 \leq i \leq m$. But $x_i \in X \subseteq L$ and L is an antichain, so that $y = x_i$. This gives

$$X \subseteq \{x_1, \ldots, x_m\}$$

which is the contradiction (since X was suppose to be infinite).

We use this observation for sets $L \subseteq I(x)$ for $x \in \mathbb{S}$.

Consider any subset $H \subseteq S$ and any $x \in \mathbb{S}$ and let $L = H \cap I(x)$. By the observation

$$Q \cap L = Q \cap H \cap I(x)$$

is finite and hence small. This shows that $Q \cap H$ is closed. In particular, for each $q \in Q$ the set

$$X_q = Q \cap \{q\}'$$

is closed, and hence

$$U_q = X_q' = Q' \cup \{q\}$$

is open. But

$$\{U_q \mid q \in Q\}$$

is an open cover of S, so that the compactness of Q gives

$$Q \subseteq Q' \cup \{q_1, \dots, q_m\}$$

for some $q_1, \dots, q_m \in Q$. Thus

$$Q \subseteq \{q_1, \dots, q_m\}$$

as required. ■

These results show that, in one sense, the space S is quite nice. It seems to be approaching a T_2-niceness. We will destroy that illusion.

The set $Q = \{\star\}$ is compact (saturated) an so produces a deflator

$$\partial_\star = \partial_Q$$

on $\mathcal{C}S$. The clan $\dagger\star$ consist of those closed set X with $\star \in X = \partial_\star(X)$. We will show that there are many of these.

By definition if $X \in \mathcal{C}S$ then

$$I(x) \cap X \text{ large } \Longrightarrow x \in X$$

holds for all $x \in \mathbb{S}$. In fact, we can improve this.

9.16 LEMMA. *For each $X \in \mathcal{C}S$ we have*

$$I(x) \cap X \textit{ large } \Longrightarrow x \in \partial_\star(X)$$

for all $x \in \mathbb{S}$.

Proof. Suppose $I(x) \cap X$ is large and consider any $\star \in U \in \mathcal{O}S$. Let

$$Y = (X \cap U)^-$$

so that we require $x \in Y$.

We have

$$X \cap U \cap Y' \subseteq Y' \cap Y = \emptyset$$

so that

$$X \subseteq U' \cup Y$$

and hence

$$I(x) \cap X \subseteq (I(x) \cap U') \cup (I(x) \cap Y)$$

holds. Since $\star \in U \in \mathcal{O}S$ we see that $I(x) \cap U'$ is small. By hypothesis $I(x) \cap X$ is large (not small), so that $I(x) \cap Y$ is large, and hence $x \in Y$ since Y is closed. ■

With this we have the result we want.

9.17 THEOREM. *For each rampant tree $\mathbb{T}$ the closed set $X = \{\star\} \cup \mathbb{T}$ is in the clan $\dagger\star$.*

Proof. Since $\star \in X$ we have $\star \in \partial_\star(X)$. Consider any other $x \in X$. Then $x \in \mathbb{T}$ and

$$I(x) \cap X = I(x) \cap \mathbb{T}$$

and this is large by the choice of $\mathbb{T}$. Thus $x \in \partial_\star(X)$ by Lemma 9.16. This shows that $X = \partial_\star(X)$ which is enough to complete the proof. ■

I am not sure if this example is a mere curiosity or whether there is something more to it. I tend to the latter view. It seems to me that the analysis of the V-points of S is intimately connected with the combinatorial properties of the tree $\mathbb{S}$. Furthermore, the same ideas can be extended to many other trees. However, I won't pursue that idea here.

9.4 A power domain example

As mentioned in the pre-amble when using V-modifications many people mither on about power domains as though this is the only useful aspect of the constructions. It is not, but to make sure those people are not left out, in this subsection we look at such an application. However, even here the whole example could be done without even mentioning power domains.

Let T be any poset. We fix this throughout this subsection, and use it to generate several different spaces.

For the first one we view T itself as a space. Thus let $A = \Upsilon T$ be the Alexandroff topology, the topology of all upper sections of T. As a space T is a bit pathological, but it does have its uses. The given comparison on T is the specialization order of the topology. For each subset $E \subset T$ the (topological) closure is just the downward closure $E^\downarrow$, the set of all $t \in T$ which lies below at least one member of E. As in any space, this is the smallest closed set that includes E. Unusually for a space, in T there is a smallest open set that includes E, namely the upward closure $E^\uparrow$ of E. This, of course, is just the saturation of E in the space. In other words, in this space open sets and saturated sets are the same. Finally, for each $t \in T$ the principal upper section

$$\uparrow t = \{t\}^\uparrow$$

is open, and these sets form a base of the topology.

In general T is not sober, but the spatial reflection

$$A \longrightarrow \mathcal{O}\mathsf{pt}(A)$$

is an isomorphism (since A is spatial), and the canonically induced map

$$T \longrightarrow \mathsf{pt}(A)$$

exhibits T as a subspace of $\mathsf{pt}(A)$. The specialization order allows us to view $\mathsf{pt}(A)$ as a poset, and then we see that it is an algebraic domain (algebraic dcpo). Furthermore, each algebraic domain can be represented in this way.

By a slight modification of Theorem 6.10 there is an insertion

$$\mathcal{L}T \longrightarrow \mathsf{pt}(VA)$$

which exhibits the space of compact lenses of T as a subspace of $\mathsf{pt}(VA)$. Here we will look at a restriction of this to a smaller family of lenses.

Before we describe this family let's have a look at $\mathcal{L}T$ for this space.

In T the closure operation $(\cdot)^-$ is just the downward closure operation $(\cdot)^\downarrow$. Hence a subset $L \subseteq T$ is a lens precisely when

$$L = L^\uparrow \cap L^\downarrow$$

and for each subset $E \subseteq T$

$$L = E^\uparrow \cap E^\downarrow$$

is a lens. On general grounds for such a generated lens we have we have

$$L^\uparrow = E^\uparrow \qquad L^\downarrow = E^\downarrow$$

and there may be many such generating sets E.

9.18 LEMMA. *In the space T a lens L is compact precisely when $L^\uparrow = E^\uparrow$ for some finite $E \subseteq T$.*

Proof. By Lemma 1.6 the lens L is compact precisely when $L^\uparrow$ is a compact set.

Each finite set E is compact, as is its saturation $E^\uparrow$. Thus if $L^\uparrow = E^\uparrow$ for some finite $E \subseteq T$, then $L^\uparrow$ and L are compact.

Conversely suppose L is a compact lens. Since

$$L^\uparrow = \bigcup\{\uparrow l \mid l \in L\}$$

and each $\uparrow l$ is open, we see there is some finite set $E \subseteq L$ with

$$L^\uparrow = \bigcup\{\uparrow e \mid e \in E\}$$

and then $L^\uparrow = E^\uparrow$, as required. ∎

This result gives us a way of generating compact lenses within T. We simply modify the finite subsets.

9.19 DEFINITION. A lens L of T is finitely generated if it has the form

$$L = E^\uparrow \cap E^\downarrow$$

for some finite $E \subset T$.

Let $\mathcal{F}T$ be the set of finitely generated lenses of T. ∎

By Lemma 9.18 we have

$$\mathcal{F}T \subseteq \mathcal{L}T$$

at the set level (and, as will will see, at the topological level). In general these two families can be distinct.

9.20 EXAMPLE. Consider the natural numbers $\mathbb{N}$ as a linearly ordered set. As an Alexandroff space $\mathbb{N}$ is compact (for the bottom 0 must be a member of some component of any open covering on $\mathbb{N}$, and then that component is the whole space). Since $\mathbb{N}^{\uparrow} = \mathbb{N} = \mathbb{N}^{\downarrow}$, the whole space is a compact lens, but it is not finitely generated. ■

For reasons that will be partly explained shortly, we look at the V-modification of T on the set $\mathcal{F}T$. This furnishes $\mathcal{F}T$ as a subspace of $\mathcal{L}T$. Thus for $U \in A = \Upsilon T$ we use

$$L \in \Diamond(U) \Longleftrightarrow L \text{ meets } U \qquad L \in \Box(U) \Longleftrightarrow L \subseteq U$$

(for $L \in \mathcal{F}T$) to extract subsets of $\mathcal{F}T$. These are the subbasic open sets of the topology on $\mathcal{F}T$.

In general a space obtained by V-modification can be rather complicated, but in this case it's a bit pathetic. To see this we first describe the specialization order on $\mathcal{F}T$. We have seen how to do this in Lemma 1.8 for the a more general situation. We use that here and extract more information for this spatial case. We need to set up a bit of notation.

Consider $L \in \mathcal{F}T$. We have

$$L = E^{\uparrow} \cap E^{\downarrow}$$

for some finite set

$$E = \{e_1, \dots, e_m\}$$

of members of T. These give us several open sets

$$E^{\uparrow} \qquad \uparrow e_1 \quad \cdots \quad \uparrow e_m$$

of T which we may use to obtain an open set

$$[\Diamond]E = \Box(E^{\uparrow}) \cap \Diamond(\uparrow e_1) \cap \cdots \cap \Diamond(\uparrow e_m)$$

of $\mathcal{F}T$. We keep this notation in the next result.

9.21 LEMMA. *For each pair K, L of finitely generated lenses the following are equivalent*

(i) $L \sqsubseteq K$

(ii) $K \subseteq L^{\uparrow}$ *and* $L \subseteq K^{\downarrow}$

(iii) $K \in [\Diamond](E)$

where $L = E^{\uparrow} \cap E^{\downarrow}$ *in part (iii).*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows by a slight modification of the proof of Lemma 1.8.

Now we have

$$K \in \Box(E^{\uparrow}) \Longleftrightarrow K \subseteq E^{\uparrow} = L^{\uparrow}$$

and

$$K \in \Diamond(\uparrow e) \iff K \text{ meets } \uparrow e \iff e \in K^{\downarrow}$$

for each $e \in E$. Thus

$$K \in [\diamond](E) \iff K \subseteq L^{\uparrow} \text{ and } E \subseteq K^{\downarrow} \iff K \subseteq L^{\uparrow} \text{ and } L^{\downarrow} = E^{\downarrow} \subseteq K^{\downarrow}$$

which leads to the equivalence $(ii) \Leftrightarrow (iii)$. ■

This gives us the specialization order on $\mathcal{F}T$. Part (iii) of this result enables us to describe the topology.

9.22 COROLLARY. *The topology of $\mathcal{F}T$ is nothing more than the upper section topology of the poset $(\mathcal{F}T, \sqsubseteq)$.*

Proof. We must show that each $\sqsubseteq$-upper section is saturated. For this it suffices to show that each principal upper section is saturated. To this end consider the principal upper section generated by $L \in \mathcal{F}T$. By the Lemma this is exactly the open set $[\diamond](E)$ where E is a generating set of L. ■

We wish to modify Theorem 6.10 to exhibit $\mathcal{F}T$ as a subspace of $\mathsf{pt}(VA)$. To do that we replace A be its isomorphic image $\mathcal{O}\mathsf{pt}(A)$, but to do that we need a description of the point space $\mathsf{pt}(A)$.

As a rule we view the points of a frame as its $\wedge$-irreducible elements, but here it is better to give this a twist.

9.23 DEFINITION. An ideal of the poset T is a non-empty lower section p which is directed, that is for each $x, y \in p$ there is some $z \in p$ with $x, y \leq z$. ■

For each $t \in T$ the principal lower section $\downarrow t$ is an ideal of T. More generally, it doesn't take too long to see that a lower section p of T is an ideal precisely when the complement $T - p$ is a $\wedge$-irreducible element of the frame $A = \Upsilon T$. Thus we may use the set S of all ideals as the carrier of the point space $\mathsf{pt}(A)$.

For each $U \in A = \Upsilon T$ we use

$$p \in \circledcirc(U) \iff p \text{ meets } U$$

(for $p \in S$) to produce a subset $\circledcirc(U)$ of S. It is straight forward to check that this imposes a topology on S such that

$$A \xrightarrow{\ \circledcirc\ } \mathcal{O}S$$

is a frame isomorphism (since A is spatial) and, as the notation suggests, this is a slightly non-standard version of the spatial reflection of A. In particular, the space S is sober. We will find that in this case this description is more convenient that the standard version.

For instance, the specialization order on S is just inclusion, that is

$$p \sqsubseteq q \iff p \subseteq q$$

for ideals p, q. This poset S is sometimes called the ideal completion of the poset T, and the topology on S is the Scott topology.

The correspondence between $A = \Upsilon T$ and $\mathcal{O}S$ can be described in a different way.

9.24 LEMMA. *For $X \in \mathcal{C}S$ let $F \subseteq T$ be given by*

$$t \in F \iff \downharpoonleft t \in X$$

(for $t \in T$). Then F is a lower section of T and $X' = \circledcirc(F')$, that is

$$p \in X \iff p \subseteq F$$

for $p \in S$.

Proof. To show that F is a lower section consider any elements s, t of T with $s \leq t \in F$. Then $\downharpoonleft s \subseteq \downharpoonleft t \in X$, so that $\downharpoonleft s \in X$ (since X is closed), and hence $s \in F$.

To prove the equivalence suppose first that $p \in X$ and consider $t \in p$, so we must show that $t \in F$. But $\downharpoonleft t \subseteq p$, so that $\downharpoonleft t \in X$, and hence $t \in F$, as required.

Conversely, suppose $p \subseteq F$ and, by way of contradiction, suppose $p \notin X$. There is some upper section U of T with $X' = \circledcirc(U)$, and then $p \in \circledcirc(U)$, so that p meets U, at t say. We have

$$t \in p \subseteq F \qquad t \in U$$

and hence

$$\downharpoonleft t \in X \qquad \downharpoonleft t \in \circledcirc(U)$$

(by the definition of F and since $\downharpoonleft t$ meets U at t). Since $X' = \circledcirc(U)$, this is a patent contradiction. ■

We would like a similar characterization of the $Q \in \mathcal{Q}S$. This is not so easy, unless Q has a special form. Let M be the minimal generating set of Q. Thus M is a set of ideals m. We say Q is principally generated if each $m \in M$ is a principal ideal, that is has the form $\downharpoonleft t$ for some $t \in T$. There is no reason why, in general, Q should be principally generated. For instance, take any ideal m an let $Q = \{m\}^{\uparrow}$. Then $M = \{m\}$, and m need not be principal.

9.25 LEMMA. *For $Q \in \mathcal{C}S$ let $G \subseteq T$ be given by*

$$t \in G \iff \downharpoonleft t \in Q$$

(for $t \in T$). Then G is an upper section of T and

$$p \text{ meets } G \Longrightarrow p \in Q$$

for $p \in S$. If Q is principally generated then this implication is an equivalence.

Proof. To show G is an upper section consider any elements $t \leq s$ of T with $t \in G$. Then $\downharpoonleft t \subseteq \downharpoonleft s$ with $\downharpoonleft t \in Q$, so that $\downharpoonleft s \in Q$ (since Q is saturated), and hence $s \in G$.

Suppose the ideal p meets G at t, say. Then $\downharpoonleft t \subseteq p$ with $\downharpoonleft t \in Q$, and hence $p \in Q$ (since Q is saturated).

For the converse we suppose Q is principally generated. For such a Q suppose $p \in Q$ for some ideal p. Since $Q \in \mathcal{Q}S$ we have some minimal generator $m \subseteq p$. This m has the form $\downharpoonleft t$ for some $t \in T$. But now $\downharpoonleft t \in Q$, so that $t \in G$. Also $t \in m \subseteq p$, to show that p meets G at t. ■

We use this idea in reverse to convert a finite subset of T into a member of $\mathcal{Q}S$.

9.26 LEMMA. *For a finite subset E of T let $Q \subseteq S$ be given by*

$$p \in Q \iff p \text{ meets } E$$

(for $p \in S$). Then $Q \in \mathcal{Q}S$, and Q is principally generated.

Proof. Trivially, Q is saturated.

To show that Q is compact consider a covering

$$Q \subseteq \bigcup\{\circledcirc(\uparrow t) \mid t \in L\}$$

where L is a lower section of T. For each $e \in E$ we have $\downarrow e \in Q$, so there is some $t \in L$ with $\downarrow e \in \circledcirc(\uparrow t)$, and hence $t \leq e$. Letting e range through E we obtain a finite subset $F \subseteq L$ with $E \subseteq F^{\uparrow}$. A few simple calculations shows that

$$Q \subseteq \bigcup\{\circledcirc(\uparrow t) \mid t \in F\}$$

to give the required result.

To show that Q is principally generated consider any $m \in M$. We have $m \in Q$, so that m meets E, to give some $e \in E$ with $\downarrow e \subseteq m$. But $\downarrow e \in Q$ (since the ideal meets E at e), so $m = \downarrow e$ by the minimality of m. ■

By putting these results together we are able to exhibit some V-points.

9.27 THEOREM. *For a finite subset E of T let $Q, X \subseteq S$ be given by*

$$p \in Q \iff p \text{ meets } E^{\uparrow} \qquad p \in X \iff p \subseteq E^{\downarrow}$$

(for $p \in S$). Then $Q \in \mathcal{Q}S, X \in \mathcal{C}S$, and the pair (Q, X) is V-point of S.

Proof. For $p \in S$ we have

$$p \text{ meets } E^{\uparrow} \iff p \text{ meets } E$$

so that $Q \in \mathcal{Q}S$ by Lemma 9.26. Similarly

$$p \subseteq E^{\downarrow} \iff p \subseteq E$$

so that $X \in \mathcal{C}S$ by Lemma 9.24.

Let M be the minimal generating set of Q. Thus, as in the proof of Lemma 9.26, we have

$$M = \{\downarrow e_1, \ldots, \downarrow e_l\}$$

for some $e_1, \ldots, e_l \in E$. For each such e we have $\downarrow e \subseteq E^{\downarrow}$ so that $\downarrow e \in X$. Thus $M \subseteq X$.

It remains to show that $Q \ltimes X$. To this end observe that for each $U \in A = \Upsilon T$ we have

$$Q \subseteq \circledcirc(U) \iff (\forall p \in Q)[p \text{ meets } U] \iff (\forall e \in E)[\downarrow e \text{ meets } U] \iff E \subseteq U$$

and hence $E^{\uparrow}$ is the smallest possible U with $Q \subseteq \circledcirc(U)$. In particular, an inclusion

$$X \subseteq (X \cap \circledcirc(E^{\uparrow}))^{-}$$

will give the required result.

Consider any $p \in X$ and any open neighbourhood $\circledcirc(V)$ of p (where $V \in A$). Then p meets V at t, say. Since $t \in p \subseteq E^{\downarrow}$ we have $t \leq e$ for some $e \in E$. Then

$$\downarrow e \in X \cap \circledcirc(E^{\uparrow}) \cap \circledcirc(V)$$

(since $\downarrow e \subseteq E^{\downarrow}$, and $\downarrow e$ meets $E^{\uparrow}$ at e, and $\downarrow e$ meets V at t) which, since the neighbourhood $\circledcirc(V)$ is arbitrary, leads to the required result. ■

There is a little bit more in Theorem 9.27. Consider any $L \in \mathcal{F}T$. We have

$$L = E^{\uparrow} \cap E^{\downarrow} \qquad L^{\uparrow} = E^{\uparrow} \qquad L^{\downarrow} = E\downarrow$$

for some finite $E \subseteq T$. Thus

$$p \in Q \Longleftrightarrow p \text{ meets } L^{\uparrow} \qquad p \in X \Longleftrightarrow p \subseteq L^{\downarrow}$$

(for $p \in S$) produces a V-point (Q, X) of S. We use this assignment in the next result.

9.28 THEOREM. *For each poset T with ideal completion S, the assignment*

$$\begin{array}{ccc} \mathcal{F}T & \xrightarrow{\psi} & \mathsf{pt}(V\mathcal{O}S) \\ L & \longmapsto & (Q, X) \end{array}$$

(as described above) is a continuous injection that exhibits $\mathcal{F}T$ as a subspace of $\mathsf{pt}(V\mathcal{O}S)$.

Proof. Firstly we must show that the assignment is injective. For this we use the principal ideals $\downarrow t$ (for $t \in T$). Suppose the pair (Q, X) arises from $L \in \mathcal{F}T$. Then for each $t \in T$ we have

$$t \in L^{\uparrow} \Longleftrightarrow \downarrow t \text{ meets } L^{\uparrow} \Longleftrightarrow \downarrow t \in Q \qquad t \in L^{\downarrow} \Longleftrightarrow \downarrow t \subseteq L^{\downarrow} \Longleftrightarrow \downarrow t \in X$$

so that $L^{\uparrow}, L^{\downarrow}$, and $L = L^{\uparrow} \cap L^{\downarrow}$ can be retrieved from (Q, X). In other words we have

$$t \in L \Longleftrightarrow \downarrow t \in Q \cap X$$

which is enough to show the required injective property.

Secondly we must show that the inverse image map $\psi^{\leftarrow}$ sets up a bijective correspondence between the topology of $\mathcal{F}T$ and that on $\mathsf{pt}(V\mathcal{O}S)$. In fact, it suffices to show that $\psi^{\leftarrow}$ sets up a bijective correspondence between the canonical subbasic open sets of the topologies. Thus we begin by describing these. Each is doubly indexed by the opens $U \in A = \Upsilon T$.

From earlier in this subsection, just after Example 9.20, for each $U \in A$ we use

$$L \in \Diamond(U) \Longleftrightarrow L \text{ meets } U \qquad L \in \Box(U) \Longleftrightarrow L \subseteq U$$

(for $L \in \mathcal{F}T$) to obtain a pair $\Diamond(U), \Box(U)$ of subbasic open sets of $\mathcal{F}T$.

By Lemma 6.6 we obtain the subbasic opens of $\mathsf{pt}(V\mathcal{O}S)$ by applying $\langle \circ \rangle, [\circ]$ to the opens of $\mathcal{O}S$. These opens have the form $\circledcirc(U)$ for $U \in A$. Thus

$$(Q, X) \in \langle \circ \rangle (\circledcirc(U)) \Longleftrightarrow X \text{ meets } \circledcirc(U) \qquad (Q, X) \in [\circ] (\circledcirc(U)) \Longleftrightarrow Q \subseteq \circledcirc(U)$$

(for $(Q, X) \in \mathsf{pt}(\mathcal{O}S)$) gives a typical subbasic open sets of $\mathsf{pt}(V\mathcal{O}S)$.

We use these to set up the required correspondence. Thus for

$$L \longleftrightarrow (Q, X)$$

and $U \in A$ we have

$$\begin{aligned} L \in \psi^{\leftarrow}(\langle\circ\rangle\,(\circledcirc(U))) &\iff X \in \langle\circ\rangle\,(\circledcirc(U)) \\ &\iff X \text{ meets } \circledcirc(U) \\ &\iff (\exists p \in S)[p \in X \text{ and } p \in \circledcirc(U)] \\ &\iff (\exists p \in S)[p \subseteq L^{\downarrow} \text{ and } p \text{ meets } U] \\ &\iff (\exists t \in T)[t \in L^{\downarrow} \text{ and } t \in U] \\ &\iff L \text{ meets } U \qquad\qquad \iff L \in \Diamond(U) \end{aligned}$$

and

$$\begin{aligned} L \in \psi^{\leftarrow}([\circ]\,(\circledcirc(U))) &\iff Q \in [\circ]\,(\circledcirc(U)) \\ &\iff Q \subseteq \circledcirc(U) \\ &\iff (\forall p \in S)[p \in Q \Longrightarrow p \in \circledcirc(U)] \\ &\iff (\forall p \in S)[p \text{ meets } L^{\uparrow} \Longrightarrow p \text{ meets } U] \\ &\iff (\forall t \in T)[t \in L^{\uparrow} \Longrightarrow t \in U] \\ &\iff L^{\uparrow} \subseteq U \qquad\qquad \iff L \in \Box(U) \end{aligned}$$

to show that

$$\psi^{\leftarrow}(\langle\circ\rangle\,(\circledcirc(U))) = \Diamond(U) \qquad \psi^{\leftarrow}([\circ]\,(\circledcirc(U))) = \Box(U)$$

and so give the required result. In each of the series of equivalences the fifth one, the step between arbitrary $p \in S$ and principal $\downarrow t \in S$, needs a moment's thought. ■

In general the space $\mathcal{F}T$ is not sober. However, its sober reflection must sit inside $\mathsf{pt}(V\mathcal{O}S)$, and in some interesting cases it is exactly $\mathsf{pt}(V\mathcal{O}S)$. This is analysed in [6].

10 A selection of questions

10.1 For section 1

10.1 **Question.** The two standard V-modifications of a sober space are based on the sets $\mathcal{Q}S, \mathcal{L}S$ of compact saturated sets and the compact lenses. Does the set of compact and closed sets form a suitable carrier?

10.2 **Question.** For a sober space S consider the pair of spaces

$$\mathcal{Q}S \hookrightarrow \mathcal{L}S$$

of Lemma 1.7. It seems that neither of these need be sober. Investigate this and, if possible, describe the sober reflections. Remember that

$$\mathcal{L}S \hookrightarrow \mathsf{pt}(V\mathcal{O}S)$$

so that the sober reflection of each lives inside $\mathsf{pt}(V\mathcal{O}S)$.

[*For me: Is there a similarity with the patch situation here?*]

10.3 Question. For each T_0 space S the insertion

$$\begin{array}{rcl} S & \hookrightarrow & \mathcal{Q}S \\ p & \longmapsto & p^{\uparrow} \end{array}$$

need not be continuous. This requires S to carry a larger topology. We add to $\mathcal{O}S$ each set $\downarrow U$ for $U \in \mathcal{O}S$ to produce a base for the larger topololgy. What is this new topology? Is it related to other standrad modifications of S? Perhaps a place to start is the Scott topology on a directed complete poset. ■

10.2 For section 2

10.4 Question. As far as a know there has been no systematic analysis of the possible structure of the blocks of a frame. I believe that such an investigation would lead to a greater understanding of frames and the way they interact with spaces.

10.3 For section 3

10.5 Question. Each block for a frame A corresponds to an admissible filter ∇ on A. Each block has a least member v_{∇}. Some blocj have a largest member, see Lemmas 2.12 and 3.12. What os the structure of the family of all these bounded blocks? If A is fitted, then this family is essentially the assembly of NA. Can something be said in a more general situation?

10.4 For section 4

10.6 Question. *Need at least one*

10.5 For section 5

10.7 Question. Develop the bundle view of $\mathsf{pt}(VA)$ describe at the end of Section 5.

10.6 For section 6

10.8 Question. A more detailed analysis of the insertion

$$\mathsf{pt}(V\mathcal{O}S) \longrightarrow \mathsf{pt}(VA)$$

should be carried out. Perhaps a few 'silly' cases are worth looking at. For instance, suppose A is quite large but S is a 1-point space. What happens?

10.9 Question. It is know, by example, that for a sober space S the V-modification $V\mathcal{O}S$ need not be spatial. However, it seems that a detailed analysis of this aspect has not been carried out. Is there a reasonable characterization of the class of spaces S for which $V\mathcal{O}S$ is spatial?

10.7 For section 7

10.10 Question. At the beginning of Section 7 I assert that for $Q \in \mathcal{Q}S$ the clan $\dagger Q$ is not very interesting as a subspace of $\mathsf{pt}(V\mathcal{O}S)$. Can you prove me wrong?

10.11 Question. Confirm or refute the following. For each sober space S and $Q \in \mathcal{Q}S$, if $X \in \dagger Q$ and $X \subseteq Q^{-}$, then $X \in \ddagger Q$.

10.12 Question. For the two step continuous map

$$\mathcal{L}S \longrightarrow \mathsf{pt}(V\mathcal{O}S) \longrightarrow \mathsf{pt}(VA)$$

which points $(\nabla, a) \in \mathsf{pt}(VA)$ arise from a lens in $\mathcal{L}S$?

10.13 Question. Theorem 7.10 gives a pair of implications. Are either of these reversible?

References

[1] R-E. Hoffmann (editor): *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere Nr. 27, Universität Bremen, 1982. (This was also published as [2].)

[2] R-E. Hoffmann and K. H. Hofmann (editors): *Continuous lattices and their applications*, vol 101 of *Lectures Notes in Pure and Applied Mathematics* (Marcel Dekker, 1985).

[3] P. T. Johnstone: The Vietoris monad on the category of locales, pp 162 – 179 of [1].

[4] P.T. Johnstone: Vietoris locales and localic semilattices, pp 155–180 of [2].

[5] P. T. Johnstone: *Stone spaces* (Cambridge University Press, 1982).

[6] E. Robinson: *Power-domains, modalities, and the vietoris monad*, Cambridge University Computer Laboratory technical report no. 98, 1986.

[7] R.A.Sexton: *A point-free and point-sensitive analysis of the patch assembly*, Ph.D thesis, University of Manchester, 2003.

[8] R.A.Sexton and H. Simmons: An ordinal indexed hierarchy of separation properties, Can be found at *//www.cs.man.ac.uk/∼hsimmons*

[9] H. Simmons: The coverage technique for partially ordered systems, Can be found at *//www.cs.man.ac.uk/∼hsimmons*