

# The topos of actions on a monoid

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A topos is a cartesian close category with certain other high order features. For each category  $\mathcal{C}$  the category of set-valued presheaves on  $\mathcal{C}$  (the yoneda completion of  $\mathcal{C}$ ) is a topos. Many other toposes can be obtained by taking appropriate quotients of such preseheaf categories.

In these notes we look at the particular case where the parent category is a monoid. This enables the presheaf mechanism to be hidden, and the category can be studied using more mainsteam algebraic methods.

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## 1 The category

Let

$$(R \cdot, 1)$$

be a monoid (unital semigroup). That is,  $R$  is a set furnished with an associative binary operation with a neutral element 1. We write

$$sr$$

(rather than  $s \cdot r$ ) for the result of applying the operation to the pair of elements  $s$  and  $r$ . Thus

$$t(sr) = (ts)r \quad s1 = s = 1s$$

(for  $r, s, t \in R$ ) are the axiomatic conditions on the structure. As usual, we refer to the structure by its carrier  $R$ . Notice that a monoid need not be commutative.

1.1 EXAMPLES. (a) Each (multiplicatively written) group is a monoid.

(b) Forgetting the additive structure of a ring produces a monoid.

(c) For any object  $X$  in any category the set of arrows

$$X \longrightarrow X$$

is a monoid. ■

Each monoid is a category with just one object and, as such, has an associated category of presheaves (it yoneda completion). This category can be described in a standard algebraic fashion.

1.2 DEFINITION. Let  $R$  be a monoid. An  $R$ -set (or more correctly a right  $R$ -set) is a set  $A$  furnished with an action

$$\begin{array}{ccc} A, R & \longrightarrow & A \\ a, r & \longmapsto & ar \end{array}$$

where

$$(as)r = a(sr) \quad a1 = a$$

for all  $a \in A$  and  $r, s \in R$ . ■

In other words, for an  $R$ -set  $A$  each  $r \in R$  indexes a 1-placed operation

$$\begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & ar \end{array}$$

on  $A$ . These operations must interact in the indicated fashion. Usually we will omit the brackets and write

$$asr$$

for both possible compounds.

1.3 EXAMPLES. (a) Let  $R$  be a ring. By forgetting the additive structure, each right  $R$ -module gives an  $R$ -set. In particular, for each field  $K$  each vector space over  $K$  gives a  $K$ -set (where the scalars are placed on the right).

(b) *Sort out the endo-monoid example* ■

(c) When  $R$  is the trivial monoid  $R = \{1\}$ , an  $R$ -set is just a set.

For a fixed monoid  $R$  we use the  $R$ -sets as the objects of a category. What are the arrows?

1.4 DEFINITION. Let  $R$  be a monoid, and let  $A, B$  be a pair of  $R$ -sets. An  $R$ -linear map

$$A \xrightarrow{f} B$$

is a function

$$f : A \longrightarrow B$$

such that

$$f(ar) = (fa)r$$

for each  $a \in A$  and  $r \in R$ . ■

Notice that in this property there are two actions involved, that on  $A$  and that on  $B$ .

Trivially, the composite of two  $R$ -linear maps is itself  $R$ -linear. Thus we may use these maps as the arrows of a category.

1.5 DEFINITION. Let  $R$  be a monoid, and let  $A, B$  be a pair of  $R$ -sets. An  $R$ -linear map  $f : A \rightarrow B$  is a function  $f : A \rightarrow B$  such that  $f(ar) = (fa)r$  for each  $a \in A$  and  $r \in R$ . The category  $\mathbf{R}$  has the  $R$ -sets as its objects and the  $R$ -linear maps as its arrows.

This is the category we will study. We will show that it is a topos and analyse some of its properties. Recall that for  $\mathbf{R}$  to be a topos we require the following.

- (0) The category  $\mathbf{R}$  has all finite limits (and finite colimits).
- (1) The category  $\mathbf{R}$  is cartesian closed.
- (2) The category  $\mathbf{R}$  has a subobject classifier.

There are variations on these axioms but all amount to the same thing. We will look at the details for (1) in Section 3, and for (2) in Section 5 (where we will also explain what subobject classification means). In the remainder of this section we look at (0).

We won't look at all finite limits (and colimits) only those that we need. Notice that the empty set  $\emptyset$  is an  $R$ -set, and this is the initial object of  $\mathbf{R}$ . Similarly, the singleton set

$$\mathbf{1} = \{\star\}$$

is an  $R$ -set with action  $\star r = \star$  for each  $r \in R$ . This is the final object of  $\mathbf{R}$ .

What about more interesting limits? In general, for this kind of algebraic category a limit is obtained by taking the set theoretic limit of the carriers and then furnishing this with a suitable action. We will use this trick a couple of times.

**1.6 CONSTRUCTION.** Let  $A, B$  be a pair of  $R$ -sets. Let  $A \times B$  be the set of all pairs  $(a, b)$  for  $a \in A$  and  $b \in B$  (the usual cartesian product). For each such pair  $(a, b)$  set

$$(a, b)r = (ar, br)$$

for each  $r \in R$ . ■

This construction produces an assignment

$$\begin{aligned} A \times B, R &\longrightarrow A \times B \\ (a, b), r &\longmapsto (ar, br) \end{aligned}$$

which, almost trivially, is an action on  $A \times B$ . Thus we have an  $R$ -set, and we find that the obvious projections

$$\begin{array}{ccc} & A \times B & \\ & \swarrow \quad \searrow & \\ A & & B \end{array}$$

are  $R$ -linear. In the routine way we check that this is a product wedge. Thus  $\mathbf{R}$  has all binary products.

Given an  $R$ -set  $A$  a subset  $X \subseteq A$  is a subobject if it is closed under the action of  $A$ , that is if

$$a \in X \implies ar \in X$$

for each  $a \in A$  and  $r \in R$ . In particular, for such a subobject  $X$  the insertion

$$X \hookrightarrow A$$

is  $R$ -linear, and monic in  $\mathbf{R}$ . We will have more to say about subobjects in Section 4.

Given a parallel pair

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

of  $R$ -linear maps, the set

$$E = \{a \in A \mid fa = ga\}$$

is a subobject of  $A$ , and the inclusion  $E \hookrightarrow A$  is the equalizer of  $f$  and  $g$ . We needn't look at the details of this for we are going to do something that is slightly more complicated, the construction of pullbacks.

1.7 CONSTRUCTION. Consider a wedge of  $R$ -linear maps, as to the left,

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ & \uparrow g & \\ & B & \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & C \\ p \uparrow & & \uparrow g \\ P & \xrightarrow{q} & B \end{array}$$

and set

$$P = \{(a, b) \mid fa = gb\}$$

to produce a subset of the product  $A \times B$ . Using the obvious projections  $p, q$  consider the square, as to the right. ■

For each  $(a, b) \in P$  and  $r \in R$  we have

$$f(ar) = (fa)r = (gb)r = g(br)$$

to show that  $P$  is a subobject of  $A \times B$ . Trivially the two projections  $p, q$  are  $R$ -linear, and the square commutes. This doesn't quite prove the following.

1.8 LEMMA. *For each wedge of  $R$ -linear maps, as in Construction 1.7, the square is a pullback.*

**Proof.** By the remarks above it remains to prove the pullback property. Consider a commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ h \uparrow & & \uparrow g \\ D & \xrightarrow{k} & B \end{array}$$

of  $R$ -linear maps. We require a unique mediating arrow

$$D \xrightarrow{m} P$$

such that

$$h = p \circ m \quad k = q \circ m$$

hold. In fact, there is only one choice, so it suffices to show that the assignment

$$\begin{array}{ccc} D & \xrightarrow{m} & P \\ d & \longmapsto & (hd, kd) \end{array}$$

is  $R$ -linear. This is a routine exercise. ■

This sets down all the basic properties of the category  $\mathbf{R}$ . In the remainder of these notes we look at the more sophisticated properties of a topos theoretic nature.

## 2 Elements

Each object of  $\mathbf{R}$  is a set  $A$  with a carried action. As a set we may talk about the set theoretic elements of  $A$ , that is the  $a \in A$ . This is not part of the internal; categorical structure of  $\mathbf{R}$ , but can be seen by us from our external position. What about the little man, Bill, who lives in  $\mathbf{R}$ . What does he think the ‘elements’ are? For any topos (or any cartesian category) there is a notion of a **global element** of an object  $A$ . This is an arrow

$$\mathbf{1} \longrightarrow A$$

from the final object to  $A$ , in other words a member of the arrow set  $[\mathbf{1}, A]$ . How, if at all, are these notions related?

For an  $R$ -set  $A$ , let us say  $a \in A$  is **total** if

$$ar = a$$

for all  $r \in R$ . Let  $\text{Tot}(A)$  be the subset of total elements of  $A$ . Since this notion is not entirely familiar, a few examples won’t go amiss.

**2.1 EXAMPLES.** The monoid  $R$  is itself an  $R$ -set. What are its total elements.

(a) Suppose  $s \in R$  is total. Then  $sr = s$  for each  $r \in R$ , and hence, in particular,  $s^2 = s$ . Thus each total element is idempotent, which can mean that  $\text{Tot}(R)$  is a quite small subset of  $R$ .

(b) Suppose  $R$  is commutative. Then  $R$  has at most one total element. For suppose both  $r, s$  are total elements. Then

$$s = sr = rs = r$$

to justify the assertion.

(c) Suppose  $R$  is a  $\wedge$ -semilattice. Then  $R$  has a total element if and only if  $R$  has a bottom. For suppose  $s \in R$  is total. Then

$$s = s \wedge r \leq r$$

for each  $r \in R$ , so  $s$  must be the bottom.

(d) Suppose  $R$  is a group. Then  $R$  has a total element if and only if  $R$  is the trivial group. For suppose  $s \in R$  is total. Then  $s$  has an inverse  $t$ , so that  $s = st = 1$ , which locate  $s$ . But now, for each  $r \in R$  we have

$$r = 1r = sr = s = 1$$

so that  $R$  is trivial.

(e) Suppose  $R$  is a field and  $A$  is a vector space over  $R$ . Then the only total element of  $A$  is the zero vector. For if  $a \in A$  is total then

$$a = a0 = 0$$

(where  $0$  is the zero vector). ■

These examples shows there can be a dramatic difference between elements and global elements for an  $R$ -set, which adds a bit of spice to the following.

**2.2 THEOREM.** *For each  $R$ -set  $A$  there is a bijective correspondence*

$$\begin{array}{ccc} \alpha & \longmapsto & \alpha(\star) \\ [\mathbf{1}, A] & & \text{Tot}(A) \\ \widehat{a} & \longleftarrow & a \end{array}$$

where  $\widehat{a}(\star) = a$  gives the lower assignment.

**Proof.** For each  $\alpha \in [\mathbf{1}, A]$  and  $r \in R$  we have

$$\alpha(\star)r = \alpha(\star r) = \alpha(\star)$$

to show that  $\alpha(\star)$  is a total element of  $A$ .

Conversely, if  $a \in \text{Tot}(A)$  then for each  $r \in R$  we have

$$\widehat{a}(\star)r = ar = a = \widehat{a}(\star)$$

to show that  $\widehat{a}$  is  $R$ -linear, and hence a member of  $[\mathbf{1}, A]$ .

This shows that the two assignments have the targets claimed. It remains to show they are an inverse pair.

Consider  $\alpha \in [\mathbf{1}, A]$  and let  $a = \alpha(\star)$ . Then

$$\widehat{a}(\star) = a = \alpha(\star)$$

to show that the composite to  $[\mathbf{1}, A]$  is the identity. The other composite is even easier to deal with. ■

This shows that the global elements are the internal way of getting at the external total elements of an  $R$ -set. Is there a similar internal way of getting at all the external elements? There is, and we use the notion of a **separator** (sometimes called a **generator**).

Consider the parent monoid  $R$ . This is an  $R$ -set with the canonical action. In particular, for each  $R$ -set  $A$  there is a set  $[R, A]$  of  $R$ -linear maps

$$R \xrightarrow{\alpha} A$$

that is function  $\alpha : R \longrightarrow A$  such that

$$\alpha(s)r = \alpha(sr)$$

for all  $r, s \in R$ . (As explained in Section 1, normally we would omit the brackets and write  $\alpha sr$  for either compound. Here we keep the brackets to bring out what is going on.)

2.3 THEOREM. For each  $R$ -set  $A$  there is a bijective correspondence

$$\begin{array}{ccc} \alpha \longmapsto & \alpha(\star) & \\ [R, A] & A & \\ \widehat{a} \longleftarrow & a & \end{array}$$

where  $\widehat{a}s = as$  (for  $s \in R$ ) gives the lower assignment.

**Proof.** We make several simple observations.

Trivially, for each  $\alpha \in [R, A]$  we have  $\alpha(1) \in A$ .

Consider any  $a \in A$  with the associated  $\widehat{a} : R \longrightarrow A$ . For each  $r, s \in R$  we have

$$\widehat{a}(s)r = (as)r = a(sr) = \widehat{a}(sr)$$

to show that  $\widehat{a}$  is  $R$ -linear.

This shows that the two assignments have the targets claimed. It remains to show they are an inverse pair.

Consider any  $\alpha \in [R, A]$  and let  $a = \alpha(1)$ . Then, for each  $r \in R$ , we have

$$\alpha(r) = \alpha(1r) = \alpha(1)r = ar = \widehat{a}(r)$$

to show that the composite to  $[R, A]$  is the identity. The other composite is even easier to deal with. ■

We will use this result many times to get at elements of  $R$ -sets and explain why certain categorical constructions do what they do. A simple example of this will illustrate what can be done.

Each arrow

$$A \xrightarrow{f} B$$

of  $\mathbf{R}$  is a function  $f : A \longrightarrow B$ , and hence each injective arrow is monic. In general when arrows are functions the converse need not be true, but it is here.

2.4 THEOREM. An arrow

$$A \xrightarrow{f} B$$

of  $\mathbf{R}$  is monic precisely when the carrying function is injective.

**Proof.** By the remarks above it suffices to show that if the arrow  $f$  is monic then the function  $f$  is injective. To this end consider  $a_1, a_2 \in A$  with  $fa_1 = fa_2$ . We require  $a_1 = a_2$ . Let

$$R \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} A$$

be the arrows which select the two elements, that is

$$\alpha_1 r = a_1 r \quad \alpha_2 r = a_2 r$$

for each  $r \in R$ . But now

$$(f \circ \alpha_1)r = f(a_1r) = (fa_1)r = (fa_2)r = f(a_2r) = (f \circ \alpha_2)r$$

for each  $r \in R$ , to show that the parallel pair

$$R \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} A \xrightarrow{f} B$$

are the same. Since  $f$  is monic this gives  $\alpha_1 = \alpha_2$ , and hence

$$a_1 = \alpha_1(1) = \alpha_2(1) = a_2$$

as required. ■

At this point you might expect a companion result showing that epic arrows are just the surjective ones. This could be done now, but it fits better with the material of Section 4.

### 3 Cartesian closure

As we are about to show, the category  $\mathbf{R}$  is cartesian closed. For each object  $A$  the endo-functor  $- \times A$  has a right adjoint  $A \Rightarrow -$  (which is sometimes written  $(-)^A$ ).

$$- \times A \dashv A \Rightarrow -$$

In more detail, for each pair of objects  $A, B$  there is an object  $(A \Rightarrow B)$ , the internal arrow object, such that for each object  $C$  there is an inverse pair of transpositions

$$\begin{array}{ccc} f \dashv & \xrightarrow{\quad} & f^\sharp \\ [C, A \Rightarrow B] & & [C \times A, B] \\ g_\flat & \longleftarrow & \dashv g \end{array}$$

between the two arrow sets. These transpositions are required to be natural for variations of  $B, C$ , and have other associated gadgets.

In this section we set up the object  $(A \Rightarrow B)$  and verify some of the required properties. This is an interesting exercise for it illustrates the difference between set theoretic elements and global elements. The object  $(A \Rightarrow B)$  is not what you expect it to be.

What can  $(A \Rightarrow B)$  be? Whatever it is, it is some set, and by Section 2 its elements are in bijective correspondence with the arrow set

$$[R, A \Rightarrow B]$$

(since  $R$  is a separator). Now, if  $\mathbf{R}$  is cartesian closed, then this arrows set is in bijective correspondence with the arrow set

$$[R \times A, B]$$

which is something we can get at quite easily.

3.1 DEFINITION. For objects  $A, B$  let  $(A \Rightarrow B)$  be the set  $[R \times A, B]$ , the set of  $R$ -linear maps from  $R \times A$  to  $B$ . ■

Thus, by definition, a typical member of  $(A \Rightarrow B)$  is a function

$$\phi : R \times A \longrightarrow B$$

such that

$$\phi(s, a)r = \phi(sr, ar)$$

for all  $r, s \in R$  and  $a \in A$ . This gives a set, but to be an object we need an action on  $A \Rightarrow B$ .

3.2 DEFINITION. For each  $R$ -linear map

$$R \times A \xrightarrow{\phi} B$$

and  $t \in R$ , let

$$\phi^t : R \times A \longrightarrow B$$

be given by

$$\phi^t(s, a) = \phi(ts, a)$$

for  $s \in R$  and  $a \in A$ . ■

This action

$$(\phi, t) \longmapsto \phi^t$$

might look a little strange, for we might expect the  $t$  to interact with both inputs to  $\phi$ . However, as we will find out, this is the one that works.

This constructed function  $\phi^t$  is  $R$ -linear since

$$\phi^t(s, a)r = \phi(tsr, a) = \phi^t(sr, ar)$$

for  $r, s \in R$  and  $a \in A$ . Thus we can at least state the required result.

3.3 LEMMA. For each objects  $A, B$  the set  $(A \Rightarrow B)$  with the assignment

$$\begin{aligned} (A \Rightarrow B) \times R &\longrightarrow (A \Rightarrow B) \\ (\phi, t) &\longmapsto \phi^t \end{aligned}$$

Proof. We must show that

$$(\phi^t)^s = \phi^{ts}$$

for each  $s, t \in R$  and  $\phi \in (A \Rightarrow B)$ . But, by a simple calculation, for each  $r \in R$  and  $a \in A$  we have

$$(\phi^t)^s(r, a) = \phi(tsr, a) = \phi^{ts}(r, a)$$

which gives us what we want. ■

Our job now is to set up and verify the required properties of the inverse pair of transpositions. The construction is straight forward.

3.4 DEFINITION. Let  $A, B, C$  be  $R$ -sets.

(#) For each  $R$ -linear map

$$C \xrightarrow{f} (A \Rightarrow B)$$

let

$$f^\sharp : C \times A \longrightarrow B$$

be given by

$$f^\sharp(c, a) = (fc)(1, a)$$

for  $a \in A, c \in C$ .

(b) For each  $R$ -linear map

$$C \times A \xrightarrow{g} B$$

let

$$g_b : C \longrightarrow (A \Rightarrow B)$$

be given by

$$(g_b c)(s, a) = g(cs, a)$$

for  $a \in A, c \in C, s \in R$ . ■

There are several things to be checked, but each one is straight forward. Let's go through these in turn.

The constructed function  $f^\sharp$  is  $R$ -linear. In other words

$$f^\sharp(c, a)r = f^\sharp(cr, ar)$$

for  $a \in A, c \in C, r \in R$ . But

$$f^\sharp(c, a)r = (fc)(1, a)r = (fc)(r, ar) = (fc)^r(1, ar) = f(cr)(1, ar) = f^\sharp(cr, ar)$$

to give the required result. The third equality uses the action on  $(A \Rightarrow B)$  and the fourth uses the given  $R$ -linearity of  $f$ .

For each  $c \in C$  the constructed function  $g_b c$  is in  $(A \Rightarrow B)$ . In other words

$$(g_b c)(s, a)r = (g_b c)(sr, ar)$$

for  $a \in A, r, s \in R$ . But

$$(g_b c)(s, a)r = g(cs, a)r = g(csr, ar) = (g_b c)(sr, ar)$$

as required.

The constructed function  $g_b$  is  $R$ -linear. In other words

$$(g_b c)^t = g_b(ct)$$

for  $C \in C, t \in R$ . To check this we evaluate each side at an arbitrary  $(s, a) \in R \times A$ . Thus

$$(g_b c)^t(s, a) = (g_b c)(ts, a) = g(cts, a) = g_b(ct)(s, a)$$

to give the required result.

The double transpose  $(\cdot)_b^\sharp$  is the identity. In other words

$$f_b^\sharp = f$$

for each  $R$ -linear map  $f : C \longrightarrow (A \Rightarrow B)$ . To check this we evaluate at  $c \in C$  and then at  $(s, a) \in R \times A$ . Thus

$$(f_b^\sharp c)(s, a) = f^\sharp(cs, a) = f(cs)(1, a) = (fc)^s(1, a) = (fc)(s, a)$$

to give the required result. At the third equality we use the given  $r$ -linearity of  $f$  and at the fourth we use the action on  $(A \Rightarrow B)$ .

The double transpose  $(\cdot)_b^\sharp$  is the identity. In other words

$$g_b^\sharp = g$$

for each  $R$ -linear map  $g : C \times A \longrightarrow B$ . To check this we evaluate at  $(c, a) \in C \times A$ . Thus

$$g_b^\sharp(c, a) = g_b c(1, a) = g(c, a)$$

to give the required result.

This doesn't yet show that  $\mathbf{R}$  is cartesian closed. We still need to show that the two assignments  $(\cdot)_b^\sharp$  and  $(\cdot)_b$  are natural.

To this end consider a pair

$$C'_2 \xrightarrow{k} C_1 \qquad B_1 \xrightarrow{l} B_2$$

of  $R$ -linear maps, as indicated. These induce a diagram

$$\begin{array}{ccccc}
 & & (\cdot)_b^\sharp & & \\
 & & \longrightarrow & & \\
 f & [C_1, (A \Rightarrow B_1)] & & [C_1 \times A, B_1] & g \\
 \downarrow & \downarrow & \longleftarrow & \downarrow & \downarrow \\
 & & (\cdot)_b & & \\
 & & \longrightarrow & & \\
 L(f) & [C_2, (A \Rightarrow B_2)] & & [C_2 \times A, B_2] & R(g) \\
 & \downarrow & \longleftarrow & \downarrow & \\
 & & (\cdot)_b^\sharp & & \\
 & & \longrightarrow & & \\
 & & (\cdot)_b & & 
 \end{array}$$

where the movements between the arrows sets are named at the side. We must show that two separate squares commute. One square goes from top left to bottom right and uses the two  $(\cdot)_b^\sharp$  as horizontal arrows. The other square goes from top right to bottom left and uses the two  $(\cdot)_b$  as horizontal arrows. Thus we want

$$L(f)_b^\sharp = R(f)_b^\sharp \qquad L(g)_b = R(g)_b$$

for arbitrary  $R$ -linear maps  $f, g$  of indicated type. (Actually one of these will suffice, since it implies the other)

Let's verify the left hand equality. To do this we evaluate both composites at an arbitrary  $(c, a) \in C_2 \times A$ . To do this we need to unravel the constructions of  $L(f)$  and  $R(g)$ .

The construction of  $R(g)$  is easy. We have

$$R(g) = l \circ g \circ (k \times id)$$

so that

$$R(g)(c, a) = l(g(kc, a))$$

and in particular we have

$$R(f^\sharp)(c, a) = l(f^\sharp(kc, a)) = l(f(kc)(1, a))$$

for  $c \in C_2, a \in A$ .

The construction of  $L(f)$  is not quite so straight forward. Whatever it is we have

$$L(f) : C_2 \longrightarrow (A \Rightarrow B_2)$$

and so takes an input  $c \in C_2$  to return a function

$$L(f)(c) : R \times A \longrightarrow B_2$$

which takes an input  $(r, a) \in R \times A$ . Observing how  $L(f)$  is built up we see that

$$L(f)(c)(r, a) = l(f(kc)(r, a))$$

for  $c \in C_2, r \in R, a \in A$ . In particular, for  $c \in C_2, a \in A$  we have

$$L(f)^\sharp(c, a) = L(f)(c)(1, a) = l(f(kc)(1, a)) = R(f^\sharp)(c, a)$$

to give the required result.

This is enough to prove the following.

**3.5 THEOREM.** *The category  $\mathbf{R}$  is cartesian closed with facilities as described above.*

One facility that isn't described above is the evaluation arrow

$$(A \Rightarrow B) \times A \xrightarrow{\text{eval}_{A,B}} B$$

attached to a pair of objects  $A, B$ . By definition, this is just the transpose

$$id_{A \Rightarrow B}^\sharp$$

of the identity arrow on  $A \Rightarrow B$ . Thus

$$\text{eval}(\phi, a) = \phi(1, a)$$

for  $\phi \in A \Rightarrow B$  and  $a \in A$ . This is almost, but not quite, what you might expect evaluation to be.

## 4 Subobjects

In line with the usual algebraic tradition, a **subobject** of an  $R$ -set  $A$  is a subset  $X \subseteq A$  which is close under the furnishings of  $A$ , that is

$$a \in X \implies ar \in X$$

holds for all  $a \in A$  and  $r \in R$ . These are the topic of this section.

**4.1 DEFINITION.** For an  $R$ -set  $A$  let  $\Omega(A)$  be the poset of subobjects of  $A$  partially ordered by inclusion. ■

The empty set  $\emptyset$  and the whole  $R$ -set  $A$  are in  $\Omega(A)$ , and it can happen that these are the only members of  $\Omega(A)$ . (It can also happen that these are not distinct.) In this section we look at the structure of  $\Omega(A)$  and the way this can change as  $A$  changes. However, before we start there is something we need to clear up.

Here we are talking about subobjects in the algebraic tradition. The categorical tradition is slightly different. Given an object  $A$  we say two monics

$$\begin{array}{ccc} & A & \\ \nearrow & & \nwarrow \\ X & & Y \end{array}$$

are equivalent if there is a (necessarily unique) arrow from  $X$  to  $Y$  such that

$$\begin{array}{ccc} & A & \\ \nearrow & & \nwarrow \\ X & \longrightarrow & Y \end{array}$$

commutes. It is not too hard to show that this imposes an equivalence relation on the monics to  $A$ . Categorically, a subobject is an equivalence class of such monics.

In Theorem 2.4 we saw that in  $\mathbf{R}$  the monics are precisely the injective arrows. A slight generalization of this shows that each monic is equivalent to an algebraic subobject. In other words, in each categorical subobject, equivalence class of monics, there is precisely one algebraic subobject. Given this it is more convenient to deal with these canonical representatives, hence we stick with subobjects in the algebraic tradition.

There is also something else you might want to clear up. Why is ‘ $\Omega$ ’ used to indicate subobject? This is one of those historical accidents that has persisted through the years. You should ask about it, but not me.

Let’s start in earnest (if he doesn’t mind).

For an object  $A$  the set  $\Omega(A)$  of subobjects is partially ordered by inclusion. This has  $A$  as its top and  $\emptyset$  as its bottom. In fact,  $\Omega(A)$  has a much richer structure.

**4.2 LEMMA.** *For each  $R$ -set  $A$  the set  $\Omega(A)$  is closed under arbitrary unions and arbitrary intersections.*

**Proof.** Let  $\mathcal{X}$  be any set of subobjects  $X$  of  $A$ . Then, for each  $a \in A, r \in R$  we have

$$\begin{aligned} a \in \bigcup \mathcal{X} &\implies (\exists X \in \mathcal{X})[a \in X] \implies (\exists X \in \mathcal{X})[ar \in X] \implies ar \in \bigcup \mathcal{X} \\ a \in \bigcap \mathcal{X} &\implies (\forall X \in \mathcal{X})[a \in X] \implies (\forall X \in \mathcal{X})[ar \in X] \implies ar \in \bigcap \mathcal{X} \end{aligned}$$

to give the required result. ■

We didn't have to sweat much to prove that but it was worth doing. Almost always in an algebraic situation, the intersection  $\bigcap \mathcal{X}$  of a family of subalgebras of a parent algebra is itself a subalgebra. However, it is rarer for the union  $\bigcup \mathcal{X}$  to be a subalgebra. Usually there is a smallest subalgebra generated by  $\bigcup \mathcal{X}$ , but this can be quite a bit larger than the union. Here the furnishings on an  $R$ -set  $A$  are rather simple, which leads to the two closure properties of  $\Omega(A)$ . This simplicity makes many of the result to come quite easy, and could allow the study of  $R$ -sets to drift into the realms of juvenile mathematics. If you compare  $R$ -sets with right modules over a ring you will see the difference.

The parent monoid  $R$  is itself an  $R$ -set and so has subobjects. What are these?

**4.3 DEFINITION.** For the monoid  $R$  and ideal (or more correctly a right ideal) is a subset  $I \subseteq R$  such that

$$s \in I \implies sr \in I$$

holds for all  $r, s \in R$ .

Let  $\Omega$  be the set of all ideals of  $R$ . ■

In other words an ideal of  $R$  is a subobject (that is, a sub- $R$ -set, *not* a submonoid). Similarly  $\Omega$  is just  $\Omega(R)$  and as such is a poset under inclusion which is closed under arbitrary unions and intersections. This gadget  $\Omega$  (which we will see is itself an  $R$ -set) plays an important role in  **$R$** . (The name 'ideal' come from the corresponding notion in ring theory which itself come from the 'ideal numbers' that are needed to correct the defective factorization properties of certain algebraic number rings.)

As we will see in all its glory, subobjects and ideals are intimately connected.

**4.4 DEFINITION.** Let  $A$  be an  $R$ -set. For each subobject  $X$  and each  $a \in A$  we use

$$s \in X : a \iff as \in X$$

(for  $s \in R$ ) to extract a subset  $X : a$  of  $R$ . This is the **residual quotient** of  $X$  by  $a$ . ■

It is easy to see that  $X : a$  is an ideal of  $R$ . Thus

$$s \in X : a \implies as \in X \implies asr \in X \implies sr \in X : a$$

for  $r, s \in R$ . Notice also that

$$a \in X \iff 1 \in X : a \iff X : a = R$$

(for  $a \in A$ ). This simple observation will become more important as we proceed.

As a particular case of this construction, for each ideal  $I$  of  $R$  and each  $s \in R$  the residual quotient  $I : s$  is also an ideal. This is given by

$$r \in I : s \iff sr \in I$$

(for  $r \in R$ ). This means that taking residual quotients can be iterated, to a certain extent.

A routine unravelling of the definitions gives the following.

4.5 LEMMA. For each subobject  $X \subseteq A$  of an  $R$ -set and each  $a \in A$  we have

$$(X : a) : r = X : ar$$

for each  $r \in R$ .

Very nice, you may say, but what is the point of that? In fact, this simple observation is at the heart of almost all the topos structure of  $\mathbf{R}$ .

4.6 THEOREM. The set  $\Omega$  of all ideal is an  $R$ -set with action

$$\begin{array}{ccc} \Omega \times R & \longrightarrow & \Omega \\ (I, s) & \longmapsto & I : s \end{array}$$

(for  $I \in \Omega, s \in R$ ).

Proof. Certainly this assignment sends each pair  $(I, s)$  to an ideal. For it to be an action we require

$$(I : s) : r = I : sr$$

(for  $I \in \Omega, r, s \in R$ ). But this is a particular case of Lemma 4.5. ■

We are going to learn quite a lot about this internal object  $\Omega$ . It has a special position within the category  $\mathbf{R}$ , and does a lot to determine the nature of  $\mathbf{R}$ . What can  $\Omega$  look like, and what are its total elements?

4.7 EXAMPLES. (a) Suppose  $I$  is a total element of  $\Omega$ , that is

$$I : r = I$$

for all  $r \in R$ . What can  $I$  be? Clearly both  $\emptyset$  and  $R$  are total members of  $\Omega$ . In fact, these are the only total members. For suppose  $I$  is non-empty. Then there is at least one  $r \in I$ , to give  $1 \in I : r = I$  and hence  $I = R$ .

(b) Suppose  $R$  is a group. Then  $\Omega = \{\emptyset, R\}$ . For suppose  $I$  is a non-empty ideal, and consider  $s \in I$ . Then  $s$  has an inverse  $r$ , so that  $1 = sr \in I$ , to give  $I = R$ . ■

In Theorem 2.4 we showed that for  $\mathbf{R}$  the monics are precisely the injective arrows. We can now prove the companion result. The technique we use anticipates a more general method which we will develop in the following sections.

4.8 THEOREM. An arrow

$$A \xrightarrow{f} B$$

of  $\mathbf{R}$  is epic precisely when the carrying function is injective.

Proof. In the usual way we see that each surjective arrow is epic. The content of the result is the converse.

Suppose

$$B \xrightarrow{f} A$$

is an epic of  $\mathbf{R}$ . To show that  $f$  is surjective consider the range

$$X = f[B]$$

of  $f$ . We show that  $X = A$ . To this end we consider a parallel pair

$$A \begin{array}{c} \xrightarrow{\chi} \\ \xrightarrow{\text{true}_A} \end{array} \Omega$$

of functions given by

$$\chi(a) = X : a \quad \text{true}_A(a) = R$$

for each  $a \in A$ . It is easy to check that each of these is  $R$ -linear. (The rather strange looking naming of these arrows will become clear in Section 5.) For each  $b \in B$  we have

$$\chi(fb) = X : fb = R$$

(since  $fb \in X$ ) so that

$$\chi \circ f = \text{true}_A \circ f$$

and hence

$$\chi = \text{true}_A$$

since  $f$  is epic. In other words, for each  $a \in A$  we have

$$X : a = \chi(a) = \text{true}_A(a) = R$$

so that  $a \in X$ , as required. ■

This gives some information about the object  $\Omega = \Omega(R)$ , and illustrates some of its uses. What about the more general case  $\Omega(A)$  for an arbitrary object  $A$ ? We have seen that  $\Omega(A)$  is a complete lattice with set theoretic operations. In fact,  $\Omega(A)$  is a frame, a point-free analogue of a topology. We need not go into the ramifications of that here, but the crucial definition and associated result are worth looking at.

**4.9 DEFINITION.** Let  $A$  be an  $R$ -set. For each pair  $X, Y$  of subobjects of  $A$  we use

$$a \in (Y \supset X) \iff Y : a \subseteq X : a$$

(for  $a \in A$ ) to extract a subset  $(Y \supset X)$  of  $A$ . ■

The symbol ‘ $\supset$ ’ is used here not to remind you of ‘superset’ but of ‘implication’ or ‘adjoint arrow’. The poset  $\Omega(A)$  is residuated. That is, as a category, it is cartesian closed.

**4.10 LEMMA.** Let  $A$  be an  $R$ -set. For each  $X, Y \in \Omega(A)$  the set  $(Y \supset X)$  is a subobject, a member of  $\Omega(A)$ , and satisfies

$$Z \subseteq (Y \supset X) \iff Z \cap Y \subseteq X$$

for  $Z \in \Omega(A)$ .

**Proof.** To show that  $(Y \supset X) \in \Omega(A)$  consider any  $a \in (Y \supset X)$  and  $s \in R$ . We want  $as \in (Y \supset X)$ , that is  $Y : as \subseteq X : as$ . But for each  $r \in R$  we have

$$r \in Y : as \implies asr \in Y \implies sr \in Y : a \subseteq X : a \implies asr \in X \implies r \in X : as$$

which leads to the required result.

We show the equivalence in two parts.

Suppose  $Z \subseteq (Y \supset X)$  and consider any  $a \in Z \cap Y$ . Then

$$Y : a \subseteq X : a \quad 1 \in Y : a$$

(using the two supersets of  $Z$ ) to give  $1 \in X : a$ , and hence  $a \in X$ .

Conversely, suppose  $Z \cap Y \subseteq X$ , consider any  $a \in Z$  and any  $s \in Y : a$ . We want  $s \in X : a$ . We have

$$a \in Z \quad as \in Y$$

so that

$$as \in Z \cap Y \subseteq X$$

and hence  $s \in X : a$ , as required. ■

Theorem 4.6 shows that for the particular case  $A = R$  the poset  $\Omega = \Omega(R)$  is an object of  $\mathbf{R}$ . In general, for an arbitrary object  $A$  the external poset  $\Omega(A)$  can *not* be viewed as an internal object. The fact that  $\Omega = \Omega(R)$  can is a special property of  $R$ . However, the object assignment

$$A \longmapsto \Omega(A)$$

does have certain functorial properties (where  $\mathbf{R}$  is not the target category).

Consider an  $R$ -linear map

$$A \xrightarrow{f} B$$

and for a subobject  $Y \in \Omega(B)$  look at the inverse image  $X = f^{-1}(Y)$  as a subset of  $A$ . Thus

$$a \in X \iff fa \in Y$$

for  $a \in A$ . Then, for each  $a \in A, r \in R$  we have

$$a \in X \implies fa \in Y \implies f(ar) = (fa)r \in Y \implies ar \in X$$

to show that  $X \in \Omega(A)$ .

**4.11 THEOREM.** *For each  $R$ -linear map*

$$A \xrightarrow{f} B$$

*the inverse image assignment*

$$\Omega(A) \xrightarrow{f^{-1}} \Omega(B)$$

*is at least monotone, and satisfies*

$$(f^{-1}Y) : a = Y : fa$$

*for each  $Y \in \Omega(B)$  and  $a \in A$ .*

**Proof.** The first part is immediate. For the second part we have, for each  $r \in R$ ,

$$r \in (f^{-1}Y) : a \iff ar \in f^{-1}Y \iff (fa)r = (far) \in Y \iff r \in Y : fa$$

to give the required result. ■

The analysis of the construction  $\Omega(\cdot)$  can be taken quite a bit further, and provides a lot of information about  $\mathbf{R}$ . We don't need to go along that path here.

## 5 Subobject classification

The characteristic property of a topos is that it has a special object  $\Omega$  and a selected global element

$$1 \xrightarrow{\text{true}} \Omega$$

which classifies subobjects in a sense to be explained shortly. For the topos  $\mathbf{R}$  this object is  $\Omega = \Omega(R)$  as described in Section 4. By Example 4.7(a) there are just two global elements of  $\Omega$ , and we name these

$$1 \xrightarrow{\text{true}} \Omega \qquad 1 \xrightarrow{\text{false}} \Omega$$

where **true** picks out  $R \in \Omega$  and **false** picks out  $\emptyset$ .

What is subobject classification? We are interested in commuting squares

$$\begin{array}{ccc} A & \xrightarrow{\chi} & \Omega \\ \uparrow & (\square) & \uparrow \text{true} \\ X & \xrightarrow{!_X} & 1 \end{array}$$

where  $A$  is an arbitrary object with some subobject  $X$ , and  $\chi$  is an  $R$ -linear map. In fact, we want to know when such a square is a pullback. Notice that for any such commuting square we have

$$a \in X \implies \chi(a) = R$$

for all  $a \in A$ . The pullback condition converts this implication into an equivalence.

First of all let's see how each subobject produces such a pullback square.

**5.1 LEMMA.** *Let  $A$  be an object with a subobject  $X \subseteq A$ . Setting*

$$\chi(a) = X : a$$

*for  $a \in A$  produces a pullback square  $(\square)$ . Furthermore*

$$a \in X \iff \chi(a) = R$$

*(for  $a \in A$ ).*

**Proof.** By Lemma 4.5 and Theorem 4.6 this  $\chi$  is an  $R$ -linear map, that is

$$\chi ar = \chi(a) : r$$

for all  $a \in A, r \in R$ . Also, for  $a \in A$  we have

$$a \in X \iff 1 \in X : a = \chi(a) \iff \chi(a) = R$$

to verify the required equivalence. This ensures that the square does commute, so it remains to show we have a pullback.

Consider any situation

$$\begin{array}{ccc}
 & A & \xrightarrow{\chi} & \Omega \\
 & \uparrow & & \uparrow \text{true} \\
 & X & \xrightarrow{!_X} & \mathbf{1} \\
 f \nearrow & & \searrow !_B & \\
 B & & & 
 \end{array}$$

where the outer cell commutes. We want a mediating arrow

$$B \xrightarrow{g} X$$

for which the two produced triangular cells commute. Any arrow from  $B$  to  $X$  will make the bottom cell commute. Since  $X \subseteq A$  there can be at most one such arrow  $g$  which will make the top cell commute. In other words it suffices to show  $fb \in X$  for each  $b \in B$ .

Consider any  $b \in B$ . We have

$$(\chi \circ f)(b) = R$$

since the outer cell commutes. Thus, using Theorem 4.11 we have

$$f^{-1}(X) : b = X : fb = \chi(fb) = R$$

so that  $b \in f^{-1}(X)$ , and hence  $fb \in X$ , as required. ■

This result show that for each object  $A$  the assignment

$$\begin{array}{ccc}
 \Omega(A) & \longrightarrow & [A, \Omega] \\
 X & \longmapsto & \chi
 \end{array}$$

is an embedding. The crucial topos condition is that this embedding must be a bijection, that is every  $R$ -linear map  $A \longrightarrow \Omega$  must arise in this canonical way from some subobject for which the associated square ( $\square$ ) is a pullback.

Consider ant  $R$ -linear map

$$A \xrightarrow{\chi} \Omega$$

with the associated wedge

$$\begin{array}{ccc} A & \xrightarrow{\chi} & \Omega \\ & & \uparrow \text{true} \\ & & \mathbf{1} \end{array}$$

using the nominated global element of  $\Omega$ . By Lemma 1.8 we can form a pullback square

$$\begin{array}{ccc} A & \xrightarrow{\chi} & \Omega \\ \uparrow & & \uparrow \text{true} \\ X & \xrightarrow{!_X} & \mathbf{1} \end{array}$$

for some arrow  $X \longrightarrow A$ . As in any category the arrow  $\mathbf{1} \longrightarrow \Omega$  is monic, hence so is the arrow  $X \longrightarrow A$ . By Theorem 2.4 this arrow is injective, and hence we may replace  $X$  by a subobject  $X \subseteq A$  (for we replace  $X$  by the range of the arrow to  $A$ ). In this way we arrive at a square ( $\square$ ) which is a pullback.

5.2 THEOREM. *Let*

$$A \xrightarrow{\chi} \Omega$$

*be an  $R$ -linear map from some object  $A$ , and suppose ( $\square$ ) is a pullback (with  $X \subseteq A$ ). Then*

$$\chi(a) = R : a$$

*for all  $a \in A$*

**Proof.** Since the square commutes we have

$$a \in X \implies \chi(a) = R$$

for each  $a \in A$ . We use the given pullback condition to turn this into an equivalence.

Suppose  $\chi(a) = R$  (for some  $a \in A$ ). Let

$$\alpha : R \longrightarrow A$$

be the  $R$ -linear map which selects  $a$ , that is

$$\alpha(r) = ar$$

for each  $r \in R$ . Then

$$(\chi \circ \alpha)(r) = \chi(ar) = \chi(a) : r = R : r = R$$

and hence we have a diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{\chi} & \Omega \\ & & \uparrow & & \uparrow \text{true} \\ & & X & \xrightarrow{!_X} & \mathbf{1} \\ \alpha & \nearrow & & & \\ R & & & & \\ & \searrow & & & \\ & & & & \mathbf{1} \\ & & & & \uparrow \text{true} \\ & & & & \Omega \end{array}$$

where the outer cell commutes. Since the inner cell is a pullback, the arrow factors uniquely through some arrow  $R \longrightarrow X$ . In other words

$$ar = \alpha(r) \in X$$

for each  $r \in R$ . In particular, with  $r = 1$  we have  $a \in X$ .

This shows that

$$a \in X \iff \chi(a) = R$$

for all  $a \in A$ . We use this to show

$$\chi(a) = X : a$$

(for  $a \in A$ ). Consider any  $a \in A$ . For each  $r \in R$  we have

$$r \in \chi(a) \iff \chi(ar) = \chi(a) : r = R \iff ar \in X \iff r \in X : a$$

to give the required result. The second equivalence uses the preliminary observation. ■

This completes the proof that  $\mathbf{R}$  is a topos. Of course, this proof is no more than a particular case that the presheaf category on any category is a topos, but the particular details are not so complicated and do give us a better feel for the inner workings of  $\mathbf{R}$ .

In the next few sections we continue this theme of looking at  $\mathbf{R}$ -instances of more general topos situations.

## 6 The subobject double adjunction

We hinted in Section 4 that the construction  $\Omega(\cdot)$  is the object assignment of a contravariant functor from  $\mathbf{R}$  to the category of frames. The behaviour on arrows is just the inverse image construction, as in Theorem 4.11. We don't need that here, but we do need a rather special property of the images morphisms not shared by all frame morphism.

6.1 LEMMA. *For each  $R$ -linear map*

$$A \xrightarrow{f} B$$

*the inverse image assignment*

$$\Omega(A) \xrightarrow{f^\leftarrow} \Omega(B)$$

*preserves the implication  $(\cdot \supset \cdot)$ , that is*

$$f^\leftarrow(Y \supset X) = (f^\leftarrow Y \supset f^\leftarrow X)$$

*for  $X, Y \in \Omega(B)$ .*

**Proof.** For each  $a \in A$  we have

$$\begin{aligned} a \in f^\leftarrow(Y \supset X) &\iff a \in (Y \supset X) \\ &\iff Y : fa \subseteq X : fa \\ &\iff (f^\leftarrow Y) : a \subseteq (f^\leftarrow X) : a \iff a \in (f^\leftarrow Y \supset f^\leftarrow X) \end{aligned}$$

to give the required result. The third equivalence uses Theorem 4.11. ■

We will not pursue these functorial properties just yet. Rather, we will concentrate on the monotone map

$$\Omega(A) \xleftarrow{f^\leftarrow} \Omega(B)$$

induced by an  $R$ -linear map

$$A \xrightarrow{f} B$$

between arbitrary  $R$ -sets  $A, B$ . We will show that this is the central arrow in a double poset adjunction.

6.2 DEFINITION. Let

$$A \xrightarrow{f} B$$

be an  $R$ -linear map. For each  $X \in \Omega(A)$  we use

$$\begin{aligned} b \in \exists_f(X) &\iff (\exists a : A)[(b = fa) \wedge (a \in X)] \\ b \in \forall_f(X) &\iff (\forall a : A, s : R)[(bs = fa) \implies (a \in X)] \end{aligned}$$

(for  $b \in B$ ) to produce subsets  $\exists_f(X), \forall_f(X)$  of  $B$ . ■

Notice that  $\exists_f(X)$  is just the direct image  $f[X]$  of  $X$  across  $f$ . The definition is written in the form to emphasize the comparison with  $\forall_f(X)$ .

6.3 LEMMA. For each  $R$ -linear map  $f$ , as above, the subsets  $\exists_f(X), \forall_f(X)$  of  $B$  are subobjects of  $B$ .

**Proof.** For each  $b \in B$  and  $s \in R$  we have

$$\begin{aligned} b \in \exists_f(X) &\implies (\exists a : A)[(b = fa) \wedge (a \in X)] \\ &\implies (\exists a : A)[(bs = fas) \wedge (as \in X)] \\ &\implies (\exists a : A)[(bs = fa) \wedge (a \in X)] \implies bs \in \exists_f(X) \end{aligned}$$

and

$$\begin{aligned} b \in \forall_f(X) &\implies (\forall a : A, r : R)[(br = fa) \implies (a \in X)] \\ &\implies (\forall a : A, r : R)[(b(sr) = fa) \implies (a \in X)] \\ &\implies (\forall a : A, r : R)[((bs)r = fa) \implies (a \in X)] \implies bs \in \forall_f(X) \end{aligned}$$

to give the required result. In the argument for  $\forall_f(X)$  the second implication holds since it is a specialization of the quantification over  $R$ . ■

This show that Definition 6.2 produces a pair of functions

$$\exists_f, \forall_f : \Omega(A) \longrightarrow \Omega(B)$$

between the posets of subobjects. As you might expect, we have more. A perusal of the quantified expressions gives the following.

6.4 LEMMA. For each  $R$ -linear map  $f$ , as above, the assignments  $\exists_f, \forall_f$  produce monotone maps

$$\Omega(A) \begin{array}{c} \xrightarrow{\exists_f} \\ \xrightarrow{\forall_f} \end{array} \Omega(B)$$

between the external posets.

With this we are in a position to state and prove the following important result.

6.5 THEOREM. Let

$$A \xrightarrow{f} B$$

be any  $R$ -linear map. The three induced monotone maps

$$\Omega(B) \begin{array}{c} \xrightarrow{\exists_f} \\ \xleftarrow{f^{\leftarrow}} \\ \xrightarrow{\forall_f} \end{array} \Omega(A)$$

form a double adjunction

$$\exists_f \dashv f^{\leftarrow} \dashv \forall_f$$

between the posets of subobjects.

**Proof.** We must show

$$\exists_f(X) \subseteq Y \iff X \subseteq f^{\leftarrow}(Y) \quad f^{\leftarrow}(Y) \subseteq X \iff Y \subseteq \forall_f(X)$$

for  $X \in \Omega(A), Y \in \Omega(B)$ .

When we remember that  $\exists_f$  is just the direct image function, the left hand equivalence is immediate, but let's look at the details just to be sure.

Suppose first that  $\exists_f(X) \subseteq Y$ , and consider any  $a \in X$ . To show  $a \in f^{\leftarrow}(Y)$  consider  $b = fa$ . Then

$$b \in \exists_f(X) \subseteq Y = Y$$

so that  $fa = b \in Y$ , to give the required result.

Conversely, suppose  $X \subseteq f^{\leftarrow}(Y)$ , and consider any  $b \in \exists_f(X)$ . To show  $b \in Y$  observe that  $b = fa$  for some

$$a \in X \subseteq f^{\leftarrow}(Y)$$

to give the required result.

For the right hand equivalence suppose first that  $f^{\leftarrow}(Y) \subseteq X$ , and consider any  $b \in Y$ . To show  $b \in \forall_f(X)$  consider any  $a \in A, s \in R$  with  $bs = fa$ . Then  $bs \in Y$ , so that

$$a \in f^{\leftarrow}(Y) \subseteq X$$

to give the required result.

Conversely, suppose  $Y \subseteq \forall_f(X)$ , and consider any  $a \in f^{\leftarrow}(Y)$ . To show  $a \in X$  consider  $b = fa$ . Then

$$b = fa \in Y \subseteq \forall_f(X)$$

and, with  $s = 1$ , we have  $bs = fa$ , so that  $a \in X$ , to give the required result. ■

This kind of external double adjunction can be set up for many different categories. However, for  $\mathbf{R}$ , and more generally for any topos, there is an internal version of the double adjunction. We will look at that in Section 8.

## 7 The internal quantifiers

In Section 5 we attached to each object  $A$  an object  $PA = (A \Rightarrow \Omega)$ , the power object of  $A$ . We checked that  $P$  is a contravariant endo-functor on  $\mathbf{R}$ . In this section we show that each object  $A$  supports three arrows

$$PA \begin{array}{c} \xrightarrow{\exists_A} \\ \xleftarrow{\forall_A} \\ \xrightarrow{\forall_A} \end{array} \Omega$$

which form an internal double adjunction.

$$\exists_A \dashv \forall_A \dashv \forall_A$$

In fact, this construction is a particular case of a more general construction which we look at in the next section. That more general setting is an internal version of the double subobject adjunction of Section 6.

This simplified version is worth looking at for two reasons. Firstly, the arrows  $\exists_A$  and  $\forall_A$  are those used to interpret the quantifiers  $(\exists x : A)$  and  $(\forall x : A)$ , so it is handy if we know something about these arrows. Secondly, parts of the more general construction are rather intricate, so it helps to see a simpler case first.

What is an ‘internal adjunction’? In this case we can make sense of the notion by taking an external view.

The set

$$PA = (A \Rightarrow \Omega) = [R \times A, \Omega]$$

carries a partial ordering, the pointwise comparison. The set  $\Omega$  is partially ordered by inclusion. This induces the comparison on  $PA$  by

$$\phi \leq \theta \iff (\forall r : R, a : A)[\phi(r, a) \subseteq \theta(r, a)]$$

for  $\phi, \theta \in PA$ . Using this we show that the three functions  $\exists_A, \forall_A, \forall_A$  are monotone and form a double poset adjunction. Thus we show

$$\exists_A(\phi) \subseteq U \iff \phi \leq \forall_A(I) \quad \forall_A(I) \leq \phi \iff I \subseteq \forall_A(\phi)$$

for all  $\phi \in PA$  and  $I \in \Omega$ .

**7.1 DEFINITION.** Let  $A$  be an  $R$ -set.

For each ideal  $I \in \Omega$  let

$$\forall_A(I) : R \times A \longrightarrow \Omega$$

be the function given by

$$\forall_A(I)(r, a) = I : r$$

for all  $r \in R, a \in A$ .

For each  $\phi \in PA$  let

$$\exists_A(\phi) \quad \forall_A(\phi)$$

be the subsets of  $R$  given by

$$x \in \exists_A(\phi) \iff (\exists a : A)[1 \in \phi(x, a)] \quad x \in \forall_A(\phi) \iff (\forall r : R, a : A)[1 \in \phi(xr, a)]$$

(for  $x \in R$ ). ■

We have several little jobs to do.

7.2 LEMMA. For each object  $A$ , the assignment  $\mathfrak{I}_A$  produces an arrow

$$PA \longleftarrow \mathfrak{I}_A \longrightarrow \Omega$$

and externally this function is monotone.

**Proof.** Trivially, for each  $I \in \Omega, r \in R, a \in A$  the value  $\mathfrak{I}_A(I)(r, a)$  is an ideal. We must show that the function

$$\mathfrak{I}_A(I) : R \times A \longrightarrow \Omega$$

is  $R$ -linear, that is

$$\mathfrak{I}_A(I)(s, a) : r = \mathfrak{I}_A(I)(sr, ar)$$

for  $r, s \in R, a \in A$ . Since

$$I : s : r = I : sr$$

this is immediate.

Thus we have a function

$$\mathfrak{I}_A : \Omega \longrightarrow PA$$

and it remains to show that this is  $R$ -linear, and hence we have an arrow. In other words, we require

$$\mathfrak{I}_A(I)^t = \mathfrak{I}_A(I : t)$$

for each  $I \in \Omega, t \in R$ . To do this we evaluate at an arbitrary  $(r, a) \in R \times A$ . Thus

$$\mathfrak{I}_A(I)^t(r, a) = \mathfrak{I}_A(I)(tr, a) = I : tr = I : t : r = \mathfrak{I}_A(I : t)(r, a)$$

as required.

This deals with the first part of the result. To complete the proof we look at the external behaviour of  $\mathfrak{I}_A$ . We require

$$I \subseteq J \implies \mathfrak{I}_A(I) \leq \mathfrak{I}_A(J)$$

for ideals  $I, J$ . But

$$I \subseteq J \implies I : r \subseteq J : r$$

for  $r \in R$ , so the required implication is immediate. ■

In Section 8 we show how this arrow  $\mathfrak{I}_A$  arises in a canonical fashion. You might like to chew on this now.

Next we have several little jobs concerning  $\exists_A$  and  $\forall_A$ . It is instructive to do these in parallel.

7.3 LEMMA. For each object  $A$  and  $\phi \in PA$ , the subsets

$$\exists_A(\phi) \quad \forall_A(\phi)$$

of  $R$  are ideals.

**Proof.** Consider any

$$x \in \exists_A(\phi) \quad x \in \forall_A(\phi)$$

and any  $r \in R$ . We must show that

$$xr \in \exists_A(\phi) \quad xr \in \forall_A(\phi)$$

and we do this by separate arguments.

( $\exists$ ) Suppose  $x \in \exists_A(\phi)$  so we have some  $a \in A$  with

$$1 \in \phi(x, a) \quad \phi(x, a) = R$$

where the right hand side is a rephrasing of the left hand side. But now  $r \in \phi(x, a)$ , so that

$$1 \in \phi(x, a) : r = \phi(xr, ar)$$

where this equality arises from the  $R$ -linearity of  $\phi$ . Thus  $xr \in \forall_A(\phi)$ , as required.

( $\forall$ ) Suppose  $x \in \forall_A(\phi)$  and, to show  $xr \in \forall_A(\phi)$ , consider any  $s \in R, a \in A$ . We want

$$1 \in \phi(xrs, a)$$

but this is an immediate consequence of the way the quantification over  $R$  is built into the construction of  $\exists_A(\phi)$ . ■

This shows that we certainly have a pair of functions

$$\exists_A : PA \longrightarrow \Omega \quad \forall_A : PA \longrightarrow \Omega$$

but, of course, we want more.

**7.4 LEMMA.** *For each object  $A$ , the assignments  $\exists_A, \forall_A$  produce arrows*

$$PA \xrightarrow{\exists_A} \Omega \quad PA \xrightarrow{\forall_A} \Omega$$

*and externally these functions are monotone.*

**Proof.** We show first that each assignment is  $R$ -linear, that is

$$\exists_A(\phi) : t = \exists_A(\phi^t) \quad \forall_A(\phi) : t = \forall_A(\phi^t)$$

for all  $\phi \in PA, t \in R$ .

( $\exists$ ) For each  $x \in R$  we have

$$\begin{aligned} x \in \exists_A(\phi) : t &\iff tx \in \exists_A(\phi) \\ &\iff (\exists a : A)[1 \in \phi(tx, a)] \\ &\iff (\exists a : A)[1 \in \phi^t(x, a)] \iff x \in \exists(\phi^t) \end{aligned}$$

to give the required result.

( $\forall$ ) For each  $x \in R$

$$\begin{aligned} x \in \forall_A(\phi) : t &\iff tx \in \forall_A(\phi) \\ &\iff (\forall r : R, a : A)[1 \in \phi(txr, a)] \\ &\iff (\forall r : R, a : A)[1 \in \phi^t(xr, a)] \iff x \in \forall(\phi^t) \end{aligned}$$

to give the required result.

Next we observe that, externally, each of these arrows is monotone, that is

$$\phi_1 \leq \phi_2 \implies \exists_A(\phi_1) \subseteq \exists_A(\phi_2) \text{ and } \forall_A(\phi_1) \subseteq \forall_A(\phi_2)$$

for all  $\phi_1, \phi_2 \in PA$ . The trick is to use unravellings, as above, with  $t = 1$  and remember how the pointwise comparison works.  $\blacksquare$

We are now in a position to state and prove the main result of this section.

**7.5 THEOREM.** *Let  $A$  be any  $R$ -set. Externally, the three induced arrows*

$$\begin{array}{ccc} & \xrightarrow{\exists_A} & \\ PA & \xleftarrow{\mathbf{!}_A} \Omega & \xrightarrow{\forall_A} \\ & \xrightarrow{\forall_A} & \end{array}$$

*are monotone and form a double poset adjunction.*

**Proof.** The external monotonicity is given by Lemma 7.2 (for the central arrow) and Lemma 7.4 (for the two outer arrows). Here we must deal with the adjunction properties that is

$$\exists_A(\phi) \subseteq I \iff \phi \leq \mathbf{!}_A(I) \quad \mathbf{!}_A(I) \leq \phi \iff I \subseteq \forall_A(\phi)$$

for  $\phi \in PA$  and  $I \in \Omega$ .

$(\exists_A \dashv \mathbf{!}_A)$  Consider  $\phi \in PA, I \in \Omega$ . We see that the relevant comparisons as on the left

$$\begin{aligned} \exists_A(\phi) \subseteq I & \quad (\forall x : R, a : A)[1 \in \phi(x, a) \implies x \in I] \\ \phi \leq \mathbf{!}_A(I) & \quad (\forall r : R, a : A)[\phi(r, a) \subseteq I : r] \end{aligned}$$

unravel to the quantified forms, as on the right. You should observe how, for the top right condition, the existential quantifier hidden in  $\exists_A(\phi)$  has become a universal quantifier. Classic stuff.

Suppose  $\exists_A(\phi) \subseteq I$ . Consider any  $r \in R, a \in A$  and any  $x \in \phi(r, a)$ . Then

$$1 \in \phi(r, a) : x = \phi(rx, ax)$$

so that  $rx \in I$  (by the assumed inclusion) to give  $x \in I : r$ , which leads to the required result.

Conversely suppose  $\phi \leq \mathbf{!}_A(I)$ . Consider any  $x \in R, a \in A$  where  $1 \in \phi(x, a)$ . The assumed comparison gives  $1 \in I : x$  so that  $x \in I$ , as required.

$(\mathbf{!}_A \dashv \forall_A)$  Consider  $\phi \in PA, I \in \Omega$ . We see that the relevant comparisons as on the left

$$\begin{aligned} \mathbf{!}_A(I) \leq \phi & \quad (\forall r : R, a : A)[I : r \subseteq \phi(r, a)] \\ I \subseteq \forall_A(\phi) & \quad (\forall r, x : R, a : A)[x \in I \implies 1 \in \phi(xr, a)] \end{aligned}$$

unravel to the quantified forms, as on the right.

Suppose  $\mathbf{!}_A(I) \leq \phi$ . Consider  $r, x \in R, a \in A$  where  $x \in I$ . Then  $xr \in I$  so that  $1 \in I : xr$  and hence  $1 \in \phi(xr, a)$  by the assumed comparison, to give the required result.

Conversely, suppose  $I \subseteq \forall_A(\phi)$ . Consider any  $x, r \in R, a \in A$  where  $x \in I : r$ . Then  $rx \in I$  so that

$$1 \in \phi(rx, ax) = \phi(r, a) : x$$

by a particular case of the assumed comparison, and hence  $x \in \phi(r, a)$ , as required,  $\blacksquare$

That concludes the construction of the quantifier adjunctions. Later, in Section 9 [To be written], we will see how the arrows  $\exists_A, \forall_A$  are used in the interpretation of the internal language of  $\mathbf{R}$ .

## 8 The internal double adjunction

In Section 6 we showed how each arrow

$$A \xrightarrow{f} B$$

leads to an external double adjunction

$$\Omega(A) \begin{array}{c} \xrightarrow{\exists_f} \\ \xleftarrow{f^{\leftarrow}} \\ \xrightarrow{\forall_f} \end{array} \Omega(B)$$

between the posets  $\Omega(A), \Omega(B)$  of subobjects. In Section 7 we produced an internal version of, what turns out to be, a particular case of this. In this section we produce an internal version

$$PA \begin{array}{c} \xrightarrow{\exists_f} \\ \xleftarrow{P(f)} \\ \xrightarrow{\forall_f} \end{array} PB$$

of the full double adjunction. Once we have done that we will be able to show how the adjunction of Section 7 arises as a particular case of the more general internal adjunction produced here.

Observe that there is a potential for confusion here since we are using the same symbols  $\exists_f, \forall_f$  for both external monotone maps and internal arrows. To avoid this, for this section only we write

$$\Omega(A) \begin{array}{c} \xrightarrow{\dot{\exists}_f} \\ \xleftarrow{f^{\leftarrow}} \\ \xrightarrow{\dot{\forall}_f} \end{array} \Omega(B)$$

for the external maps and keep the undecorated symbols for the internal arrows. However, this convention is not needed for a while (not until after Theorem 8.6).

We wish to convert the given arrow  $f$  into an internal double adjunction.

$$\exists_f \dashv P(f) \dashv \forall_f$$

What is an ‘internal adjunction’? As we observed in Section 7, both the objects

$$PA = (A \Rightarrow \Omega) = [R \times A, \Omega] \quad PB = (B \Rightarrow \Omega) = [R \times B, \Omega]$$

carry the pointwise comparison and so externally can be viewed as posets. We will check that all three maps are monotone and form posets adjunctions. Thus, eventually we show

$$\exists_f(\phi) \leq \psi \iff \phi \leq P(f)(\psi) \quad P(f)(\psi) \leq \phi \iff \psi \leq \forall_f(\phi)$$

for all  $\phi \in PA$  and  $\psi \in PB$ .

We begin by sorting out the behaviour of  $P(f)$ . Recall that for  $\psi \in PB$  the function  $P(f)(\psi)$  is the composite

$$R \times A \xrightarrow{id \times f} R \times B \xrightarrow{\psi} \Omega$$

so that we have

$$P(f)(\psi)(s, a) = \psi(s, fa)$$

for each  $s \in R, a \in A$ . We could check directly that  $P(f)(\psi)$  is a member of  $PA$ , that is

$$P(f)(\psi)(s, a) : r = P(f)(\psi)(sr, ar)$$

for  $r, s \in R, a \in A$ . However, this is a simple consequence of the given  $R$ -linearity of  $f$ .

The external properties of  $P(f)$  do not seem to be a consequence of general topos facts.

### 8.1 LEMMA. *The assignment*

$$P(f) : PB \longrightarrow PB$$

is monotone, that is

$$\psi_1 \leq \psi_2 \implies P(f)(\psi)_1 \leq P(f)(\psi)_2$$

for all  $\psi_1, \psi_2 \in PB$ .

**Proof.** Suppose  $\psi_1 \leq \psi_2$ , that is

$$\psi_1(s, b) \subseteq \psi_2(s, b)$$

for all  $s \in R, b \in B$ . In particular, for  $s \in R, a \in A$

$$P(f)(\psi)_1(s, a) = \psi_1(s, fa) \subseteq \psi_2(s, fa) = P(f)(\psi)_2(s, a)$$

to give  $P(f)(\psi)_1 \leq P(f)(\psi)_2$ . ■

We now turn to the construction of  $\exists_f$  and  $\forall_f$ .

### 8.2 DEFINITION. Let

$$A \xrightarrow{f} B$$

be an  $R$ -linear map.

For each  $\phi \in PA$  and  $s \in R, b \in B$  let

$$\exists_f(\phi)(s, b) \quad \forall_f(\phi)(s, b)$$

be the subsets of  $R$  given by

$$\begin{aligned} x \in \exists_f(\phi)(s, b) &\iff (\exists a : A)[(bx = fa) \wedge (1 \in \phi(sx, a))] \\ x \in \forall_f(\phi)(s, b) &\iff (\forall r : R, a : A)[(bxr = fa) \Rightarrow (1 \in \phi(sxr, a))] \end{aligned}$$

(for  $x \in R$ ). ■

As usual we have several little jobs to do. We follow thwe same general path as in Section 7 but with a couple of extra steps.

**8.3 LEMMA.** *For each  $R$ -linear map  $f$ , as above, and for each  $\phi \in PA$  and  $s \in R, b \in B$ , the subsets*

$$\exists_f(\phi)(s, b) \quad \forall_f(\phi)(s, b)$$

*of  $R$  are ideals.*

**Proof.** Consider any

$$x \in \exists_f(\phi)(s, b) \quad x \in \forall_f(\phi)(s, b)$$

and any  $r \in R$ . We must show that

$$xr \in \exists_f(\phi)(s, b) \quad xr \in \forall_f(\phi)(s, b)$$

and we do this by separate arguments.

( $\exists$ ) Suppose  $x \in \exists_f(\phi)(s, a)$  so we have some  $a \in A$  with

$$bx = fa \quad \text{and} \quad \phi(sx, a) = R$$

(where the second component is a rephrasing of the official component). But now we have  $r \in \phi(sx, a)$ , so that

$$1 \in \phi(sx, a) : r = \phi(sxr, ar)$$

where this equality arises from the  $R$ -linearity of  $\phi$ . Thus, using the  $R$ -linearity of  $f$ , we have

$$bxr = far \quad \text{and} \quad 1 \in \phi(sxr, ar)$$

to give the required result (with  $ar$  as the appropriate witness).

( $\forall$ ) Suppose  $x \in \forall_f(\phi)(s, b)$  and, to show  $xr \in \forall_f(\phi)(s, b)$  consider any  $t \in R, a \in A$  with  $bxrt = fa$ . We want  $1 \in \phi(sxtr, a)$ . But this is an immediate consequence of  $x \in \forall_f(\phi)(s, b)$  (using  $rt$  to instantiate  $r$ ).  $\blacksquare$

This shows that for each  $\phi \in PA$  we have a pair of functions

$$\exists_f(\phi) : R \times B \longrightarrow \Omega \quad \forall_f(\phi) : R \times B \longrightarrow \Omega$$

but, of course, we want much more.

**8.4 LEMMA.** *For each  $R$ -linear map  $f$ , as above, and each  $\phi \in PA$ , the assignments  $\exists_f(\phi), \forall_f(\phi)$  are  $R$ -linear, and so are members of  $PA$ .*

**Proof.** We must show that

$$\exists_f(\phi)(s, b) : t = \exists_f(\phi)(st, bt) \quad \forall_f(s, b) : t = \forall_f(st, bt)$$

for all  $s, t \in R, b \in B$ .

( $\exists$ ) For each  $x \in R$  both of

$$x \in \exists_f(\phi)(s, b) : t \quad x \in \exists_f(\phi)(st, bt)$$

unravel as

$$(\exists a : A)[(btx = fa) \wedge (1 \in \phi(stx, a))]$$

to give the required result.

( $\forall$ ) For each  $x \in R$  both of

$$x \in \forall_f(\phi)(s, b) : t \quad x \in \forall_f(\phi)(st, bt)$$

unravel as

$$(\forall r : R, a : A)[(btxr = fa) \Rightarrow (1 \in \phi(stxr, a))]$$

to give the required result. ■

This shows that we have a pair of functions

$$\exists_f : PA \longrightarrow PB \quad \forall_f : PA \longrightarrow PB$$

but we still want more.

**8.5 LEMMA.** *For each  $R$ -linear map  $f$ , as above, the assignments  $\exists_f, \forall_f$  produce arrows*

$$PA \xrightarrow{\exists_f} PB \quad PA \xrightarrow{\forall_f} PB$$

*and externally these functions are monotone.*

**Proof.** We show first that each of the two constructions produces an arrow.

( $\exists$ ) We require

$$\exists_f(\phi)^t = \exists_f(\phi^t)$$

for each  $\phi \in PA$  and  $t \in R$ . In other words we require

$$\exists_f(\phi)^t(s, b) = \exists_f(\phi^t)(s, b)$$

for each  $s \in R, b \in B$ . Thus, since

$$\exists_f(\phi)^t(s, b) = \exists_f(\phi)(ts, b)$$

we require

$$x \in \exists_f(\phi)(ts, b) \iff x \in \exists_f(\phi^t)(s, b)$$

for  $x \in R$ . We find that both sides of this equivalence unravels to

$$(\exists a : A)[(bx = fa) \wedge (1 \in \phi(tsx, a))]$$

to give the required result.

( $\forall$ ) We require

$$\forall_f(\phi)^t = \forall_f(\phi^t)$$

for each  $\phi \in PA$  and  $t \in R$ . In other words we require

$$\forall_f(\phi)^t(s, b) = \forall_f(\phi^t)(s, b)$$

for each  $s \in R, b \in B$ . Thus, since

$$\forall_f(\phi)^t(s, b) = \forall_f(\phi)(ts, b)$$

we require

$$x \in \forall_f(\phi)(ts, b) \iff x \in \forall_f(\phi^t)(s, b)$$

for  $x \in R$ . We find that both sides of this equivalence unravels to

$$(\forall r : A, a : A)[(bxr = fa) \Rightarrow (1 \in \phi(tsxr, a))]$$

to give the required result.

Secondly, we show that the two external maps are monotone, that is

$$\phi_1 \leq \phi_2 \implies \exists_f(\phi_1) \leq \exists_f(\phi_2) \quad \phi_1 \leq \phi_2 \implies \forall_f(\phi_1) \leq \forall_f(\phi_2)$$

for  $\phi_1, \phi_2 \in PA$ .

( $\exists$ ) Suppose  $\phi_1 \leq \phi_2$  so that

$$\phi_1(s, a) \subseteq \phi_2(s, a)$$

for each  $s \in R, b \in B$ . Consider any

$$x \in \exists_f(\phi_1)(s, b)$$

for arbitrary  $s \in R, b \in B$ . There is some  $a \in A$  with

$$bx = fa \quad 1 \in \phi_1(sx, a) \subseteq \phi_2(sx, a)$$

and hence

$$x \in \exists_f(\phi_2)(s, b)$$

which leads to the required result.

( $\forall$ ) Suppose  $\phi_1 \leq \phi_2$  so that

$$\phi_1(s, a) \subseteq \phi_2(s, a)$$

for each  $s \in R, b \in B$ . Consider any

$$x \in \forall_f(\phi_1)(s, b)$$

for arbitrary  $s \in R, b \in B$ . There are  $r \in R, a \in A$  with

$$bxr = fa \quad 1 \in \phi_1(sxr, a) \subseteq \phi_2(sxr, a)$$

and hence

$$x \in \forall_f(\phi_2)(s, b)$$

which leads to the required result. ■

We are getting there slowly. At least now we are in a position to state and prove the adjunction result.

8.6 THEOREM. *Let*

$$A \xrightarrow{f} B$$

*be an  $R$ -linear map. Externally, the three induced arrows*

$$\begin{array}{ccc} & \xrightarrow{\exists_f} & \\ PA & \xleftarrow{P(f)} \xrightarrow{\quad} & PB \\ & \xleftarrow{\forall_f} & \end{array}$$

*are monotone and form a double poset adjunction.*

**Proof.** The external monotonicity is given by Lemma 8.1 (for the central arrow) and Lemma 8.5 (for the two outer arrows). Here we must deal with the adjunction properties

$$\exists_f \dashv P(f) \dashv \forall_f$$

that is

$$\exists_f(\phi) \leq \psi \iff \phi \leq P(f)(\psi) \quad P(f)(\psi) \leq \phi \iff \psi \leq \forall_f(\phi)$$

for  $\phi \in PA$  and  $\psi \in PB$ . Here, as at the beginning of this section  $P(f)(\psi) = P(f)\psi$ .

( $\exists_f \dashv P(f)$ ) Consider  $\phi \in PA, \psi \in PB$ . We see that the relevant comparisons as on the left

$$\begin{array}{ll} \exists_f(\phi) \leq \psi & (\forall s, x : R, b : B, a : A)[bx = fa1 \in \phi(sx, a) \implies x \in \psi(s, b)] \\ \phi \leq P(f)(\psi) & (\forall s : R, a : A)[\phi(s, a) \subseteq \psi(s, fa)] \end{array}$$

unravel to the quantified forms, as on the right. It may look as though there are too many quantifiers in the top right ‘conditions, but remember there is an existential quantifier over  $R$  hidden in the inclusion of the right hand condition, and this converts into a universal quantifier!

Suppose  $\exists_f(\phi) \leq \psi$ . Then, as a particular case (obtained by replacing ‘ $a$ ’ by ‘ $ax$ ’) we have

$$(\forall s, x : R, b : B, a : A) \left[ \begin{array}{l} b = fa \\ 1 \in \phi(sx, ax) \end{array} \implies x \in \psi(s, b) \right]$$

so that

$$(\forall s, x : R, a : A)[(1 \in \phi(sx, ax)) \implies (x \in \psi(s, b))]$$

and hence

$$(\forall s, x : R, a : A)[(x \in \phi(s, a)) \implies (x \in \psi(s, b))]$$

which a more explicit form of  $\phi \leq P(f)(\psi)$ .

Conversely suppose  $\phi \leq P(f)(\psi)$ . Consider any  $s, x \in R, b \in B, a \in A$  where

$$bx = fa \quad 1 \in \phi(sx, a)$$

hold. The second of these with the assumed comparison gives

$$1 \in \psi(sx, fa) = \psi(sx, bx)$$

so that  $x \in \psi(s, b)$ , as required.

$(P(f) \dashv \forall_f)$  Consider  $\phi \in PA, \psi \in PB$ . We see that the relevant comparisons as on the left

$$\begin{aligned} P(f)(\psi) &\leq \phi & (\forall r : R, a : A)[\psi(r, fa) \subseteq \phi(r, a)] \\ \psi &\leq \forall_f(\phi) & (\forall r, s, x : R, b : B, a : A) \left[ \begin{array}{l} bxr = fa \\ x \in \psi(s, b) \end{array} \implies 1 \in \phi(sxr, a) \right] \end{aligned}$$

unravel to the quantified forms, as on the right.

Suppose  $P(f)(\psi) \leq \phi$ . Consider  $r, s, x \in R, b \in B, a \in A$  where

$$bxr = fa \quad x \in \psi(s, b)$$

hold. From the second of these we have  $xr \in \psi(s, b)$ , so that

$$1 \in \psi(sxr, bxr)$$

which gives

$$1 \in \psi(sxr, fa)$$

and hence

$$1 \in \phi(sxr, a)$$

by the assumed comparison, as required.

Suppose  $\psi \leq \forall_f(\phi)$ . Consider any  $rx \in R, a \in A$  where

$$x \in \psi(r, fa)$$

holds. To use the assumed comparison we set

$$r' := 1 \quad s' := r \quad x' := x \quad b' := fa \quad a' := ax$$

so that

$$b'x'r' = fax = fa' \quad x' \in \psi(s', b')$$

and hence

$$1 \in \phi(s'x'r', a') = \phi(rx, ax)$$

to give

$$x \in \phi(r, a)$$

as required. ■

This completes the construction of the internal double adjunction. In the remainder of this section we do two things. We show how this internal double adjunction is related to the external double adjunction of Section 6. We show how a particular case of this internal double adjunction leads to that of Section 7.

How is the internal double adjunction

$$\begin{array}{ccc} & \xrightarrow{\exists_f} & \\ PA & \xleftarrow{P(f)} \xrightarrow{\quad} & PB \\ & \xrightarrow{\forall_f} & \end{array}$$

related to the external double adjunction

$$\Omega(A) \begin{array}{c} \xrightarrow{\dot{\exists}_f} \\ \xleftarrow{f^{\leftarrow}} \\ \xrightarrow{\dot{\forall}_f} \end{array} \Omega(B)$$

between the posets of subobjects? (Remember that in this section we use the dot to distinguish the external maps from the internal arrows.) The precise answer to the question is: It isn't. At least it isn't related to this particular case. You will see why shortly.

The arrows internal double adjunction relate elements

$$\phi \in PA \quad \psi \in PB$$

in one direction or the other. These are  $R$ -linear maps

$$R \times A \xrightarrow{\phi} \Omega \quad R \times B \xrightarrow{\psi} \Omega$$

that is, they are characters on

$$R \times A \quad R \times B$$

respectively. As such each classifies a subobject

$$\Phi \in \Omega(R \times A) \quad \Psi \in \Omega(R \times B)$$

and it is to the instance

$$\Omega(A) \begin{array}{c} \xrightarrow{\dot{\exists}_{id \times f}} \\ \xleftarrow{(id \times f)^{\leftarrow}} \\ \xrightarrow{\dot{\forall}_{id \times f}} \end{array} \Omega(B)$$

of the external double adjunction that the internal one is related.

Starting from one side or the other we are going to follow the various paths

$$\begin{array}{ccc} [R \times A, \Omega] & & [R \times B, \Omega] \\ \phi \downarrow & & \psi \downarrow \\ \Phi & & \Psi \\ \dot{\exists} \downarrow & & \dot{\forall} \downarrow \\ \Omega(R \times A) & \begin{array}{c} \xrightarrow{\dot{\exists}} \\ \xleftarrow{(\cdot)^{\leftarrow}} \\ \xrightarrow{\dot{\forall}} \end{array} & \Omega(R \times B) \end{array}$$

to produce three functions across the top. We will find that these are exactly the functions

$$\exists_f \dashv P(f) \dashv \forall_f$$

of the internal double adjunction. (For convenience, we have omitted some of the details of the lower functions.)

We will need to know how to translate between character and subobject on either side. Let's write down all the details for easy reference.

$$\begin{aligned}
(r, a) \in \Phi &\iff 1 \in \phi(r, a) & (s, b) \in \Psi &\iff 1 \in \psi(s, b) \\
\phi(r, a) &= \Phi : (r, a) & \psi(s, b) &= \Psi : (s, b) \\
x \in \phi(r, a) &\iff (rx, ax) \in \Phi & x \in \psi(s, b) &\iff (sx, bx) \in \Psi
\end{aligned}$$

Here, as usual,  $r, s, x \in R, a \in A, b \in B$ .

We have, more or less, already seen how the central map  $(id \times f)^{\dashv}$  produces the arrow  $P(f)$ . However, to illustrate the general method let's look at the details.

**8.7 LEMMA.** *Consider a pair*

$$\phi \iff \Phi \quad \Psi \iff \psi$$

*of gadgets, as above. If  $\Phi = (id \times f)^{\dashv}(\Psi)$  then  $\phi = P(f)(\psi)$ .*

**Proof.** For each  $x, r \in R, a \in A$  the various correspondences give

$$x \in \phi(r, a) \iff (rx, ax) \in \Phi \iff (rx, fax) \in \Psi \iff x \in \psi(r, fa)$$

which leads to the required result. ■

We will use the same technique to generate both  $\exists_f, \forall_f$ . Of course, we now go in the other direction, and there will be a bit more content in the calculations.

**8.8 THEOREM.** *Consider a pair*

$$\phi \iff \Phi \quad \Psi \iff \psi$$

*of gadgets, as above. Then both*

- *If  $\Psi = \dot{\exists}_{id \times f}(\Phi)$  then  $\psi = \exists_f(\phi)$*
- *If  $\Psi = \dot{\forall}_{id \times f}(\Phi)$  then  $\psi = \forall_f(\phi)$*

*hold.*

**Proof.** We look at the two cases in turn.

( $\exists$ ) For each  $s, x \in R, b \in B$  we have

$$\begin{aligned}
x \in \psi(s, b) &\iff (sx, bx) \in \Psi \\
&\iff (\exists(r, a) : R \times A)[((r, fa) = (sx, bx)) \wedge (r, a) \in \Phi] \\
&\iff (\exists r : R, a : A)[(r = sx) \wedge (fa = bx) \wedge (r, a) \in \Phi] \\
&\iff (\exists a : A)[(bx = fa) \wedge 1 \in \phi(sx, a)] &\iff x \in \exists_f(\phi)(s, b)
\end{aligned}$$

as required. The crucial second equivalence uses the construction of  $\dot{\exists}_{id \times f}(\Phi)$ .

( $\forall$ ) For each  $s, x \in R, b \in B$  we have

$$\begin{aligned}
x \in \psi(s, b) &\iff (sx, bx) \in \Psi \\
&\iff (\forall(t, a) : R \times A, r : R)[(t, fa) = (sx, bx)r \implies (t, a) \in \Phi] \\
&\iff (\forall t, r : R, a : A)[(t = sxr) \wedge (fa = bxr) \implies (sxr, a) \in \Phi] \\
&\iff (\forall r : R, a : A)[(bxr = fa) \implies 1 \in \phi(sxr, a)] &\iff x \in \forall_f(\phi)(s, b)
\end{aligned}$$



This shows how the central arrow of the internal quantifier double adjunction arises. The same idea with the technique of Theorem 8.8 will produce the other two arrows.

Starting from an arbitrary  $\phi \in PA$  we follow the two paths

$$\begin{array}{ccc}
 [R \times A, \Omega] & & [R \times 1, \Omega] \longrightarrow \Omega \\
 \phi \downarrow & & \psi \dashv \longrightarrow I \\
 \Phi & & \Psi \\
 \Omega(R \times A) \xrightarrow{\exists} \Omega(R \times 1) & & \Omega(R \times A) \xrightarrow{\forall} \Omega(R \times 1)
 \end{array}$$

to obtain some idea  $I \in \Omega$ .

**8.9 THEOREM.** *Consider gadgets  $\phi, \Phi, \Psi, \psi, I$  as in the diagram above. Both*

- *If  $\Psi = \exists_{id \times !A}(\Phi)$  then  $I = \exists_A(\phi)$*
- *If  $\Psi = \forall_{id \times !A}(\Phi)$  then  $I = \forall_A(\phi)$*

*hold.*

**Proof.** We look at the two cases on turn, and we use some of the details of the proof of Theorem 8.8 Remember also that  $\star x = \star$  and  $\star$  is the only value returned by  $f$ .

( $\exists$ ) For each  $x \in R$  we have

$$\begin{aligned}
 x \in I &\iff x \in \psi(1, \star) \\
 &\iff (x, \star x) \in \Psi \\
 &\iff (\exists a : A)[(\star x = fa) \wedge 1 \in \phi(x, a)] \\
 &\iff (\exists a : A)[1 \in \phi(x, a)] &\iff x \in \exists_A(\phi)(s, b)
 \end{aligned}$$

as required.

( $\forall$ ) For each  $x \in R$  we have

$$\begin{aligned}
 x \in I &\iff x \in \psi(1, \star) \\
 &\iff (x, \star x) \in \Psi \\
 &\iff (\forall r : R, a : A)[(\star xr = fa) \implies 1 \in \phi(xr, a)] \\
 &\iff (\forall r : R, a : A)[1 \in \phi(xr, a)] &\iff x \in \forall_A(\phi)(s, b)
 \end{aligned}$$

as required. ■

It all seems to make sense now doesn't it?

## 9 Some uses of the internal language

[To be written]