The point-free approach to sheafification

Harold Simmons

Department of Computer Science, The University, Manchester, England
hsimmons@cs.man.ac.uk

Abstract

Let Ω be an arbitrary frame, and consider the category $\mathsf{Psh}(\Omega)$ of presheaves on Ω and the and the category $\mathsf{Set}(\Omega)$ of Ω-sets. There are various other associated categories and functors between them. I describe these with emphasis on the functor which separates a presheaf and the functor that sheafifies a presheaf.

These are notes from a short course (of 6 hours) given in August and September 2001. They are little more than a first draft, and may contain some errors. *In particular the notes contain some comments in this kind of type face indicating that there is something still to be done or sorted out.*

If you find these notes useful or think there are parts that could be improved, please feel free to contact me.

This version was compiled on September 21, 2001.

Contents

1 Introduction

I assume you are already familiar with point-sensitive sheaves. If you are not, you can still read these notes, but you may miss some of the reasons for doing certain things.

I will remind you of the salient facts. I will, of course, assume that you know what the words mean.

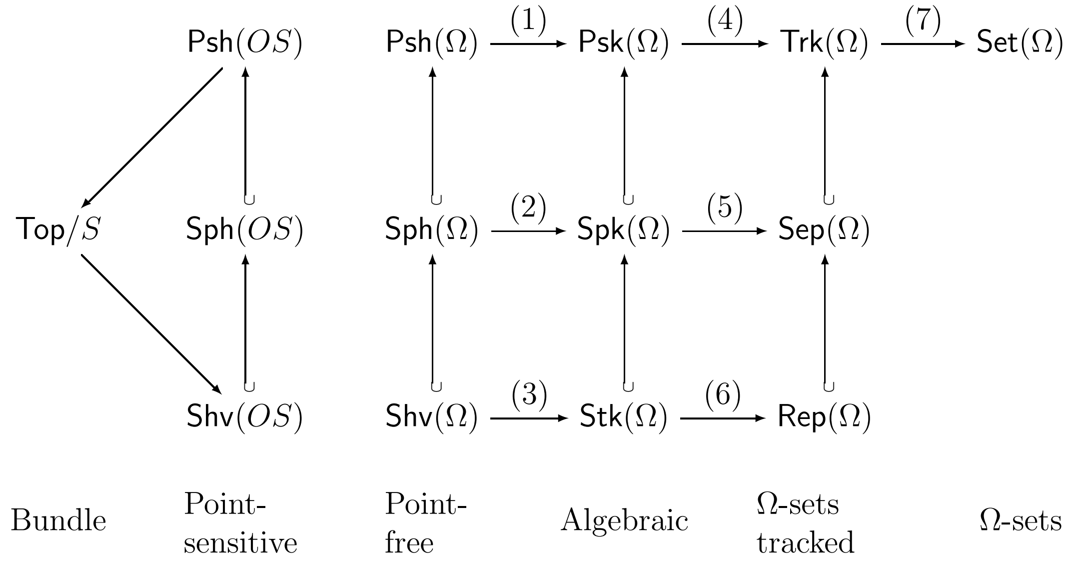


Table 1: The various categories

Let Top be the category of topological spaces and continuous maps. Given such a space S we can form the comma category Top/S of S-bundles, continuous maps $R \longrightarrow S$ from an arbitrary space to the fixed base space S. The space S carries its topology OS which we may view as a poset, and hence as a rather trivial kind of category. Each bundle determined a presheaf on OS (that is, a Set-valued contravariant functor on OS). In fact, the presheaves that arise in this way are rather special; they are sheaves. It turns out that this notion can be defined entirely in terms of OS without any reference to the points S.

Take a look at the left hand part of Table 1. There we find four categories

$$\mathsf{Top}/S \quad \mathsf{Psh}(OS) \quad \mathsf{Sph}(OS) \quad \mathsf{Shv}(OS)$$

connected by four functors two of which are inclusions. We have met Top/S already. Of the others $\mathsf{Psh}(OS)$ and $\mathsf{Shv}(OS)$ are, respectively, the category of presheaves and the category of sheaves where the arrows are just natural transformations. The functor $\mathsf{Top}/S \longrightarrow \mathsf{Shv}(OS)$ is the obvious way of converting a bundle into a sheaf. There is a less obvious functor $\mathsf{Psh}(OS) \longrightarrow \mathsf{Top}/S$ which converts a presheaf into its bundle of stalks. The interesting fact is that the composite of these two functors provides a reflection of presheaves into sheaves, it is the sheafification functor.

The intermediate category $\mathsf{Sph}(OS)$ of separated presheaves is not often isolated explicitly, but it is instructive if we do so here.

As mentioned above the sheaves can be isolated within presheaves without mention of the points of S. So why do we need the base space S?

Recall that a frame Ω is a certain kind of complete lattice. In particular, it has suprema $\bigvee X$ for all subsets $X \subseteq \Omega$, and binary infima $x \wedge y$ for all $x, y \in \Omega$. The crucial feature is that it satisfies the Frame Distributive Law

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

for all $X \subseteq \Omega$ and $a \in \Omega$. Each topology OS is a frame, but there are many other examples. For instance, each complete boolean algebra is a frame, and may not look anything like a power set.

Let Ω be any frame. The definitions of presheaf, separated presheaf, and sheaf can be set up on Ω. (In fact, they can be set up on any category provided this is furnished with certain extra facilities to form a site.) There is no problem forming the categories $\mathsf{Psh}(\Omega), \mathsf{Sph}(\Omega), \mathsf{Shv}(\Omega)$ and the functorial inclusions to form the left hand column of the right hand block of the table. When $\Omega = OS$ this column is exactly the point-sensitive column to its left.

What is not so clear is that $\mathsf{Shv}(\Omega)$ is reflective in $\mathsf{Psh}(\Omega)$. These notes show that it is and indicate the crucial role played by the separated presheaves.

There are several ways to construct the reflector. The method we use here (and which is the whole purpose of these notes) arose in an area which, on the face of it, has nothing whatsoever to do with sheaves.

Non-standard models of set theory are often constructed using a rather intricate and combinatorial technique know as forcing. In order to understand this Scott reworked the various constructions using boolean valued set. Here we mimic the construction of the Zermelo hierarchy with the real world set of truth values (true and false) replaced by an arbitrary complete boolean algebra. (Given a forcing construction the appropriate boolean algebra can be obtained from the poset of forcing conditions.) This idea was reworked by Higgs into a categorical format in [5], where he describe the category of boolean valued sets.

In fact, Higgs did more. He did not merely work over some arbitrary complete boolean algebra, but over an arbitrary frame Ω. Thus he produced the category $\mathsf{Set}(\Omega)$ of Ω-valued sets appearing at the top right of the table. The surprising outcome is that the two categories $\mathsf{Shv}(\Omega)$ and $\mathsf{Set}(\Omega)$ are equivalent, and so we have a connection between a universe of generalized sets as Zermelo might have perceived and a universe of varying sets as exemplified by the notion of a topos and promoted by Lawvere. (As pointed out by Higgs, this connection was in the air before [5] was written, but the full details had not been set down before.)

These notes are base on the published article [4], especially chapter II, and the hand-written notes [3]. I show how the construction of the sheafification functor can be reduced to an undergraduate topic (if ever that is needed). The idea is to move from the point-free column of the table through two more columns of categories to finish up at $\mathsf{Set}(\Omega)$. At each step to the right we reformulate the notions to simplify the ideas and hide the irrelevant features. This gives some insight into the nature of sheaves and into the world of $\mathsf{Set}(\Omega)$.

As well as the six inclusions, the right hand part of the table contains seven functors (1) — (7) which we will construct (almost in this order). We will find that each of (1), (2), (3), (5), and (6) is an equivalence (in fact, an isomorphism), but (4) and (7) are not.

We work relative to some arbitrary (but fixed) frame Ω. For much of the time this could be any complete lattice, but every now and then we invoke the FDL. These crucial steps will be indicated in this manner.

The three articles [3], [4], [5] provide many more details and several examples of the use of $\mathsf{Set}(\Omega)$. A concise introduction to sheaves is given in pages 169 – 175 of [2], and [1] chapter 3 is a rather careful account of point-free sheaves. More advanced material can be found in [6].

2 Presheaves and sheaves

As indicated in the introduction we fix some frame Ω, which can be quite arbitrary, and work relative to this.

To begin let's recall some basic notions.

2.1 DEFINITION. A presheaf on Ω is a family

$$A(x)$$

of sets indexed by the elements $x \in \Omega$, together with a family

$$A(x) \xrightarrow{A(y,x)} A(y)$$

of functions indexed by the comparisons $y \leq x$ in Ω. These functions must satisfy

$$A(x,x) = id_{A(x)} \qquad A(z,x) = A(z,y) \circ A(y,x)$$

for all $z \leq y \leq x$ in Ω.

We often write $A(\cdot)$ to indicate such a presheaf, and we write

$$\begin{array}{ccc} A(x) & \xrightarrow{A(y,x)} & A(y) \\ a & \longmapsto & a|y \end{array}$$

for the connecting maps. We think of these as 'restriction maps'. ■

In other words, a presheaf on Ω is just a Set-valued contravariant functor on the poset Ω (viewed as a category). In particular, this definition makes sense for any poset Ω. However, some of the next notions make use of the completeness of Ω.

2.2 DEFINITION. Let $A(\cdot)$ be a presheaf.

(a) A is separated if for each $X \subseteq \Omega$, and each $a, b \in A(\bigvee X)$,

$$(\forall x \in X)[a|x = b|x] \Longrightarrow a = b]$$

holds.

(b) Let $X \subseteq \Omega$. An X-selection is a family of elements

$$a(x) \in A(x)$$

indexed by $x \in X$. Such a selection is coherent if

$$a(x)|(x \wedge y) = a(y)|(x \wedge y)$$

holds for all $y, x \in X$.

(c) A is collated if for each $X \subseteq \Omega$, and each coherent X-selection $a(\cdot)$, there is at least one $a \in A(\bigvee X)$ with

$$a(x) = a|x$$

for each $x \in X$. We call such an a a collation of $a(\cdot)$.

(d) A is a sheaf if it is both separated and collated. ■

Before we go any further a selection of examples won't go amiss.

2.3 EXAMPLES. (a) The standard example of a preseaf is formed by the sections of a continuous map. Let S be any space and consider its topology $\Omega = OS$. Let

$$R \xrightarrow{\ \phi\ } S$$

be any continuous map. (In this context, such a map is often called a bundle over S.) For each $U \in \Omega$ a U-section of ϕ is a continous map

$$\sigma : U \longrightarrow T$$

such that the composite $\phi \circ \sigma$ is just the insertion of U into S. Let $A(U)$ be the set of all such U-sections, and for $V \subseteq U$ from Ω let

$$\begin{array}{rcl} A(U) & \xrightarrow{\ A(V,U)\ } & A(V) \\ \sigma & \longmapsto & \sigma|V \end{array}$$

be the actual restriction map. A few simple calculations shows that this is a sheaf.

(b) Let S be any space with topology $\Omega = OS$. For each $U \in \Omega$ let

$$B(U) = \text{the set of bounded, continuous maps } U \longrightarrow \mathbb{R}$$

with the obvious restriction functions. This produces a presheaf on Ω which is separated, but need not be collated.

(c) Each frame Ω can carry many presheaves. Here are three special ones. For each $a \in \Omega$ let $\mathcal{L}(a)$ be the set of lower sections X of Ω with $X \subseteq \downarrow a$. In particular $\mathcal{L}(\top)$ is the set $\mathcal{L}\Omega$ of all lower sections of Ω. For each $a \in \Omega$ let

$$\Omega(a) = \mathcal{L}(a) \qquad \Omega\langle a\rangle = \{X \in \mathcal{L}(a) \mid \bigvee X = a\} \qquad \Omega[a] = \downarrow a$$

(so that $\Omega[a] \subseteq \Omega\langle a\rangle \subseteq \Omega(a)$). For $b \leq a$ (from Ω) we use

$$\begin{array}{rcl} \Omega(a) & \longrightarrow & \Omega(b) \\ X & \longmapsto & X|b \end{array} \qquad \begin{array}{rcl} \Omega\langle a\rangle & \longrightarrow & \Omega\langle b\rangle \\ X & \longmapsto & X|b \end{array} \qquad \begin{array}{rcl} \Omega[a] & \longrightarrow & \Omega[b] \\ x & \longmapsto & x \wedge b \end{array}$$

where

$$X|b = \{x \in X \mid x \leq b\} = \{x \wedge b \mid x \in X\} = X \wedge b$$

to obtain three presheaves. Here we find that $\Omega(\cdot)$ is collated but need not be separated, whereas $\Omega[\cdot]$ is a sheaf. ■

As remarked above the presheaves on Ω are just the Set-valued contravariant functors on Ω. These form the objects of a category where the arrows are the corresponding natural transformations. The sheaves on Ω give a full subcategory, and there is the intermediate category of separated presheaves. Let's give these names.

2.4 DEFINITION. Let $\mathsf{Psh}(\Omega), \mathsf{Sph}(\Omega$, and $\mathsf{Shv}(\Omega)$ be, respectively, the category of presehaves, the category of separated presheaves, and the category of sheaves, all on Ω and with natural transformations as arrows. ■

Thus we have two full insertions

$$\mathsf{Shv}(\Omega) \hookrightarrow \mathsf{Sph}(\Omega) \hookrightarrow \mathsf{Psh}(\Omega)$$

and we wish to analyse the relationships between these categories. This, of course, is the left hand column of the right part of Table 1.

The first important fact is that $\mathsf{Shv}(\Omega)$ is reflective in $\mathsf{Psh}(\Omega)$. In other words, the inclusion has a left adjoint

$$\mathsf{Psh}(\Omega) \rightleftarrows \mathsf{Shv}(\Omega)$$

usually called the sheafification functor.

When Ω is spatial, that is when $\Omega = OS$ for some space S, we find there an adjunction

$$\mathsf{Psh}(\Omega) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathsf{Top}/S$$

using the comma category Top/S of bundles over S. Furthermore, the composite $\Lambda \circ \Gamma$ provides the sheafification functor.

When Ω is not spatial this method is not available. In fact, even in the spatial case, for some purposes the bundle structure of a presheaf is not important and is a distraction. (The are other situation when the bundle structure is the central object of interest, and the functorial aspect are a way of analysing this.)

For a general Ω sheafification can be achieved by at least two different methods which are quite elementary, but rather messy, and in one case a bit mysterious. Here we will look at a third method which is much cleaner and provide new insight into the nature of sheaves.

One method is a bit like the bundle construction where Set/S is replaced by some point-free analogue. This is described in [6], pages 524 and 525, Exercises 9, 10, 11, 12. With hindsight it seems there might be some connection with that construction and the one described here. However, I am not aware of such an analysis. It is possible that section 1.7 of [1] might help.

[`Another look at the GvdB paper might be worthwhile.`]

A second method is certainly related to the one described here, but it has this mysterious component. It is worth remembering some of the salient facts about this construction without going into the details.

(I can't remember seeing an account of sheafification for this simple case. Most accounts deal with a more general situation. The following description is extracted from [6] pages 128 – 134.)

Consider any presheaf $A(\cdot)$. To sheafify A we need to separate it and then fill in the missing collations. Consider any $u \in \Omega$, and let $\langle u \rangle$ be the set of subsets $X \subseteq \Omega$ with $\bigvee X = u$. In other words, $\langle u \rangle$ is just $\Omega\langle u \rangle$ as described in Example 2.3(c). Let $A\langle u \rangle$ be the set of coherent X-selections for varying $X \in \langle u \rangle$. Thus, as u varies through Ω we obtain all possible coherent X-selections. Notice that A is a sheaf precisely when for each u each member of $A\langle u \rangle$ is realized by exactly one member of $A(u)$. We try to achieve this.

For $a(\cdot), b(\cdot) \in A\langle u \rangle$ indexed by X, Y respectively, we say $a(\cdot)$ and $b(\cdot)$ are equivalent and write $a(\cdot) \sim b(\cdot)$ if

$$(\forall z \in Z)[a(z) = b(z)]$$

for some $Z \in \langle u \rangle$ with $Z \subseteq X \cap Y$. We see that this is an equivalence relation on $A[u]$.

Let

$$A^+(u) = A\langle u \rangle / \sim$$

(the set of $\sim$-blocks). We convert $A^+(\cdot)$ into a presheaf.

Consider $v \le u$ from Ω, and consider $a(\cdot) \in A\langle u \rangle$ indexed by X. Let $Y = X \wedge v \in \langle u \rangle$ and set

$$b(y) = a(y)$$

for $y \in Y$ to obtain $b(\cdot) \in A\langle u \rangle$. (In short, $b(\cdot) = a(\cdot)|Y$.) A few calculations shows that

$$\begin{array}{ccc} A^+(u) & \longrightarrow & A^+(v) \\ a(\cdot)/\sim & \longmapsto & b(\cdot)/\sim \end{array}$$

is well defined and provides the required connecting maps. Furthermore, we find that

$$\begin{array}{ccc} A(u) & \longrightarrow & A^+(u) \\ a(\cdot) & \longmapsto & a(\cdot)/\sim \end{array}$$

is the u-component of a morphism

$$A(\cdot) \xrightarrow{\ \eta\ } A^+(\cdot)$$

from the given presheaf to the constructed presheaf.

All this is straight forward, rather tedious, and not very enlightening.

With a bit more effort we can prove the following.

2.5 LEMMA. *Let A be a presheaf.*

(i) The presheaf A^+ is separated.

(ii) The presheaf A is separated if and only if $A \longrightarrow A^+$ is monic.

(iii) if A is separated then A^+ is collated (and hence is a sheaf).

(iv) The presheaf A is a sheaf if and only if $A \longrightarrow A^+$ is an isomorphism.

This shows us how to sheafify. Given a presheaf A, by (i) A^+ is separated, and by (iii) A^{++} is a sheaf. Furthermore, we have the required universality.

2.6 THEOREM. *For each morphism*

$$A \xrightarrow{\ f\ } B$$

from a presheaf A to a sheaf B, there is a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & A^+ & \end{array}$$

for some unique morphism $f^\sharp$.

What is going on here?

On the face of it there is an obvious way to convert a presheaf A into a sheaf. We first separate it by slicing out an appropriate equivalence relation (similar to the one used in the construction of A^+). Once we have this separated presheaf, we collate it by filling in the missing collations (using the coherent X-selections as representations). This construction seems to be doing a bit of both.

The crucial fact is that $(\cdot)^+$ does *not* reflect a presheaf into a separated presheaf, it also partly collates. There are separated sheaves which are not collated, so the map

$$A(u) \longrightarrow A^+(u)$$

need not be surjective. What happens if we modify the construction of $(\cdot)^+$ so that we use only those coherent X-selections that are already realized. This gives a surjective slice $A(u) \longrightarrow A^\times(u) \subseteq A^+(u)$ of each $A(u)$. It seems 'obvious' that this produces the separated reflection of A, but the proof of this is not immediate.

This poses two questions.

(?) Can we find two construction where the first separates a presheaf and the second collates a separated presheaf? In other words, we are looking for left adjoints to the

$$\mathsf{Psh}(\Omega) \rightleftarrows \mathsf{Sph}(\Omega) \rightleftarrows \mathsf{Shv}(\Omega)$$

two inclusion functors. The composite left adjoints will be the sheafification functor. Of course, we want elementary descriptions of these two constructions, and the description of the composite may look like the functor outline above.

(?) Can we find one elementary construction which immediately converts a presheaf into a sheaf? This, of course, should be simpler than just the composite of two constructions.

In these notes we will see that both of these can be done. Moreover, the traditional way of handling sheafification does not give us much guidance as to how this should be done. The constructions we finally produce are almost routine.

You may be wondering why we bother with the intermediate category of separated presheaves. There are two reasons. Eventually these gadgets become interesting in themselves. We won't get to that stage in these notes, but it does no harm to remember we might someday. The more immediate reason is that looking at separated presheaves does help to explain what is going on.

Exercises

2.1 Show that the presheaf obtained from a bundle, as in Examples 2.3(a), is a sheaf.

Let

$$R \xrightarrow{\ \phi\ } S \qquad T \xrightarrow{\ \psi\ } S$$

be a pair of bundles over a space S, and let $A(\cdot), B(\cdot)$ be the corresponding presheaves on $\Omega = OS$. Show that each commuting triangle as on the left

$$\begin{array}{ccc} R & \xrightarrow{\ \theta\ } & T \\ & \searrow_{\phi} \quad {}_{\psi}\swarrow & \\ & S & \end{array} \qquad\qquad A(\cdot) \longrightarrow B(\cdot)$$

where θ is continuous induces a presheaf morphism as on the right.

Does every presheaf morphisms $A(\cdot) \longrightarrow B(\cdot)$ arise in this way? [*What is the answer*]

2.2 Show that Example 2.3(b) does produce a separated presheaf which need not be collated.

Can you find a necessary and sufficient on the space S for this preseheaf to be collated?

2.3 Consider the three presheaves $\Omega(\cdot), \Omega\langle\cdot\rangle, \Omega[\cdot]$ of Examples 2.3(c).

(a) Show that, in fact, all three are presheaves.

(b) Show that if $\Omega(\cdot)$ is separated then each element of Ω is compact.

(c) Show that $\Omega(\cdot)$ is collated.

(d) Show that $\Omega[\cdot]$ is a sheaf.

(e) Analyse the separation and collation properties of $\Omega\langle\cdot\rangle$.

(f) Show that

$$\begin{array}{ccc} \Omega(a) & \longrightarrow & \Omega[a] \\ X & \longmapsto & \bigvee X \end{array}$$

gives a presheaf morphism.

(g) Can you find a universal property that this morphism has?

(h) How is the presheaf $\Omega\langle\cdot\rangle$ related to the other two?

2.4 Consider the construction $A \longrightarrow A^{\times}$ suggested towards the end of this section. Decide whether or not this does produce the separated reflection.

3 Prestacks and presheaves

Our first job is to reformulate the notion of a presheaf to hide its inner workings but to expose its essential properties. This will give us three categories which are essentially the same as $\mathsf{Psh}(\Omega), \mathsf{Sph}(\Omega), \mathsf{Shv}(\Omega)$ but which are easier to deal with, for they are more 'algebraic' and they carry around less clutter.

A display over Ω is just a function

$$A \xrightarrow{[\![\cdot]\!]} \Omega$$

from an arbitrary set. We sometimes say $[\![\cdot]\!]$ displays or indexes the set A over Ω. (This notation might look a bit excentric, but it will make sense shortly.)

We can turn such displays into a category. A morphism

$$(A, [\![\cdot]\!]) \xrightarrow{f} (B, [\![\cdot]\!])$$

between two such displays is a function

$$A \xrightarrow{f} B$$

such that

$$[\![fa]\!] = [\![a]\!]$$

for each $a \in A$. In other words, the obvious triangle commutes. (Naturally, I assume you are capable of understanding such an identity without having to decorate the two different displays.)

Each display (as above) gives us a block decomposition

$$\mathcal{A} = (A(x) \mid x \in \Omega)$$

of A where

$$a \in A(x) \Longleftrightarrow [\![a]\!] = x$$

(for $a \in A, x \in \Omega$). Conversely, given any Ω-indexed family $\mathcal{A}$ of sets (as above) we may take the disjoint union

$$A = \coprod \mathcal{A}$$

to create a display.

These two constructions are inverse to each other (up to natural equivalence).

3.1 DEFINITION. A prestack on Ω is a display with a compatible action

$$\begin{array}{ccc} A & \xrightarrow{\text{extent}} & \Omega \\ a & \longmapsto & [\![a]\!] \end{array} \qquad \begin{array}{ccc} A, \Omega & \xrightarrow{\text{action}} & A \\ a, x & \longmapsto & ax \end{array}$$

where

$$a[\![a]\!] = a \qquad [\![ax]\!] = [\![a]\!] \wedge x \qquad (ax)y = a(x \wedge y)$$

for all $a \in A, x, y \in \Omega$.

A prestack morphism

$$A \xrightarrow{\ f\ } B$$

between two prestacks is a function (as indicated) such that

$$[\![fa]\!] = [\![a]\!] \qquad f(ax) = (fa)x$$

(for $a \in A, x \in \Omega$). ■

The reason for the name 'extent' will become clear.

It is entirely routine to show that prestacks and prestack morphisms form a category. Let's formally give this a name.

3.2 DEFINITION. Let $\mathsf{Psk}(\Omega)$ be the category of prestacks and prestack morphism, both over Ω.

This gives us the top component of the central column of the right hand part of Table 1. Why is this category interesting? Because it gives us a more amenable version of the category of presheaves on Ω.

3.3 THEOREM. *The two categories* $\mathsf{Psh}(\Omega)$ *and* $\mathsf{Psk}(\Omega)$ *are canonically equivalent.*

Proof. Given a presheaf $A(\cdot)$ over Ω we first display it

$$A \xrightarrow{\llbracket \cdot \rrbracket} \Omega$$

in the usual way, so that

$$\llbracket a \rrbracket = x \iff a \in A(x)$$

(for $a \in A, x \in \Omega$). To give this an action consider any $a \in A, x \in \Omega$, and set

$$ax = a|y = A(y,x)(a) \qquad \text{where } y = \llbracket a \rrbracket \wedge x$$

so that

$$\begin{array}{ccc} A(\llbracket a \rrbracket) & \xrightarrow{A(y, \llbracket a \rrbracket)} & A(y) \\ a & \longmapsto & ax \end{array}$$

shows where these various elements live. In particular, we have $ax \in A(y)$ and hence

$$\llbracket ax \rrbracket = y = x \wedge \llbracket a \rrbracket$$

as required. Taking any $x = \llbracket a \rrbracket$ we have $y = \llbracket a \rrbracket$, so that

$$a\llbracket a \rrbracket = A(y,y)(a) = a$$

to give the two left hand axioms.

For any $a \in A$ and $x, y \in \Omega$ let

$$u = \llbracket a \rrbracket \qquad v = \llbracket ax \rrbracket = u \wedge x \qquad w = \llbracket axy \rrbracket = u \wedge x \wedge y$$

so that traipsing round

$$\begin{array}{ccccc} A(u) & & \xrightarrow{A(w,u)} & & A(w) \\ & {\scriptstyle A(v,u)} \searrow & & \nearrow {\scriptstyle A(w,v)} & \\ & & A(v) & & \end{array}$$

we get

$$(ax)y = A(w,v)(ax) = (A(w,v) \circ A(v,u))(a) = A(w,u)(a) = a(x \wedge y)$$

as required for the last axiom.

This shows how to convert each presheaf into a prestack. A few more routine calculations shows that the construction is functorial.

Conversely, given a prestack (A, α) we use

$$a \in A(x) \iff \llbracket a \rrbracket = x$$

(for $x \in \Omega, a \in A$) to extract subsets of A. For $y \leq x$ (in Ω) and $a \in A(x)$ we have

$$\llbracket ay \rrbracket = \llbracket a \rrbracket \wedge y = x \wedge y = y$$

so that

$$\begin{array}{ccc} A(x) & \longrightarrow & A(y) \\ a & \longmapsto & ay \end{array}$$

is a connecting function. We check that this converts A into a presheaf in a functorial fashion.

Finally, we see that the two constructions are inverse to each other. ■

This sets up the functor (1) of the table. Notice that we have proved something stronger that the formal definition of an equivalence. Some people would say that the two categories are isomorphic. Rather than give the functor a name we will say the categories are 'canonically' equivalent, as in the statment of Theorem 3.3.

Our aim is to describe some of the properties of $\mathsf{Psk}(\Omega)$. In particular, we want to describe the analogues of the separated presheaves and the sheaves, and to set up the required reflections in terms of prestacks. When we extract these subcategories of $\mathsf{Psk}(\Omega)$ we will find that the equivalence (1) restricts to two more equivalences (2) and (3).

The category $\mathsf{Psk}(\Omega)$ has some interesting properties which can be dealt with in detail elsewhere, but a couple are indicated in the Exercises.

Exercises

3.1 Show that $a\top = a$ hold for each element a of a prestack.

[*Are there any more simple identities*]

3.2 For each set Z we may set $A(x) = Z$ with the identity function as the connecting maps to obtain a constant presheaf. Describe this as a prestack.

3.3 Describe each of the presheaves of Examples 2.3(c) as prestacks.

3.4 Let $A(\cdot), B(\cdot)$ be a pair of presheaves with corresponding prestacks A, B, respectively. Show that the two arrow sets

$$\mathsf{Psh}[A(\cdot), B(\cdot)] \qquad \mathsf{Psk}[A, B]$$

are in bijective correspondence.

Complete the proof of Theorem 3.3. In other words, show that the two constructions are functorial.

3.5 (a) Show that $\mathsf{Psk}(\Omega)$ has a final object, and this is just the frame Ω structured in a certain way.

(b) For this final object describe the morphisms $\Omega \longrightarrow A$ to an arbitrary prestack A.

3.6 Show that in $\mathsf{Psk}(\Omega)$ a morphism is monic exactly when it is injective.

4 Separated and collated prestacks

To work with prestacks rather than presheaves we need to know which are separated and which are collated. The separation property is almost obvious.

4.1 DEFINITION. A prestack A is separated if for each $a, b \in A$ and $v \in \Omega, X \subseteq \Omega$ with

$$v \leq \bigvee X \qquad v \leq [\![a]\!] \wedge [\![b]\!]$$

the implication

$$(\forall x \in X)[ax = bx] \Longrightarrow av = bv$$

holds. ■

It doesn't take too long to prove the following.

4.2 LEMMA. *Let A be a prestack with associated presheaf $A(\cdot)$. Then $A(\cdot)$ is separated if and only if A is separated.*

Proof. Suppose first that $A(\cdot)$ is separated and consider any situation

$$v \leq \bigvee X \qquad v \leq u = [\![a]\!] \wedge [\![b]\!]$$

where

$$(\forall x \in X)[ax = bx]$$

(for $a, b \in A$ and $u \in \Omega, X \subseteq \Omega$). Let

$$c = av \qquad d = bv$$

so that

$$[\![c]\!] = [\![a]\!] \wedge v = v \qquad [\![d]\!] = [\![c]\!] \wedge v = v$$

that is $c, d \in A(v)$. Let $Y = X \wedge v$ so that both

$$\bigvee Y = \bigvee X \wedge v = v$$

(by the FDL). For each $y \in Y$, since $y \leq v$, we have

$$cy = avy = a(v \wedge y) = ay \qquad dy = bvy = b(v \wedge y) = by$$

and hence

$$(\forall y \in Y)[cy = dy]$$

(since $Y \subseteq X$). Thus, since $A(\cdot)$ is separated, we have

$$av = c = d = bv$$

as required.

Conversely suppose that A is separated and consider any situation

$$a, b \in A(v) \qquad v = \bigvee X$$

where

$$(\forall x \in X)[ax = bx]$$

(for $a, b \in A$ and $u \in \Omega, X \subseteq \Omega$). Since

$$[\![a]\!] = v = [\![b]\!]$$

and A is separated, we have

$$a = av = bv = b$$

as required. ■

As this proof shows, the definition of a separated prestack is little more than a rephrasing of the corresponding definition of a separated presheaf. There is a bit more in Definition 4.1 but only to make the notion more convenient.

4.3 DEFINITION. Let $\mathsf{Spk}(\Omega)$ be the category of separated prestacks and prestack morphisms. ■

We are now moving down the central column of Table 1, and Lemma 4.2 gives the following.

4.4 THEOREM. *The two categories* $\mathsf{Sph}(\Omega)$ *and* $\mathsf{Spk}(\Omega)$ *are canonically equivalent.*

This sets up the functor (2) of the table. It is just the restriction of (1) to $\mathsf{Sph}(\Omega)$.

The notion of collation is a bit more enthralling.

We are concerned with families $F \subseteq A$ of a given prestack.

4.5 DEFINITION. A family $F \subseteq A$ of a prestack is coherent if

$$a[\![b]\!] = b[\![a]\!]$$

holds for all $a, b \in F$. ■

Notice that if F is coherent in A and $a, b \in F$ have the same extent $x = [\![a]\!] = [\![b]\!]$ then

$$a = ax = bx = b$$

and hence F can be indexed as

$$F = (a(x) \mid x \in X)$$

for some $X \subseteq \Omega$. With this observation we see that coherent families of A is exactly the coherent selections from the corresponding presheaf, and the characterization of the collated prestacks is obvious. However, we don't need to go via the associated presheaf.

4.6 DEFINITION. Each prestack A carries a comparison given by

$$b \leq a \Longleftrightarrow b = a[\![b]\!]$$

(for $a, b \in A$). ■

Notice that

$$b \le a \Longrightarrow [\![b]\!] \le [\![a]\!]$$

but, in general, the converse does not hold.

It is an easy exercise to show that this comparison is a partial ordering. The collation properties of A are concerned with the neture of this poset.

4.7 LEMMA. *A prestack A is collated if and only if each coherent family $F \subseteq A$ has an upper bound.*

Proof. Suppose first that A is collated and consider any coherent family

$$F = (a(x) \mid x \in X)$$

of A. Since A is collated there is some $a \in A$ with $[\![a]\!] = \bigvee X$ and

$$a(x) = ax$$

for each $x \in X$. In particular $a(x) \le a$ for each such x, and hence a is an upper bound for F.

Conversely, suppose the coherent family F, as above, has an upper bound b, say. Thus $a(x) \le b$ to give

$$a(x) = bx$$

for each $x \in X$. Let $u = \bigvee X$, and consider $c = bu$. For each $x \in X$ we have $x \le [\![b]\!]$, to give $u \le [\![b]\!]$, and hence $[\![c]\!] = u$. Also

$$cx = (bu)x = (b(u \wedge x) = bx = a(x)$$

(for each $x \in X$), so that c is a collation of F. ■

The collation of a prestack is concerned with the existence of mere upper bounds for certain families, the coherent families. A completeness property of a poset is concerned with the existence of *least* upper bounds for certain families. Thus as an aside we can strengthen this collation property to a completeness property (and hope that it might be useful).

4.8 DEFINITION. A stack is a prestack which is coherently complete, that is each coherent family has a least upper bound.

Let $\mathsf{Stk}(\Omega)$ be the category of stacks and prestack morphisms. ■

In this section we have set up the central column of Table 1, together with the two functors (1) and (2). We still have to set up the functors (3). To do that we need some further analysis of separation and collation.

Exercises

4.1 Show that the comparison carried by a prestack is a partial ordering.

[*Could do with a few more.*]

5 Completeness properties of a prestack

Each prestack A is a poset and its collation properties are concerned with the existence of certain upper bounds within this poset. Let's investigate this further.

Consider any family $F \subseteq A$ and $b \in A$. As usual, we write

$$F \le b$$

and say F is bounded above by b (or some similar phrase) if $a \le b$ for each $a \in F$. We need various related notions

5.1 DEFINITION. For a family $F \subseteq A$ we say

F is bounded	if	F is bounded above
F is consistent	if	each finite subset of F is bounded
F is 2-consistent	if	each doubleton subset of F is bounded
F is coherent	if	$(\forall a, b \in F)[a[\![b]\!] = b[\![a]\!]]$

respectively ■

This notion of coherence is the same as that introduced in Definition 4.5.

5.2 LEMMA. *For each family $F \subseteq A$ the three implications*

$$F \text{ is bounded} \Longrightarrow F \text{ is consistent} \Longrightarrow F \text{ is 2-consistent} \Longrightarrow F \text{ is coherent}$$

hold.

Proof. Only the last (right hand) implication is not immediate. To prove that consider $a, b \in F$ and suppose $a, b \le c$ for some c. Let $x = [\![a]\!], y = [\![b]\!]$, so that

$$a = cx \qquad b = cy$$

and hence

$$ay = (cx)y = c(x \wedge y) = c(y \wedge x) = (cy)x = bx$$

as required. ■

We are interested in upper bounds, minimal upper bounds, and least upper bounds for certain families. Of course, if a family has a minimal upper bound, then it is certainly bounded. In an arbitrary poset the converse can fail, but not here. Before we prove that we introduce a bit of terminology.

For $F \subseteq A$ we call the subset

$$[\![F]\!] = X = \{[\![a]\!] \mid a \in F\}$$

of Ω the support of F.

5.3 LEMMA. *Suppose $F \le b$ (where $F \subseteq A, b \in A$), and let $u = \bigvee X$ where $X = [\![F]\!]$. Then*

$$F \le bu \le b$$

and bu is a minimal upper bound for F.

Proof. Let $y = [\![b]\!]$. For each $a \in F$ with $x = [\![a]\!]$ we have $a \le b$ so that $a = bx$, and hence $x \le y$. Thus $u \le y$ and $[\![bu]\!] = u$, so that

$$(bu)x = b(u \wedge x) = bx = a$$

and hence

$$F \le bu \le b$$

holds.

Now consider any $F \le c \le bu$. For each $a \in F$ we have

$$x = [\![a]\!] \le [\![c]\!] \le [\![bu]\!] = u$$

and hence $[\![c]\!] = u$ (by taking the supremum in Ω over all $a \in F$). Thus

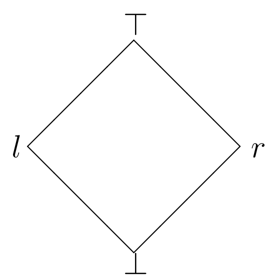
$$c = (bu)[\![c]\!] = bu$$

to show the minimality of bu. ■

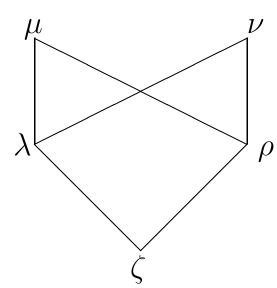
This result with Lemma 4.7 shows that a prestack is collated exactly when each coherent family has a minimal upper bound. However, collation need not produce suprema.

Let's look at a simple example.

5.4 EXAMPLE. Consider the four element boolean algebra $\Omega = \mathcal{P}\{\lambda, \rho\}$ where $\lambda \neq \rho$. We may picture this as shown on the left where $\top = \{\lambda, \rho\}, l = \{\lambda\}, r = \{\rho\}, \bot = \emptyset$.



The base Ω



The prestack A

To produce a presheaf on Ω let $\lambda, \rho, \mu, \nu, \zeta$ be pairwise distinct (where λ and ρ are the same as before). Let

$$A(\top) = \{\mu, \nu\} \quad A(l) = \{\lambda\} \quad A(r) = \{\rho\} \quad A(\bot) = \{\zeta\}$$

with the obvious restriction maps. As a prestack we have

$$\begin{aligned} A &= \{\mu\,, \nu\,, \lambda\,, \rho\,, \zeta\} \\ [\![\cdot]\!] &= \ \top\,, \top\,, l\,, r\,, \bot \end{aligned}$$

where the extent of each element is indicated. By a tedious calculation we find that the associated poset on A is as shown to the right above. In particular, $\{\lambda, \rho\}$ is a coherent family where each of μ and ν is a minimal upper bound.

This exhibits the collations of one coherent family and (by Lemma 5.5 below) shows that A is not separated. There are several other coherent families, but each one contains its collation (where in each of these other cases this is the least upper bound).

Note that the family $\{\mu, \nu\}$ is *not* coherent, since

$$\mu\top = \mu \neq \nu = \nu\top$$

and $[\![\mu]\!] = \top = [\![\nu]\!]$. ■

Collation is concerned with the existence of minimal upper bounds. Of course, it is nicer if these are least upper bounds.

5.5 LEMMA. *A prestack is separated if and only each bound family has a least upper bound.*

Proof. Suppose first that the prestack A is separated. Consider any family $F \subseteq A$, and suppose b is an upper bound for F. By Lemma 5.3 we may suppose that b is a minimal upper bound. We show that this b is the least upper bound.

Let $X = [\![F]\!]$ and let $u = \bigvee X$. We have $x \leq [\![b]\!]$ for each $x \in X$, and hence $u \leq [\![b]\!]$. By Lemma 5.3 we have $F \leq bu \leq b$, and hence $b = bu$ (by the minimality of b).

Consider any upper bound c of F. We require $b \leq c$. We have $F \leq cu \leq c$ and $u \leq [\![b]\!] \wedge [\![c]\!]$. For each $x \in X$ we have $x = [\![a]\!]$ for some $a \in F$, and then

$$bx = a = cx$$

(since $a \leq b, c$). Thus

$$(\forall x \in X)[bx = cx]$$

and hence $b = c$ (since A is separated)

$$b = bu = cu = c[\![b]\!]$$

to give $b \leq c$, as required.

Conversely, suppose that each bounded family has a least upper bound. Consider any $b, c \in A$ and any $X \subseteq \Omega, v \in \Omega$ with

$$v \leq \bigvee X \qquad v \leq y \wedge z \qquad (\forall x \in X)[bx = cx]$$

where $y = [\![b]\!], z = [\![c]\!]$. For each $x \in X$ let

$$a(x) = bx = cx$$

and consider the family $F = \{a(x) \mid x \in X\}$. Since

$$[\![a(x)]\!] = [\![bx]\!] = [\![b]\!] \wedge x = x$$

we have $a(x) \leq b, c$, to give

$$F \leq b \qquad F \leq c$$

and hence, with $u = \bigvee X$, we have

$$F \leq bu \leq b \qquad F \leq cu \leq c$$

where both bu and cu are minimal upper bounds of F, by Lemma 5.3. Both of these are the least upper bound, so that $bu = cu$ to to give $bv = cv$ (since $v \leq u$), as required. ■

Remember (by Definition 4.8) a prestack is a stack if and only if it is coherently complete. What has this got to do with sheaves?

5.6 THEOREM. *A prestack is a stack if and only it corresponds to a sheaf.*

Proof. Suppose first that the prestack A is a stack. In particular, each coherent family has an upper bound (since it has a least upper bound), and hence A is collated by Lemma 4.8. Each coherent family has a least upper bound, so that is the only possible minimal upper bound, and hence A is separated by Lemma 5.5.

Conversely, suppose that A corresponds to a sheaf, and consider any coherent family F. By Lemma 4.8 we see that F has an upper bound, and hence F has a minimal upper bound by Lemma 5.3, and by Lemma 5.5 this is the least upper bound. ■

This sets up the functor (3) of the table. More importantly it shows that we can forget the internal structure of a presheaf, and suggest ways of sheafifying a prestack. We go through some kind of completion process.

Exercises–*To be done*

6 Ω-sets

[*It would be nice to have a name for these that did not include 'Ω'*]

The notion of a prestack hides a lot of the irrelevant inner details of a presheaf and, as we have seen, much can be done with the extent and action. However, when we want to view a prestack as a generalize set this action should not be seen. We can hide it by converting the extent into a binary relation.

6.1 DEFINITION. An Ω-set is a set A furnished with an Ω-valued function

$$A \times A \xrightarrow{[\![\cdot = \cdot]\!]} \Omega$$

which is symmetric and transitive in the sense that both

$$[\![a = b]\!] = [\![b = a]\!] \qquad [\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$$

hold for all $a, b, c \in A$. This is called the equality of A. ■

For each such A and $a \in A$ we let

$$[\![a]\!] = [\![a = a]\!]$$

and call this the extent of a. An element a is global is $[\![a]\!] = \top$ (so, in general there are non-global elements). Notice how the extent function $[\![\cdot]\!]$ displays A over Ω.

Later we will see that many Ω-sets arise naturally from prestacks. But before that let's look at some abstract examples.

6.2 EXAMPLES. The frame Ω can be used to produce several Ω-sets.

(a) Let

$$[\![x = y]\!] = x \wedge y$$

for $x, y \in \Omega$. This gives an Ω-set which, as we will see, arises from a constant prestack. Note that the extent function is just the identity function on Ω.

(b) Let

$$[\![x = y]\!] = (x \leftrightarrow y)$$

for $x, y \in \Omega$. This is an Ω-set which does not arise from any prestack. Note that each element is global.

(c) Let A be the set of ordered pairs (a, e) where $a \le e$ for $a, e \in \Omega$. Let

$$[\![(a,e) = (b,f)]\!] = e \wedge (a \leftrightarrow b) \wedge f$$

for two such pairs. This is an Ω-set where the second projection (from the pairs) is the extent function. ■

Each Ω-set has a couple of associated gadgets, one of which is the extent, and the other isn't.

6.3 DEFINITION. Given an Ω-set A we set

$$[\![a \equiv b]\!] = [\![a]\!] \vee [\![b]\!] \supset [\![a = b]\!] = \big([\![a]\!] \supset [\![a = b]\!]\big) \wedge \big([\![b]\!] \supset [\![a = b]\!]\big)$$

(for $a, b \in A$) to obtain the associated **equivalence** on A. ■

A proof of the following is left as an exercise.

6.4 LEMMA. *For each Ω-set A the equivalence revalues A as an Ω-set in which each element is global. Furthermore*

(i) $[\![a]\!] \wedge [\![a \equiv b]\!] \le [\![b]\!]$

(ii) $[\![a \equiv b]\!] = [\![a]\!] \vee [\![b]\!] \supset [\![a \equiv b]\!]$

(iii) $[\![a = b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!]$

hold for all $a, b \in A$.

Property (ii) shows that we can not iterate this construction. In fact, from the construction we see that the equality and the equivalence agree precisely when each element is global.

Property (iii) shows that the furnishing of an Ω-set can be decomposed into the extent and the equivalence. We can also go the other way.

6.5 LEMMA. *Let $(A, [\![\cdot \equiv \cdot]\!])$ be an Ω-set in which each element is global, and let $[\![\cdot]\!]$ be a display on A which is compatible with $[\![\cdot \equiv \cdot]\!]$ in the sense that (i) and (ii) of Lemma 6.4 hold. Then (iii) furnishes A as an Ω-set with $[\![\cdot]\!]$ as its extent function and $[\![\cdot \equiv \cdot]\!]$ as its equivalence.*

In some ways it is neater to define the notion of an Ω-set in terms of the extent $[\![\cdot]\!]$ and the equivalence $[\![\cdot \equiv \cdot]\!]$. However, it is more common to work with the equality $[\![\cdot = \cdot]\!]$.

We wish to view each prestack as an Ω-set. Each prestack already has an extent, so we need to augment this in some way. The following result looks as though it is an application of Lemmas 6.4 and 6.5, but it's not.

6.6 LEMMA. *Let A be a prestack, and let*

$$[\![a \sim b]\!] = \bigvee\{x \in \Omega \mid ax = bx\} \qquad [\![a = b]\!] = [\![a]\!] \wedge [\![a \sim b]\!] \wedge [\![b]\!]$$

for each $a, b \in A$. Then $[\![\cdot = \cdot]\!]$ structures A as an Ω-set where the two extents agree.

Proof. Trivially, both $[\![\cdot \sim \cdot]\!]$ and $[\![\cdot = \cdot]\!]$ are symmetric.

We show that both $[\![\cdot \sim \cdot]\!]$ and $[\![\cdot = \cdot]\!]$ is transitive. Consider $a, b, c \in A$ and $x, y \in \Omega$ such that

$$ax = bx \qquad by = cy$$

hold. Then

$$a(x \wedge y) = (ax)y = (bx)y = b(x \wedge y) = (by)x = (cy)x = c(x \wedge y)$$

and hence

$$[\![a \sim b]\!] \wedge [\![b \sim c]\!] = \bigvee\{x \wedge y \mid ax = bx, by = cy\} \leq \bigvee\{z \mid az = cz\} = [\![a \sim c]\!]$$

by the FDL. Thus

$$[\![a = b]\!] \wedge [\![b = c]\!] = [\![a]\!] \wedge [\![b]\!] \wedge [\![c]\!] \wedge [\![a \sim b]\!] \wedge [\![b \sim c]\!] \leq [\![a]\!] \wedge [\![c]\!] \wedge [\![a \sim c]\!] = [\![a = c]\!]$$

as required.

Finally, for each $a \in A$ we have $[\![a \sim a]\!] = \top$ so that

$$[\![a]\!] = [\![a = a]\!]$$

and hence for a prestack the given extent and the constructed extent agree. ■

This proof shows that $(A, [\![\cdot \sim \cdot]\!]$ is an Ω-set in which each element is global. Thus, it seems, we could use Lemma 6.5 to move to the Ω-set $(A, [\![\cdot = \cdot]\!]$. Unfortunately, in general $[\![\cdot \sim \cdot]\!]$ is not[1] the equivalence $[\![\cdot \equiv \cdot]\!]$ of the Ω-set.

There are a couple of observations about prestacks that are useful from time to time.

6.7 LEMMA. *Let A be a prestack. Then*

$$\begin{array}{ll} (i) & [\![a = b]\!] = \bigvee\{z \in \Omega \mid z \leq [\![a]\!] \wedge [\![b]\!] \text{ and } az = bz\} \\ (ii) & [\![a = b]\!] \leq [\![a \sim b]\!] \leq [\![a \equiv b]\!] \\ (iii) & [\![ax = by]\!] = [\![a = b]\!] \wedge (x \wedge y) \end{array}$$

hold for all $a, b \in A$ and $x, y \in \Omega$.

Proof. (i) We have

$$[\![a = b]\!] = [\![a]\!] \wedge \bigvee\{z \in \Omega \mid az = bz\} \wedge [\![b]\!]$$

so the identity follows by a two uses of the FDL.

(ii) We have $[\![a = b]\!] \leq [\![a \sim b]\!]$, by definition, so it suffices to verify the right hand comparison. Consider any $x \in \Omega$ with $ax = bx$. Then

$$[\![a]\!] \wedge x = [\![ax]\!] = [\![bx]\!] = [\![b]\!] \wedge x$$

[1] In [3] it is claimed that $[\![\cdot \sim \cdot]\!]$ is the equivalence $[\![\cdot \equiv \cdot]\!]$, but this is false. In [4] this equality is proved for separated stacks. Earlier drafts of these notes confused $[\![\cdot \sim \cdot]\!]$ and $[\![\cdot \equiv \cdot]\!]$. It may be that some of this confusion has slipped through.
There is something to be sorted out here.

so that

$$[\![a]\!] \wedge x \leq [\![b]\!] \qquad [\![b]\!] \wedge x \leq [\![a]\!]$$

and hence

$$[\![a]\!] \wedge [\![a \sim b]\!] \leq [\![b]\!] \qquad [\![b]\!] \wedge [\![a \sim b]\!] \leq [\![a]\!]$$

hold. Thus

$$[\![a]\!] \wedge [\![a \sim b]\!] = [\![a]\!] \wedge [\![a \sim b]\!] \wedge [\![b]\!] = [\![a = b]\!] \qquad [\![b]\!] \wedge [\![a \sim b]\!] = [\![b]\!] \wedge [\![a \sim b]\!] \wedge [\![a]\!] = [\![a = b]\!]$$

and hence

$$[\![a \sim b]\!] \leq ([\![a]\!] \supset [\![a = b]\!]) \wedge ([\![b]\!] \supset [\![a = b]\!]) = [\![a \equiv b]\!]$$

as required.

(iii) It suffices to show

$$[\![ax = b]\!] = [\![a = b]\!] \wedge x$$

(and then use this twice). Let $u = [\![a]\!] \wedge [\![b]\!]$. Since $[\![ax]\!] = [\![a]\!] \wedge x$ we have

$$\begin{aligned} [\![ax = b]\!] &= \bigvee\{ \quad z \quad \mid (ax)z = bz \text{ and } z \leq u \wedge x\} \\ [\![a = b]\!] \wedge x &= \bigvee\{y \wedge x \mid \quad ay = by \quad \text{ and } y \leq u\} \end{aligned}$$

where the lower one follows by the FDL.

Consider any z from the upper collection. Then

$$(ax)z = a(x \wedge z) = az$$

so that $y = z$ belongs to the lower collection, to give

$$[\![ax = b]\!] \leq [\![a = b]\!] \wedge x$$

and we require the converse comparison.

Consider any y from the lower collection, and let $z = y \wedge x$. Then

$$(ax)z = a(x \wedge z) = a(x \wedge y) = (ay)x = (by)x = b(y \wedge x) = bz$$

so that z belongs to the upper collection, and this leads to the required result. ■

You should observe that something can be lost when we pass from a prestack to an Ω-set. Even if we know that an Ω-set arises from a prestack, it may not be possible to retrieve the action. This is because different actions with the same extent can produce the same Ω-set.

[*Set an exercise on this*]

Eventually we will turn this construction from prestack to Ω-set into a functor from $\mathsf{Psk}(\Omega)$ into a category not yet defined.

Exercises

6.1 (a) Show that the Ω-set of Examples 6.2(a) does arise from a prestack.

(b) Show that the construction of Examples 6.2(b) gives an Ω-set which does not arise from any prestack.

(c) Show that the construction of Examples 6.2(c) gives an Ω-set which does also arises from any prestack.

(d) Describe the equivalence $[\![\cdot \equiv \cdot]\!]$ and the auxiliary relation $[\![\cdot \sim \cdot]\!]$ of each of these examples.

6.2 Prove Lemmas 6.4 and 6.5.

Show that

$$[\![a = b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!]$$

holds for all elements a, b of an Ω-set.

6.3 Show that if each element of an Ω-set is global, then it can not arise from a prestack (unless it is empty or Ω is trivial).

6.4 Show that for each prestack A the identities

(i) $[\![a = b]\!] = [\![a]\!] \wedge [\![a \sim b]\!] \wedge [\![b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!]$

(ii) $[\![a \equiv b]\!] = ([\![a]\!] \supset [\![a = b]\!]) \wedge ([\![b]\!] \supset [\![a = b]\!])$

(iii) $[\![a \equiv b]\!] = ([\![a]\!] \supset [\![a \sim b]\!]) \wedge ([\![b]\!] \supset [\![a \sim b]\!]) \wedge ([\![a]\!] \leftrightarrow [\![b]\!])$

hold for all $a, b \in A$.

6.5 Find an example of a prestack where the two relations $[\![\cdot \sim \cdot]\!], [\![\cdot \equiv \cdot]\!]$ are distinct.

6.6 Let A, B be a pair of Ω-sets.

(a) Let

$$A \otimes B = \{(a, b) \mid a \in A, b \in B \text{ with } [\![a]\!] = [\![b]\!]\}$$

and set

$$[\![(a_1, b_1) = (a_2, b_2)]\!] = [\![a_1 = a_2)]\!] \wedge [\![b_1 = b_2)]\!]$$

for such pairs.

Show that this furnishes $A \otimes B$ as an Ω-set.

Suppose each of A, B arises from a prestack. Does $A \otimes B$ arise from a prestack?

(b) Let

$$A \oplus B = \{(a, b) \mid a \in A, b \in B \text{ with } [\![a]\!] \wedge [\![b]\!] = \perp$$

and set

$$[\![(a_1, b_1) = (a_2, b_2)]\!] = [\![a_1 = a_2)]\!] \vee [\![b_1 = b_2)]\!]$$

for such pairs.

Show that this furnishes $A \oplus B$ as an Ω-set.

Suppose each of A, B arises from a prestack. Does $A \oplus B$ arise from a prestack?

[*I don't know the answer to either (a) or (b).*]

7 Ω-morphisms

The construction which converts a prestack into an Ω-set, as described in Lemma 6.6, seems rather canonical, and should be a functor. Before we can verify this we need to make the Ω-sets the obejcts of a category by producing the appropriate arrows. This can be done in two ways to produce a naive category and a more sophisticated category.

7.1 DEFINITION. A tracking morphism (or tracking function)

$$A \xrightarrow{\ f\ } B$$

between Ω-sets A and B is a function f, as indicated, which is extensional and strict in the sense that

$$(e) \quad [\![a_1 = a_2]\!] \le [\![fa_1 = fa_2]\!] \qquad (s) \quad [\![fa]\!] \le [\![a]\!]$$

hold for all $a, a_1, a_2 \in A$. ■

Almost trivially the composite of two tracking morphisms is itself a tracking morphism. Thus we obtain the category which is the top of the right hand column of Table 1.

7.2 DEFINITION. Let $\mathsf{Trk}(\Omega)$ be the category of Ω-sets and tracking morphisms. ■

Of course we need some examples of this notion. These are not hard to find.

7.3 LEMMA. *Each morphism*

$$A \xrightarrow{\ f\ } B$$

between prestacks is extensional and strict, and so is a tracking morphism between the corresponding Ω-sets.

Proof. In terms of the extent $[\![\cdot]\!]$ the given function f satisfies

$$[\![fa]\!] = [\![a]\!] \qquad f(ax) = (fa)x$$

for all $a \in A, x \in \Omega$. In particular, f is strict.

Consider $a_1, a_2 \in A$. For each $x \in \Omega$ with $a_1x = a_2x$ we have

$$(fa_1)x = f(a_1x) = f(a_2x) = (fa_2)x$$

and hence

$$[\![a_1 \sim a_2]\!] \le [\![fa_1 \sim fa_2]\!]$$

(using the auxiliary relation of A). This gives

$$[\![a_1 = a_2]\!] = [\![a_1]\!] \wedge [\![a_2]\!] \wedge [\![a_1 \sim a_2]\!] \le [\![fa_1]\!] \wedge [\![fa_2]\!] \wedge [\![fa_1 \sim fa_2]\!] = [\![fa_1 = fa_2]\!]$$

to show that f is extensional. ■

This with Lemma 6.6 gives us the conversion functor.

7.4 THEOREM. *There is a canonical functor*

$$\mathsf{Psk}(\Omega) \longrightarrow \mathsf{Trk}\Omega)$$

from prestacks to Ω-sets and tracked morphisms.

This is the functor (4) of Table 1. Although we describe it as canonical it is not an equivalence. There are Ω-sets that do not arise from a prestack, and there are tracking morphisms that do not arise from a prestack morphism. [*At least I think there are such tracking morphisms.*] The definition of $\mathsf{Trk}(\Omega)$ is just a bit too loose, but we will soon tighten this. Although $\mathsf{Trk}(\Omega)$ looks like the obvious category to analyse, it turns out to be the wrong one. It is little more that a stepping stone to the right category, which has some unusual features.

7.5 DEFINITION. An Ω-morphism (or simple a morphism)

$$A \xrightarrow{\quad F \quad} B$$

between a pair A, B of Ω-sets is a function

$$B \times A \xrightarrow{\quad F \quad} \Omega$$

which is extensional, functional, and total in the sense that

(e) $[\![b_2 = b_1]\!] \wedge F(b_1, a_1) \wedge [\![a_1 = a_2]\!] \leq F(b_2, a_2)$

(f) $F(b_1, a) \wedge F(b_2, a) \leq [\![b_1 = b_2]\!]$

(t) $[\![a]\!] = \bigvee\{F(b, a) \mid b \in B\}$

hold for all $a, a_1, a_2 \in A$ and $b_1, b_2 \in B$. In particular

$$F(b, a) \leq [\![b]\!] \wedge [\![a]\!]$$

for all $a, \in A$ and $b \in B$. ■

With this definition it is not at all clear that we obtain a category of Ω-sets, nor even how we form the composites of two morphism. Eventually we will explian this, but before that let's look at an example which partly explains what is going on.

7.6 LEMMA. *For each tracking morphism*

$$f : A \longrightarrow B$$

between two Ω-sets the assignment

$$\begin{array}{rcl} B \times A & \xrightarrow{\quad F \quad} & \Omega \\ (b, a) & \longmapsto & [\![b = fa]\!] \end{array}$$

is an Ω-morphism.

Proof. We check the required conditions (e,f,t) in turn.
(e) For all $a, a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have

$$[\![b_2 = b_1]\!] \wedge [\![b_1 = fa_1]\!] \wedge [\![a_1 = a_2]\!] \leq [\![b_2 = b_1]\!] \wedge [\![b_1 = fa_1]\!] \wedge [\![fa_1 = fa_2]\!] \leq [\![b_2 = fa_2]\!]$$

where the first comparison holds by the extionality of f, and the second holds by the transitivity of equality.

(f) For all $a \in A$ and $b_1, b_2 \in B$ we have

$$[\![b_1 = fa]\!] \wedge [\![b_2 = fa]\!] = [\![b_1 = fa]\!] \wedge [\![fa = b_2]\!] \leq [\![b_1 = b_2]\!]$$

by the properties of equality.

(t) The two given properties of f ensure that $[\![fa]\!] = [\![a]\!]$ for each $a \in A$. Also, for $b \in B$, we have

$$F(b, a) = [\![b = fa]\!] \leq [\![fa]\!] = [\![a]\!]$$

to verify one of the required comparisons. The other comparison follows by taking the particular case $b = fa$. ■

We often say that f tracks F (or F is tracked by f). This gives us plenty of examples of morphisms, but we still have a bit of work to do.

7.7 DEFINITION. Let

$$A \xrightarrow{\quad F \quad} B \qquad B \xrightarrow{\quad G \quad} C$$

be a pair of morphism. The Ω-valued function

$$K : C \times A \longrightarrow \Omega$$

given by

$$K(c, a) = \bigvee \{G(c, b) \wedge F(b, a) \mid b \in B\}$$

(for $a \in A, c \in C$) is called the composite $G \circ F$ of F before G. ■

Of course, a mere definition isn't enough.

7.8 LEMMA. *The composite of two Ω-morphisms is itself an Ω-morphism.*

Proof. Let us continue with the notation of Definition 7.7. so we must check that $K = G \circ F$ is extensional, functional, and total.

(e) For $a_1, a_2 \in A, c_1, c_2 \in C$ let

$$v = [\![c_2 = c_1]\!] \wedge K(c_1, a_1) \wedge [\![a_1 = a_2]\!]$$

so that

$$\begin{aligned} v &= \bigvee \left\{ [\![c_2 = c_1]\!] \wedge G(c_1, b) \wedge F(b, a_1) \wedge [\![a_1 = a_2]\!] \mid b \in B \right\} \\ &\leq \bigvee \left\{ G(c_2, b) \wedge F(b, a_2) \mid b \in B \right\} \qquad = \quad (G \circ F)(c_2, a_2) \end{aligned}$$

as required. The first equality uses the FDL, and the comparison uses the given properties of F and G.

(f) For $a \in A, c_1, c_2 \in C$ the FDL gives

$$K(c_1, a) \wedge K(c_2, a) = \bigvee \{G(c_1, b_1) \wedge F(b_1, a) \wedge G(c_2, b_2) \wedge F(b_2, a) \mid b_1, b_2 \in B\}$$

and the a few manipulations give the required result.

(t) For $a \in A, c \in C$ we have

$$\begin{aligned}
\bigvee\{K(c,a) \mid c \in C\} &= \bigvee\{G(c,b) \wedge F(b,a) \mid b \in B, c \in C\} \\
&= \bigvee\left\{\bigvee\{G(c,b) \mid c \in C\} \wedge F(b,a) \mid b \in B\right\} \\
&= \bigvee\left\{[\![b]\!] \wedge F(b,a) \mid b \in B\right\} \qquad = [\![a]\!]
\end{aligned}$$

to give the required result. You should work out the justifications for each step. ■

With this we have most of the proof of the following.

7.9 THEOREM. *The classes of* Ω*-sets and* Ω*-morphisms form a category* $\mathsf{Set}(\Omega)$.

Proof. Given three morphisms

$$A \xrightarrow{\ F\ } B \qquad B \xrightarrow{\ G\ } C \qquad C \xrightarrow{\ H\ } D$$

we find that both the compounds

$$H \circ G \circ F$$

(when bracketed) unravel to

$$(H \circ G \circ F)(d,a) = \bigvee\{H(d,c) \wedge G(c,b) \wedge F(b,a) \mid c \in C, b \in B\}$$

(for $d \in D, a \in A$), to verify the required associativity. This calculation uses the FDL.

We find that

$$\begin{array}{rcl} A \times A & \longrightarrow & \Omega \\ (a_1, a_2) & \longmapsto & [\![a_1 = a_2]\!] \end{array}$$

acts as the identity arrow on A. In other words the identity morphism on A is tracked by the identity function on A. ■

Our aim is to convert each prestack into an Ω-set in a functorial fashion. With the category $\mathsf{Trk}(\Omega)$ we have already got part of the way. The last crucial component is the following.

7.10 LEMMA. *Let*

$$A \xrightarrow{\ f\ } B \qquad B \xrightarrow{\ g\ } C$$

be a pair of tracking morphisms between Ω*-sets* A, B, C*, and suppose these track the morphisms*

$$A \xrightarrow{\ F\ } B \qquad B \xrightarrow{\ G\ } C$$

respectively. Then the composite $g \circ f$ *tracks the composite morphism* $G \circ F$.

Proof. Let $h = g \circ f$ and $H = G \circ F$. We have

$$H(c,a) = \bigvee\{[\![c = gb]\!] \wedge [\![b = fa]\!] \mid b \in B\}$$

for $a \in A, b \in B, c \in C$. Thus

$$H(c,a) \leq \bigvee\{[\![c = gb]\!] \wedge [\![gb = ha]\!] \mid b \in B\} \leq [\![c = ha]\!]$$

by Lemma 7.6(e). Also, with $b = fa$ we have

$$H(c,a) \geq [\![c = ha]\!] \wedge [\![fa]\!] = [\![c = ha]\!]$$

since

$$[\![c = ha]\!] \leq [\![ha]\!] \leq [\![fa]\!]$$

by Lemma 7.6(s). ■

With this we have the second step in the convertion

7.11 THEOREM. *There is a canonical functor*

$$\mathsf{Trk}(\Omega) \longrightarrow \mathsf{Set}(\Omega)$$

from prestacks with tracking morphism to Ω-sets with Ω-morphisms.

This is just the functor (7) of table 1. With this we have the top rwo of functors of the right hand part of the table. We will see how these help to provide the answers to the questions at the end of section 2. However, there are still some surprises in store.

The functor (7) has an unusual feature. It can convert a tracking function f into an Ω-isomorphism even though f is neither injective nor surjective. The step from $\mathsf{Trk}(\Omega)$ to $\mathsf{Set}(\Omega)$ is rather subtle.

Exercises

7.1 Let $f : A \longrightarrow B$ be a function between two Ω-sets A, B.

(a) Show that f is a tracking morphism if and only if it is operational and total, that is if

$$[\![a_1 \equiv a_2]\!] \leq [\![fa_1 \equiv fa_2]\!] \qquad [\![fa]\!] = [\![a]\!]$$

hold for all $a, a_1, a_2 \in A$.

(b) Show that if

$$F(b,a) = [\![b = fa]\!]$$

(for $a \in A, b \in B$) defines an Ω-morphism $A \longrightarrow B$ then f must be a tracking morphism.

7.2 Show that for each Ω-sets the two Ω-sets $A \otimes B$ and $A \oplus B$ (as constructed in Exercise 6.6) provide the product and coproduct in $\mathsf{Set}(\Omega)$.

[`How does this relate to product and sum in` $\mathsf{Psk}(\Omega)$]

8 Separated Ω-sets

Where have we got to in the construction of Table 1? We can forget the left hand part and concentrate on the right hand part. Of this we have a wedge

$$\begin{array}{ccccc}
\mathsf{Psk}(\Omega) & \longrightarrow & \mathsf{Trk}(\Omega) & \longrightarrow & \mathsf{Set}(\Omega) \\
\uparrow & & & & \\
\mathsf{Spk}(\Omega) & & & & \\
\uparrow & & & & \\
\mathsf{Stk}(\Omega) & & & &
\end{array}$$

and we are missing two corners $\mathsf{Sep}(\Omega)$ and $\mathsf{Rep}(\Omega)$ with the connecting functors.

Both of the top functors have their deficiencies. In particular, not every Ω-set arises from a prestack, and it seems that not ever tracking morphism arises from a prestack morphism. However, when we move to separated gadets these correspondences become tighter. Before we can show that we need to sort out what a separated Ω-set is.

We know that each prestack carries a comparison which is a partial ordering. There is a similar comparison on each Ω-set.

8.1 DEFINITION. Each Ω-set A carries a comparison given by

$$b \sqsubseteq a \iff [\![b]\!] = [\![b = a]\!]$$

(for $a, b \in A$). ■

Almost trivially this comparison is a pre-ordering (where the transitivity is ensured by the transitivity of equality). However, it need not be a partial ordering. As usual with a pre-order it is useful to consider the induced equivalence relation.

8.2 DEFINITION. For each Ω-set A let $\cdot \cong \cdot$ be the equivalence relation on A given by

$$a \cong b \iff [\![a]\!] = [\![a = b]\!] = [\![b]\!]$$

(for $a, b \in A$). We call this the congruence of A.

We say A is reduced if this equivalence is equality. ■

Although the congruence is defined in terms of the carried equality, it is sometimes easier to work in terms of the carried equivalence.

8.3 LEMMA. *For each Ω-set A*

$$a \cong b \iff [\![a]\!] = [\![b]\!] \le [\![a \equiv b]\!]$$

holds for all $a, b \in A$.

Proof. If $a \cong b$ then

$$[\![a \equiv b]\!] = [\![a]\!] \vee [\![b]\!] \supset [\![a = b]\!] = \top$$

to give the first required implication.

Conversely, if $[\![a]\!] = [\![b]\!] \leq [\![a \equiv b]\!]$ then

$$[\![a = b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!] = [\![a]\!] = [\![b]\!]$$

to complete the proof. ■

Notice that if $a \cong b$ then $[\![a \equiv b]\!] = \top$.

The two carried comparisons are related by the following.

8.4 LEMMA. *Let A be a prestack.*

(i) The implication

$$b \leq a \Longrightarrow b \sqsubseteq a$$

holds for all $a, b \in A$.

(ii) A is separated exactly when A is reduced (as an Ω-set).

(iii) When A is separated the two comparisons $\leq$ and $\sqsubseteq$ agree.

Proof. In this proof we use the auxiliary relation $[\![\cdot \sim \cdot]\!]$ as defined in Lemma 6.6.

(i) Suppose $b \leq a$, so that $b = ay$ where $y = [\![b]\!]$. In particular, $y \leq [\![a]\!]$ and $by = b = ay$ so that $y \leq [\![b \sim a]\!]$, to give

$$[\![b]\!] = y \leq [\![a]\!] \wedge [\![a \sim b]\!] \wedge [\![b]\!] = [\![a = b]\!] \leq [\![b]\!]$$

and hence $b \sqsubseteq a$.

(ii) Suppose first that A is separated and consider $a \cong b$ in A. Let $u = [\![a]\!] = [\![b]\!]$ and consider $Y \subseteq \Omega$ given by

$$y \in Y \Longleftrightarrow ay = by$$

so that $u \leq [\![a \sim b]\!] = \bigvee Y$. Let $X = u \wedge Y$ so that

$$x \in X \Longleftrightarrow x \in Y \text{ and } x \leq u$$

and let $v = \bigvee X$. We have $v \leq u$. Also $ax = bx$ for each $x \in X$, so that $av = bv$, since A is separated. But, by the FDL, we have

$$v = \bigvee X = u \wedge \bigvee Y = u$$

and hence

$$a = au = av = bv = bu = b$$

as required.

Conversely, suppose that A is reduced and consider any situation

$$(\forall x \in X)[ax = bx] \qquad v \leq \bigvee X \qquad v \leq [\![a]\!] \wedge [\![b]\!]$$

(as in Definition 4.1). Following the proof of Lemma 4.2 we set $c = av$ and $d = bv$, so that $[\![c]\!] = v = [\![d]\!]$. For each $x \in X$ we have

$$cx = a(v \wedge x) = (ax)v = (bx)v = b(x \wedge v) = dx$$

and hence

$$v \leq \bigvee X \leq [\![c \sim d]\!] \leq [\![c \equiv d]\!]$$

to give $c \cong d$ (by Lemma 8.3). But A is reduced, so that $c = d$, as required.

(iii) By (i) and (ii) it suffices to show that

$$b \sqsubseteq a \Longrightarrow b \leq a$$

holds when A is reduced. Thus, consider $b \sqsubseteq a$ and let

$$y = [\![b]\!] = [\![b = a]\!]$$

so that $y \leq [\![a]\!]$ and

$$[\![ay]\!] = [\![a]\!] \wedge y = y \qquad [\![ay = b]\!] = [\![a = b]\!] \wedge y = y$$

(using Lemma 6.7), and hence $ay \cong b$. But A is reduced so that

$$b = ay = a[\![b]\!]$$

to give $b \leq a$, as required. ■

This result shows that, provided we know that an Ω-set arises from a prestack, we can determine whether or not it is separated entirely from its Ω-structure. Our problem is to isolate the (separated) prestacks within the Ω-sets.

8.5 DEFINITION. An Ω-set A is **stacked** if for each $a \in A$ and $y \leq [\![a]\!]$ there is some $b \sqsubseteq a$ with $[\![b]\!] = y$. ■

The action on a prestack immediately gives the following.

8.6 LEMMA. *When viewed as an Ω-set each prestack is stacked.*

With this out of the way we can make the crucial definition.

8.7 DEFINITION. An Ω-set A **separated** if it is stacked and reduced. ■

By Lemmas 8.6 and 8.4 and we see that each separated prestack gives rise to a separated Ω-set. In fact, we can go much further.

8.8 THEOREM. *Each separated prestack gives rise to a separated Ω-set.*
Furthermore, each separated Ω-set arises from a unique separated prestack.

Proof. The first assertion follows by Lemmas 8.6 and 8.4.

For the second assertion suppose A be any separated Ω-set. Using the given extent $[\![\cdot]\!]$ we must furnish A with a compatible action which induces the given equality. We must also show that there is only one such compatible action.

Consider $a \in A$ and $x \in \Omega$. Since A is stacked there is at least one $b \sqsubseteq a$ with $[\![b]\!] = [\![a]\!] \wedge x$. Note that $[\![b]\!] = [\![a = b]\!]$. Consider any other $c \sqsubseteq a$ with $[\![c]\!] = [\![a]\!] \wedge x = [\![b]\!]$. Then

$$[\![a = b]\!] = [\![b]\!] = [\![c]\!] = [\![a = c]\!]$$

so that

$$[\![b]\!] = [\![c]\!] = [\![b = a]\!] \wedge [\![a = c]\!] \leq [\![b = c]\!]$$

to give $b \cong c$, and hence $b = c$ since A is reduced.

This sets up an action

$$\begin{array}{rcl} A \times \Omega & \longrightarrow & A \\ a, x & \longmapsto & ax \end{array}$$

where

$$ax = \text{ the unique } b \sqsubseteq a \text{ with } [\![b]\!] = [\![a]\!] \wedge x$$

and which, almost trivially, furnishes A as a prestack.

As an aside, note that if A does arise from a prestack, then we have retrieved the original action.

This constructed prestack gives us a second Ω-set ΞA carried by A, and furnished with some equality $[\![\cdot - \cdot]\!]'$. We show this agrees with the original equality $[\![\cdot = \cdot]\!]$.

Consider any $a, b \in A$. By the construction of Lemma 6.6 and the FDL we see that

$$[\![a = b]\!]' = \bigvee \{x \in \Omega \,|\, x \leq [\![a]\!] \wedge [\![b]\!] \text{ and } ax = bx\}$$

holds. Consider any such x (as in the definition of $[\![a = b]\!]'$). Using Lemma 6.7 we have

$$x = [\![a]\!] \wedge x = [\![a = a]\!] \wedge x = [\![ax = a]\!]$$

with $x = [\![bx = b]\!]$ (by the same argument) so that (since $ax = bx$)

$$x = [\![a = ax]\!] \wedge [\![bx = b]\!] \leq [\![a = b]\!]$$

and hence

$$[\![a = b]\!]' \leq [\![a = b]\!]$$

(by taking the supremum over all x).

For the converse comparison let

$$z = [\![a = b]\!] \leq [\![a]\!] \wedge [\![b]\!]$$

(so that $z \leq [\![a = b]\!]'$ is required). We have

$$[\![az]\!] = [\![a]\!] \wedge z = z$$

and $[\![bz]\!] = z$ by a similar argument. Also, by Lemma 6.7 we have

$$[\![az = bz]\!] = [\![a = b]\!] \wedge z = z$$

to show that

$$[\![az]\!] = [\![bz]\!] = [\![az = bz]\!] = z$$

and hence $az \cong bz$. But A is reduced so that $az = bz$ and hence

$$[\![a = b]\!] = z \leq [\![a]\!] \wedge [\![b]\!] \wedge [\![a \sim b]\!]' = [\![a = b]\!]'$$

as required.

A couple more simple calculations show that this is the only prestack which gives rise to the parent Ω-set. ■

Just as we isolated the category of separated prestacks, so we can isolate the category of separated Ω-sets. However, we must take a bit of care with the arrows

8.9 DEFINITION. Let $\mathsf{Sep}(\Omega)$ be the category of separated Ω-sets and tracking morphisms. ■

Thus $\mathsf{Sep}(\Omega)$ is a full subcategory of $\mathsf{Trk}(\Omega)$ and Theorem 8.8 gives a a functor

$$\mathsf{Spk}(\Omega) \longrightarrow \mathsf{Sep}(\Omega)$$

which is 'bijective on objects'. In fact, we can do better.

8.10 THEOREM. *Let A, B be separated prestacks and view both as separated Ω sets. Then the two arrow sets*

$$\mathsf{Spk}(\Omega)[A, B] \qquad \mathsf{Set}(\Omega)[A, B]$$

are in bijective correspondence. In fact, they are precisely the same set of functions.

Proof. The new content of this result is that each tracking morphism

$$A \xrightarrow{\quad f \quad} B$$

between a pair of separated prestacks A and B is a prestack morphisms. We have

$$[\![a_1 = a_2]\!] \leq [\![fa_1 = fa_2]\!] \qquad [\![fa]\!] = [\![a]\!]$$

for each $a, a_1, a_2 \in A$, and we must show that

$$f(ax) = (fa)x$$

holds for each $x \in \Omega$. We show

$$f(ax) \cong (fa)x$$

and then remember that B is reduced.

We have

$$[\![f(ax)]\!] = [\![ax]\!] = [\![a]\!] \wedge x \qquad [\![(fa)x]\!] = [\![(fa)]\!] \wedge x[\![a]\!] \wedge x$$

so that

$$[\![f(ax) = (fa)x]\!] = [\![a]\!] \wedge x$$

will suffice. But, a couple of uses of 6.7 gives

$$[\![f(ax) = (fa)x]\!] = [\![f(ax) = fa]\!] \wedge x \geq [\![ax = a]\!] \wedge x = [\![a = a]\!] \wedge x = [\![a]\!] \wedge x$$

and the converse comparison is immediate. ■

Let's state the immediate consequence of Theorems 8.8 and 8.10.

8.11 COROLLARY. *The two categories* $\mathsf{Spk}(\Omega)$ *and* $\mathsf{Sep}(\Omega)$ *are canonically equivalent.*

This equivalence is the functor (5) of Table 1. Some people would say that $\mathsf{Spk}(\Omega)$ and $\mathsf{Sep}(\Omega)$ are canonically isomorphic. In fact, if we step outside the closed world of category theory, then there is good reasons to say these categories are 'the same'.

Let us now return to the problem of separating a prestack. We want to show that the inclusion

$$\mathsf{Spk}(\Omega) \hookrightarrow \mathsf{Psk}(\Omega)$$

has a left adjoint to the inclusion. Now that we know about the comparisons and equivalence carried by a prestack the required construction is almost routine.

Consider an arbitrary prestack A. Since the congruence $\cong$ is an equivalence relation, we may form the block slice

$$A \xrightarrow{\ \zeta\ } A/\cong$$

of A (that is, the set of equivalence classes of A). We structure this as a prestack.

For $a_1, a_2 \in A$ we have

$$\zeta a_1 = \zeta a_2 \Longrightarrow a_1 \cong a_2 \Longrightarrow [\![a_1]\!] = [\![a_2]\!]$$

so we may set

$$[\![\zeta a]\!] = [\![a]\!]$$

(for $a \in A$) to put a display on $A/\cong$.

A small argument of the kind used in the proof of Lemma 8.4 show that

$$a_1 \cong a_2 \Longrightarrow a_1 x \cong a_2 x$$

(for $a_1, a_2 \in A$ and $x \in \Omega$) and hence we may set

$$(\zeta a)x = \zeta(ax)$$

(for $a \in A$ and $x \in \Omega$) to obtain a potential action on $A/\cong$.

By construction we have

- $(\zeta a)[\![\zeta a]\!] = (\zeta a)[\![a]\!] = \zeta(a[\![a]\!]) = \zeta a$
- $[\![(\zeta a)x]\!] = [\![\zeta(ax)]\!] = [\![ax]\!] = [\![a]\!] \wedge x = [\![\zeta a]\!] \wedge x$
- $((\zeta a)x)y = (\zeta(ax))y = \zeta((ax)y) = \zeta(a(x \wedge y)) = (\zeta a)(x \wedge y)$

(for $a \in A$ and $x, y \in \Omega$) to show that with these furnishings $A/\cong$ is a prestack.

Also

$$[\![\zeta a]\!] = [\![a]\!] \qquad \zeta(ax) = (\zeta a)x$$

(for $a \in A$ and $x \in \Omega$) to show that

$$A \xrightarrow{\ \zeta\ } A/\cong$$

is a prestack morphism.

With several more simple calculations we obtain a proof of the following.

8.12 THEOREM. *For each prestack A the associated prestack $A/\cong$ is separated. Furthermore, for each prestack morphism*

$$A \xrightarrow{\ f\ } B$$

to a separated prestack B, there is a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle\zeta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & A/\cong & \end{array}$$

for a unique morphism $f^\sharp$.

For each prestack A we may view both A and $A/\cong$ as Ω-sets. The morphism ζ tracks an Ω-morphism between these two. The next result is perhaps a little surprising.

We call ζ the separating morphism of A.

8.13 LEMMA. *For each prestack A the separating morphism*

$$A \xrightarrow{\ \zeta\ } A/\cong$$

tracks an isomorphism between the corresponding Ω-sets.

Proof. We need to exhibit a morphism

$$A/\cong \longrightarrow A$$

which is inverse to the morphism tracked by ζ.

Before we define this morphism we need an observation.

Consider $b, c \in A$ with $\zeta b = \zeta c$ Then

$$[\![b]\!] = [\![b = c]\!] = [\![c]\!]$$

and hence

$$[\![a = b]\!] \leq [\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$$

for each $a \in A$. By symmetry this gives

$$[\![a = b]\!] = [\![a = c]\!]$$

and hence we may set

$$Z(a, \zeta b) = [\![a = b]\!]$$

(for $a, b \in A$) to obtain a function

$$Z : A \times A/\cong \longrightarrow \Omega$$

which we show is the required inverse morphism.

We must show that Z is extensional, functional, and total. Only the extensionality is not immediate.

We first check that

$$[\![\zeta a = \zeta b]\!] = [\![a = b]\!]$$

holds for each $a, b \in A$. For each $x \in \Omega$ we have

$$\zeta(ax) = (\zeta a)x \qquad [\![\zeta a]\!] = [\![a]\!]$$

with similar equalities for b. In particular

$$[\![a \equiv b]\!] \leq [\![\zeta a \equiv \zeta b]\!]$$

and hence

$$[\![a = b]\!] \leq [\![\zeta a = \zeta b]\!]$$

holds. For the converse comparison consider any $x \leq [\![a]\!] \wedge [\![b]\!]$ with $(\zeta a)x = (\zeta b)x$. Then $\zeta(ax) = \zeta(bx)$, to give $ax \cong bx$, and hence

$$x = [\![a]\!] \wedge x = [\![ax]\!] = [\![ax = bx]\!] = [\![a = b]\!] \wedge x$$

to give $x \leq [\![a = b]\!]$. Thus

$$[\![\zeta a = \zeta b]\!] = [\![a]\!] \wedge [\![b]\!][\![\zeta a \equiv \zeta b]\!] \leq [\![a = b]\!]$$

(by taking the supremum over all x), as required.

With this we have

$$[\![a_2 = a_1]\!] \wedge Z(a_1, \zeta b_1) \wedge [\![\zeta b_1 = \zeta b_2]\!] = [\![a_2 = a_1]\!] \wedge [\![a_1 = b_1]\!] \wedge [\![b_1 = b_2]\!] \leq [\![a_2 = b_2]\!] = Z(a_2, \zeta b_2)$$

to verify the required extensionality.

It remains to check that the two composites

$$Z \circ \zeta \qquad \zeta \circ Z$$

are the identity morphisms on A and $A/\cong$, respectively. Here, of course, ζ is viewed as a morphism, not a tracking function.

For $a, b \in A$ we have

$$(Z \circ \zeta)(a, b) = \bigvee \{Z(a, \zeta c) \wedge [\![\zeta c = \zeta b]\!] \mid c \in A\} = \bigvee \{[\![a = c]\!] \wedge [\![c = b]\!] \mid c \in A\} \leq [\![a = b]\!]$$

and selecting $c = a$ gives the converse comparison, so that $(Z \circ \zeta)(a, b) = [\![a = b]\!]$, as required.

For $a, b \in A$ we have

$$(\zeta \circ Z)(a, b) = \bigvee \{[\![\zeta a = \zeta c]\!] \wedge Z(c, \zeta b) \mid c \in A\} = \bigvee \{[\![a = c]\!] \wedge [\![c = b]\!] \mid c \in A\} = [\![a = b]\!]$$

by a similar argument, to complete the proof. ■

The step from tracking morphisms to Ω-morphisms seems to obliterate some distinctions we might want to keep. In general the the separating function ζ is not injective, so do we really want to view it as an isomorphism? The answer to that depends on where you want to live!

8.1 Write out the details of the proof of Theorem 8.12.

8.2 Let A be a prestack viewed as an Ω-set. Show that the following are equivalent.

(i) A is separated.

(ii) $(\forall a, b \in A)\big[a[\![a = b]\!] = b[\![b = a]\!]\big]$

(iii) $(\forall a, b \in A)\big[a[\![a \equiv b]\!] = b[\![b \equiv a]\!]\big]$

(iv) $(\forall a, b \in A)\big[[\![a \equiv b]\!] = \top \Longrightarrow a = b\big]$

Furthermore, show that if A is separated then the axiliary relation and the equivalence agree.

8.3 Show that each tracking morphism

$$A \xrightarrow{\quad f \quad} B$$

from a prestack to a separated prestack is a prestack morphism.

8.4 Does the slicing construction provide a reflection of $\mathsf{Trk}(\Omega)$ into $\mathsf{Sep}(\Omega)$?
I haven't looked at this

9 Replete Ω-sets

We have a nice correspondence between separated presheaves and separated Ω-sets, and this suggests two questions.

How can we extract those separated Ω-sets which are the canonical images of sheaves? What is the role of Ω-morphisms (as opposed to tracking morphisms) in this?

We have seen a partial answer to this second question in Lemma 8.13. By using Ω-morphisms we find that certain tracking morphisms give isomorphisms where this is not indicated by the nature of the tracking function. The full significance of this will become clearer later.

By Lemma 5.5 we know that each separated prestack is a sheaf if and only if as a poset it is sufficiently complete. This completeness can be reformulated.

9.1 DEFINITION. For an Ω-set A a singleton is a function

$$s : A \longrightarrow \Omega$$

such that

$$s(a) \wedge [\![a = b]\!] \le s(b) \qquad s(a) \wedge s(b) \le [\![a = b]\!]$$

holds for all $a, b \in A$. ■

Note that $s(a) \le [\![a]\!]$.

There are canonical examples of singletons.

9.2 EXAMPLE. For each $a \in A$ setting

$$a^\frown(c) = [\![c = a]\!]$$

(for $c \in A$) defines a singleton $a^\frown$. ■

The set of all singletons on A carries a natural comparison, the pointwise comparison. This is a partial order. We have seen part of this before.

9.3 LEMMA. *The equivalence*

$$b^\frown \leq a^\frown \Longleftrightarrow b \sqsubseteq a$$

holds for all $a, b \in A$.

Proof. Assuming $b^\frown \leq a^\frown$ we have

$$[\![b]\!] = b^\frown(b) \leq a^\frown(b) = [\![a = b]\!]$$

which gives $b \sqsubseteq a$.

Conversely, if $b \sqsubseteq a$ then

$$b^\frown(c) = [\![c = b]\!] \leq [\![b]\!] = [\![a = b]\!]$$

(for each $c \in A$) so that

$$b^\frown(c) \leq [\![a = b]\!] \wedge [\![b = c]\!] \leq [\![c = a]\!] = a^\frown(c)$$

as required. ■

As a consequence of this we have

$$a \cong b \Longleftrightarrow a^\frown = b^\frown$$

and, in particular, the same singleton may arise from different elements.

9.4 COROLLARY. *The presheaf is reduced if and only if*

$$a^\frown = b^\frown \Longrightarrow a = b$$

holds for all $a, b \in A$.

Of course, if every singleton of A had the form $a^\frown$ for some $a \in A$, then the notion wouldn't be much use.

9.5 DEFINITION. Let A be an Ω-set.

A singleton s of A is **realized** if $s = a^\frown$ for some $a \in A$.

A is **replete** if each singleton is realized by a unique element. ■

(Scott and the later writers use the word 'complete' in place of 'replete'. However, 'complete' is also used in too many other situations.)

Observe that a replete Ω-set is reduced. However, the converse if far from true. By the end of the next section we will see that replete Ω-sets are exactly the canonical images of sheaves. However, a proof of that needs quite a bit of preparation. In this section we will consider how we may remedy a non-replete Ω-set.

To do this we must convert each Ω-set into a replete Ω-set.

9.6 DEFINITION. Let A be an Ω-set, let $\mathfrak{S}(A)$ be the set of singletons on A, and set

$$[\![s = t]\!] = \bigvee\{s(b) \wedge t(b) \mid b \in A\}$$

for each $s, t \in \mathfrak{S}(A)$. ■

This is an important construction, so we will spend some time analysing it.

9.7 LEMMA. *For each Ω-set A the set $\mathfrak{S}(A)$ furnished with $[\![\cdot = \cdot]\!]$ (of Definition 9.6) is an Ω-set, and*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & \mathfrak{S}(A) \\ a & \longmapsto & a^{\frown} \end{array}$$

is a tracking morphism.

Proof. Much of this is routine calculation.

Trivially, the defined $[\![\cdot = \cdot]\!]$ on $\mathfrak{S}(A)$ is symmetric. For transitivity several uses of the FDL gives

$$\begin{aligned} [\![r = s]\!] \wedge [\![s = t]\!] &= \bigvee\{r(a) \wedge s(a) \wedge s(b) \wedge t(b) \mid a, b \in A\} \\ &\leq \bigvee\{r(a) \wedge [\![a = b]\!] \wedge t(b) \mid a, b \in A\} \\ &\leq \bigvee\{r(a) \wedge t(a) \mid a \in A\} \qquad = \quad [\![r = t]\!] \end{aligned}$$

for all singletons r, s, t. Note how the singleton properties are used in the middle of the calculation.

To verify the tracking properties of η observe that

$$[\![a^{\frown} = s]\!] = s(a)$$

for each $a \in A$, $s \in \mathfrak{S}(A)$. Again this is a simple calculation with the singleton properties. In particular

$$[\![\eta a = \eta b]\!] = [\![a^{\frown} = \eta b]\!] = b^{\frown} a = [\![a = b]\!]$$

and then

$$[\![\eta a]\!] = [\![\eta a = \eta a]\!] = [\![a = a]\!] = [\![a]\!]$$

which is more than enough to show that η is a tracking morphism. ■

For later use it is worth recording a couple of observation used in this proof.

9.8 LEMMA. *For each $a, b \in A$ and $s \in \mathfrak{S}(A)$ both*

$$[\![a^{\frown} = s]\!] = s(a) \qquad [\![a^{\frown} = b^{\frown}]\!] = [\![a = b]\!]$$

hold.

Notice also that for each singleton $s \in \mathfrak{S}(A)$ we have

$$[\![s]\!] = [\![s = s]\!] = \bigvee\{s(a) \mid a \in A\} = \bigvee s[A]$$

that is the extent of s is the supremum of its range.

The set $\mathfrak{S}(\Omega)$ carries (at least) two comparisons. Because it is an Ω-set it carries the associated comparison $\sqsubseteq$, and because it is a set of functions it carries the pointwise comparison $\leq$ (which is a partial ordering). Do we need to take care not to confuse these?

9.9 LEMMA. *The equivalence*

$$t \sqsubseteq s \iff t \leq s$$

holds for all singletons (of a parent Ω-set A).

Proof. Suppose that $t \sqsubseteq s$, then for each $a \in A$ we have

$$\begin{aligned} t(a) &= t(a) \wedge [\![t]\!] \\ &= t(a) \wedge [\![s = t]\!] \\ &= \bigvee\{t(a) \wedge t(b) \wedge s(b) \mid b \in A\} \\ &\leq \bigvee\{[\![a = b]\!] \wedge s(b) \mid b \in A\} \qquad \leq \quad s(a) \end{aligned}$$

to give $t \leq s$. Notice how the hypothesis is used for the second equality.

Conversely, given $t \leq s$ we have

$$[\![t = s]\!] = \bigvee\{t(a) \wedge s(a) \mid a \in A\} = \bigvee\{t(a) \mid a \in A\} = [\![t]\!]$$

to give $t \sqsubseteq s$. ■

The pointwise comparison of singletons is a partial ordering, which gives the following consequence.

9.10 COROLLARY. *For each Ω-set A the associated Ω-set $\mathfrak{S}(A)$ of singletons is reduced.*

In Lemma 8.13 we saw a surjective tracking morphism which, in general, is not injective but still tracks an isomorphism. We now have an example of a tracking morphism which may be neither surjective not injective yet still tracks an isomorphism.

9.11 THEOREM. *For each Ω-set A the associate Ω-set $\mathfrak{S}(A)$ is replete and*

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathfrak{S}(A) \\ a & \longmapsto & a^\frown \end{array}$$

tracks an isomorphism.

Proof. To show that $\mathfrak{S}(A)$ is replete consider any singleton

$$\rho : \mathfrak{S}(A) \longrightarrow \Omega$$

on $\mathfrak{S}(A)$, so that

$$\rho(s) \wedge [\![s = t]\!] \leq \rho(t) \qquad \rho(s) \wedge \rho(t) \leq [\![s = t]\!]$$

for all $s, t \in \mathfrak{S}(A)$ (singletons on A). Let

$$r : A \longrightarrow \Omega$$

be the function given by

$$r(a) = \rho(a^\frown)$$

for each $a \in A$. We show this is a singleton on A.

For $a, b \in A$ we have

$$r(a) \wedge [\![a = b]\!] = \rho(a\widehat{\ }) \wedge a\widehat{\ } = b\widehat{\ } \leq \rho(b\widehat{\ }) = r(b)$$

and

$$r(a) \wedge r(b) = \rho(a\widehat{\ }) \wedge \rho(b\widehat{\ }) \leq [\![a\widehat{\ } = b\widehat{\ }]\!] = [\![a = b]\!]$$

as required.

Finally (for the proof of the first part) we show that

$$\rho(s) = [\![s = r]\!]$$

for each $s \in \mathfrak{S}(A)$, and hence ρ is realized by r. Remembering Lemma 9.8, for each $a \in A$ we have

$$r(a) \wedge s(a) = \rho(a\widehat{\ }) \wedge [\![a\widehat{\ } = s]\!] \leq \rho(s)$$

so that

$$\rho(s) \wedge s(a) = \rho(s) \wedge [\![a\widehat{\ } = s]\!] \leq \rho(a\widehat{\ }) = r(a)$$

to give

$$\begin{aligned} [\![s = r]\!] &= \bigvee\{s(a) \wedge r(a) \mid a \in A\} \\ &= \bigvee\{\rho(s) \wedge s(a) \wedge r(a) \mid a \in A\} \\ &= \bigvee\{\rho(s) \wedge s(a) \mid a \in A\} \\ &= \rho(s) \wedge \bigvee\{s(a) \mid a \in A\} \\ &= \rho(s) \wedge [\![s]\!] \qquad = \quad \rho(s) \end{aligned}$$

as required (to complete the proof of the first part).

For the proof of the second part we need to exhibit a morphism

$$\mathfrak{S}(A) \longrightarrow A$$

which is inverse to η. Of course, in general, we can not expect this to be tracked.

To this end consider the evaluation function

$$A \times \mathfrak{S}(A) \xrightarrow{\ E\ } \Omega$$

given by

$$E(a, s) = s(a)$$

(for $a \in A, s \in \mathfrak{S}(A)$). We first check that this is extensional, functional, and total, and so is a morphism. In fact, these three properties are almost trivial, and only the first needs a few moment's thought. For $a, b \in A, s, t \in \mathfrak{S}(A)$ we have

(e) $[\![b = a]\!] \wedge E(a, s) \wedge [\![s = t]\!] = [\![b = a]\!] \wedge s(a) \wedge [\![s = t]\!] \leq t(b)$

(f) $E(a, s) \wedge E(b, s) = s(a) \wedge s(b) \leq [\![a = b]\!]$

(t) $[\![s]\!] = \bigvee\{E(a, s) \mid a \in A\}$

where the last comparison of (e) follows by a simple calculation.

It remains to check that the two composites

$$E \circ \eta \qquad \eta \circ E$$

are the identity morphisms on A and $\mathfrak{S}(A)$, respectively. Here, of course, η is viewed as a morphism, not a tracking function.

For $a, b \in A$ we have

$$\begin{aligned}(E \circ \eta)(a,b) &= \bigvee\{E(a,s) \wedge [\![s = \eta b]\!] \mid s \in \mathfrak{S}(A)\} \\ &= \bigvee\{s(a) \wedge [\![s = b^\frown]\!] \mid s \in \mathfrak{S}(A)\} \\ &= \bigvee\{s(a) \wedge s(b) \mid s \in \mathfrak{S}(A)\} \qquad \leq \ [\![a = b]\!]\end{aligned}$$

and by considering $s = a^\frown$ we obtain the converse comparison, so that

$$(E \circ \eta)(a,b) = [\![a = b]\!]$$

as required.

For $s, t \in \mathfrak{S}(A)$ we have

$$(\eta \circ E)(s,t) = \bigvee\{[\![s = \eta a]\!] \wedge E(a,t) \mid a \in A\} = \bigvee\{s(a) \wedge t(a) \mid a \in A\} = [\![s = t]\!]$$

which finally completes the proof. ■

We call

$$A \xrightarrow{\ \eta\ } \mathfrak{S}(A)$$

the repletion of A. But why is this interesting? We answer that in the next section.

9.1 Show that for each prestack A, when viewed as an Ω-set the assignment

$$\begin{array}{ccc}\mathfrak{S}(A/\cong) & \longrightarrow & \mathfrak{S}(A) \\ t & \longmapsto & t \circ \zeta\end{array}$$

is a tracking isomorphism, where ζ is the separating morphism of A.

10 Sheafification

Perhaps you can now see where all this is going. Even so, you may be in for a surprise. We must fill in the remaining corner $\mathsf{Rep}(\Omega)$ of Table 1, and show how this leads to solutions of the question of section 2.

10.1 DEFINITION. Let A be an Ω-set. A family $F \subseteq A$ is compatible if

$$[\![a]\!] \wedge [\![b]\!] \leq [\![a \equiv b]\!]$$

holds for all $a, b \in F$. ■

Note that if F is compatible then

$$[\![a = b]\!] \leq [\![a]\!] \wedge [\![b]\!] \leq [\![a]\!] \wedge [\![b]\!] \wedge [\![a \equiv b]\!] = [\![a = b]\!]$$

and hence we have the stronger property

$$[\![a = b]\!] = [\![a]\!] \wedge [\![b]\!]$$

for all $a, b \in F$.

Observe also that any subfamily of a compatible family is itself compatible. Thus we probably want to deal only with those compatible families which are 'maximal' in some appropriate sense.

10.2 LEMMA. *Let A be a prestack and let $F \subseteq A$. Then*

$$F \text{ coherent } \Longrightarrow F \text{ compatible}$$

and this is an equivalence when A is separated.

Proof. Consider any $a, b \in F$. We let

$$x = [\![a]\!] \quad y = [\![b]\!] \quad z = x \wedge y$$

and we keep this notation throughout the proof. Note that

$$[\![ay]\!] = [\![a]\!] \wedge y = z = [\![bx]\!]$$

(where the last equality follows by a similar argument).

Assuming F is coherent we have $ay = bx$, and hence $az = bz$, so that $z \leq [\![a \equiv b]\!]$, which shows that F is compatible.

Assuming F is compatible, Lemma 6.7 gives

$$[\![ay = bx]\!] = [\![a = b]\!] \wedge z = z$$

and hence

$$[\![ay]\!] = [\![ay = bx]\!] = [\![bx]\!]$$

to give $ay \cong bx$. When A is separated, this gives $ay = bx$, to show that F is coherent. ■

We also need another class of examples. For an Ω-set A we set

$$F(s) = \{a \in A \mid s(a) = [\![a]\!]\}$$

to obtain a family of A.

10.3 LEMMA. *Let A be an Ω-set.*

(i) For each singleton s the family $F(s)$ is compatible

(ii) The implication

$$t \leq s \Longrightarrow F(t) \subseteq f(s)$$

holds for all singletons s, t.

(iii) If A is stacked then the implication of (ii) is an equivalence.

Proof. (i) Consider $a, b \in F(s)$. Then

$$[\![a]\!] \wedge [\![b]\!] \leq s(a) \wedge s(b) \leq [\![a = b]\!]$$

to show that $F(s)$ is compatible.

(ii) Suppose $t \leq s$ and consider any $a \in F(t)$. Then

$$[\![a]\!] = t(a) \leq s(a) \leq [\![a]\!]$$

so that $s(a) = [\![a]\!]$ to show that $a \in F(s)$.

(ii) Suppose A is stacked and $F(t) \subseteq F(s)$. Consider any $a \in A$, let

$$y = t(a) \leq [\![a]\!]$$

so we require $y \leq s(a)$. Since A is stacked we have

$$y = [\![b]\!] = [\![b = a]\!]$$

for some $b \in A$. But now

$$[\![b]\!] = t(a) \wedge [\![b = a]\!] \leq t(b)$$

and hence $b \in F(t) \subseteq F(s)$ to give $s(b) = y$. With this we have

$$y = s(b) \wedge [\![b = a]\!] \leq s(a)$$

as required. ■

In a sense the singletons give us all compatible families.

10.4 LEMMA. *Suppose F is a compatible family of the Ω-set A. Then $F \subseteq F(s)$ for some singleton s.*

Proof. Consider the function

$$s : A \longrightarrow \Omega$$

defined by

$$s(a) = \bigvee \{[\![a = c]\!] \mid c \in F\}$$

(for $a \in A$). We show this is a suitable singleton.

Consider arbitrary $a, b \in A$.

By the FDL we have

$$s(a) \wedge [\![a = b]\!] = \bigvee \left\{[\![a = c]\!] \wedge [\![a = b]\!] \mid c \in F\right\} \leq \bigvee \left\{[\![b = c]\!] \mid c \in F\right\} = s(b)$$

which is the first requirement for s to be a singleton.

For each $c, d \in F$ we have

$$[\![a = c]\!] \wedge [\![b = d]\!] \leq [\![c]\!] \wedge [\![d]\!] = [\![c = d]\!]$$

(by the compatibility) so that

$$[\![a=c]\!] \wedge [\![b=d]\!] = [\![a=c]\!] \wedge [\![c=d]\!] \wedge [\![d=b]\!] \leq [\![a=b]\!]$$

and hence the FDL gives

$$s(a) \wedge s(b) = \bigvee \big\{ [\![a=c]\!] \wedge [\![b=d]\!] \mid c, d \in A \big\} \leq [\![a=b]\!]$$

which is the second requirement for s to be a singleton.

Finally for $a \in F$ another use of the compatibility and the FDL gives

$$s(a) = \bigvee \big\{ [\![a=c]\!] \mid c \in A \big\} = \bigvee \big\{ [\![a]\!] \wedge [\![c]\!] \mid c \in A \big\} = [\![a]\!] \wedge \bigvee \big\{ [\![c]\!] \mid c \in A \big\} = [\![a]\!]$$

to show that $a \in F(s)$, as required. ■

By the results of section 5 we know that a prestack is a stack (and so arises from a sheaf) if and only is it is separated and each coherent family is bounded above. This bounding property has an analogue for Ω-sets.

10.5 DEFINITION. A **head** for a compatible family F of an Ω-set A is some element $b \in A$ such that $c \sqsubseteq b$ holds for each $c \in F$. ■

These ideas lead to a useful characterization of repleteness.

10.6 THEOREM. *An Ω-set A is replete if and only if it is stacked, reduced, and each compatible family has a head.*

Proof. Suppose first that A is replete.

To show that A is stacked consider any $y \leq [\![a]\!]$ (where $a \in A, y \in \Omega$). Let

$$s(c) = [\![c=a]\!] \wedge y$$

for each $c \in A$. A simple calculation shows that s is a singleton, and hence $s = b\hat{}$ for some $b \in A$. In other words

$$[\![c=b]\!] = [\![c=a]\!] \wedge y$$

for each $c \in A$. In particular

$$[\![a=b]\!] = [\![a=a]\!] \wedge y = y$$

and hence

$$[\![b]\!] = [\![b=b]\!] = [\![b=a]\!] \wedge y = [\![b=a]\!]$$

so that $b \sqsubseteq a$ with $[\![b]\!] = y$, as required.

By Corollary 9.5 we see that A is reduced.

Consider any compatible $F \subseteq A$. By Lemma 10.4 we have $F \subseteq F(s)$ for some singleton s and hence, since A is replete, there is some $a \in A$ such that

$$c \in F \Longrightarrow c \in F(s) \Longrightarrow [\![c]\!] = s(c) = a\hat{}c = [\![c=a]\!] \Longrightarrow c \sqsubseteq a$$

and hence a is a head for F.

Secondly, suppose that A is stacked, reduced, and each compatible family has a head. Consider any singleton s with the associated compatible family $F(s)$. Consider any head b for $F(s)$, that is

$$s(c) = [\![c]\!] \Longrightarrow [\![c]\!] \leq [\![c = b]\!] \leq [\![b]\!]$$

holds (for all $c \in A$). Let

$$x = \bigvee \{[\![c]\!] \mid c \in C\}$$

so that $x \leq [\![b]\!]$. Since A is stacked we have

$$[\![a]\!] = x = [\![a = b]\!]$$

for some $a \in A$. We show that $s = a\widehat{\ }$ which, since A is reduced, will complete the proof.

Observe that, for each $c \in F(s)$ we have

$$[\![c]\!] \leq x = [\![a]\!] = [\![a = b]\!] \qquad [\![c]\!] \leq [\![b = c]\!]$$

so that

$$[\![c]\!] \leq [\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$$

and hence for each $d \in A$

$$[\![d = a]\!] \wedge [\![c]\!] = [\![d = a]\!] \wedge [\![a = c]\!] \wedge s(c) = [\![d = c]\!] \wedge s(c) \leq s(d)$$

(since s is a singleton). Thus, using the FDL, we have

$$[\![d = a]\!] = [\![d = a]\!] \wedge [\![a]\!] = \bigvee \{[\![d = a]\!] \wedge [\![c]\!] \mid c \in C\} \leq s(d)$$

which gives $a\widehat{\ } \leq s$.

Conversely, for each $d \in A$ we have $s(d) \leq [\![d]\!]$ so that, since A is stacked, there is some $c \in A$ such that

$$[\![c]\!] = s(d) = [\![c = d]\!]$$

holds. In particular, $c \in F(s)$, so that

$$s(d) = [\![c]\!] \leq [\![a = c]\!] \wedge [\![c = d]\!] \leq [\![a = d]\!]$$

which gives $s \leq a\widehat{\ }$, and hence $s = a\widehat{\ }$. ■

By Corollary 8.11 we have a reasonably tight matching between separated prestacks and separated Ω-sets. As object these are essentially the same. However, some of the Ω-morphisms between these may not be tracked. We can now improve that matching. In fact, there is very little to do except gather together the appropriate results.

We begin with an analogue of Theorem 8.8.

10.7 THEOREM. *Each stack gives rise to a replete Ω-set.*
Furthermore, each replete Ω-set arises from a unique stack.

Proof. Consider first any stack A. This is a separated prestack so, when viewed as an Ω-set it is stacked and reduced. Consider any compatible family $F \subseteq A$. By Theorem 10.6 it suffices to exhibit a head for F. By Lemma 10.2 we see that F is coherent, and hence (since A is collated) there is some $b \in A$ with $c \leq b$ for each $c \in F$. By Lemma 8.4 we have $c \sqsubseteq b$ for each $c \in C$, and hence b is a head for F.

Secondly, consider any replete Ω-set A. By Theorems 10.6 and 8.8 we see that A arises from a unique separated prestack. We must show that this is collated. Consider any coherent family $F \subseteq A$. By Lemma 5.3 it suffices to exhibit a bound for F. By Lemma 10.2 we see that F is compatible, and hence (since A is a replete) there is some $b \in A$ with $c \sqsubseteq b$ for each $c \in F$. By Lemma 8.4 we have $c \leq b$ for each $c \in C$, and hence b is a bound for F. ■

At this point we can formally define the remaining category.

10.8 DEFINITION. Let $\mathsf{Rep}(\Omega)$ be the category of replete Ω-sets and tracking morphisms. ■

By construction $\mathsf{Rep}(\Omega)$ is a full subcategory of $\mathsf{Sep}(\Omega)$. This with the following immediate consequence of Theorem 8.10 completes the construction of Table 1.

10.9 THEOREM. *The two categories* $\mathsf{Stk}(\Omega)$ *and* $\mathsf{Rep}(\Omega)$ *are canonically equivalent.*

We have a three functors

$$\mathsf{Rep}(\Omega) \hookrightarrow \mathsf{Sep}(\Omega) \hookrightarrow \mathsf{Trk}(\Omega) \longrightarrow \mathsf{Set}(\Omega)$$

where the two left hand ones are full insertions, but the right hand one isn't so tight. However, the next result shows that the composite functor is full.

10.10 THEOREM. *Each* Ω*-morphism, as on the left*

$$A \xrightarrow{\quad F \quad} B \qquad\qquad A \xrightarrow{\quad f \quad} B$$

where B *is replete is induced by a unique tracking function, as on the right.*

Proof. Consider $a \in A$ and, for this a consider $s : B \longrightarrow \Omega$ given by

$$s(b) = F(b, a)$$

(for $b \in B$). A couple of simple calculations (using the morphism properties of F) shows that this is a singleton. Thus, since B is replete, there is a unique element $fa \in B$ with $s = (fa)\widehat{}$. This sets up a function $f : A \longrightarrow B$ such that

$$F(b, a) = [\![b = fa]\!]$$

for all $a \in A, b \in B$. A couple more calculations shows that f is a tracking morphism which F, and is the only such tracking morphism. ■

We can now answer the questions posed in section 2. We can give a 2-step, a 1-step, *and* a 0-step description of sheafification.

How do we sheafify a presheaf A? Of course, we view A as a prestack. By Theorem 5.6 the problem is to convert A into a stack in a universal way. By Theorem 10.7 we are looking for a relete Ω-set.

Consider any prestack morphism

$$A \xrightarrow{\ f\ } B$$

to a stack B.

We know that B is separated so, by Theorem 8.12, we obtain a factorization $f^\sharp$

$$\begin{array}{ccccc} A & \xrightarrow{\zeta_A} & A/\cong & \xrightarrow{\eta_\bullet} & \mathfrak{S}(A/\cong) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\sharp} & & \\ B & \xrightarrow[id_B]{} & B & & \end{array}$$

through the slice morphism ζ_A of A. We compare this with the repletion $\eta_\bullet$ of $A/\cong$.

By Theorem 9.11 the morphism $\eta_\bullet$ tracks an Ω-isomorphism Z to give an Ω-morphism

$$\mathfrak{S}(A/\cong) \xrightarrow{\ Z\ } A/\cong \xrightarrow{\ F^\sharp\ } B$$

using the Ω-morphism $F^\sharp$ tracked by $f^\sharp$. We know that B is replete so, by Theorem 10.10, the morphism $F^\sharp$ is tracked a unique prestack morphism f^*. This gives us a commuting diagram

$$\begin{array}{ccccc} A & \xrightarrow{\zeta_A} & A/\cong & \xrightarrow{\eta_\bullet} & \mathfrak{S}(A/\cong) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\sharp} & & \downarrow{\scriptstyle f^*} \\ B & \xrightarrow[id_B]{} & B & \xrightarrow[id_B]{} & B \end{array}$$

to show that the composite $\eta_\bullet \circ \zeta_A$ is the required 2-step sheafification of A.

A particular case of this

$$\begin{array}{ccccc} A & \xrightarrow{\zeta_A} & A/\cong & \xrightarrow{\eta_\bullet} & \mathfrak{S}(A/\cong) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_A^\sharp} & & \downarrow{\scriptstyle \eta_A^*} \\ \mathfrak{S}(A) & \xrightarrow[id]{} & \mathfrak{S}(A) & \xrightarrow[id]{} & \mathfrak{S}(A) \end{array}$$

is induced by the repletion η_A of A. We may check that the resulting morphism η_A^* is

$$\begin{array}{ccc} \mathfrak{S}(A/\cong) & \xrightarrow{\eta_A^*} & \mathfrak{S}(A) \\ t & \longmapsto & t \circ \zeta_A \end{array}$$

and (by Exercise 9.1) this is an isomorphism. Thus η_A is a 1-step sheafification of A.

We can do even better. By Theorem 9.11 we know that η_A tracks an Ω-isomorphism. In other words, once we view the prestack A as an Ω-set we essentially have the external structure of its sheafification *without doing anything*. Only when we require the internal structure do we need to pass to the algebra of singletons. With a bit of licence we can call this a 0-step sheafification. Merely by immersing $\mathsf{Psh}(\Omega)$ into $\mathsf{Set}(\Omega)$ we can achieve all sheafifications at once.

Depending on who you are (or where you live) there are two perspectives of Ω-sets.

From outside the category $\mathsf{Set}(\Omega)$ we can see the internal structure of each Ω-set A. In particular, we can perceive a difference between A and $\mathfrak{S}(A)$, and between many Ω-sets that $\mathsf{Set}(\Omega)$ regards as essentially the same (that is, isomorphic). From inside $\mathsf{Set}(\Omega)$ the view seems rather limited since it misses things that might be important. However, it turns out that for some purposes these hidden facets are a distraction.

This dichotomy is not just philosophical clap-trap (although it does give plenty of scope for that). It is made precise when we start to interpret higher order languages within $\mathsf{Set}(\Omega)$.

Exercises

10.1 Show that $\mathsf{Rep}(\Omega)$ is a reflective subcategory of each of $\mathsf{Sep}(\Omega)$, $\mathsf{Trk}(\Omega)$, and $\mathsf{Set}(\Omega)$.

References

[1] Francis Borceux: Handbook of Categorical Algebra, vol. 3.

[2] P. T. Johnstone: Stone spaces.

[3] Michael P. Fourman: Sheaves over cHa and their Logic, Handwritten notes from a course given in Utrecht, 1977.

[4] M. P. Fourman and D. S. Scott: Sheaves and their Logic (especially chapter II), in Springer LNM vol. 753.

[5] D. Higgs: Categorical approach to boolean-valued set theory, Typewritten notes from who-knows-where.

[6] S. Mac Lane and I. Moerdijk: Sheaves in Geometry and Logic, Springer 1992.

Some solutions

For section 2

2.1 *To be done.* ■

2.2 *To be done.* ■

2.3 (a) This is straight forward.

(b) Consider any $a \in \Omega$ and any lower section X with $\bigvee X = a$. We use X both as a member of $\Omega(a)$ and as an index set. Consider the two members X and $\downarrow a$ of $\Omega(a)$. For each $x \in X$ we have

$$(\downarrow a)|x = \downarrow(a \wedge x) = \downarrow x = X|x$$

so that X and $\downarrow a$ agree locally (on X). If $\Omega(\cdot)$ is separated, then $X = \downarrow a$, and hence that $a \in X$. Thus each covering of a must contain a. (This a rather strong form of compactness.)

(c) Consider any $X \subseteq \Omega$ and let

$$\mathcal{L} = \{L(x) \mid x \in X\}$$

be a coherent X-selection of lower sections. Thus

$$L(x) \leq x \qquad L(x)|y = L(y)|x$$

for all $x, y \in X$. Let $L = \bigcup \mathcal{L}$ so that $L \leq \bigvee X$ and $L(x) \subseteq L|x$ for each $x \in X$. We show that $L(x) = L|x$.

Consider any $z \in L|x$. Thus $z \leq x$ and $z \in L$, so that $z \in L(y)$ for some $y \in X$. But now

$$z \in L(y)|x = L(x)|y \subseteq L(x)$$

as required.

(d) To show that $\Omega[\cdot]$ is separated, consider any $X \subseteq \Omega$ and any $a, b \in \Omega[\bigvee]$ with $a|x = b|x$ for each $x \in X$. Then

$$a, b \leq \bigvee X \qquad a \wedge x = b \wedge x$$

for each such x, so that

$$a = a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\} = \bigvee\{b \wedge x \mid x \in X\} = b \wedge \bigvee X = b$$

by the FDL.

To show that $\Omega[\cdot]$ is collated, consider any $X \subseteq \Omega$ and any coherent X-selection $\{a(x) \mid x \in x\}$. Then

$$a(x) \leq x \qquad a(x) \wedge y = a(y) \wedge x$$

for all $x, y \in X$. Let

$$a = \bigvee\{a(y) \mid y \in X\}$$

so that $a \in \Omega[\bigvee X]$. Also, for each $x \in X$

$$a|x = a \wedge x = \bigvee\{a(y) \wedge x \mid y \in X\} = \bigvee\{a(x) \wedge y \mid y \in X\} = a(x) \wedge \bigvee X = a(x)$$

as required.

(e) By the argument of part (b) we see that if $\Omega\langle\cdot\rangle$ is separated then each member of Ω is compact. By the argument of part (c) we see that $\Omega\langle\cdot\rangle$ is collated.

(f) We must show that

$$\begin{array}{ccc} \Omega(a) & \longrightarrow & \Omega[a] \\ \downarrow & & \downarrow \\ \Omega(b) & \longrightarrow & \Omega[b] \end{array}$$

commutes for each $b \leq a$. In other words, we must show that

$$(\bigvee X) \wedge b = \bigvee(X \wedge b)$$

holds for each $X \in \Omega(a)$. This follows by a use of the FDL.

(g) The morphism

$$\Omega(\cdot) \xrightarrow{\eta_{(\cdot)}} \Omega[\cdot]$$

is the sheafification of $\Omega(\cdot)$. In fact, in this case it has a stronger universal property. For each morphism

$$\Omega(\cdot) \xrightarrow{f_{(\cdot)}} A(\cdot)$$

to a separated presheaf $A(\cdot)$, there is a unique morphism $f^\sharp_{(\cdot)}$ (as indicated) such that the triangle

$$\begin{array}{ccccc} \Omega(\cdot) & & \xrightarrow{f_{(\cdot)}} & & A(\cdot) \\ & \searrow^{\eta_{(\cdot)}} & & \nearrow_{f^\sharp_{(\cdot)}} & \\ & & \Omega[\cdot] & & \end{array}$$

commutes.

Given such a morphism $f_{(\cdot)}$ we know that the square

$$\begin{array}{ccc} \Omega(a) & \xrightarrow{f_a} & A(a) \\ \downarrow & & \downarrow \\ \Omega(b) & \xrightarrow[f_b]{} & A(b) \end{array} \qquad f_b(X \wedge b) = f_a(X)|b$$

commutes for each $b \leq a$. In other words, for each $X \in \Omega(a)$ the equality to the right hold. Define

$$f^\sharp_a : \Omega[a] \longrightarrow A(a)$$

by

$$f_a^\sharp x = f_a(\downarrow x)$$

for $x \leq a$. We must check various properties.

Consider the square

$$\begin{array}{ccc} \Omega[a] & \xrightarrow{f_a^\sharp} & A(a) \\ \downarrow & & \downarrow \\ \Omega[b] & \xrightarrow[f_b^\sharp]{} & A(b) \end{array}$$

for $b \leq a$. For each $x \in \Omega[a] = \downarrow a$ we have

$$f_b^\sharp(x \wedge b) = f_b(\downarrow(x \wedge b)) = f_b((\downarrow x) \wedge b) = f_a(\downarrow x)|b = f_a^\sharp(x)|b$$

so that the square commutes, and $f_{(\cdot)}^\sharp$ is a presheaf morphism.

Next we show that

$$\begin{array}{ccc} \Omega(a) & \xrightarrow{\quad f_a \quad} & A(\cdot) \\ & {\scriptstyle \eta_a} \searrow \quad \nearrow {\scriptstyle f_a^\sharp} & \\ & \Omega[a] & \end{array}$$

commutes for each $a \in \Omega$. Consider $X \in \Omega(a)$ and let $z = \bigvee X$. For each $x \in X$ we have

$$f_a(X)|x = f_x(X \wedge x) = f_x(\downarrow x) = f_a(\downarrow z)|x$$

where the first equality follows by the given property of $f_{(\cdot)}$ (described above), and the third by the same argument with X replaced by $\downarrow z$. Thus

$$f_a^\sharp(\bigvee X) = f_a(\downarrow z) = f_a(X)$$

to show that the triangle does commute.

Finally, we must show that $f_{(\cdot)}^\sharp$ is the only possible such morphism. But if

$$g_a \circ \eta_a = f_a$$

(for $a \in \Omega$) then

$$g_a(\bigvee X) = f_a(X)$$

(for each $x \in \Omega(a)$) and hence

$$g_a x = f_a(\downarrow x)$$

(by considering $X = \downarrow x$), to give the required result.

(h) We may check that the insertion $\Omega\langle a\rangle \subseteq \Omega(a)$ is natural in a, and so we have a presheaf morphism $\Omega\langle\cdot\rangle \longrightarrow \Omega(\cdot)$. The sheafification of $\Omega\langle\cdot\rangle$ is the constant sheaf with constant value $\mathbf{1}$. ■

Is it the case that for any collated presheaf A the construct A^+ is both the separated reflection and the sheafification on A? ■

2.4 *To be done.* ■

For section 3

3.1 Consider any $a \in A$ (an arbitrary prestack) and let $u = [\![a]\!]$. Since $a = au$ we have

$$a\top = (au)\top = a(u \wedge \top) = au = a$$

as required. ■

3.2 We have $A(x) = Z$ for each $x \in \Omega$. To display this we must take a *disjoint* union of copies of Z. Thus we let A be the set of all pair (z, a) where $z \in Z$ and $a \in \Omega$. With this we may check that

$$[\![z, a]\!] = a \qquad (z, a)x = (z|x, a \wedge x)$$

is the appropriate extent and action. ■

3.3 For $\Omega(\cdot)$ we must make sure we get a correct display of this. Thus (in the manner of Solution 3.2) we take the set of all pairs

$$(X, a)$$

for $a \in \Omega$ and $X \in \mathcal{L}(\downarrow a)$. With this we find that

$$[\![X, a]\!] = a \qquad (X, a)x = (X \wedge x, a \wedge x)$$

is the appropriate extent and action.

For $\Omega\langle\cdot\rangle$ the construction is easier since each $X \in \mathcal{L}\Omega$ belongs to precisely one $\Omega\langle a\rangle$, namely $a = \bigvee X$. Thus we take $\mathcal{L}\Omega$ with

$$[\![X]\!] = \bigvee X \qquad Xx = X \wedge x$$

as the appropriate extent and action.

For $\Omega[\cdot]$ we take Ω with

$$[\![a]\!] = a \qquad ax = a \wedge x$$

as the appropriate extent and action (for $a, x \in \Omega$). ■

3.4 Suppose

$$A(\cdot) \xrightarrow{\ f_{(\cdot)}\ } B(\cdot)$$

is a presheaf morphism. Thus the square

$$\begin{array}{ccc} A(z) & \xrightarrow{f_z} & B(x) \\ \downarrow & & \downarrow \\ A(y) & \xrightarrow[f_y]{} & B(y) \end{array}$$

commutes for all $y \leq z$ from Ω. The two vertical arrows are the restriction maps. Consider any $a \in A$. We have $a \in A([\![a]\!])$ so we may set

$$fa = f_{[\![a]\!]}a \in B([\![a]\!])$$

to produce a function

$$f : A \longrightarrow B$$

which we show is a prestack morphism.

By construction $fa \in B(\llbracket a \rrbracket)$, and hence $\llbracket fa \rrbracket = \llbracket a \rrbracket$ for each $a \in A$.

Consider any $a \in A$ and $x \in \Omega$. Let $z = \llbracket a \rrbracket$ and $y = z \wedge x$, so that

$$ax = a|y \qquad \llbracket ax \rrbracket = y \qquad fa = f_z a \qquad f(ax) = f_y(a|y)$$

hold. Now consider the commuting square above. We have

$$f(ax) = f_y(a|y) = (f_z a)|y = (fa)|y = (fa)x$$

to show that f is a prestack morphism.

For the converse construction suppose

$$A \xrightarrow{\quad f \quad}$$

is a prestack morphism. For each $x \in \Omega$ and $a \in A(x)$ we have

$$\llbracket fa \rrbracket = \llbracket a \rrbracket = x$$

so that f restricts to a function

$$f_x : A(x) \longrightarrow B(x)$$

which we show is natural in x. Consider any $y \leq z$ from Ω and $a \in A(z)$. We have

$$a|y = ay$$

so that

$$f_y(a|y) = f_y(ay) = f(ay) = (fa)y = (fa)|y = (f_z a)|y$$

as required.

The remainder of the solution is a collection of similar simple calculations. ■

3.5 (a) It is easy to check that the constant presheaf with value $\mathbf{1}$ is the final object of $\mathsf{Psh}(\Omega$. This converts into the final object of $\mathsf{Psk}(\Omega)$. The general construction is given in Solution 3.2. However, here we have $Z = \mathbf{1} = \{\bullet\}$ so we don't need the first component of the pairs. Thus we take Ω with

$$\llbracket a \rrbracket = a \qquad ax = a \wedge x$$

as the appropriate extent and action (for $a, x \in \Omega$). This is nothing more than $\Omega[\cdot]$ viewed as a prestack. (You might like to prove that, as presheafs, $\Omega[\cdot]$ and the constant $\mathbf{1}$ presheaf are isomorphic.

We can prove directly that Ω is the final object of $\mathsf{Psk}(\Omega)$. Given any prestack morphisms

$$A \xrightarrow{\quad f \quad} \Omega$$

we must have

$$fa = [\![FA]\!] = [\![a]\!]$$

(for $a \in A$). Furthermore, for $u \in \Omega$, we have

$$f(au) = [\![au]\!] = [\![a]\!] \wedge u = (fa)u$$

so that this function is always a morphism.

(b) We must describe the morphisms

$$\Omega \xrightarrow{\ f\ } A$$

for an arbitrary A.

Consider any global element $a \in A$, and set

$$fx = ax$$

for $x \in \Omega$. Then

$$[\![fx]\!] = [\![ax]\!] = [\![a]\!] \wedge x = x \qquad f(ux) = a(u \wedge x) = (au)x = (fu)x$$

$$(h \circ f_a)u = h(au) = (ha)u = (hb)u = h(bu) = (h \circ f_b)u$$

for $x, u \in \Omega$, to show that this f is morphisms.

Conversely, consider any morphism f and let $a = f\top$. Then

$$[\![a]\!] = [\![f\top]\!] = [\![\top]\!] = \top$$

to show that a is global. Also, for $x \in \Omega$

$$fx = f(\top \wedge x) = (f\top)x = ax$$

so that f is the morphism determined by a. ■

3.6 Since the morphisms of $\mathsf{Psk}(\Omega)$ are functions, each injective morphism is monic. For the converse we use the standard separating technique. (Here a separator is what is sometimes called a generator. It has nothing to do with the separation property of sheaves.)

It seems that $\mathsf{Psk}(\Omega)$ does not have a separator, but it does have a separating family. Consider the final object Ω as described in Solution 3.5. This separates the global elements of each prestack. To deal with other elements consider the lower section $W = {\downarrow}w$ for an arbitrary $w \in \Omega$. This is a prestack in its own right with the induced furnishings

$$[\![a]\!] = a \qquad ax = a \wedge x$$

for $a \in W, x \in \Omega$. By reworking Solution 3.5 we see that each morphism

$$W \xrightarrow{\ f\ } A$$

has the form

$$fu = au$$

for some $a \in A$ with $[\![a]\!] = w$ (the top of W), and each such a gives a morphism f_a.

Now consider a monic arrow

$$A \xrightarrow{\;h\;} C$$

and consider $a, b \in A$ with $ha = hb$. Then

$$[\![a]\!] = [\![ha]\!] = [\![hb]\!] = [\![b]\!] = w \qquad (say)$$

to give a parallel pair of arrows

$$W \underset{f_b}{\overset{f_a}{\rightrightarrows}} A \xrightarrow{\;h\;} C$$

which we may compose with h, as indicated. For each $u \in W$ we have

$$(h \circ f_a)u = h(au) = (ha)u = (hb)u = h(bu) = (h \circ f_b)u$$

to give

$$h \circ f_a = h \circ f_b$$

and hence $f_a = f_b$, since h is monic. Thus $a = f_a w = f_b w = b$, as required. ■

For section 4

4.1 Since $a = a[\![a]\!]$ (for each $a \in A$) the defined relation is reflexive.

Suppose $b \leq a$. Then $[\![b]\!] = [\![a[\![b]\!]]\!] = [\![a]\!] \wedge [\![b]\!]$, and hence $[\![b]\!] \leq [\![a]\!]$. In particular, if $b \leq a \leq b$ then $[\![b]\!] = [\![a]\!]$ and hence

$$b = a[\![b]\!] = a[\![a]\!] = a$$

to show that the relation is antisymetric.

Finally, suppose that $c \leq b \leq a$. Then $[\![c]\!] \leq [\![b]\!] \leq [\![a]\!]$ so that

$$a[\![c]\!] = a([\![b]\!] \wedge [\![c]\!]) = (a[\![b]\!])[\![c]\!]) = b[\![c]\!]) = c$$

and hence $c \leq a$, to show that the relation is transitive. ■

For section 5

There are no exerecise yet

For section 6

6.1 (a) Consider the final prestack Ω, as described in Solution 3.5. We find that

$$[\![a \sim b]\!] = \bigvee\{x \in \Omega \mid a \wedge x = b \wedge x\} = (a \leftrightarrow b)$$

(for $a, b \in \Omega$) which gives

$$[\![a = b]\!] = a \wedge b \qquad [\![a \equiv b]\!] = (a \leftrightarrow b) = [\![a \sim b]\!]$$

and so produces that Ω-set of Examples 6.2(a).

(b) Since each element is global, this is an instance of Solution 6.3.

(c) Consider the set $\mathbf{\Omega}$ of all pairs (a, e) of elements $a \leq e$ of Ω with

$$[\![a, e]\!] = e \qquad (a, e)x = (a \wedge x, e \wedge x)$$

as the extent and action. It is easy to check that this is a prestack. A few calculations give

$$\begin{aligned}
[\![(a, e) \sim (b, f)]\!] &= (a \leftrightarrow b) \wedge (e \leftrightarrow f) \\
[\![(a, e) = (b, f)]\!] &= e \wedge (a \leftrightarrow b) \wedge f \\
[\![(a, e) \equiv (b, f)]\!] &= [\![a \sim b]\!]
\end{aligned}$$

where the last equality follows by a bit of manipulation.

(d) This has been done already. ■

6.2 To prove Theorem 6.4 observe that (for all $a, b \in A$)

$$[\![a \equiv b]\!] \leq [\![a]\!] \supset [\![a = b]\!]$$

so that

$$[\![a]\!] \wedge [\![a \equiv b]\!] \leq [\![a = b]\!]$$

and we will use this several times.

(i) Since $[\![a = b]\!] \leq [\![b]\!]$ this follows by the observation above.

(ii) Let

$$z = [\![a]\!] \vee [\![b]\!] \supset [\![a = b]\!] = ([\![a]\!] \supset [\![a = b]\!]) \wedge ([\![b]\!] \supset [\![a = b]\!])$$

so that

$$z \leq [\![a \equiv b]\!]$$

is required (since the other comparison is immediate). We have

$$z \wedge [\![a]\!] \leq [\![a]\!] \wedge [\![a \equiv b]\!] \leq [\![a = b]\!]$$

to give

$$z \leq ([\![a]\!] \supset [\![a = b]\!])$$

and the other comparison follows by interchanging a and b.

(iii) We have

$$[\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \leq [\![a = b]\!] \leq [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!]$$

as required.

This doesn't complete the proof, for we still have to show that $(A, [\![\cdot \equiv \cdot]\!])$ is an Ω-set with all elements global.

Trivially, $[\![\cdot \equiv \cdot]\!]$ is symmetric.

Now

$$[\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b \equiv c]\!] \leq [\![a = b]\!] \wedge [\![b \equiv c]\!] = [\![a = b]\!] \wedge [\![b]\!] \wedge [\![b \equiv c]\!] \leq [\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$$

so that

$$[\![a \equiv b]\!] \wedge [\![b \equiv c]\!] \leq ([\![a]\!] \supset [\![a = c]\!]$$

holds. In a similar way we have

$$[\![a \equiv b]\!] \wedge [\![b \equiv c]\!] \leq ([\![c]\!] \supset [\![a = c]\!]$$

to give

$$[\![a \equiv b]\!] \wedge [\![b \equiv c]\!] \leq [\![a \equiv c]\!]$$

as required.

Finally

$$[\![a \equiv b]\!] = ([\![a]\!] \supset [\![a = a]\!]) = \top$$

as required.

This completes the proof of Theorem 6.4.

Next we must prove Theorem 6.5.

Trivially, the defined assignment $[\![\cdot = \cdot]\!]$ is symmetric.

We have

$$[\![a = b]\!] \wedge [\![b = c]\!] = [\![a]\!] \wedge [\![b]\!] \wedge [\![c]\!] \wedge [\![a \equiv b]\!] \wedge [\![b \equiv c]\!] \leq [\![a]\!] \wedge [\![c]\!] \wedge [\![a \equiv c]\!] = [\![a = c]\!]$$

to show that $[\![\cdot = \cdot]\!]$ is transitive.

Also

$$[\![a = a]\!] = [\![a]\!] \wedge [\![a]\!] \wedge [\![a \equiv a]\!] = [\![a]\!]$$

(since $[\![a = a]\!] = \top$), so that the given extent $[\![\cdot]\!]$ is also the extent of the constructed $(A, [\![\cdot = \cdot]\!])$.

It remains to show that

$$z = ([\![a]\!] \supset [\![a = b]\!]) \wedge ([\![b]\!] \supset [\![a = b]\!])$$

is just the given $[\![a \equiv b]\!]$. But

$$z \wedge [\![a]\!] \leq [\![a = b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!] = [\![a]\!] \wedge [\![a \equiv b]\!] \leq [\![a \equiv b]\!]$$

by the given property (i). This with a similar comparison where a and b are interchanged gives

$$z \leq ([\![a]\!] \supset [\![a \equiv b]\!]) \wedge ([\![b]\!] \supset [\![a \equiv b]\!]) = ([\![a]\!] \vee [\![b]\!] \supset [\![a \equiv b]\!]) = [\![a \equiv b]\!]$$

by the given property (ii). Another use of (i) gives

$$[\![a]\!] \wedge [\![a \equiv b]\!] \leq [\![a]\!] \wedge [\![a \equiv b]\!] \wedge [\![b]\!] = [\![a = b]\!]$$

so that

$$[\![a \equiv b]\!] \leq ([\![a]\!] \supset [\![a = b]\!])$$

which, with a symmetric comparison, gives

$$[\![a \equiv b]\!] \leq z$$

as required. This completes the proof of Theorem 6.5. ■

6.3 Suppose the prestack gives rise to an Ω-set in which each element is global. Consider and $a \in A$ and $x \in \Omega$. Then

$$\top = [\![ax]\!] = [\![a]\!]x = x$$

so that either $x = \top$ (and so Ω is trivial) or there is no such a (and so A is empty). ■

6.4 (i) The first equality is the definition of $[\![\cdot = \cdot]\!]$, and the second is a general property of Ω-sets.

(ii) This is a general property of Ω-sets.

(iii) Using the definition of $[\![\cdot = \cdot]\!]$ we have

$$[\![a]\!] \supset [\![a = b]\!] = \big([\![a]\!] \supset [\![a \sim b]\!]\big) \wedge \big([\![a]\!] \supset [\![b]\!]\big)$$

and hence the required result follws from the definition of $[\![\cdot \equiv \cdot]\!]$. ■

6.5 Let

$$\Omega = [0, 1]$$

be the real interval viewed as a frame. (The linearity of Ω will simplify some of the later calculations.) Consider the prestack $\Omega(\cdot)$ of Example 2.3(c) viewed as an Ω-set as in Solution 3.3. The elements are pairs

$$(X, a)$$

where $0 \leq a \leq 1$ and X is a lower section of Ω below a. We have

$$[\![X, a]\!] = a \qquad (X, a)u = (X \wedge u, a \wedge u)$$

for $u \in \Omega$. The auxilairay relation is given by

$$[\![(X, a) \sim (Y, b)]\!] = \bigvee \{\!|(X, a) \sim (Y, b)|\!\}$$

where

$$u \in \{\!|(X, a) \sim (Y, b)|\!\} \iff X \wedge u = Y \wedge u \text{ and } a \wedge u = b \wedge u$$

is the set used. Let's consider the case $a = b < 1$. Thus

$$u \in \{\!|(X, a) \sim (Y, b)|\!\} \iff X \wedge u = Y \wedge u$$

holds. Suppose $X \neq Y$ with $x \in X - Y$, say. Consider any $u \geq x$. Then $x \in X \wedge u$ but $x \notin Y \wedge u$, so that $u \notin \{\!|(X, a) \sim (Y, b)|\!\}$. From this we see that

$$\{\!|(X, a) \sim (Y, b)|\!\} = X \cap Y$$

so that

$$[\![(X,a) \sim (Y,b)]\!] = \bigvee(X \cap Y) = e$$

(say). With

$$X = [0,a] \qquad Y = [0,a)$$

we have $e = a < 1$. But

$$[\![(X,a) \equiv (Y,b)]\!] = (a \supset e) = 1$$

to show that $[\![\cdot \sim \cdot]\!]$ and $[\![\cdot \equiv \cdot]\!]$ are distinct. ■

6.6 *To be done.* ■

For section 7

7.1 (a) Suppose first that f is a tracking morphism, that is

$$[\![a_1 = a_2]\!] \le [\![fa_1 = fa_2]\!] \qquad [\![fa]\!] \le [\![a]\!]$$

for all $a, a_1, a_2 \in A$. Then

$$[\![a]\!] = [\![a = a]\!] \le [\![fa = fa]\!] = [\![fa]\!]$$

to give $[\![fa]\!] = [\![a]\!]$. Next, remembering the construction of $[\![\cdot \equiv \cdot]\!]$ we have

$$[\![fa_1]\!] \wedge [\![a_1 \equiv a_2]\!] = [\![a_1]\!] \wedge [\![a_1 \equiv a_2]\!] \le [\![a_1 = a_2]\!] \le [\![fa_1 = fa_2]\!]$$

to give

$$[\![a_1 \equiv a_2]\!] \le ([\![fa_1]\!] \supset [\![fa_1 = fa_2]\!]) \qquad [\![a_1 \equiv a_2]\!] \le ([\![fa_2]\!] \supset [\![fa_1 = fa_2]\!])$$

where the second comparison follows by symmetry. These lead to

$$[\![a_1 \equiv a_2]\!] \le [\![fa_1 \equiv fa_2]\!]$$

as required.

Conversely, suppose we have this last comparison together with $[\![fa]\!] = [\![a]\!]$. Then

$$[\![a_1 = a_2]\!] = [\![a_1]\!] \wedge [\![a_1 \equiv a_2]\!] \wedge [\![a_2]\!] \le [\![fa_1]\!] \wedge [\![fa_1 \equiv fa_2]\!] \wedge [\![fa_2]\!] = [\![fa_1 = fa_2]\!]$$

to show that f is a tracking morphism.

(b) We have

$$F(b,a) = [\![b = fa]\!] \le [\![b]\!] \wedge [\![fa]\!]$$

for all $a \in A, b \in B$. Since F is total we have

$$[\![a]\!] = \bigvee\{F(b,a) \mid b \in B\} \le [\![fa]\!] \qquad [\![fa]\!] = F(fa,a) \le [\![a]\!]$$

to give $[\![fa]\!] = [\![a]\!]$ (for each $a \in A$). Consider $a_1, a_2 \in A$. With $b_1 = b_2 = fa_1$ the functionality

$$[\![b_2 = b_1]\!] \wedge [\![b_2 = fa_1]\!] \wedge [\![a_2 = a_1]\!] \le [\![b_2 = fa_2]\!]$$

becomes

$$[\![fa_1]\!] \wedge [\![a_2 = a_1]\!] \leq [\![fa_1 = fa_2]\!]$$

and hence

$$[\![a_1 = a_2]\!] = [\![a_1 = a_2]\!] \wedge a_1 = [\![a_1 = a_2]\!] \wedge fa_1 \leq [\![fa_1 = fa_2]\!]$$

as required. ■

7.2 *To be done.* ■

For section 8

8.1 *To be done.* ■

8.2 Given $a, b \in A$ let

$$u = [\![a = b]\!] \qquad v = [\![a \equiv b]\!]$$

so that

$$u = [\![a]\!] \wedge [\![b]\!] \wedge v \qquad v = ([\![a]\!] \supset u) \wedge ([\![b]\!] \supset u)$$

and

$$[\![a]\!] \wedge v \leq [\![b]\!] \qquad [\![b]\!] \wedge v \leq [\![a]\!] \qquad [\![a]\!] \wedge v = u = [\![b]\!] \wedge v$$

hold.

(i)⇒(ii). We have

$$[\![au]\!] = [\![a]\!] \wedge u = u \quad [\![bu]\!] = [\![b]\!] \wedge u = u \quad [\![au = bu]\!] = [\![a = b]\!] \wedge u = u$$

so that $au \cong bu$. If A is separated, then a is reduced, and hence $au = bu$.

(ii)⇒(iii). Since $a = a[\![a]\!]$ we have

$$av = a([\![a]\!] \wedge v) = au$$

which leads to the required result.

(iii)⇒(iv). This is trivial.

(iv)⇒(i). Since

$$a \cong b \Longrightarrow [\![a \equiv b]\!] = \top$$

this is immediate.

If A is separated then (iii) gives

$$[\![a \equiv b]\!] \leq [\![a \sim b]\!]$$

and the converse comparison always holds. ■

8.3 This is done in the proof of Theorem 8.10 (where the fact that A is separated is not used). ■

8.4 *To be looked at.* ■

For section 9

9.1 This is a series of simple calculations.

First we must check that $t \circ \zeta$ is a singleton of A for $t \in \mathfrak{S}(A/\cong)$. For each $a, b \in A$ we have

$$t(\zeta a) \wedge [\![\zeta a = \zeta b]\!] \leq t(\zeta b) \qquad t(\zeta a) \wedge t(\zeta b) \leq [\![\zeta a = \zeta b]\!]$$

(since t is a singleton), and we know that

$$[\![\zeta a = \zeta b]\!] = [\![a = b]\!]$$

which ensures that $t \circ \zeta$ is singleton.

Next we have

$$[\![t_1 = t_2]\!] = \bigvee \{t_1(\zeta a) \wedge t_2(\zeta a) \mid a \in A\} = \bigvee \{(t_1 \circ \zeta)a \wedge (t_2 \circ \zeta)a \mid a \in A\}$$

which is enough to show that the assignment is a tracking morphism. Furthermore, if we can show that the assigment is a bijection, then this equality will ensure that it is a tracking isomorphism.

Since ζ is surjective, the assigment $t \longmapsto t \circ \zeta$ is injective. For surjectivity consider any singleton s of A. It suffices to show that

$$t(\zeta a) = s(a)$$

(for $a \in A$) gives a well-defined singleton t of $A/\cong$. Consider $a, b \in A$ with $\zeta a = \zeta b$. Thus $a \cong b$ to give $[\![a]\!] = [\![a = b]\!] = [\![b]\!]$ and hence

$$sa = s(a) \wedge [\![a]\!] = s(a) \wedge [\![a = b]\!] \leq s(b)$$

to give $s(a) = s(b)$ (by a symmetric argument). This shows that t is well-defined. Finally, using the observation above, for $a, b \in A$ we have

$$s(a) \wedge [\![a = b]\!] = t(\zeta a) \wedge [\![\zeta a = \zeta b]\!] \leq t(\zeta b) = s(b)$$
$$s(a) \wedge s(b) = t(\zeta a) \wedge t(\zeta b) \leq [\![\zeta a = \zeta b]\!] = [\![a = b]\!]$$

to show that s is a singleton. ■

For section 10

10.1 *To be done.* ■