A coverage construction of the reals and the irrationals

Harold Simmons

School of Mathematics
The University, Manchester, England
hsimmons@manchester.ac.uk

Abstract

I modify the standard coverage construction of the reals to obtain the irrationals. However, this causes a jump in ordinal complexity from $\omega + 1$ to Ω.

The coverage technique has its origins in the generation of Gabriel and Grothendieck topologies. Later the technique was modified for use in point-free topology, and for the generation of genuine topological spaces. In the setting the technique takes the form of a rather simple kind of relation

$$r \vdash U$$

between elements r of a poset S and lower section U of S. We think of each $r \in S$ as a name for a basic open set, and each $U \in \mathcal{L}S$ (the family of all lower section of S) as a name for an arbitrary open set of the space under construction. We wish to read this relation as

'the basic open set named by r is included in the open set named by U'

(which more or less tells us what the general properties of $\vdash$ should be).

Each such relation is specified by certain postulated primitive instances. We may think of S together with these postulated instances as a presentation of the topological space under construction. However, notice that, as yet, I haven't said what the points of the space are. That is why the construction is point-free, it doesn't need the points.

The coverage technique is often used in a constructive setting, where some care is taken over the set theoretic and logical principles employed. However, when we stand outside this restricted environment we see there are certain hidden complexities.

For instance, each $U \in \mathcal{L}S$ names an open set, but each open can have many different names. What is the relationship between $U, V \in \mathcal{L}S$ which name the same open set? It turns out that for each $U \in \mathcal{L}S$ there is a unique largest $U^+ \in \mathcal{L}S$ which names the same open, but moving from U to U^+ can involve a long ordinal iteration. This attaches an ordinal measure to each coverage system.

Each coverage relation is determined by its postulated primitive instances. Given two such relations $\vdash_1, \vdash_2$ we can form the syntactic join by simply using primitive instances of either kind. It can happen that although the two components $\vdash_1, \vdash_2$ are rather simple, the syntactic join is far more complicated. The jump in ordinal measure can be quite large.

There are several standard uses of this technique, including the construction of Cantor space, Baire space, and the space of real numbers. The ordinal measure of each of these is known, but not often stated explicitly. For the reals it is $\omega + 1$. As far as I know, nobody has produce the space of irrationals by this technique. Here I obtain a suitable coverage by slightly modifying that for the reals. However, even though the syntactic change is slight, the change is complexity is quite dramatic. The ordinal measure is Ω, the least uncountable ordinal. This also illustrates that the syntactic join of two coverage relations can be far more complicated than the two components.

There is also a more algebraic way of describing the coverage technique. The poset $\mathcal{L}S$ of lower sections of a poset S is the free $\bigvee$-semilattice generated by S. Suppose we require a certain $\bigvee$-quotient of $\mathcal{L}S$. We may specify such a quotient by requiring that certain intervals $V \subseteq U$ of $\mathcal{L}S$ should collapse, that is identify V and U. Of course, this may cause many more intervals to collapse. After some routine algebraic analysis we see that such a specification is equivalent to a certain kind of function on $\mathcal{L}S$. In fact, such functions and coverage relations are essentially the same thing. The difference is only a matter of notation. For each specification it is the associated function that must be iterated to produce $(\cdot)^+$ on $\mathcal{L}S$. This algebraic view leads to several other techniques which are not so clear from the syntactic perspective.

In this paper I first outline the coverage technique, and give some of the point-free results that are needed later. I then show how this enables us to produce both the reals and the irrationals. I do these two construction in unison to demonstrate the similarities, and to suggest - incorrectly - that they they have roughly the same complexity. Finally, I calculate the closure ordinal of each construction. I work entirely classically. I use the classical ordinals as a measure of complexity. I also invoke the Cantor-Bendixson properties of the real numbers to determine the closure ordinal for the irrationals. There is, however, one non-constructive aspect I do draw attention to. Several different choice principles are used in this paper, and this has an impact on the different complexities. Each use is HIGHLIGHTED in this way.

My interest in this was re-awakened by the work of Nicola Gambino [6], especially chapter 4 where he analyses the CHOICE content of various coverage constructions, and in a constructive setting.

Contents

1 Background material

I give a rather terse account of the coverage technique and the use of frames. A slower account is given in [8].

1.1 Coverage relations

The term 'coverage' is used for several different but closely related notions. I will take what is perhaps the most general notion, and impose further restrictions as needed.

Let $(S, \leq)$ be a poset. Let $\mathcal{L}S, \Upsilon S$ be, respectively, the family of lower, upper sections of S. Each of these is a topology on S, and each is the corresponding family of closed sets of the other. Also $\mathcal{L}S$ is the free $\bigvee$-semilattice generated by S. More details of this are given in Lemma 1.6. The coverage technique is used to produce quotients of $\mathcal{L}S$.

1.1 DEFINITION. Let S be a poset.

A poser on S is a relation $s \Vdash E$ between elements $s \in S$ and subsets $E \subseteq S$.

A coverage on S is a relation $r \vdash U$ between elements $r \in S$ and lower section $U \in \mathcal{L}S$ satisfying

$$\frac{r \in U}{r \vdash U}\ (\text{inflationary}) \qquad \frac{r \leq s \quad s \vdash V \quad V \subseteq U}{r \vdash U}\ (\text{monotone})$$

for all $r, s \in S$ and $U, V \in \mathcal{L}S$. ■

A poser is so called because it poses a problem which requires a solution.

1.2 DEFINITION. Let $\Vdash$ be a poser on a poset S. A solution to the posed problem is a monotone function f from S to a $\bigvee$-semilattice A

$$S \xrightarrow{\ f\ } A \qquad s \Vdash E \Longrightarrow f(s) \leq \bigvee f[E]$$

such that the implication holds for all $s \in S, E \in \mathcal{P}S$. ■

Think of S as the generators of an unknown $\bigvee$-semilattice. If $s \Vdash E$ in S then in the semilattice the element named by s must lie below the supremum of the set of elements named by E. We require a universal solution to this problem, see Subsection 1.2.

1.3 DEFINITION. Let $\Vdash$ be a poser on a poset S. The associated coverage of $\Vdash$ is the relation $\vdash$ generated by the rules

$$\frac{r \in U}{r \vdash U} \qquad \frac{r \leq s \quad s \Vdash E \quad E \subseteq U}{r \vdash U}$$

for $r, s \in S$ and $U \in \mathcal{L}S, E \in \mathcal{P}S$. ■

Each coverage is a particular kind of poser, and so requires a solution, as in Definition 1.2. The following partly explains why we pass from a poser to its associated coverage. Further reasons are given Theorem 1.11.

1.4 LEMMA. *A poser and its associated coverage on a poset have the same solutions.*

A coverage $\vdash$ on a poset S produces a $\bigvee$-quotient $(\mathcal{L}S)_\vdash$ of $\mathcal{L}S$, namely the family of all $U \in \mathcal{L}S$ such that $s \in U$ whenever $s \vdash U$. This is the universal solution to the problem. Here we want this solution to be a frame, see Subsection 1.2. For that we require an extra condition on the coverage.

We use '$\downarrow$' to indicate the downwards closure of an element of a subset.

1.5 DEFINITION. A coverage $\vdash$ on a poset S is stable if it has the equivalent properties

$$\frac{r \vdash U}{r \vdash U \cap \downarrow r} \qquad \frac{r \leq s \vdash U}{r \vdash U \cap \downarrow r}$$

for $r, s \in S$ and $U \in \mathcal{L}S$. ■

1.2 Frame theory

Let $\boldsymbol{Pos}$, $\boldsymbol{Sup}$, $\boldsymbol{Frm}$, and $\boldsymbol{Top}$ be the categories of, respectively, posets with monotone maps, $\bigvee$-semilattices with $\bigvee$-preserving monotone maps, frames (see later in this subsection), and topological spaces with continuous maps.

There is an obvious functor

$$\boldsymbol{Pos} \longleftarrow \boldsymbol{Sup}$$

and this has a left adjoint. For a poset S the family $\mathcal{L}S$ of lower sections of S is a complete lattice. There is a monotone map

$$\begin{array}{ccc} S & \xrightarrow{\ i\ } & \mathcal{L}S \\ s & \longmapsto & \downarrow s \end{array}$$

and this reflect S into $\boldsymbol{Sup}$.

1.6 LEMMA. *For each poset S and monotone map*

$$S \xrightarrow{\ f\ } A$$

to a $\bigvee$-semilattice A, there is a unique $\boldsymbol{Sup}$-arrow

$$\mathcal{L}S \xrightarrow{\ f^\sharp\ } A$$

such that $f = f^\sharp \circ i$.

For each poset S the complete poset $\mathcal{L}S$ has other properties, in particular it is a frame.

1.7 DEFINITION. A frame

$$(A, \leq, \bigvee, \bot, \wedge, \top)$$

is a complete lattice which satisfies the Frame Distributive Law

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

for $a \in A, X \subseteq A$. A frame morphism is a monotone function which preserves arbitrary suprema and finitary meets. ■

For each topological space S the carried a topology $\mathcal{O}S$ is a frame. For each continuous map ϕ

$$T \xrightarrow{\ \phi\ } S \qquad\qquad \mathcal{O}S \xrightarrow{\ \phi^{\leftarrow}\ } \mathcal{O}T$$

the inverse image functions $\phi^{\leftarrow}$ is a frame morphism. This gives a contravariant functor

$$\boldsymbol{Frm} \longleftarrow \boldsymbol{Top}$$

which has a contravariant adjoint. Each frame A has a point space $\mathsf{pt}(A)$, together with a surjective frame morphism to the carried topology.

$$A \longrightarrow \mathcal{O}(\mathsf{pt}(A))$$

We use a particular instance of this in Section 2 where we describe the point space of $\mathcal{L}S$.

Consider a poser $\Vdash$ on a poset S. This has an extreme solution; we collapse S to a singleton. We want a universal solution, one through which each other solution must pass. Whatever this is it must have the form

$$S \xrightarrow{\ i\ } \mathcal{L}S \xrightarrow{\ g\ } A$$

where g is a surjective $\bigvee$-semilattice morphism. Thus we are looking for a certain $\bigvee$-quotient of $\mathcal{L}S$. Here we also require this solution to be a frame.

The universal solution of a poser, or its equivalent coverage, on a poset S certainly produces a $\bigvee$-quotient of LS, but this need not be a frame. I will give a simple example of this in Subsection 1.4 when we have the appropriate notions. To obtain a frame we use a special kind of coverage, a stable coverage. This is more easily seen from an algebraic perspective.

1.3 Nuclei and inflators

The universal algebra of frames is best done using certain functions on frames.

1.8 DEFINITION. Let A be a frame.

An inflator on A is an inflationary and monotone function $f : A \longrightarrow A$, that is

$$x \leq f(x) \qquad x \leq y \Longrightarrow f(x) \leq f(y)$$

for $x, y \in A$.

A closure operation on A is an inflator f which is idempotent, that is $f^2 = f$.

An inflator f is stable if

$$f(x) \wedge y \leq f(x \wedge y)$$

for $x, y \in A$.

A pre-nucleus is an inflator f such that

$$f(x) \wedge f(y) = f(x \wedge y)$$

for $x, y \in A$.

A nucleus is a pre-nucleus which is idempotent (and hence a closure operation). ■

Each nucleus is a pre-nucleus, and each pre-nucleus is stable. This is the most common terminology, but sometimes a stable inflator is called a pre-nucleus. Notice that because of monotonicity the comparison in the definition of pre-nucleus can be improved to an equality. However, the comparison in the definition of stable inflator can not be improved.

Each morphism $f^* = f$ between frames

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

has a right adjoint f_*, and the composite $k = f_* \circ f^*$ is a nucleus on A. This composite k is the kernel nucleus of f, and it is characterized by

$$y \leq k(x) \Longleftrightarrow f(y) \leq f(x)$$

for $x, y \in A$. Every nucleus k on A arises in this way. The set A_k of elements fixed by A is a frame and

$$\begin{array}{ccc} A & \xrightarrow{k^*} & A_k \\ x & \longmapsto & k(x) \end{array}$$

is a frame morphism with kernel k.

1.9 DEFINITION. Let f be an inflator on the frame A. The ordinal iterates

$$(f^\alpha \mid \alpha \in \mathbb{O}\mathrm{rd})$$

of f are generated by

$$f^0(x) = x \qquad f^{\alpha+1}(x) = f(f^\alpha(x)) \qquad f^\lambda(x) = \bigvee\{f^\alpha(x) \mid \alpha \le \lambda\}$$

for each ordinal α, limit ordinal λ, and $x \in A$. ■

The family of all inflators on A is closed under composition and pointwise suprema. Thus the ordinal iterates of an inflator produce an ascending chain of inflators

$$f^0 \le f = f^1 \le f^2 \le \cdots \le f^\alpha \le \cdots \quad (\alpha \in \mathbb{O}\mathrm{rd})$$

under the pointwise comparison. There is a smallest ordinal θ, the rank of f, with $f^\alpha = f^\theta$ for all $\alpha \ge \theta$, We write f^∞ for this closure when the value of θ is not important or not known. The closure f^∞ of an inflator need not be a nucleus.

1.10 LEMMA. *The composite of two pre-nuclei is a pre-nucleus.*
If f is a pre-nucleus, then so is each ordinal iterate f^α.
If f is a pre-nucleus, then f^∞ is a nucleus.
The composite of two stable inflators is stable.
If f is a stable inflator, then f^ω is a pre-nucleus.
If f is a stable inflator, then each limit ordinal iterate f^λ is a pre-nucleus.
If f is a stable inflator, then f^∞ is a nucleus.

1.4 Inflators and coverages

Each poset S gives a frame $\mathcal{L}S$, and we have two kinds of gadgets associated with S, coverages on S and inflators on $\mathcal{L}S$. These are the same thing.

1.11 THEOREM. *Let S be a poset. There is a bijective correspondence between coverages $\vdash$ on S and inflators f on $\mathcal{L}S$. This is given by*

$$s \vdash U \Longleftrightarrow s \in f(U)$$

for $s \in S$ and $U \in \mathcal{L}S$. Furthermore, $\vdash$ is stable precisely when f is stable.

Think of a pair $(S, \Vdash)$ as a presentation of the universal solution to the posed problem. We know, in principle, how to obtain this solution. We fill out $\Vdash$ to its associated coverage $\vdash$ to obtain an inflator f on $\mathcal{L}S$, and then take the lower sections of S fixed by f. An arbitrary inflator f on $\mathcal{L}S$ generates a closure operation f^∞ on $\mathcal{L}S$, and gives us a $\bigvee$-quotient $(\mathcal{L}S)_f$ of $\mathcal{L}S$. This need not be a frame, since f^∞ need not be a nucleus.

For example, let S be any set viewed as a discrete poset. Thus $\mathcal{L}S = \mathcal{P}S$, the power set of S. Let f be any topological closure operation on $\mathcal{P}S$. Then $(\mathcal{L}S)_f = \mathcal{C}S$ is the corresponding family of closed subsets. This need not be a frame.

Let $\Vdash_1$ and $\Vdash_2$ be a pair of posers on S with induced inflators f_1 and f_2 on $\mathcal{L}S$. The syntactic join of these two posers is the poser $\Vdash$ given by

$$s \Vdash E \Longleftrightarrow s \Vdash_1 E \text{ or } \Vdash_2 E$$

for $s \in S$ and $E \subseteq S$. This induces an inflator f on $\mathcal{L}S$ which is just the pointwise join

$$f = f_1 \dot\vee f_2$$

of the component inflators, that is

$$f(U) = f_1(U) \cup f_2(U)$$

for $U \in \mathcal{L}S$. Each of f, f_1, f_3 has a closure $f^\infty, f_1^\infty, f_3^\infty$. We find that

$$f^\infty = f_1^\infty \vee f_3^\infty$$

the join in the assembly of all closure operations on $\mathcal{L}S$. In general, this is *not* the pointwise join; that can be much smaller.

1.5 Composites of nuclei

For each pair j, k of nuclei on a frame A there is a smallest nucleus $j \vee k$ above both j and k. In general this is not easy to compute, even when the two components j and k are simple. In this section we obtain two results, Theorems 1.13 and 1.15 which are specifically designed for use in Subsections 4.1 and 4.2.

It is convenient to fix some notation.

1.12 CONTEXT. Let j be a nucleus and let f be a stable inflator on the frame A. Suppose f^ω is a nucleus. Let

$$g = j \dot\vee f \qquad \ell = j \circ f^\omega$$

to produce two stable inflators. ■

It is not hard to see that

$$g^\infty = j \vee f^\omega = \ell^\infty$$

that is both g^∞ and ℓ^∞ are the least nucleus above j and f^∞. We wish to estimate the two ranks.

There is a simple case.

1.13 THEOREM. *For the Context 1.12, suppose $f \circ j \leq j \circ f$. Then*

$$g^{\omega+1} = \ell = j \circ f^\omega$$

and this is the join $j \vee f^\omega$.

Proof. We have $j \leq g$ and $f \leq g$, to give $f^\omega \leq g^\omega$, and hence $\ell \leq g \circ g^\omega = g^{\omega+1}$. Our problem is to tighten this bound. By a simple induction we have $f^s \circ j \leq j \circ f^s$ for each $s < \omega$. With this, for each $x \in A$ we have

$$(f^\omega \circ j)(x) = f^\omega(j(x)) = \bigvee\{(f^s(j(x)) \mid s < \omega\} \leq (j \circ f^\omega)(x)$$

using the derived comparison. This gives

$$l^2 = j \circ f^\omega \circ j \circ f^\omega \leq j \circ j \circ f^\omega \circ f^\omega = j \circ f^\omega = l$$

since both j and f^ω are nuclei. Thus l is a nucleus. Finally, since $j \leq l$ and $f \leq l$ we have

$$g^{\omega+1} \leq g^\infty \leq \ell^\infty = \ell \leq g^{\omega+1}$$

which gives the required result. ■

We use this result when constructing the reals. It shows that $\omega + 1$ is the rank of the relevant stable inflator. The constuction of the irrationals is more complicated. We do *not* have a comparison $f \circ j \leq j \circ f$ (for the relevant nucleus and inflator), and we find that the rank of both g and l is Ω, the least uncountable ordinal.

1.14 LEMMA. *For the Context 1.12 we have* $g^\delta = \ell^\delta$ *where* $\delta = \omega^\omega$.

Proof. Trivially we have $g \leq \ell$ and a simple induction gives $g^\alpha \leq \ell^\alpha$ for all ordinals α. In particular, we have $g^\delta \leq \ell^\delta$, and it remains to verify the converse comparison.

For $x \in A$ we have

$$(g^\omega \circ g)(x) = g^\omega(g(x)) = \bigvee\{g^s(g(x)) \mid s < \omega\} = \bigvee\{g^{s+1}(x) \mid s < \omega\} = g^\omega(x)$$

so that $g^\omega \circ g = g^\omega$.

Trivially we have $j \leq g$ and $f^\omega \leq g^\omega$ so that $\ell \leq g \circ g^\omega = g^{\omega+1}$. A simple induction now gives

$$\ell^s \leq g^{\omega\cdot s+1} = g \circ g^{\omega\cdot s}$$

for each $s < \omega$, and hence

$$\ell^\omega = \bigvee\{l^s \mid s < \omega\} \leq g^{\omega^2}$$

since $\omega \cdot s + 1 \leq \omega^2$ for each $s < \omega$. A simple ordinal induction now gives

$$\ell^{\omega\cdot\alpha} \leq g^{\omega^2\cdot\alpha}$$

for each ordinal α, and hence $\ell^\delta \leq g^\delta$ since $\omega \cdot \delta = \delta = \omega^2 \cdot \delta$. ■

This does *not* show that g and ℓ have the same rank, but we do have the following.

1.15 THEOREM. *For the Context 1.12, if the rank of* ℓ *is* Ω*, then so is that of* g.

Proof. Suppose the rank of ℓ is Ω, and let θ be the rank of g. Using Lemma 1.14 a simple induction gives

$$g^{\delta\cdot\alpha} = \ell^{\delta\cdot\alpha}$$

for each ordinal α. In particular

$$g^\Omega = \ell^\Omega$$

(since $\delta \cdot \Omega = \Omega$), and hence $\theta \leq \Omega$.

By way of contradiction suppose $\theta < \Omega$, and consider any $\theta < \alpha < \Omega$. Then we have $\theta \leq \delta \cdot \theta < \delta \cdot \alpha$, so that

$$g^{\delta\cdot\theta} = g^{\delta\cdot\alpha} \qquad \ell^{\delta\cdot\theta} \lneq \ell^{\delta\cdot\alpha}$$

which is contradictory. ■

2 A general programme

For a poset S the frame $\mathcal{L}S$ of lower sections of S might seem a rather pathetic example of a topology. We describe one method of turning $\mathcal{L}S$ into something more interesting.

As remarked in Subsection 1.2, each frame A has a point space $\mathsf{pt}(A)$ together with a surjective frame morphism

$$A \longrightarrow \mathcal{O}(\mathsf{pt}(A))$$

indexing the carried topology. In general the points of A can be viewed in several different ways. When $A = \mathcal{L}S$ these points are best seen as the filters on S.

2.1 DEFINITION. Let S be a poset. A **filter** on S is a non-empty upper section π which is downward directed, that is for each $r, s \in \pi$ there is some $t \in \pi$ with $t \leq r, s$. ■

Let $\mathcal{F}S$ be any collections of filters on S. For each $U \in \mathcal{L}S$ let $f(U) \subseteq \mathcal{F}S$ given by

$$\pi \in f(U) \Longleftrightarrow \pi \text{ meets } U$$

for $\pi \in \mathcal{F}S$. The range $\mathcal{O}(\mathcal{F}S) = f[\mathcal{L}S]$ of f is a topology on $\mathcal{F}S$, and

$$\mathcal{L}S \xrightarrow{\ f\ } \mathcal{O}(\mathcal{F}S)$$

is a surjective frame morphism. As well as the family $\mathcal{O}(\mathcal{F}S)$ of open sets there is the associated family $\mathcal{C}(\mathcal{F}S)$ of closed sets. Sometimes we use the dual complement of f

$$\Upsilon S \xrightarrow{\ \overline{f}\ } \mathcal{C}(\mathcal{F}S)$$

to the family of closed set of the filter space given by

$$\pi \in \overline{f}(X) \Longleftrightarrow \pi \subseteq X \qquad \overline{f}(X) = f(X')'$$

for $X \in \Upsilon S$ and $\pi \in \mathcal{F}S$.

Since f is a frame morphism it has a kernel nucleus ϕ (on $\mathcal{L}S$) given by

$$V \subseteq \phi(U) \Longleftrightarrow f(V) \subseteq f(U)$$

for $U, V \in \mathcal{L}S$. Since f is surjective, this nucleus determines $\mathcal{O}(\mathcal{F}S)$ up to isomorphism. The dual complement $\overline{\phi}$ of ϕ (on ΥS) is given by

$$\overline{\phi}(X) \subseteq Y \Longleftrightarrow \overline{f}(X) \subseteq \overline{f}(Y) \qquad \overline{\phi}(X) = \phi(X')'$$

for $X, Y \in \Upsilon S$. This has another description.

2.2 LEMMA. *For the situation described above we have*

$$\overline{\phi}(X) = \bigcup \{\pi \in \mathcal{F}S \mid \pi \subseteq X\}$$

for each $X \in \Upsilon S$.

Proof. For $X \in \Upsilon S$ let Z be the indicated union of filters. For each $Y \in \Upsilon S$ we have

$$\overline{\phi}(X) \subseteq Y \iff \overline{f}(X) \subseteq \overline{f}(Y) \iff (\forall \pi \in \mathcal{F}S)[\pi \in X \implies \pi \in Y] \iff Z \subseteq Y$$

to give the required result. ■

The point space $\mathsf{pt}(\mathcal{L}S)$ of $\mathcal{L}S$ is the space $\mathcal{F}S$ of *all* filters on S. What about the space $\mathcal{N}S$ of all non-principal filters on S?

To set up $\mathcal{O}(\mathcal{N}S)$ as a quotient of $\mathcal{L}S$ we use the precursor of all inflators.

Consider a T_0 space S with topology $\mathcal{O}S$ and family $\mathcal{C}S$ of closed subsets. For $X \in \mathcal{C}S$ let $\boldsymbol{lim}(X)$ be the set of limit points, the non-isolated points, of X, those $s \in X$ for which there is no $U \in \mathcal{O}S$ with $X \cap U = \{s\}$. This is a closed subset of X. The set X is perfect if $\boldsymbol{lim}(X) = X$. The set $\boldsymbol{lim}(X)$ need not be perfect, but by iterating through the ordinals we obtain the perfect part $\boldsymbol{lim}^\infty(X)$ of X. For $U \in \mathcal{O}S$ we set

$$\boldsymbol{der}(U) = \boldsymbol{lim}(U')'$$

to obtain the CB-pre-nucleus $\boldsymbol{der}$ on $\mathcal{O}S$.

Now return to the poset case S. The topology $\mathcal{O}S$ is $\mathcal{L}S$, and $\mathcal{C}S$ is the family ΥS of upper sections of S. For $X \in \Upsilon S$ the isolated points of X are the minimal members of X. The set X is perfect if it has no minimal members. The CB-process $\boldsymbol{lim}$ repeatedly throws away minimal members.

2.3 LEMMA. *Let S be a poset with associated space $\mathcal{N}S$ of non-principal filters. Then*

$$\boldsymbol{lim}^\infty(X) = \bigcup\{\pi \in \mathcal{N}S \mid \pi \subseteq X\}$$

for each $X \in \mathcal{C}S$.

Proof. Given a $X \in \mathcal{C}S$ let

$$Y = \bigcup\{\pi \in \mathcal{N}S \mid \pi \subseteq X\} \qquad Z = \boldsymbol{lim}^\infty(X)$$

so that $Y = Z$ is required.

To show that Y is perfect and hence $Y \subseteq Z$, consider any $s \in Y$. Then $s \in \pi \subseteq X$ for some $\pi \in \mathcal{N}S$. But π is non-principal and has no minimal elements, to give some $t \in \pi \subseteq X$ with $t \lneq s$. Thus s is not minimal in Y, and so is not isolated in Y.

Conversely to show that $Z \subseteq Y$ we use the fact that Z is perfect. Consider any $s \in Z$. Then s is not a minimal element of Z and hence there is some $t \lneq s$ with $t \in Z$. By iterating this process we obtain a strictly descending chain

$$s = s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_i \geq \cdots \quad (i < \omega)$$

of elements of Z. Using this chain we see that

$$r \in \pi \iff (\exists i < \omega)[s_i \leq r]$$

gives some $\pi \in \mathcal{N}S$ with $s \in \pi \subseteq Z \subseteq X$, and hence $s \in Y$, as required. ■

Notice the use of DEPENDENT CHOICE in this proof.

2.4 THEOREM. *Let S be a poset with associated space $\mathcal{N}S$ of non-principal filters.*

$$\mathcal{L}S \xrightarrow{\ f\ } \mathcal{O}(\mathcal{N}S)$$

The kernel of the canonical quotient f is just the CB-nucleus $\boldsymbol{der}^\infty$.

The topology of every space can be presented (in many ways) as a quotient of the frame $\mathcal{L}S$ of lower sections of some poset S. For instance, we can always take the topology for S, but usually there are more interesting generating posets. In this way we may view any topological space as the space $\mathcal{F}S$ of certain filters on some poset S via a surjective frame morphism

$$\mathcal{L}S \xrightarrow{\ f\ } \mathcal{O}(\mathcal{F}S)$$

as above. This space is determined by a certain nucleus ϕ on $\mathcal{L}S$. Suppose we can generate ϕ as the closure l^∞ of a certain inflator l on $\mathcal{L}S$. Then we have a concise description (S, l) of the space, and the closure ordinal ∞ is some kind of measure of the complexity of the construction.

To produce the space from S we need to kill the unwanted filters on S. This process was first named in [4] where it is shown that a use of $\boldsymbol{der}$ (or $\boldsymbol{lim}$) can produce interesting examples. Here we use more powerful inflators.

3 The reals and the irrationals

Starting from the rationals $\mathbb{Q}$ as a linearly ordered set, we produce the reals and the irrationals as spaces. We use $\mathbb{Q}$ to construct a simple poset $\mathbb{S}$. We easily write down posers which combine to give the spaces, but proving they do this takes longer.

We locate the reals and the irrationals as subspaces of the point space of $\mathcal{L}\mathbb{S}$, that is we show how to view each real as a filter on $\mathbb{S}$. This gives us a pair of nuclei ρ (for the reals) and ι (for the irrationals) on $\mathcal{L}\mathbb{S}$. The posers combine to give a pair of stable coverages on $\mathbb{S}$, and hence we have a pair $\boldsymbol{real}$ and $\boldsymbol{irrt}$ of stable inflators on $\mathcal{L}\mathbb{S}$. We show that $\boldsymbol{real}^\infty = \rho$ and $\boldsymbol{irrt}^\infty = \iota$, and later we determine the two closure ordinals.

The inflators $\boldsymbol{real}$ and $\boldsymbol{irrt}$ kill the unwanted points of $\mathcal{L}\mathbb{S}$. Most are easy to eliminate, but then differences appear. For the reals a simple COMPACTNESS argument will do, but for the irrationals we need something like a BAIRE CATEGORY argument.

The construction of the reals has been known for many years. The first documented description seems to be in [4], but the construction did not originate there. In essence it is the Dedekind completion method with a few twiddly bits. A rather terse account is given on pages 123 and 124 of [7]. In the account here I determine various ordinal bounds, and consequently take a little longer.

The construction of the irrationals, or at least its fine details, appears to be new here.

3.1 The parent poset

Rather than the full reals and irrationals we construct the real and irrational intervals

$$\mathbb{I} = (0, 1) \qquad \mathbb{J} = \mathbb{I} \cap \text{‘Irrationals’}$$

as spaces. This way we don't have to mess around with points at infinity. To help you remember which is which, $\mathbb{I}$ stands for Interval whereas $\mathbb{J}$ is a bit Jarring.

$\mathbb{I} = (0,1) \cap$ 'Reals'	$\mathbb{J} = \mathbb{I} \cap$ 'Irrationals'
$\mathbb{K} = \mathbb{I}$	$\mathbb{K} = \mathbb{J}$
$\chi = \rho$	$\chi = \iota$
$\boldsymbol{inf} = \boldsymbol{lap}$	$\boldsymbol{inf} = \boldsymbol{spl}$
$\boldsymbol{gen} = \boldsymbol{real}$	$\boldsymbol{gen} = \boldsymbol{irrt}$
$\mathcal{K}\mathbb{S} = \mathcal{I}\mathbb{S}$	$\mathcal{K}\mathbb{S} = \mathcal{J}\mathbb{S}$

Table 1: Some unified notation

We let a, b, l, r, m, n range over $\mathbb{Q}$, often as pairs $\langle a, b\rangle, \langle l, r\rangle, \langle m, n\rangle$. We let p, q range over $\mathbb{R}$, sometimes as an interval (p, q).

3.1 DEFINITION. Let $\mathbb{S}$ be the set of all ordered pairs $\langle a, b\rangle$ of rationals with $0 \leq a < b \leq 1$. We compare such pairs by

$$\langle l, r\rangle \subseteq \langle a, b\rangle \iff a \leq l < r \leq b$$

to obtain a poset. ■

Strictly speaking each member $\langle a, b\rangle$ of $\mathbb{S}$ is just a pair of markers, but we secretly think of it as the corresponding real interval (a, b).

3.2 DEFINITION. Let $\Vdash_{\text{out}}$ $\Vdash_{\text{lap}}$ $\Vdash_{\text{spl}}$ be the posers on $\mathbb{S}$ whose only instances are

$\langle a, b\rangle \Vdash_{\text{out}} Z$ where $Z = \{\langle l, r\rangle \mid a < l < r < b\}$
$\langle a, b\rangle \Vdash_{\text{lap}} Z$ where $Z = \{\langle a, b\rangle\}$ or $\{\langle a, n\rangle, \langle m, b\rangle\}$ for some $a \leq m < n \leq b$
$\langle a, b\rangle \Vdash_{\text{spl}} Z$ where $Z = \{\langle a, b\rangle\}$ or $\{\langle a, m\rangle, \langle m, b\rangle\}$ for some $a < m < b$

for rationals a, b, m, n. ■

The first alternative in the definition of $\Vdash_{\text{lap}}$ is included to give a direct comparison with $\Vdash_{\text{spl}}$. Each of these posers has an associated coverage $\vdash_{\text{out}}$ $\vdash_{\text{lap}}$ $\vdash_{\text{spl}}$ and, in fact, $\Vdash_{\text{out}}$ is already a coverage. A proof of the following is easy.

3.3 LEMMA. *Each of the three coverages* $\vdash_{\text{out}}, \vdash_{\text{lap}}, \vdash_{\text{spl}}$ *is stable.*

We locate the spaces $\mathbb{I}$ and $\mathbb{J}$ within $\mathsf{pt}(\mathcal{L}\mathbb{S})$, the space of filters on $\mathbb{S}$. The two constructions are similar with some differences. We look at the two cases in unison, so we introduce a unified notation as in Table 1. We let $\mathbb{K} = \mathbb{I}$ or $\mathbb{K} = \mathbb{J}$ depending on which case we are thinking of.

3.4 DEFINITION. For $p \in \mathbb{K}$ let $\widetilde{p}$ be the set of pairs $\langle a, b\rangle \in \mathbb{S}$ with $a < p < b$. ■

Observe that $\widetilde{p}$ is a filter on $\mathbb{S}$, and so gives a point of $\mathcal{L}\mathbb{S}$. We may check that $p \longmapsto \widetilde{p}$ is a topological embedding of $\mathbb{K}$ in $\mathsf{pt}(\mathcal{L}\mathbb{S})$. The idea is to kill the other points of $\mathcal{L}\mathbb{S}$. We don't work with the embedding $p \longmapsto \widetilde{p}$, but with the induced frame morphism.

3.5 DEFINITION. Let

$$\mathcal{L}\mathbb{S} \xrightarrow{\;k\;} \mathcal{O}\mathbb{K} \qquad\qquad \Upsilon\mathbb{S} \xrightarrow{\;\overline{k}\;} \mathcal{C}\mathbb{K}$$

be the surjective frame morphism and its dual complement given by

$$p \in k(U) \Longleftrightarrow \widetilde{p} \text{ meets } U \qquad p \in \overline{k}(X) \Longleftrightarrow \widetilde{p} \subseteq X$$

for $p \in \mathbb{K}, U \in \mathcal{L}\mathbb{S}, X \in \Upsilon\mathbb{S}$. Let χ be the kernel of k, that is the nucleus on $\mathcal{L}\mathbb{S}$ given by

$$V \subseteq \chi(U) \Longleftrightarrow k(V) \subseteq k(U)$$

for $U, V \in \mathcal{L}\mathbb{S}$. ■

This gives the next line of Table 1. We let

$$\rho = \chi \text{ for } \mathbb{K} = \mathbb{I} \qquad \iota = \chi \text{ for } \mathbb{K} = \mathbb{J}$$

to obtain two nuclei $\rho \le \iota$ on $\mathcal{L}\mathbb{S}$. We describe both of these in terms of inflators on $\mathcal{L}\mathbb{S}$.

We should check that $\{k(U) \mid U \in \mathcal{L}\mathbb{S}\}$ is precisely the metric topology on $\mathbb{K}$. Consider any $p \in k(U)$ where $U \in \mathcal{L}\mathbb{S}$. Then $\langle a, b\rangle \in U$ for some rationals $a < p < b$, and hence

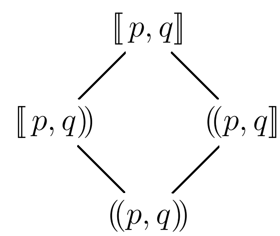
$$p \in (a, b) \subseteq k(U)$$

to show that $k(U)$ is metric open. Conversely, for rationals $0 \le a < b \le 1$ let U be the principal lower section of $\mathbb{S}$ generated by $\langle a, b\rangle$. Then $k(U) = (a, b)$ by a simple argument.

It is useful to have a concrete description of the filters on $\mathbb{S}$.

3.6 DEFINITION. For each pair of reals $0 \le p \le q \le 1$ consider the subsets of $\mathbb{S}$

$$\begin{array}{lll} [\![p,q]\!] & \text{given by} & \langle a,b\rangle \in [\![p,q]\!] \Longleftrightarrow a \le p \le q \le b \\ [\![p,q)\!) & \text{given by} & \langle a,b\rangle \in [\![p,q)\!) \Longleftrightarrow a \le p \le q < b \\ (\!(p,q]\!] & \text{given by} & \langle a,b\rangle \in (\!(p,q]\!] \Longleftrightarrow a < p \le q \le b \\ (\!(p,q)\!) & \text{given by} & \langle a,b\rangle \in (\!(p,q)\!) \Longleftrightarrow a < p \le q < b \end{array}$$



for $(l, r) \in \mathbb{S}$. ■

Each of these is an upper section of $\mathbb{S}$. A few are empty, but most are filters. Thus

$$[\![p,1)\!) = (\!(p,1)\!) = \emptyset = (\!(0,q)\!) = (\!(0,q]\!]$$

but all others are non-empty. For each $0 < p \le q < 1$ we have

$$(\!(p,q)\!) = [\![p,q)\!) \cap (\!(p,q]\!]$$

with inclusions as indicated in the diagram. If either p or q is irrational then certain inclusions collapse to equalities. When both p, q are rational the top filter $[\![p,q]\!]$ is principal. For $0 < p \le q < 1$ each of

$$[\![p,q)\!) \quad (\!(p,q)\!) \quad (\!(p,q]\!]$$

is non-principal. In particular, for each $p \in \mathbb{I}$ the filter $\widetilde{p} = (\!(p,p)\!)$ is non-principal.

The points of $\mathcal{L}\mathbb{S}$ are precisely the filters on $\mathbb{S}$. Using thye notation of Definition 3.6 we can say what these filters are.

3.7 THEOREM. *For each filter* π *on* $\mathbb{S}$ *one of*

$$\pi = [\![p,q]\!] \quad \pi = [\![p,q)\!) \quad \pi = (\!(p,q]\!] \quad \pi = (\!(p,q)\!)$$

holds for unique reals $0 \le p \le q \le 1$.

Proof. Consider a filter π on $\mathbb{S}$. Each pair $s = \langle a,b\rangle \in \pi$ has a closure $s^- = [a,b]$ as an interval. Using the filter properties, the family $\{s^- \mid s \in \pi\}$ of all these closures has the finite intersection property. By the appropriate COMPACTNESS result

$$\bigcap \{s^- \mid s \in \pi\}$$

is a non-empty closed interval. Let

$$p = \sup\{a \mid (\exists b)[\langle a,b\rangle \in \pi]\} \qquad q = \inf\{b \mid (\exists a)[\langle a,b\rangle \in \pi]\}$$

so that $(\!(p,q)\!) \subseteq \pi \subseteq [\![p,q]\!]$. We check that π must be one of the four listed filters. ■

This shows that the point space of $\mathcal{L}\mathbb{S}$ is full of spikes and bumps. We want to remove all these points except the $\widetilde{p}$ where p is real or irrational depending on the case. As a taster, let's have a look at the CB-gadgets.

3.8 DEFINITION. For each $U \in \mathcal{L}\mathbb{S}$ and $X \in \Upsilon\mathbb{S}$ let $\boldsymbol{der}(U)$ and $\boldsymbol{lim}(X)$ be the subsets of $\mathbb{S}$ given by

$$\langle a,b\rangle \in \boldsymbol{der}(U) \iff (\forall a \le l < r \le b)[(a < l \text{ or } r < b) \Longrightarrow \langle l,r\rangle \in U]$$
$$\langle a,b\rangle \in \boldsymbol{lim}(X) \iff (\exists a \le l < r \le b)[(a < l \text{ or } r < b) \text{ and } \langle l,r\rangle \in X]$$

for $\langle a,b\rangle \in \mathbb{S}$. In both cases the quantification is over $\mathbb{Q}$, that is $l, r \in \mathbb{Q}$. ■

It is not too difficult to check that $\boldsymbol{lim}$ is the classical Cantor-Bendixson derivative on $\Upsilon\mathbb{S}$, and $\boldsymbol{der}$ is its dual complement. On general grounds we know that $\boldsymbol{der}$ is a pre-nucleus. Here it has a much stronger property.

3.9 LEMMA. *The derivative* $\boldsymbol{der}$ *on* $\mathcal{L}\mathbb{S}$ *is a nucleus.*

Proof. It suffices to show that $\boldsymbol{der}$ is idempotent. Consider any $\langle a,b\rangle \in \boldsymbol{der}^2(U)$ where $U \in \mathcal{L}\mathbb{S}$, We require $\langle a,b\rangle \in \boldsymbol{der}(U)$. To this end consider any rationals $a \le l < r \le b$ where $a < l$ or $r < b$. We require $\langle l,r\rangle \in U$.

Suppose $a < l$. Since $\mathbb{Q}$ is a dense linear order, there are rationals $a < m < l < r \le n \le b$. But now $\langle m,n\rangle \in \boldsymbol{der}(U)$, and hence $\langle l,r\rangle \in U$, as required. The other case is dealt with in the same way. ■

The trick in this proof is the denseness of the rationals. We use this property quite a lot, often without mentioning the fact.

3.2 Three inflators

We use various combinations of inflators each of which is more powerful that $\boldsymbol{der}$. The following inflators corresponding to the posers of Definition 1.12, hence the notation.

3.10 DEFINITION. For each $U \in \mathcal{L}\mathbb{S}$ let

$$\boldsymbol{out}(U) \qquad \boldsymbol{lap}(U) \qquad \boldsymbol{spl}(U)$$

be the subsets of $\mathbb{S}$ given by

$$\begin{aligned}
\langle a,b\rangle \in \boldsymbol{out}(U) &\iff (\forall a < l < r < b)[\langle l,r\rangle \in U] \\
\langle a,b\rangle \in \boldsymbol{lap}(U) &\iff (\exists a \le m < n \le b)[\langle a,n\rangle, \langle m,b\rangle \in U] \\
\langle a,b\rangle \in \boldsymbol{spl}(U) &\iff (\exists a < m < b)[\langle a,m\rangle, \langle m,b\rangle \in U]
\end{aligned}$$

for $\langle a,b\rangle \in \mathbb{S}$. Here the quantification is over $\mathbb{Q}$, that is $l,r,m,n \in \mathbb{Q}$. ■

In this subsection we look at the basic properties of these three gadgets. In particular, we show that each is an inflator on $\mathcal{L}\mathbb{S}$, and so each has a dual complement

$$\overline{\boldsymbol{out}} \qquad \overline{\boldsymbol{lap}} \qquad \overline{\boldsymbol{spl}}$$

operating on $\Upsilon\mathbb{S}$. These are given by

$$\begin{aligned}
\langle a,b\rangle \in \overline{\boldsymbol{out}}(X) &\iff (\exists a < l < r < b)[\langle l,r\rangle \in X] \\
\langle a,b\rangle \in \overline{\boldsymbol{lap}}(X) &\iff (\forall a \le m < n \le b)[\langle a,n\rangle \in X \text{ or } \langle m,b\rangle \in X] \\
\langle a,b\rangle \in \overline{\boldsymbol{spl}}(X) &\iff (\forall a < m < b)[\langle a,m\rangle \in X \text{ or } \langle m,b\rangle \in X]
\end{aligned}$$

For $X \in \Upsilon\mathbb{S}$ and $\langle a,b\rangle \in \mathbb{S}$. As in Definition 3.10, the quantification is over $\mathbb{Q}$.

3.11 LEMMA. *The operation* $\boldsymbol{out}$ *is a nucleus on* $\mathcal{L}\mathbb{S}$ *with* $\boldsymbol{der} \le \boldsymbol{out} \le \rho$*.*

Proof. Almost trivially, $\boldsymbol{out}$ is an inflator on $\mathcal{L}\mathbb{S}$.
To see that it is a pre-nucleus consider

$$\langle a,b\rangle \in \boldsymbol{out}(U) \cap \boldsymbol{out}(V)$$

for $U,V \in \mathcal{L}\mathbb{S}$, and consider any rational $a < l < r < b$. We have both

$$\langle a,b\rangle \in \boldsymbol{out}(U) \qquad \langle a,b\rangle \in \boldsymbol{out}(V)$$

so that $\langle l,r\rangle \in U \cap V$ to give the required result.

To show that $\boldsymbol{out}$ is idempotent consider $\langle a,b\rangle \in \boldsymbol{out}^2(U)$ where $U \in \mathcal{L}\mathbb{S}$. We require $\langle a,b\rangle \in \boldsymbol{out}(U)$. To this end consider any rationals $a < l < r < b$, so that $\langle l,r\rangle \in U$ is required. But there are rationals

$$a < m < l < r < n < b$$

and then $\langle m,n\rangle \in \boldsymbol{out}(U)$ (since $\langle a,b\rangle \in \boldsymbol{out}^2(U)$) to give $\langle l,r\rangle \in U$, as required.

The comparison $\boldsymbol{der} \le \boldsymbol{out}$ is immediate.

To show $\boldsymbol{out} \le \rho$ we use the morphism k of Definition 3.5 for the case $\mathbb{K} = \mathbb{I}$. Thus ρ is the nucleus of this k. Consider $U \in \mathcal{L}\mathbb{S}$ and let $V = \boldsymbol{out}(U)$. An inclusion $k(V) \subseteq k(U)$ will give $V \subseteq \rho(U)$, which is the required result.

Consider any $p \in k(V)$. Then $\widetilde{p}$ meets V, to give rationals $a < p < b$ with $\langle a,b\rangle \in V = \boldsymbol{out}(U)$. Consider any rationals $a < l < r < b$. Then $\langle l,r\rangle \in U$, to show that $\widetilde{p}$ meets U, and hence $p \in k(U)$, as required. ■

We have $\boldsymbol{der} \le \boldsymbol{out}$, but these two are different.

3.12 EXAMPLE. We have $\overline{\boldsymbol{out}} \le \boldsymbol{lim}$, and we now show that these are different. A simple calculation gives

$$\boldsymbol{out}(\llbracket p, q \rrbracket) = (\!(p, q)\!)$$

for all reals $0 < p \le q < 1$. In particular $\boldsymbol{out}(\llbracket p, p \rrbracket) = \widetilde{p}$ for each rational $0 < p < 1$. For such a p consider rational $a = p < b < 1$. Then

$$\langle a, b \rangle \in \boldsymbol{lim}(\llbracket p, p \rrbracket) - \widetilde{p}$$

to illustrate the difference, ■

We now look at $\boldsymbol{lap}$ and $\boldsymbol{spl}$ in unison. For this we use the next line of Table 1. Let

$$\boldsymbol{inf} = \boldsymbol{lap} \qquad \boldsymbol{inf} = \boldsymbol{spl}$$

as appropriate for the case. Notice that for $U \in \mathcal{L}\mathbb{S}$ we have

$$(a, b) \in \boldsymbol{inf}(U)$$

precisely when there are intervals

$$(a, n), (m, b) \in U$$

with

$$(\mathbb{I}) \quad a \le m < n \le b \qquad\qquad a < m = n < b \quad (\mathbb{J})$$

as appropriate for the case. For both cases we decompose an interval (a, b) into two parts. In the $\boldsymbol{lap}$ case these parts must overlap, but in the $\boldsymbol{spl}$ case they merely abut.

3.13 LEMMA. *Each of* $\boldsymbol{lap}$ *and* $\boldsymbol{spl}$ *is a stable inflator on* $\mathcal{L}\mathbb{S}$ *with* $\boldsymbol{der} \le \boldsymbol{lap} \le \boldsymbol{spl}$.

Proof. Let $\boldsymbol{inf}$ be either $\boldsymbol{lap}$ or $\boldsymbol{spl}$.

Consider $U \in \mathcal{L}\mathbb{S}$. To show $U \subseteq \boldsymbol{inf}(U)$ consider any $\langle a, b \rangle \in U$ and take any rationals $a \le m \le n \le b$ as in $(\mathbb{I}, \mathbb{J})$. We have

$$\langle a, n \rangle, \langle m, b \rangle \subseteq \langle a, b \rangle \in U \in \mathcal{L}\mathbb{S}$$

to give $\langle a, n \rangle, \langle m, b \rangle \in U$ and hence $\langle a, b \rangle \in \boldsymbol{inf}(U)$.

Consider $U \in \mathcal{L}\mathbb{S}$. To show $\boldsymbol{inf}(U) \in \mathcal{L}\mathbb{S}$. Consider $\langle l, r \rangle \subseteq \langle a, b \rangle \in \boldsymbol{inf}(U)$. Thus there are rationals

$$a \le l < r \le b \qquad a \le m \le n \le b$$

with $\langle a, n \rangle, \langle m, b \rangle \in U$ and where m, n are restricted by $(\mathbb{I}, \mathbb{J})$ as appropriate for the case. There are three possibilities

$$r \le n \qquad l < m \le n < r \qquad m \le l$$

depending how l, r, m, n sit in (a, b). The two outer possibilities give

$$\langle l, r \rangle \subseteq \langle a, n \rangle \in U \qquad \langle l, r \rangle \subseteq \langle m, b \rangle \in U$$

respectively, and hence $\langle l, r \rangle \in U \subseteq \boldsymbol{inf}(U)$. The central possibility gives a decomposition to show $\langle l, r \rangle \in \boldsymbol{inf}(U)$.

This shows that $\boldsymbol{inf}$ is an operation on $\mathcal{L}\mathbb{S}$, and it is immediate that it is an inflator.

To show that $\boldsymbol{inf}$ is stable consider $\langle a,b\rangle \in \boldsymbol{inf}(U)\cap V$ for $U,V \in \mathcal{L}\mathbb{S}$. Since $\langle a,b\rangle \in \boldsymbol{inf}(U)$ we have

$$\langle a,n\rangle, \langle m,b\rangle \in U$$

for appropriate rationals $a \le m \le n \le b$. Since $\langle a,b\rangle \in V$ this gives

$$\langle a,n\rangle, \langle m,b\rangle \in U\cap V$$

and hence $\langle a,b\rangle \in \boldsymbol{inf}(U\cap V)$, as required.

To show that $\boldsymbol{der} \le \boldsymbol{lap}$ consider $\langle a,b\rangle \in \boldsymbol{der}(U)$ where $U \in \mathcal{L}\mathbb{S}$. Consider rationals $a < m < n < b$. We show that $\langle a,n\rangle, \langle m,b\rangle \in U$. With $l = a$ and $r = n$ we have $a \le l < r < b$, so that $\langle a,n\rangle = \langle l,r\rangle \in U$. With $l = m$ and $r = b$ we get $\langle m,b\rangle \in U$.

The comparison $\boldsymbol{lap} \le \boldsymbol{spl}$ is immediate. ■

As the following examples show, neither $\boldsymbol{lap}$ nor $\boldsymbol{spl}$ is a pre-nucleus.

3.14 EXAMPLES. (a) Let U,V be the sets of pairs given by

$$\langle a,b\rangle \in U \iff 1/3 \notin (a,b) \qquad \langle a,b\rangle \in V \iff 2/3 \notin (a,b)$$

for $\langle a,b\rangle \in \mathbb{S}$. Trivially we have $U,V \in \mathcal{L}\mathbb{S}$. The intersection $U\cap V$ is the set of pairs which 'contain' neither $1/3$ nor $2/3$. We have

$$\langle 0,1\rangle \in \boldsymbol{spl}(U)\cap \boldsymbol{spl}(V) \qquad \langle 0,1\rangle \notin \boldsymbol{spl}(U\cap V)$$

by two splittings in the left hand case, and since any splitting of $\langle 0,1\rangle$ must pick up either $1/3$ or $2/3$ in the right hand case. This shows that $\boldsymbol{spl}$ is not a pre-nucleus.

(b) Let U,V be the sets of intervals given by

$$\langle a,b\rangle \in U \iff |\,(a,b)\cap\{1/4,1/2\}| \le 1 \qquad \langle a,b\rangle \in V \iff |\,(a,b)\cap\{1/2,3/4\}| \le 1$$

for $\langle a,b\rangle \in \mathbb{S}$, where $|\cdot|$ indicates cardinality. In both cases the condition on $\langle a,b\rangle$ is that the (real) interval contains at most one of two nominated rationals. The decompositions

$$0 < 1/4 < 1/2 < 1 \qquad 0 < 1/2 < 3/4 < 1$$

show that

$$\langle 0,1\rangle \in \boldsymbol{lap}(U) \qquad \langle 0,1\rangle \in \boldsymbol{lap}(V)$$

respectively. By way of contradiction, suppose $\langle 0,1\rangle \in \boldsymbol{lap}(U\cap V)$ and let $0 \le m < n \le 1$ be a witnessing decomposition. We have

$$\langle 0,n\rangle \in U \qquad \langle m,1\rangle \in V$$

so that $n \le 1/2 \le m$ which, since $m < n$, is nonsense. Thus $\boldsymbol{lap}$ is not pre-nucleus. ■

Each of $\boldsymbol{lap}$ and $\boldsymbol{spl}$ is a stable inflator and therefore each closure $\boldsymbol{lap}^\infty$ and $\boldsymbol{spl}^\infty$ is a nucleus. The closure ordinal is not too large, it is no more than ω.

3.15 LEMMA. *Each of $\boldsymbol{lap}^\omega$ and $\boldsymbol{spl}^\omega$ is a nucleus.*

Proof. Let $\boldsymbol{inf}$ be either $\boldsymbol{lap}$ or $\boldsymbol{spl}$. Thus $\boldsymbol{inf}$ is a stable inflator and hence $\boldsymbol{inf}^\omega$ is a pre-nucleus. Thus the required result follows from an inclusion

$$\boldsymbol{inf}^{\omega+1}(U) \subseteq \boldsymbol{inf}^\omega(U)$$

for arbitrary $U \in \mathcal{L}\mathbb{S}$. Consider any $\langle a, b\rangle \in \boldsymbol{inf}^{\omega+1}(U)$. We have

$$\langle a, n\rangle, \langle m, b\rangle \in \boldsymbol{inf}^\omega(U)$$

for some rationals $a \leq m \leq n \leq b$ appropriate for the case. Since $\boldsymbol{inf}^\omega(U)$ is a pointwise union we have

$$\langle a, n\rangle, \langle m, b\rangle \in \boldsymbol{inf}^r(U)$$

for some $r < \omega$, and hence $\langle a, b\rangle \in \boldsymbol{inf}^{r+1}(U) \subseteq \boldsymbol{inf}^\omega(U)$ to give the required result. ■

The following shows that the rank of each of $\boldsymbol{lap}$ and $\boldsymbol{spl}$ is precisely ω.

3.16 EXAMPLE. For each strictly positive real number l let $V(l)$ be the set of all pairs $\langle a, b\rangle \in \mathbb{S}$ with $(b - a) < l$. Trivially we have $V(l) \in \mathcal{L}\mathbb{S}$ with $V(l) = \mathbb{S}$ if $1 < l$. We show

$$\boldsymbol{inf}(V(l)) = V(2l)$$

and hence no finite iterate $\boldsymbol{inf}^r$ is a nucleus.

Consider first $\langle a, b\rangle \in \boldsymbol{inf}(V(l))$. There are rationals m, n with

$$a \leq m \leq n \leq b \qquad (n - a) < l \quad (b - m) < l$$

with

$$(\mathbb{I}) \quad a \leq m < n \leq b \qquad\qquad a < m = n < b \quad (\mathbb{J})$$

as appropriate for the case. A small calculation gives $(b-a) < 2l$, and hence $\langle a, b\rangle \in V(2l)$.

Conversely, consider $\langle a, b\rangle \in V(2l)$. Since $(b - a) < 2l$ we have

$$(b - a) + 2\delta < 2l$$

for all sufficiently small rational δ, including 0. Let

$$p = (a + b)/2 \qquad m = p - \delta \qquad n = p + \delta$$

so that $a < m \leq p \leq n < b$ with $(n-a), (b-m) < l$ to give $(n-a), (b-m) \in V(l)$. Thus $\langle a, b\rangle \in \boldsymbol{lap}(V(l)) \subseteq \boldsymbol{spl}(V(l)$, as required. ■

Each of $\boldsymbol{lap}$ and $\boldsymbol{spl}$ does part of the job we want doing.

3.17 LEMMA. *We have both* $\boldsymbol{lap}^\omega \leq \rho$ *and* $\boldsymbol{spl}^\omega \leq \iota$.

Proof. We use the morphism k and it kernel χ of Definition 3.5. Since χ is a nucleus it suffices to show $\boldsymbol{inf} \leq \chi$.

Consider any $U \in \mathcal{L}\mathbb{S}$ and let $V = \boldsymbol{inf}(U)$. We show

$$(\forall p \in \mathbb{K})[\widetilde{p} \text{ meets } V \Longrightarrow \widetilde{p} \text{ meets } U]$$

so that $k(V) \subseteq k(U)$, and hence $V \subseteq \chi(U)$, as required.

Consider any $p \in \mathbb{K}$ where $\widetilde{p}$ meets V. There are rationals $0 \le a < p < b \le 1$ with $\langle a, b\rangle \in \boldsymbol{inf}(U)$. This gives rationals $a \le m \le n \le b$ with $\langle a, n\rangle, \langle m, b\rangle \in U$ and where the appropriate one of $(\mathbb{I}, \mathbb{J})$ holds.

For the $\mathbb{I}$-case the real p can not be equal to both m and n. Thus we have one of

$$a < p < n \qquad m < p < b$$

to give one of

$$\langle a, n\rangle \in U \cap \widetilde{p} \qquad \langle m, b\rangle \in U \cap \widetilde{p}$$

and hence $\widetilde{p}$ meets U.

For the $\mathbb{J}$-case the irrational p can not be equal to the rational $m = n$. Thus again we have one of

$$a < p < n \qquad m < p < b$$

to show that $\widetilde{p}$ meets U. ■

From Lemma 3.11 we have $\boldsymbol{out} \le \rho \le \iota$, and we know that $\boldsymbol{out}$ is not as big as ρ. Since $\boldsymbol{lap} \le \rho$ and $\boldsymbol{spl} \le \iota$ we can use either of these inflators to boost $\boldsymbol{out}$.

3.18 DEFINITION. For each $U \in \mathcal{L}\mathbb{S}$ let

$$\boldsymbol{real}(U) = \boldsymbol{out}(U) \cup \boldsymbol{lap}(U) \qquad \boldsymbol{irrt}(U) = \boldsymbol{out}(U) \cup \boldsymbol{spl}(U)$$

to produce a pair $\boldsymbol{real}, \boldsymbol{irrt}$ of stable inflators on $\mathcal{L}\mathbb{S}$. ■

We may check that $\boldsymbol{real}$ is the inflator corresponding to the associated coverage of the syntactic join of the two posers $\Vdash_{\text{out}}$ and $\Vdash_{\text{lap}}$. Similarly, $\boldsymbol{irrt}$ is the inflator corresponding to the associated coverage of the syntactic join of the two posers $\Vdash_{\text{out}}$ and $\Vdash_{\text{spl}}$.

This gives us the penultimate line of Table 1. We let

$$\boldsymbol{gen} = \boldsymbol{real} \qquad \boldsymbol{gen} = \boldsymbol{irrt}$$

as appropriate for the case. By Lemmas 3.11 and 3.17 we have $\boldsymbol{gen}^\infty \le \chi$. We require a converse comparison.

3.3 The squeezing arguments

The proofs of $\boldsymbol{real}^\infty = \rho$ and $\boldsymbol{irrt}^\infty = \iota$ are similar, but now the difference become more important. With a bit of give and take it is still possible to unify the proofs.

3.19 DEFINITION. We extract two subfamilies $\mathcal{I}\mathbb{S}$ and $\mathcal{J}\mathbb{S}$ of $\Upsilon\mathbb{S}$ as follows.

Let $\mathcal{I}\mathbb{S}$ be the family of those closed sets $Z \in \Upsilon\mathbb{S}$ for which both

$$\begin{aligned} \langle a, b\rangle \in Z &\Longrightarrow \quad (\exists a < l < r < b)[\langle l, r\rangle \in Z] \\ \langle a, b\rangle \in Z &\Longrightarrow (\forall a \le m < n \le b)[\langle a, n\rangle \in Z \text{ or } \langle m, b\rangle \in Z] \end{aligned}$$

hold for each $\langle a, b\rangle \in \mathbb{S}$. Here the quantifications are over $\mathbb{Q}$.

Let $\mathcal{J}\mathbb{S}$ be the family of those closed sets $Z \in \Upsilon\mathbb{S}$ for which both

$$\begin{aligned} (a, b) \in Z &\Longrightarrow (\exists a < l < r < b)[\langle l, r\rangle \in Z] \\ (a, b) \in Z &\Longrightarrow \quad (\forall a < m < b)[\langle a, m\rangle \in Z \text{ or } \langle m, b\rangle \in Z] \end{aligned}$$

hold for each $\langle a, b\rangle \in S$. Again the quantifications are over $\mathbb{Q}$. ■

In both cases the first clause is the same. The second clauses are different and reflect the difference between $\boldsymbol{lap}$ (for $\mathcal{I}\,\mathbb{S}$) and $\boldsymbol{spl}$ (for $\mathcal{J}\,\mathbb{S}$). In particular, we have $\mathcal{J}\,\mathbb{S} \subseteq \mathcal{I}\,\mathbb{S}$.

We can now complete the unified notation of Table 1. We let

$$\mathcal{K}\,\mathbb{S} = \mathcal{I}\,\mathbb{S} \qquad \mathcal{K}\,\mathbb{S} = \mathcal{J}\,\mathbb{S}$$

as appropriate for the case.

3.20 LEMMA. *For each case $\mathcal{K}\,\mathbb{S}$ is the family of those $Z \in \Upsilon\mathbb{S}$ with $\overline{\boldsymbol{gen}}(Z) = Z$.*

To prove $\boldsymbol{gen}^\infty = \chi$ we first show that for each situation $\langle a, b\rangle \in Z \in \mathcal{K}\,\mathbb{S}$ there is a witness $p \in \mathbb{K}$ with $\langle a, b\rangle \in \widetilde{p} \subseteq Z$. We squeeze the interval (a, b) to one of zero length, and a simple COMPACTNESS argument provides p in the real case. For the irrational case we must ensure that the witness is irrational. We use a variant of the BAIRE CATEGORY argument to omit each rational as a possible witness. In both cases the trick is to reduce the length $(b - a)$ of $\langle a, b\rangle$ by a suitable amount.

3.21 LEMMA. (One-step splitting) *For each situation $\langle a, b\rangle \in Z \in \mathcal{K}\,\mathbb{S}$ we have*

$$\langle a', b'\rangle \in Z \qquad (b' - a') \leq 2/3(b - a)$$

for some rationals $a < a' < b' < b$.

Furthermore, in the irrational case we can ensure $q \notin (a', b')$ for any given rational q.

Proof. Before the unified argument, let's show how to omit a given $q \in \mathbb{Q}$ in the irrational case. We start from some $\langle a, b\rangle \in Z \in \mathcal{J}\,\mathbb{S}$. If $q \leq a$ or $b \leq q$, then we are done. If $a < q < b$ then, by the second property of $Z \in \mathcal{J}\,\mathbb{S}$, one of $\langle a, q\rangle, \langle q, b\rangle$ is in Z.

Now for the unified argument. Since $\langle a, b\rangle \in Z \in \mathcal{K}\,\mathbb{S}$ the first property of $Z \in \mathcal{K}\,\mathbb{S}$ gives rationals l, r with $a < l < r < b$ and $\langle l, r\rangle \in Z$. By the second property of $Z \in \mathcal{K}\,\mathbb{S}$ we have rationals $l < m \leq n < r$ where one of $\langle l, n\rangle, \langle m, r\rangle$ is in Z. In the real case we take an equal splitting of (l, r), so that both $\langle l, n\rangle$ and $\langle m, r\rangle$ have length no more than $(2/3)^{\text{rds}}$ that of $\langle l, r\rangle$. In the irrational case we let $m = n$ be the mid point of (l, r), so that both $\langle l, n\rangle$ and $\langle m, r\rangle$ have length no more than half that of $\langle l, r\rangle$.

In both case we obtain $l \leq a' < b' \leq r$ with $\langle a', b'\rangle \in Z$ and

$$(b' - a') \leq 2/3(r - l) \leq 2/3(b - a)$$

to give the required result. (Remember that $1/2 < 2/3$.) ■

Next we iterate this one-step splitting.

3.22 LEMMA. (Witnessing) *For each situation $\langle a, b\rangle \in Z \in \mathcal{K}\,\mathbb{S}$ there is at least one real $p \in \mathbb{K}$ with $\langle a, b\rangle \in \widetilde{p} \subseteq Z$.*

Proof. Starting from the given situation $\langle a, b\rangle \in Z \in \mathcal{K}\,\mathbb{S}$ we may iterate a use of Lemma 3.21 to produce a pair of strict ω-chains

$$a = a_0 < a_1 < \cdots < a_i < \cdots\cdots < b_i < \cdots < b_1 < b_0 = 0$$

where

$$\langle a_i, b_i\rangle \in Z \qquad (b_i - a_i) \leq (2/3)^i(b - a)$$

for each $i < \omega$. This is a use of DEPENDENT CHOICE. By the COMPLETENESS of $\mathbb{I}$ there is a unique *real* $p \in \mathbb{I}$ such that

$$\sup\{a_i \mid i < \omega\} = p = \inf\{b_i \mid i < \omega\}$$

holds. In particular, $\langle a_i, b_i \rangle \in \widetilde{p}$ for each i. Furthermore, for each $\langle l, r \rangle \in \widetilde{p}$, that is with $l < p < r$, there is some index i with $l < a_i < b_i < r$ and hence $\langle l, r \rangle \in Z$.

This is the proof for the real case. For the irrational case we need the witness p to be irrational. Use any enumeration $(q_i \mid i < \omega)$ of $\mathbb{Q}$. When constructing $\langle a_{i+1}, b_{i+1} \rangle$ from $\langle a_i, b_i \rangle$ we use Lemma 3.21 to ensure $q_i \notin (a_{i+1}, b_{i+1})$. Now p can not be rational. ■

With these preliminaries the proof of the following is straight forward.

3.23 THEOREM. *We have* $\boldsymbol{gen}^\infty = \chi$.

Proof. We work with the unified notation of Table 1 and the set up of Definition 3.5. We know that $\boldsymbol{gen}^\infty \le \chi$, and it suffices to show $\chi(U) \subseteq \boldsymbol{gen}^\infty(U)$ for each $U \in \mathcal{L}\mathbb{S}$. To this end let $V = \chi(U)$, so that $k(V) \subseteq k(U)$, and $V \subseteq \boldsymbol{gen}^\infty(U)$ is required. Let

$$X = U' \quad Y = V' \quad Z = \boldsymbol{gen}^\infty(U)'$$

so that we have

$$\overline{k}(X) \subseteq \overline{k}(Y) \quad Z \subseteq X \quad Z \in \mathcal{J}\mathbb{S}$$

and $Z \subseteq Y$ is required. Consider $\langle a, b \rangle \in Z$. By Lemma 3.22 we have $\langle a, b \rangle \in \widetilde{p} \subseteq Z$ for some $p \in \mathbb{K}$. But now

$$p \in \overline{k}(Z) \subseteq \overline{k}(X) \subseteq \overline{k}(Y)$$

to give $\langle a, b \rangle \in \widetilde{p} \subseteq Y$ and hence $\langle a, b \rangle \in Y$, as required. ■

4 The various ranks

We determine the ranks of ***real*** and ***irrt***. We work in the Context 1.12. There is a simple case for ***real*** and a more complicated case for ***irrt***.

4.1 Rank of *real*

By Theorem 3.23 we have $\rho = \boldsymbol{real}^\infty$ where $\boldsymbol{real} = \boldsymbol{out} \dot{\vee} \boldsymbol{lap}$ with ***out*** a nucleus, by Lemma 3.11, and where $\boldsymbol{lap}^\omega$ is a nucleus, by Lemma 3.15. We prove the following.

4.1 THEOREM. *We have*

$$\rho = \boldsymbol{real}^{\omega+1} = \boldsymbol{out} \circ \boldsymbol{lap}^\omega$$

where the rank of ***real*** *is* $\omega + 1$ *and the final use of* ***out*** *is necessary.*

With

$$j = \boldsymbol{out} \qquad f = \boldsymbol{lap} \qquad g = \boldsymbol{real}$$

we work in Context 1.12. The following shows that this is the simple case.

4.2 LEMMA. *We have* $\boldsymbol{lap} \circ \boldsymbol{out} \le \boldsymbol{out} \circ \boldsymbol{lap}$.

Proof. Consider any $\langle a,b\rangle \in \boldsymbol{lap}(\boldsymbol{out}(U))$ where $U \in \mathcal{L}\mathbb{S}$. There are rationals m, n with $a \le m < n \le b$ and $\langle a,n\rangle, \langle m,b\rangle \in \boldsymbol{out}(U)$. We require $\langle a,b\rangle \in \boldsymbol{out}(lap(U))$. To this end consider any rationals $a < l < r < b$, so that $\langle l,r\rangle \in \boldsymbol{lap}(U)$ is required, that is we require rationals $l \le p < q \le r$ with $\langle l,q\rangle, \langle p,r\rangle \in U$. We have

$$a \le m < n \le b \qquad a < l < r < b$$

and by considering the relative positions of m, n, l, r we see there are three cases

$$r < n \quad l \le m < n \le r \quad m < l$$

to be dealt with. For the left hand case we have $a < l < r < n$ with $\langle a,n\rangle \in \boldsymbol{out}(U)$, so that $\langle l,r\rangle \in U$, and we may take $p = l$ and $q = r$. The right hand case is similar. For the central case we take any rationals

$$a < l \le m < p < q < n \le r < b$$

and remember $\langle a,n\rangle, \langle m,b\rangle \in \boldsymbol{out}(U)$ to obtain the required result. ■

This result with Theorem 1.13 shows that

$$\rho = \boldsymbol{real}^{\infty} = \boldsymbol{real}^{\omega+1} = \boldsymbol{out} \circ \boldsymbol{lap}^{\omega}$$

which is part of Theorem 4.1. It remains to show that $\boldsymbol{real}^{\omega} \ne \rho \ne \boldsymbol{lap}^{\omega}$ and for that we produce a particular $W \in \mathcal{L}\mathbb{S}$ with $\boldsymbol{real}^{\omega}(W) = \boldsymbol{lap}^{\omega}(W) \ne \mathbb{S}$ but with $\rho(W) = \mathbb{S}$.

4.3 DEFINITION. Let $(q(i) \mid i \in \mathbb{Z})$ be a 2-way sequence of rationals with

$$\lim_{-\infty \leftarrow i} q(i) = 0 \qquad \lim_{i \to \infty} q(i) = 1$$

and $0 < q(i) < q(j) < 1$ for $i < j \in \mathbb{Z}$. For each $s < \omega$ let $U(s)$ be given by

$$\langle a,b\rangle \in U(s) \iff (\exists i)[q(i) < a < b < q(i+1+2^s)]$$

for rationals $0 \le a < b \le 1$. ■

This gives us an ascending chain

$$U(0) \subseteq U(1) \subseteq \cdots \subseteq U(s) \subseteq \cdots \quad (s < \omega)$$

of members of $\mathcal{L}\mathbb{S}$. We let

$$U(\omega) = \bigcup \{U(s) \mid s < \omega\}$$

to produce $U(\omega) \in \mathcal{L}\mathbb{S}$. Observe that

$$\langle 0,1\rangle \notin U(\omega) \qquad \boldsymbol{out}(U(\omega) = \mathbb{S}$$

which, after a few calculations, will enable us to take $W = U(1)$.

4.4 LEMMA. *We have*

$$\boldsymbol{lap}(U(s)) = U(s+1) \qquad \boldsymbol{out}(U(s)) \subseteq U(s+1)$$

for all $0 < s < \omega$.

Proof. Consider first $\langle a, b\rangle \in \boldsymbol{lap}(U(s))$. There are rationals $a \leq m < n \leq b$ with

$$q(i) < a < n < q(i+1+2^s) \qquad q(j) < m < b < q(j+1+2^s)$$

for some $i, j \in \mathbb{Z}$. From these comparisons we have

$$q(i) < a < b < q(j+1+2^s) \qquad q(j) < m < n < q(i+1+2^s)$$

so that $j \leq i + 2^s$ which gives $j + 2^s \leq i + 2^{s+1}$ and hence

$$q(i) < a < b < q(i+1+2^{s+1})$$

which leads to $\langle a, b\rangle \in U(s+1)$, as required.

Conversely, suppose $\langle a, b\rangle \in U(s+1)$, so that

$$q(i) < a < b < q(i+1+2^{s+1})$$

for some $i \in \mathbb{Z}$. We take the largest such i, that is with $q(i) < a \leq q(i+1)$. If $b < q(i+1+2^s)$ then $\langle a, b\rangle \in U(s)$, and we are done. Thus we may suppose $q(i+1+2^s) \leq b$. Since $i+1 < i+1+2^s$ this gives

$$q(i) < a \leq q(i+1) \leq q(i+2^s) < q(i+1+2^s) \leq b < q(i+1+2^{s+1})$$

and we may exploit the gap. We take any rational pair

$$q(i+2^s) < m < n < q(i+1+2^s)$$

and show these form an overlapping decomposition to ensure $\langle a, b\rangle \in \boldsymbol{lap}(U(s))$.

From the index i we have $\langle a, n\rangle \in U(s)$. For the proof that $\langle m, b\rangle \in U(s)$ consider the unique j such that $q(j) < m \leq q(j+1)$. We have $i + 2^s \leq j$, so that

$$i + 1 + 2^{s+1} \leq j + 1 + 2^s$$

and hence

$$b < b(i+1+2^{s+1}) \leq q(j+1+2^s)$$

which leads to the required result.

The proves the left hand equality. Notice that we did not need $s \neq 0$.

For the right hand inclusion, for arbitrary s consider $\langle a, b\rangle \in \boldsymbol{out}(U(s))$. Consider also the unique index i with $q(i) < a \leq q(i+1)$. If $b < q(i+1+2^s)$ then $\langle a, b\rangle \in U(s)$ and we are done. Thus we may suppose that

$$q(i) < a \leq q(i+1) < q(i+2) \leq q(i+1+2^s) \leq b$$

holds. Fix l with $q(i+1) < l < q(i+2)$.

For each $l < r < b$ we have $\langle l, r\rangle \in U(s)$, so that $r < q(i+2+2^s)$. Thus, by letting r approach b, we see that $b \leq q(i+2+2^s)$. When $s \neq 0$, we have $i+2+2^s < i+1+2^{s+1}$, and hence $\langle a, b\rangle \in U(s+1)$, to give the required result. ■

With this we have

$$\boldsymbol{out}(U(s)) \subseteq U(s+1) = \boldsymbol{lap}(U(s))$$

so that

$$\boldsymbol{real}(U(s)) = U(s+1) = \boldsymbol{lap}(U(s))$$

for each $0 < s < \omega$. Thus

$$\boldsymbol{real}^s(U(1)) = U(s+1) = \boldsymbol{lap}^s(U(1))$$

by a simple induction, to give

$$\boldsymbol{real}^\omega(U(1)) = U(\omega) = \boldsymbol{lap}^\omega(U(1))$$

whereas

$$\rho(U(1)) = \boldsymbol{out}(\boldsymbol{lap}^\omega(U(1))) = \boldsymbol{out}(U(\omega)) = \mathbb{S}$$

to produce the required example.

4.2 Rank of *irrt*

We show that the rank of ***irrt*** is Ω, the first uncountable ordinal. With

$$j = \boldsymbol{out} \qquad f = \boldsymbol{spl} \qquad g = \boldsymbol{irrt}$$

we work in Context 1.12. This particular instance of ℓ is worth naming.

4.5 DEFINITION. Let $\boldsymbol{\ell} = \boldsymbol{out} \circ \boldsymbol{spl}^\omega$ to obtain a pre-nucleus on $\mathcal{L}\,\mathbb{S}$. ■

The following shows that, unlike the ***real*** case, we can not simply apply Lemma 1.13.

4.6 EXAMPLE. Consider $U \in \mathcal{L}\,\mathbb{S}$ given by

$$\langle a, b\rangle \in U \iff b < 1/2 \text{ or } 1/2 < a$$

(for $\langle a, b\rangle \in \mathbb{S}$). We show

$$\boldsymbol{spl}(U) = U \qquad \boldsymbol{out}(U) \neq \mathbb{S} \qquad \boldsymbol{spl}(\boldsymbol{out}(U) = \mathbb{S}$$

and hence $\boldsymbol{spl} \circ \boldsymbol{out} \not\leq \boldsymbol{out} \circ \boldsymbol{spl}$.

To check that $\boldsymbol{spl}(U) \subseteq U$ (and hence equality) consider any $\langle a, b\rangle \in \boldsymbol{spl}(U)$. There is a rational $a < m < b$ with $\langle a, m\rangle, \langle m, b\rangle \in U$, so that neither of

$$a \leq 1/2 \leq m \qquad m \leq 1/2 \leq b$$

can hold. But now either $1/2 < a$ or $b < 1/2$ to give $(a, b) \in U$, as required.

We may check that

$$\langle a, b\rangle \in \boldsymbol{out}(U) \iff b \leq 1/2 \text{ or } 1/2 \leq a$$

for $\langle a, b\rangle \in \mathbb{S}$. In particular, $(1/4, 3/4) \notin \boldsymbol{out}(U)$, so that $\boldsymbol{out}(U) \neq \mathbb{S}$. Also we have $(0, 1/2), 1/2, 1) \in \boldsymbol{out}(U)$ so that $(0, 1) \in \boldsymbol{spl}(\boldsymbol{out}(U))$ to give $\boldsymbol{spl}(\boldsymbol{out}(U)) = \mathbb{S}$. ■

Consider any $U \in \mathcal{L}\,\mathbb{S}$. This generates an ascending chain

$$U = \boldsymbol{\ell}^0(U) \subseteq \boldsymbol{\ell}^1(U) \subseteq \cdots \subseteq \boldsymbol{\ell}^\alpha(U) \subseteq \cdots \quad (\alpha < \Omega)$$

through $\mathcal{L}\,\mathbb{S}$. Suppose these lower sections are all different. A use of CHOICE produces a member of $(\boldsymbol{\ell}^{\alpha+1}(U) - \boldsymbol{\ell}^\alpha(U))$ for each $\alpha < \Omega$. This gives uncountably many distinct members of the countable $\mathbb{S}$. Thus for each $U \in \mathcal{L}\,\mathbb{S}$ there is some ordinal $\upsilon = \upsilon(U)$ such that $\boldsymbol{\ell}^\alpha(U) = \boldsymbol{\ell}^\upsilon$ for all $\upsilon \leq \alpha < \Omega$. Hence the rank of $\boldsymbol{\ell}$ is no bigger than Ω.

To show the rank of $\boldsymbol{\ell}$ is not strictly smaller than Ω we use the CB-process ***lim*** on $\mathcal{C}\,\mathbb{I}$, the family of metric closed subsets of $\mathbb{I}$. For this we need a bit of background information.

4.7 DEFINITION. For each $X \in \mathcal{C}\,\mathbb{I}$ and ordinal α, let $X(\alpha) = \boldsymbol{lim}^{\alpha}(X)$. ■

This attaches to each $X \in \mathcal{C}\,\mathbb{I}$ a descending chain

$$X = X(0) \supseteq X(1) \supseteq \cdots \supseteq X(\alpha) \supseteq \cdots \quad (\alpha < \Omega)$$

with

$$X(\lambda) = \bigcap\{X(\alpha) \mid \alpha < \lambda\}$$

for each limit ordinal λ. The chain stabilizes at a countable ordinal. As X ranges through $\mathcal{C}\,\mathbb{I}$ these closure ordinals cofinally exhaust Ω. We use a special family of these sets.

4.8 DEFINITION. Let $\mathfrak{X}$ be the family of closed sets $X \in \mathcal{C}\,\mathbb{I}$ with $X \subseteq \mathbb{Q}$. ■

Each $X \in \mathfrak{X}$ is a set of rationals where the limit of each cauchy sequence taken from X is also in X (and hence rational). It can be checked that $\mathfrak{X}$ is the family of closed sets of a rather curious topology on $(0,1) \cap \mathbb{Q}$. There is a large supply of members of $\mathfrak{X}$.

4.9 PROPOSITION. *For each ordinal $\alpha < \Omega$ we have $X(\alpha) \neq X(\alpha+1)$ for some $X \in \mathfrak{X}$.*

The operators $\boldsymbol{\ell}$, $\boldsymbol{lim}$ live in different places

$$\boldsymbol{\ell} : \mathcal{L}\,\mathbb{S} \longrightarrow \mathcal{L}\,\mathbb{S} \qquad \boldsymbol{lim} : \mathcal{C}\,\mathbb{I} \longrightarrow \mathcal{C}\,\mathbb{I}$$

so we need something to connect them.

4.10 DEFINITION. For each $X \in \mathcal{C}\,\mathbb{I}$ and each $s < \omega$ let $\eth^{(s)}(X) \subseteq \mathbb{S}$ be given by

$$\langle a, b\rangle \in \eth^{(s)}(X) \iff |(a,b) \cap X| < 2^s$$

(for $\langle a, b\rangle \in \mathbb{S}$). Here, $|\cdot|$ indicates cardinality. Let

$$\eth^{(\omega)}(X) = \bigcup\{\eth^{(s)}(X) \mid s < \omega\}$$

and let $\eth = \eth^{(0)}$. ■

Trivially we have

$$\eth^{(\bullet)} : \mathcal{C}\,\mathbb{I} \longrightarrow \mathcal{L}\,\mathbb{S}$$

for each index $\bullet$, and each $X \in \mathcal{C}\,\mathbb{I}$ generates an ascending chain

$$\eth(X) = \eth^{(0)}(X) \subseteq \eth^{(1)}(X) \subseteq \cdots \subseteq \eth^{(s)}(X) \subseteq \cdots \subseteq \eth^{(\omega)}(X) \quad (s < \omega)$$

of lower sections in $\mathcal{L}\,\mathbb{S}$ with $\eth^{(\omega)}(X)$ as the the union. The two extremes are the important components, and these are given by

$$\begin{aligned} \langle a, b\rangle \in \eth^{(\omega)}(X) &\iff (a,b) \cap X \text{ is finite} \\ \langle a, b\rangle \in \eth(X) &\iff (a,b) \cap X \text{ is empty} \end{aligned}$$

(for $\langle a, b\rangle \in \mathbb{S}$). Notice that we have

$$Y \subseteq X \iff \eth(X) \subseteq \eth(Y)$$

for $X, Y \in \mathcal{C}\,\mathbb{I}$, to show that the assignment $\eth$ is injective. This will be important later.

The major part of this subsection is a proof of the following.

4.11 THEOREM. *For each $X \in \mathfrak{X}$ we have*

$$(\boldsymbol{\ell}^{\alpha+1} \circ \eth)(X) = (\eth \circ \boldsymbol{lim}^{1+\alpha})(X)$$

for each ordinal α.

Observe the two exponents here. When α is finite we have $\alpha + 1 = 1 + \alpha$. When α is infinite we have $1 + \alpha = \alpha$, and there is a slight hiccough at each limit level. A partial explanation of this is given by Lemma 4.22.

Before we start the proof of Theorem 4.11 let's use it to achieve our main aim.

4.12 THEOREM. *The rank of $\boldsymbol{\ell}$ and **irrt** is Ω, the least uncountable ordinal.*

Proof. By Theorem 1.15 it suffices to show that the rank of $\boldsymbol{\ell}$ is not countable.

By way of contradiction suppose the rank is countable, so that $\boldsymbol{\ell}^{\alpha} = \boldsymbol{\ell}^{\alpha+1}$ for all sufficiently large countable ordinals α. It suffices to consider only infinite countable ordinals α, so that $1 + \alpha = \alpha$.

For each $X \in \mathfrak{X}$ two uses of Theorem 4.11 gives

$$\eth(X(\alpha)) = (\boldsymbol{\ell}^{\alpha+1} \circ \eth)(X) = (\boldsymbol{\ell}^{\alpha+2} \circ \eth)(X) = \eth(X(\alpha + 1))$$

and hence

$$X(\alpha) = X(\alpha + 1)$$

since $\eth$ is injective. Proposition 4.9 provides the required contradiction. ■

It remains to prove Theorem 4.11. We develop a series of comparisons between the various component operators. For the first few of these we do not need to restrict to $\mathfrak{X}$.

4.13 LEMMA. *We have $\eth^{(\omega)} \leq \eth \circ \boldsymbol{lim}$.*

Proof. Consider any $X \in \mathcal{C}\,\mathbb{I}$ and any $\langle a, b\rangle \in \eth^{(\omega)}(X)$. The intersection $(a, b) \cap X$ is finite, and we must show that $(a, b) \cap \boldsymbol{lim}(X)$ is empty. To this end consider any real x with $a < x < b$. We require $x \notin \boldsymbol{lim}(X)$. If $x \notin X$ then we are done. Otherwise x is a member of the finite set $(a, b) \cap X$. But now there are rationals $a < l < x < r < b$ with $(l, r) \cap X = \{x\}$ to show that x is isolated in X, and hence $x \notin \boldsymbol{lim}(X)$. ■

Next we compare ***out*** and ***spl*** with $\eth^{(\bullet)}$.

4.14 LEMMA. *We have*

$$\boldsymbol{out} \circ \eth^{(s)} = \eth^{(s)} \qquad \boldsymbol{spl} \circ \eth^{(s)} \leq \eth^{(s+1)}$$

for each $s < \omega$.

Proof. For the left hand equality consider any $X \in \mathcal{C}\,\mathbb{I}$. We must show

$$\boldsymbol{out}(\eth^{(s)}(X)) \subseteq \eth^{(s)}(X)$$

(for the other inclusion is trivial). We deal with the complements of these lower sections.

Consider a non-member $\langle a, b\rangle \notin \eth^{(s)}(X)$ of the right hand side. Thus

$$|(a, b) \cap X| \geq 2^s$$

to give members of X

$$a < x(1) < x(2) < \cdots < x(2^s) < b$$

(which need not be rational). Consider rationals

$$a < l < x(1) < x(2^s) < r < b$$

so that

$$\langle l, r\rangle \notin \eth^{(s)}(X)$$

and hence $\langle a, b\rangle \notin \boldsymbol{out}(\eth^{(s)}(X))$, as required.

For the right hand comparison consider any $\langle a, b\rangle \in \boldsymbol{spl}(\eth^{(s)}(X))$ for some $s < \omega$ and $X \in \mathcal{C}\,\mathbb{I}$. Then either $\langle a, b\rangle \in \eth^{(s)}(X) \subseteq \eth^{(s+1)}(X)$ and we are done, or we have

$$\langle a, m\rangle, \langle m, b\rangle \in \eth^{(s)}(X)$$

for some rational $a < m < b$ and both

$$1 + |(a, m) \cap X| \le 2^s \qquad 1 + |(m, b) \cap X| \le 2^s$$

hold. But

$$(a, b) \cap X = \big((a, m) \cap X\big) \cup \{m\} \cup \big((m, b) \cap X\big)$$

so that

$$2 + |(a, b) \cap X| = (1 + |(a, m) \cap X|) + 1 + (1 + |(m, b) \cap X|) \le 2^s + 1 + 2^s = 1 + 2^{s+1}$$

to give $\langle a, b\rangle \in \eth^{(s+1)}(X)$. ■

From the right hand comparison we have $\boldsymbol{spl}^{\omega} \circ \eth^{(s)} \le \eth^{(\omega)}$. We can improve this.

4.15 LEMMA. *We have*

$$\boldsymbol{out} \circ \eth^{(\omega)} \le \eth \circ \boldsymbol{lim} \qquad \boldsymbol{spl}^{\omega} \circ \eth^{(\omega)} = \eth^{(\omega)}$$

Proof. Since $\boldsymbol{out}$ is monotone a use of Lemma 4.13 and then Lemma 4.14 gives

$$\boldsymbol{out} \circ \eth^{(\omega)} \le \boldsymbol{out} \circ \eth \circ \boldsymbol{lim} = \eth \circ \boldsymbol{lim}$$

for the left hand comparison.

For the right hand equality it suffices to show $\boldsymbol{spl}^{\omega} \circ \eth^{(\omega)} \le \eth^{(\omega)}$. To do that we first show $\boldsymbol{spl} \circ \eth^{(\omega)} \le \eth^{(\omega)}$.

Consider any $\langle a, b\rangle \in \boldsymbol{spl}(\eth^{(\omega)}(X))$ where $X \in \mathcal{C}\,\mathbb{I}$. There is a rational $a < m < b$ with

$$\langle a, m\rangle, \langle m, b\rangle \in \eth^{(\omega)}(X)$$

and hence, since $\eth^{(\omega)}(X)$ is a pointwise union, we have

$$\langle a, m\rangle, \langle m, b\rangle \in \eth^{(s)}(X)$$

for some $s < \omega$. But now Lemma 4.14 gives

$$\langle a, b\rangle \in \boldsymbol{spl}(\eth^{(s)}(X)) \subseteq \eth^{(s+1)}(X) \subseteq \eth^{(\omega)}(X)$$

as required.

Iterating this preliminary observation gives

$$\boldsymbol{spl}^s \circ \eth^{(\omega)} \leq \eth^{(\omega)}$$

for each $s < \omega$.

Finally consider any $\langle a, b\rangle \in \boldsymbol{spl}^\omega(\eth^{(\omega)}(X))$ where $X \in \mathcal{C}\,\mathbb{I}$. Then, since $\boldsymbol{spl}^\omega(\cdot)$ is a pointwise union, we have some $s < \omega$ with

$$\langle a, b\rangle \in \boldsymbol{spl}^s(\eth^{(\omega)}(X)) \subseteq \eth^{(\omega)}(X)$$

to give the required result. ■

The obvious way to improve this result doesn't work.

4.16 EXAMPLE. We produce a set $X \in \Upsilon\mathbb{S}$ with $(\boldsymbol{out} \circ \eth^{(\omega)})(X) \neq \eth^{(\omega)}(X)$. Let

$$0 < q(0) < q(1) < \cdots < q(i) < \cdots \quad (i < \omega)$$

be a strictly increasing sequence of rationals with limit 1/2. Let

$$X = \{q(i) \mid i < \omega\} \cup \{1/2\}$$

to obtain $X \in \mathcal{C}\,\mathbb{I}$. A simple argument gives

$$\langle a, b\rangle \in \eth^{(\omega)}(X) \Longleftrightarrow b < 1/2 \text{ or } 1/2 \leq a$$

for $\langle a, b\rangle \in \mathbb{S}$. In particular, we have $(0, 1/2) \notin \eth^{(\omega)}(X)$, and hence it will be enough to show $(0, 1/2) \in (\boldsymbol{out} \circ \eth^{(\omega)})(X)$. But, from above, for all rationals $0 < a < b < 1/2$ we have $\langle a, b\rangle \in \eth^{(\omega)}(X)$, to give the required result. ■

Using both parts of Lemma 4.15 we have

$$\boldsymbol{\ell} \circ \eth^{(\omega)} = \boldsymbol{out} \circ \boldsymbol{spl}^\omega \circ \eth^{(\omega)} = \boldsymbol{out} \circ \eth^{(\omega)} \leq \eth \circ \boldsymbol{lim}$$

and this can be improved.

4.17 LEMMA. *We have* $\quad \boldsymbol{\ell} \circ \eth^{(\omega)} = \eth \circ \boldsymbol{lim}$.

Proof. By the remarks above it suffices to show $\eth \circ \boldsymbol{lim} \leq \boldsymbol{\ell} \circ \eth^{(\omega)}$. To this end consider any $X \in \Upsilon\mathbb{S}$ and any $\langle a, b\rangle \notin \boldsymbol{\ell}(\eth^{(\omega)}(X))$. We produce some $x \in \boldsymbol{lim}(X)$ with $a < x < b$, and hence $\langle a, b\rangle \notin \eth(\boldsymbol{lim}(X))$.

Let $\mathcal{I}nf$ be the family of all pairs $\langle l, r\rangle \in \mathbb{S}$ with $a < l < r < b$ and where $(l, r) \cap X$ is infinite. Using Lemma 4.15 we have

$$\langle a, b\rangle \notin (\boldsymbol{\ell} \circ \eth^{(\omega)})(X) = (\boldsymbol{out} \circ \boldsymbol{spl}^\omega \circ \eth^{(\omega)})(X) = (\boldsymbol{out} \circ \eth^{(\omega)})(X)$$

and hence there are rationals $a < l < r < b$ with $\langle l, r\rangle \notin \eth^{(\omega)}(X)$, to show that $\langle l, r\rangle \in \mathcal{I}nf$, and hence $\mathcal{I}nf$ is non-empty.

We use a squeezing argument on $\mathcal{I}nf$. Consider $\langle l, r\rangle \in \mathcal{I}nf$. Let m be the mid-point of (l, r). One of

$$(l, m) \cap X \qquad (m, r) \cap X$$

must be infinite. Thus we produce $l \leq l^+ < r^+ \leq r$ with $\langle l^+, r^+\rangle \in \mathcal{I}nf$ and where $\langle l^+, r^+\rangle$ has half the length of $\langle l, r\rangle$. By iteration (that is a use of DEPENDENT CHOICE) we produce a strictly descending chain

$$I = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_i \supsetneq \quad (i < \omega)$$

of open intervals where each $I_i \cap X$ is infinite and where the length(I_i) reduces to zero. Furthermore, we have the closure $I^- \subseteq [l, r] \subseteq (a, b)$. Using the closed sets $I_i^- \cap X$ a COMPACTNESS argument gives some

$$x \in \bigcap\{I_i^- \mid i < \omega\} \cap X$$

and hence it suffices to show that $x \in \boldsymbol{lim}(X)$

By way of contradiction, suppose x is isolated in X. Then

$$(m, n) \cap X = \{x\}$$

for some rational $m < n$. For sufficiently large i we have $x \in I_i^-$ and

$$\text{length}(I_i) < \min\left\{\frac{n-x}{2}, \frac{x-m}{2}\right\}$$

to give $I_i^- \subseteq (m, n)$, which is the contradiction (since $I_i \cap X$ is infinite). ■

The weaker form of Lemma 4.17 can be iterated.

4.18 LEMMA. *For each ordinal non-zero α we have* $\boldsymbol{\ell}^\alpha \circ \eth^{(\omega)} \leq \eth \circ \boldsymbol{lim}^\alpha$.

Proof. Consider any $X \in \Upsilon\mathbb{S}$. We show

$$(\boldsymbol{\ell}^\alpha \circ \eth^{(\omega)})(X) \subseteq (\eth \circ \boldsymbol{lim}^\alpha)(X)$$

by induction on α.

Lemma 4.17 give the base case, $\alpha = 1$.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$\begin{aligned}
(\boldsymbol{\ell}^{\alpha+1} \circ \eth^{(\omega)})(X) &= (\boldsymbol{\ell} \circ \boldsymbol{\ell}^\alpha \circ \eth^{(\omega)})(X)\\
&\subseteq (\boldsymbol{\ell} \circ \eth \circ \boldsymbol{lim}^\alpha)(X)\\
&\subseteq (\boldsymbol{\ell} \circ \eth^{(\omega)} \circ \boldsymbol{lim}^\alpha)(X)\\
&\subseteq (\eth \circ \boldsymbol{lim} \circ \boldsymbol{lim}^\alpha)(X) \quad = \quad (\eth \circ \boldsymbol{lim}^{\alpha+1})(X)
\end{aligned}$$

using the induction hypothesis and Lemma 4.17.

For the induction leap to a limit ordinal λ we remember the notation of Definition 4.7. We have $X(\lambda) \subseteq X(\alpha)$ for each $\alpha < \lambda$, and hence

$$\eth(X(\alpha)) \subseteq \eth(X(\lambda))$$

since $\eth$ is antitone. With this observation we have

$$(\boldsymbol{\ell}^\lambda \circ \eth^{(\omega)})(X) = \bigcup\{(\boldsymbol{\ell}^\alpha \circ \eth^{(\omega)})(X) \mid \alpha < \lambda\} \subseteq \bigcup\{\eth(X(\alpha)) \mid \alpha < \lambda\} \subseteq \eth(X(\lambda)$$

as required. Here the first inclusion follows by the induction hypothesis. ■

The next three preliminaries depend on the family $\mathfrak{X}$.

4.19 LEMMA. *For each $s < \omega$ we have*

$$\boldsymbol{spl} \circ \eth^{(s)} = \eth^{(s+1)}$$

when restricted to $\mathfrak{X}$.

Proof. By Lemma 4.14 it suffices to show

$$\eth^{(s+1)}(X) \subseteq (\boldsymbol{spl} \circ \eth^{(s)})(X)$$

for $X \in \mathfrak{X}$ and $s < \omega$.

Consider $\langle a, b\rangle \in \eth^{(s+1)}(X)$. If $\langle a, b\rangle \in \eth^{(s)}(X)$ then we are done. Thus we may suppose

$$2^s \leq |(a, b) \cap X| < 2^{s+1}$$

and hence we have a listing

$$(a, b) \cap X = \{x(1) < x(2) < \cdots < x(l)\}$$

for some l with $2^s \leq l < 2^{s+1}$. Let $m = x(2^s)$, and remember that this is rational, so that both $\langle a, m\rangle$ and $\langle m, b\rangle$ are in $\mathbb{S}$. We have

$$|(a, m) \cap X| = 2^s - 1 \qquad |(m, b) \cap X| = l - 2^s < s^{s+1} - 2^s = 2^s$$

so that $\langle a, m\rangle, \langle m, b\rangle \in \eth^{(s)}(X)$, and hence $\langle a, b\rangle \in \boldsymbol{spl}(\eth^{(s)}(X))$, as required. ■

By restricting to $\mathfrak{X}$ we can improve the right hand part of Lemma 4.15.

4.20 LEMMA. *We have*

$$\boldsymbol{spl}^{\omega} \circ \eth = \boldsymbol{spl}^{\omega} \circ \eth^{(\omega)} = \eth^{(\omega)}$$

when restricted to $\mathfrak{X}$.

Proof. Using Lemma 4.15 we have

$$(\boldsymbol{spl}^{\omega} \circ \eth)(X) \subseteq (\boldsymbol{spl}^{\omega} \circ \eth^{(\omega)})(X) = \eth^{(\omega)}(X)$$

so an inclusion

$$\eth^{(\omega)}(X) \subseteq (\boldsymbol{spl}^{\omega} \circ \eth)(X)$$

will suffice. Consider any $\langle a, b\rangle \in \eth^{(\omega)}(X)$. There is some $s < \omega$ with

$$\langle a, b\rangle \in \eth^{(s)}(X) = (\boldsymbol{spl}^{s} \circ \eth)(X) \subseteq (\boldsymbol{spl}^{\omega} \circ \eth)(X)$$

where the equality follows by iterated use of Lemma 4.19. ■

Remembering that $\boldsymbol{\ell} = \boldsymbol{out} \circ \boldsymbol{spl}^{\omega}$, this with Lemma 4.17 gives the following.

4.21 COROLLARY. *We have*

$$\boldsymbol{\ell} \circ \eth = \boldsymbol{\ell} \circ \eth^{(\omega)} = \eth \circ \boldsymbol{lim}$$

when restricted to $\mathfrak{X}$.

The final preliminary is partly the cause of the hiccough at limit levels.

4.22 LEMMA. *We have*

$$\boldsymbol{spl} \circ \boldsymbol{\ell}^{\lambda} = \boldsymbol{\ell}^{\lambda} \qquad \boldsymbol{\ell}^{\lambda+1} = \boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda}$$

for each limit ordinal λ.

Proof. Consider any $\langle a, b\rangle \in (\boldsymbol{spl} \circ \boldsymbol{\ell}^{\lambda})(U)$ where $U \in \mathcal{L}\mathbb{S}$. We have

$$\langle a, m\rangle, \langle m, b\rangle \in \boldsymbol{\ell}^{\lambda}(U)$$

for some $a < m < b$, so that

$$\langle a, m\rangle, \langle m, b\rangle \in \boldsymbol{\ell}^{\alpha}(U)$$

for some $\alpha < \lambda$, to give

$$\langle a, b\rangle \in (\boldsymbol{spl} \circ \boldsymbol{\ell}^{\alpha})(U) \subseteq \boldsymbol{\ell}^{\alpha+1}(U) \subseteq \boldsymbol{\ell}^{\lambda}(U)$$

as required for the left hand equality.

An induction gives $\boldsymbol{spl}^{s} \circ \boldsymbol{\ell}^{\lambda} = \boldsymbol{\ell}^{\lambda}$ for each $s < \omega$, so that $\boldsymbol{spl}^{\omega} \circ \boldsymbol{\ell}^{\lambda} = \boldsymbol{\ell}^{\lambda}$ and hence

$$\boldsymbol{\ell}^{\lambda+1} = \boldsymbol{\ell} \circ \boldsymbol{\ell}^{\lambda} = \boldsymbol{out} \circ \boldsymbol{spl}^{\omega} \circ \boldsymbol{\ell}^{\lambda} = \boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda}$$

as required for the right hand equality. ■

We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. Fix $X \in \mathfrak{X}$. We proceed by induction over α.

The base case, $\alpha = 0$, holds by Corollary 4.21.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$\begin{aligned}(\boldsymbol{\ell}^{\alpha+2} \circ \eth)(X) &= (\boldsymbol{\ell} \circ \boldsymbol{\ell}^{\alpha+1} \circ \eth)(X)\\ &= (\boldsymbol{\ell} \circ \eth \circ \boldsymbol{lim}^{1+\alpha})(X)\\ &= (\eth \circ \boldsymbol{lim} \circ \boldsymbol{lim}^{1+\alpha})(X) \ = (\eth \circ \boldsymbol{lim}^{1+\alpha+1})(X)\end{aligned}$$

using the induction hypothesis and Corollary 4.21 applied to $X(1 + \alpha)$.

For the induction leap to a limit ordinal λ we have $1 + \lambda = \lambda$ so that

$$\begin{aligned}(\boldsymbol{\ell}^{\lambda+1} \circ \eth)(X) \ &= \ (\boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda} \circ \eth)(X)\\ &\subseteq \ (\boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda} \circ \eth^{(\omega)})(X)\\ &\subseteq \ (\boldsymbol{out} \circ \eth \circ \boldsymbol{lim}^{\lambda})(X) \ = \ (\eth \circ \boldsymbol{lim}^{\lambda})(X)\end{aligned}$$

by Lemmas 4.22, 4.18, and 4.14. Thus it suffices to show that

$$\eth(X(\lambda)) \subseteq (\boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda} \circ \eth)(X)$$

holds.

Consider any $\langle a, b\rangle \in \eth(X(\lambda))$. The intersection $(a, b) \cap X(\lambda)$ is empty. By way of contradiction suppose $\langle a, b\rangle \notin (\boldsymbol{out} \circ \boldsymbol{\ell}^{\lambda} \circ \eth)(X)$. There are rational $a < l < r < b$ with $\langle l, r\rangle \notin (\boldsymbol{\ell}^{\lambda} \circ \eth)(X)$ that is

$$\langle l, r\rangle \notin (\boldsymbol{\ell}^{\alpha+1} \circ \eth)(X)$$

for each $\alpha < \lambda$. The induction hypothesis gives $\langle l, r\rangle \notin \eth(X(1 + \alpha))$ and hence

$$(l, r) \cap X(\alpha) \neq \emptyset$$

for each $\alpha < \lambda$. (A few moment's though shows that the '+1' can be absorbed.) By a use of CHOICE this gives a λ-indexed family $x(\cdot)$ of reals, in fact rationals, with

$$l < x(\alpha) < r \qquad x(\alpha) \in X(\alpha)$$

for each $\alpha < \lambda$. We refine this sequence in two ways.

Firstly, since λ is countable, there is an ascending ω-chain

$$\alpha(0) < \alpha(1) < \cdots < \alpha(s) < \cdots \quad (s < \omega)$$

with limit λ. We now use the subsequence $x(\alpha(\cdot))$.

Since each $x(\alpha)$ is a member of (l, r) this subsequence itself has a convergent subsequence by some kind of CHOICE. In other words there is a function

$$s_{(\cdot)} : \mathbb{N} \longrightarrow \mathbb{N}$$

with

$$i < j \Longrightarrow s_i < s_j$$

for $i, j \in \mathbb{N}$, and with

$$\lim_{i \to \infty} x(\alpha(s_i)) = x$$

for some real x. Notice that $s_{(\cdot)}$ eventually gets larger than any given natural number.

When can the limit x live? Certainly we have $a < l \leq x \leq r < b$ and hence $x \notin X(\lambda)$ since $(a, b) \cap X(\lambda)$ is empty. We use this to obtain the contradiction.

For each $i < j$ we have

$$x(\alpha(s_j)) \in X(\alpha(s_j)) \subseteq X(\alpha(s_i))$$

and hence

$$x \in \boldsymbol{lim}(X(\alpha(s_i))) = X(\alpha(s_i) + 1)$$

for all i. For each $\alpha < \lambda$ there is some $s < \omega$ with $\alpha < \alpha(s)$, and hence some $i \in \mathbb{N}$ with $\alpha < \alpha(s_i)$. This gives

$$x \in X(\alpha(s_i) + 1) \subseteq X(\alpha)$$

and hence

$$x \in \bigcap \{X(\alpha) \mid \alpha < \lambda\} = X(\lambda)$$

to give the required contradiction. ■

In this account I have made very few references to the literature on this topic. That is because most of the details that I needed are not there. Nevertheless, some publications should be mentioned.

From an historical point of view [3], [4], and [7] are important (but the first two are not easy reading). Some years later much of the machinery has been re-invented, see [1, 2] and the papers cited there.

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