Fruitful and helpful ordinal functions

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Abstract

I give a self contained account of helpful ordinals functions on all finite levels. These gadgets can be used to generate ordinal notations 'from below' at least as far as the Howard ordinal.

Contents

1 Preamble

In [7] I compared two methods of generating notations for countable ordinals: the standard method, due to Bachmann, using a collapsing function; and an alternative, less standard but historically older method.

In the review [5] of [7] the reviewer quite rightly criticized me for omitting certain proofs from [7]. The paper doesn't even contain the definition of a crucial central notion. This paper corrects that omission, and with [7] forms a self contained account.

Let me outline what this paper and [7] are about.

Let Ω be the least uncountable ordinal. We wish to name as many ordinals $\alpha < \Omega$ as possible. Let

$$\Omega^+ = \Omega^{\Omega^{\Omega^{\cdot^{\cdot^{\cdot}}}}} = \epsilon_{\Omega+1}$$

be the next critical ordinal beyond Ω.

In [7] I described the standard method of generating notations as 'from above'. By that I merely meant that larger ordinals, up to Ω^+, are used to index the generation of smaller ordinals, well below Ω. The collapsing function

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$$

sends larger ordinals to smaller ordinals.

I described the alternative method of generating notations as 'from below'. By that I merely meant that ordinals already generated are used to index the next phase of generation. As each phase peters out some new gadget has to be conjured up to keep the process going. It was Veblen in [8] who first used this method to generate ordinal notations, but the idea goes right back to Archimedes in [1] who used this method to generate notations for natural numbers.

The new gadgets I employed are certain fixed point extractors, and to produce these I used what I called helpful functions. It is an account of helpful functions that is missing from [7]. This paper fills that gap.

The main results of this paper are contained in Section 3 and 4. Let me give a brief description of that material.

Let $\mathbb{O}\mathrm{rd}$ be the set of countable ordinals. Thus $\mathbb{O}\mathrm{rd} = [0, \Omega)$. From now on in this paper 'ordinal' means 'countable ordinal', a member of $\mathbb{O}\mathrm{rd}$. Let

$$\mathbb{O}\mathrm{rd}' = (\mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd})$$

be the set of all ordinal functions. The notion of a normal function $f : \mathbb{O}\mathrm{rd}'$ is a central component of any method of generating ordinal notations. Essentially we have to harvest the fixed points of such a function f. For what we do here it turns out that the class of normal functions is slightly too small. Thus we use the slightly larger class $\mathbb{F}\mathrm{ruit} \subseteq \mathbb{O}\mathrm{rd}'$ of fruitful functions. These functions $f \in \mathbb{F}\mathrm{ruit}$ are fruitful because they provide many fixed points, all of which are critical. The class $\mathbb{F}\mathrm{ruit}$ is defined and discussed in the first part of Section 3.

In general, one fruitful (or normal) function is not enough. To generate a decent stretch of ordinals we need several fruitful functions. The class $\mathbb{H}\mathrm{elp} \subseteq \mathbb{O}\mathrm{rd}'$ of helpful functions makes it easier to grow the required fruitful functions. Each helpful function $h \in \mathbb{H}\mathrm{elp}$ can be iterated

$$\alpha \longmapsto h^{\alpha}\zeta$$

through $\mathbb{O}\mathrm{rd}$ (for any given input ζ) and this is a fruitful function. This format makes the arithmetic of the fruitful function easier to handle. The class $\mathbb{H}\mathrm{elp}$ is defined and discussed in the latter part of Section 3.

This method of generating ordinals 'from below' depends on producing enough members of $\mathbb{H}\mathrm{elp}$. To do that we use certain higher order functions. These are also called helpful because the defining properties of the whole family have a certain uniformity. This larger family is described in Section 4.

There are two other sections.

Section 2 gathers together all the bits and pieces that we need. Some of this is a repeat of Section 2 of [7].

Finally, in Section 5, I give a brief historical account of the various phases in the development of ordinal notations. I indicate how each phase can be seen in terms of helpful functions, each phase using such functions at higher and higher levels.

To conclude this preamble I indicate where the proofs missing from [7] can be found in this paper.

Result from [7]		is proved here as:
2.6	(a)	2.4 and 4.3
2.6	(b)	4.1(Help1)
2.6	(c)	3.12(b) and 4.5(b)
2.7	(1)	3.12(c)
2.7	(>1)	4.5(c)
2.9		3.9(b) and 4.5
2.13		4.7

2 Background material

Let $\mathbb{O}\mathrm{rd}$ be the set of countable ordinals. Except for a brief mention towards the end of the paper, all the ordinals we meet belong to $\mathbb{O}\mathrm{rd}$. Thus it is safe to let 'ordinal' mean 'countable ordinal'. Naturally, we use various ordinal functions

$$f : \mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd}$$

as well as higher level versions of such functions. To handle these we set up a bit of notation.

For an arbitrary set $\mathbb{S}$ let

$$\mathbb{S}' = (\mathbb{S} \longrightarrow \mathbb{S})$$

the set of functions on $\mathbb{S}$. This construction $(\cdot)'$ can be iterated.

2.1 DEFINITION. The chain $\mathbb{O}\mathrm{rd}^{(\cdot)}$ of spaces is generated by

$$\mathbb{O}\mathrm{rd}^{(0)} = \mathbb{O}\mathrm{rd} \qquad \mathbb{O}\mathrm{rd}^{(r+1)} = \mathbb{O}\mathrm{rd}^{(r)\prime}$$

for each $r < \omega$ (where $\mathbb{S}'$ is $\mathbb{S} \longrightarrow \mathbb{S}$ for each set $\mathbb{S}$). ■

Thus $\mathbb{O}\mathrm{rd}^{(0)}$ is just the space $\mathbb{O}\mathrm{rd}$ of ordinals, and $\mathbb{O}\mathrm{rd}^{(1)}$ is the space $\mathbb{O}\mathrm{rd}'$ of ordinal functions, and so on. It seems that members of $\mathbb{O}\mathrm{rd}^{(l+2)}$ are rarely used, but we will meet several in this paper. Notice that this space $\mathbb{O}\mathrm{rd}^{(l+2)}$ can be decomposed as

$$\mathbb{O}\mathrm{rd}^{(l+2)} = \mathbb{O}\mathrm{rd}^{(l+1)} \to \mathbb{O}\mathrm{rd}^{(l)} \to \cdots \to \mathbb{O}\mathrm{rd}' \to \mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd}$$

(where punctuating brackets should be inserted in the obvious way). In particular, each function $G : \mathbb{O}\mathrm{rd}^{(l+2)}$ must receive successive inputs

$$g : \mathbb{O}\mathrm{rd}^{(l+1)},\ g_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ g_1 : \mathbb{O}\mathrm{rd}',\ \zeta : \mathbb{O}\mathrm{rd}$$

to produce

$$Gg : \mathbb{O}\mathrm{rd}^{(l+1)},\ Ggg_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ Ggg_l \cdots g_1 : \mathbb{O}\mathrm{rd}'$$

and then return its eventual output $Ggg_l \cdots g_1\zeta \in \mathbb{O}\mathrm{rd}$.

The space $\mathbb{O}\mathrm{rd}$ is linearly ordered and carries an actual supremum operation $\bigvee$ which converts each countable subset $X \subseteq \mathbb{O}\mathrm{rd}$ into its least upper bound $\bigvee X$. There is a formal way to lift this operation to higher levels.

2.2 DEFINITION. (base) For each non-empty, countable subset $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}'$ the function $\bigvee \mathcal{G} : \mathbb{O}\mathrm{rd}'$ is given by

$$\left(\bigvee \mathcal{G}\right)\zeta = \bigvee \{g\zeta \mid g \in \mathcal{G}\}$$

(for $\zeta \in \mathbb{O}\mathrm{rd}$). We call this function $\bigvee \mathcal{G}$ the pointwise supremum of $\mathcal{G}$.

(raise) For each $l < \omega$ and each non-empty, countable subset $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}^{(l+2)}$ the function $\bigvee \mathcal{G} : \mathbb{O}\mathrm{rd}^{(l+2)}$ is given by

$$\left(\bigvee \mathcal{G}\right)g = \bigvee \{Gg \mid G \in \mathcal{G}\}$$

(for $g \in \mathbb{O}\mathrm{rd}^{(l+1)}$). We call this function $\bigvee \mathcal{G}$ the pointwise supremum of $\mathcal{G}$. ■

Do not be mislead by this construction. To explain what is going on let us temporarily write $\bigvee^{(l)}$ for the gadget constructed on $\mathbb{O}\mathrm{rd}^{(l)}$. Thus $\bigvee^{(0)}$ is the actual supremum operation on $\mathbb{O}\mathrm{rd}$, and then $\bigvee^{(1)}, \bigvee^{(2)}, \ldots, \bigvee^{(l)}, \ldots$ are constructed in turn by recursion on l. The notation is *not* intended to suggest there is a partial ordering on $\mathbb{O}\mathrm{rd}^{(l)}$ of which $\bigvee^{(l)}$ is the supremum operation. Nevertheless, later we will find a subclass $\mathbb{H}^{(l)} \subseteq \mathbb{O}\mathrm{rd}^{(l)}$ which can be partially ordered with $\bigvee^{(l)}$ as a supremum operation.

For $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}^{(l+2)}$ the construction of $\bigvee^{(l+2)}\mathcal{G}$ can be unravelled as

$$\left(\bigvee^{(l+2)}\mathcal{G}\right) g g_l \cdots g_1 \zeta = \bigvee \{ G g g_l \cdots g_1 \zeta \mid G \in \mathcal{G} \}$$

using the actual supremum operation on $\mathbb{O}\mathrm{rd}$.

Ordinal iterations of function $g \in \mathbb{O}\mathrm{rd}'$ are standard fare. The pointwise supremum enables us to lift this to higher levels.

2.3 DEFINITION. (a) For each $l < \omega$ and each $g : \mathbb{O}\mathrm{rd}^{(l+1)}$, the ordinal iterates $g^\bullet$ of g are generated by

$$g^0 = id \qquad g^{\alpha+1} = g \circ g^\alpha \qquad g^\lambda = \bigvee \{ g^\alpha \mid \alpha < \lambda \}$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$ and limit ordinal $\lambda \in \mathbb{O}\mathrm{rd}$. (Here id is the identity function on $\mathbb{O}\mathrm{rd}^{(l)}$.)

(b) For each $l < \omega$ a class $\mathbb{S} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$ is smooth if $f \circ g \in \mathbb{S}$ for each $f, g \in \mathbb{S}$, and $\bigvee \mathcal{G} \in \mathbb{S}$ for each non-empty and countable $\mathcal{G} \subseteq \mathbb{S}$. ∎

Again, for the moment, treat this as nothing more than a definition. In general the ordinal iterates of g (as defined here) may not behave as you think they should. The notion of a smooth class is a way of calming down some of the wilder behaviour.

2.4 LEMMA. *Suppose $\mathbb{S} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$ is smooth and $g \in \mathbb{S}$. Then $g^\alpha \in \mathbb{S}$ for each non-zero ordinal α.*

We will construct several smooth classes, most at high levels. However, for the first examples we stick with $\mathbb{O}\mathrm{rd}'$.

3 Low level functions

In this section we look at standard ordinal functions of type $\mathbb{O}\mathrm{rd}'$. We isolate two classes of such functions. The class of fruitful functions $\mathbb{F}\mathrm{ruit} \subseteq \mathbb{O}\mathrm{rd}'$ form a rather mild variation of the usual notion of a normal function. This class $\mathbb{F}\mathrm{ruit}$ is more amenable and, in particular, it is smooth. The class $\mathbb{H}\mathrm{elp} \subseteq \mathbb{O}\mathrm{rd}'$ of helpful functions on this level is also smooth. It is the interaction between $\mathbb{F}\mathrm{ruit}$ and $\mathbb{H}\mathrm{elp}$ that interests us here.

Thus there are three main classes of ordinal functions that we meet: general functions, fruitful functions, and helpful functions. Usually we write

f for a fruitful function $\qquad$ g for a general function $\qquad$ h for a helpful function

to indicate which kind of function is being used. This is a convenient informal convention which, of course, may be broken at times. Fruitful functions are so called because they have lots of fixed points each of which is critical. The helpful functions enable us to produce fruitful functions, and hence generate critical ordinals.

(Actually, this convention came about because for a long time I couldn't remember the difference between fruitful and helpful – which I was then calling something else. I had to invent this little trick to keep my sanity. I'm not sure it worked.)

To begin the analysis we first isolate a smooth class $\mathbb{IM} \subseteq \mathbb{O}\mathrm{rd}$ which includes both $\mathbb{F}$ruit and $\mathbb{H}$elp.

We are interested in various combinations of standard property of functions $g : \mathbb{O}\mathrm{rd}'$. Most of these have names.

3.1 DEFINITION. A function $g : \mathbb{O}\mathrm{rd}'$ is, respectively

(i)	inflationary	if	$\alpha \leq g\alpha$
(si)	strictly inflationary	if	$\alpha < g\alpha$
(m)	monotone	if	$\beta \leq \alpha \Rightarrow g\beta \leq g\alpha$
(sm)	strictly monotone	if	$\beta < \alpha \Rightarrow g\beta < g\alpha$
(b)	big	if	$\omega^\alpha \leq g\alpha$ (except possibly for $\alpha = 0$)
(sb)	strictly big	if	$g\alpha$ is critical
(c)	continuous	if	$g(\bigvee A) = \bigvee g[A]$

for all ordinals α, β, and each non-empty countable set A of ordinals.

Let $\mathbb{IM}$ be the class of functions which are both inflationary and monotone. ■

The five properties (i, si, m, sm, c) are standard. The two properties (b, sb) are not often named, but often used as a technical convenience. It doesn't take long to see that the three implications

$$si \Rightarrow i \qquad sm \Rightarrow i + m \qquad i + sb \Rightarrow b$$

hold. If g is continuous then

$$g\lambda = \bigvee\{g\alpha \mid \alpha < \lambda\}$$

for each limit ordinal λ. In general this is not enough to ensure continuity, but it is for monotone functions. Luckily we are concerned almost entirely with the class $\mathbb{IM}$ of functions.

As with any class of monotone functions the class $\mathbb{IM}$ can be partially ordered using the pointwise comparison.

$$f \leq g \Longleftrightarrow (\forall \alpha \in \mathbb{O}\mathrm{rd})[f\alpha \leq g\alpha]$$

With this comparison, for each non-empty countable subset $\mathcal{G} \subseteq \mathbb{IM}$ the pointwise supremum $\bigvee \mathcal{G}$ is the actual supremum.

3.2 LEMMA. *The class* $\mathbb{IM}$ *is smooth. Furthermore*

$$\beta \leq \alpha \Longrightarrow g^\beta \leq g^\alpha$$

holds for each $g \in \mathbb{IM}$ *and ordinals* α, β.

Proof. The first part is easy, and the second part follows by induction on α (making use of the inflationary property of g). ■

By intention each smooth class is closed under (non-zero) ordinal iterates. When $g \in \mathbb{IM}$ the family $\{g^\alpha \mid \alpha \in \mathbb{O}\mathrm{rd}\}$ of iterates is an ascending chain

$$id = g^0 \leq g = g^1 \leq g^2 \leq \cdots \leq g^\alpha \leq \cdots \quad (\alpha \in \mathbb{O}\mathrm{rd})$$

which helps with certain calculations. For instance, we have the following.

3.3 LEMMA. *If $g \in \mathbb{IM}$ then*

$$g^\alpha \circ g^\beta = g^{\beta+\alpha} \qquad (g^\beta)^\alpha = g^{\beta\times\alpha}$$

for all $\alpha, \beta \in \mathbb{O}\mathrm{rd}$.

Proof. Both of these are proved by induction on α. Let's look at the leap to a limit ordinal λ for the second identity. Thus we require

$$(g^\beta)^\lambda\zeta = g^{\beta\times\lambda}\zeta$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. We have

$$(g^\beta)^\lambda\zeta = \bigvee\{(g^\beta)^\alpha\zeta \mid \alpha < \lambda\} = \bigvee\{g^{\beta\times\alpha}\zeta \mid \alpha < \lambda\} \qquad g^{\beta\times\lambda}\zeta = \bigvee\{g^\gamma\zeta \mid \gamma < \beta\times\lambda\}$$

where the second equality uses the induction hypothesis. Also

$$\beta\times\lambda = \bigvee\{\beta\times\alpha \mid \alpha < \lambda\}$$

(by construction of ordinal multiplication). The comparison

$$(g^\beta)^\lambda\zeta \leq g^{\beta\times\lambda}\zeta$$

is immediate (since $\beta\times\alpha \leq \beta$ for $\alpha < \lambda$). For the converse consider any $\gamma < \beta$. There is some $\alpha < \lambda$) with $\gamma \leq \beta$, and then Lemma 3.2 gives

$$g^\gamma\zeta \leq g^{\beta\times\alpha}\zeta \leq (g^\beta)^\lambda\zeta$$

which leads to the required result. ■

We have made a bit of a meal of this proof to highlight the required properties of g.

As a consequence of this result some limit iterates of functions in $\mathbb{IM}$ are constant for long periods, and so can not be strictly monotone.

3.4 EXAMPLE. Let $g \in \mathbb{IM}$ and suppose λ is additively critical. For each ordinal ζ and ordinal $\alpha < \lambda$ we have

$$g^\lambda(g^\alpha\zeta) = (g^\lambda \circ g^\alpha)\zeta = g^{\alpha+\lambda}\zeta = g^\lambda\zeta$$

and hence g^λ is constant between ζ and $g^\alpha\zeta$.

The situation is even more dramatic if g is continuous. In this case $g^\lambda = g^\mu$ for some $\mu \geq \lambda\cdot\omega$. ■

We can now isolate the fruitful and the helpful functions.

3.5 DEFINITION. An ordinal function $f \in \mathbb{O}\mathrm{rd}'$ is fruitful if it is inflationary, monotone, big, and continuous. Let $\mathbb{F}\mathrm{ruit}$ be the class of fruitful functions.

An ordinal function $g \in \mathbb{O}\mathrm{rd}'$ is normal if it is strictly monotone, big, and continuous.

An ordinal function $h \in \mathbb{O}\mathrm{rd}'$ is helpful if it is strictly inflationary, monotone, and strictly big. Let $\mathbb{H}\mathrm{elp}$ be the class of helpful functions. ■

These normal functions are just the usual normal functions that are big. It turns out that *strict* monotonicity is rather too restrictive, so we use the larger class $\mathbb{F}\mathrm{ruit}$ of fruitful functions. Each such function g belongs to $\mathbb{IM}$, and Example 3.4 shows that not all the iterates are normal. We rectify that by releasing our grip on normality and becoming fruity.

A proof of the following is straight forward.

3.6 LEMMA. *Each of the classes* $\mathbb{F}\mathrm{ruit}$ *and* $\mathbb{H}\mathrm{elp}$ *is smooth.*

Why are the classes $\mathbb{F}\mathrm{ruit}$ and $\mathbb{H}\mathrm{elp}$ useful? To answer that we introduce a particular second level function.

3.7 DEFINITION. Let $\boldsymbol{Fix} : \mathbb{O}\mathrm{rd}''$ be the function given by

$$\boldsymbol{Fix} f\zeta = f^{\omega}(\zeta + 1)$$

for each function $f : \mathbb{O}\mathrm{rd}'$ and ordinal ζ. ■

We have

$$\boldsymbol{Fix} f\zeta = \bigvee\{f^{r}(\zeta + 1) \mid r < \omega\}$$

and this makes sense for any function $f : \mathbb{O}\mathrm{rd}'$. However, we use $\boldsymbol{Fix}$ only on $f \in \mathbb{F}\mathrm{ruit}$. For such f we see that $\boldsymbol{Fix}$ is a fixed point extractor.

3.8 LEMMA. *For each* $f \in \mathbb{F}\mathrm{ruit}$ *and* $\zeta \in \mathbb{O}\mathrm{rd}$*, the value* $\boldsymbol{Fix} f\zeta$ *is the least ordinal* ν *such that* $\zeta < \nu = f\nu$*. Furthermore, this value* ν *is critical.*

Proof. For the given $f \in \mathbb{F}\mathrm{ruit}$ and ordinal ζ, let $\nu = \boldsymbol{Fix} f\zeta$. Let

$$\zeta[r] = f^{r}(\zeta + 1)$$

for $r < \omega$, so that

$$\zeta < \zeta[0] \leq \cdots \leq \zeta[r] \leq \cdots$$

since f is inflationary and monotone, with

$$\nu = \bigvee\{\zeta[r] \mid r < \omega\}$$

by unravelling the definition of $\boldsymbol{Fix}$. Since f is continuous this gives

$$f\nu = \bigvee\{f\zeta[r] \mid r < \omega\} = \bigvee\{\zeta[r+1] \mid r < \omega\} = \nu$$

to show that ν is a fixed point of f.

Let μ be any fixed point of f with $\zeta < \mu$. Then $\zeta[0] \le \mu$ and hence since f is monotone a simple induction gives

$$\zeta[r] \le f\mu = \mu$$

for each $r < \omega$. Thus $\nu \le \mu$.

Finally, since f is big we have

$$\zeta < \nu \le \omega^\nu \le f\nu = \nu$$

to show that ν is critical. ■

Much of the standard material on ordinal notations is about extracting fixed points, so we can see why ***Fix*** might be useful.

Fruitful and helpful functions work hand in hand.

3.9 LEMMA. *(a) For each $f \in \mathbb{F}\mathrm{ruit}$ the function **Fix** f is helpful.*

(b) For each $h \in \mathbb{H}\mathrm{elp}$ and ordinal ζ, the ordinal function $\alpha \mapsto h^\alpha\zeta$ is normal.

Proof. (a) For fruitful f let $h =$ ***Fix*** f. For $\zeta \in \mathbb{O}\mathrm{rd}$ let $\nu = h\zeta$. By Lemma 3.8, we have $\zeta < \nu = f\nu$ with a certain minimality on ν. In particular, h is strictly inflationary.

Consider any $\zeta \le \eta$ and let $\mu = h\eta$. Then $\zeta \le \eta < \mu = f\mu$ and hence $\nu \le \mu$ by the minimality of ν. This shows that h is monotone.

Since $\nu \ne 0$, we have $\omega^\nu \le f\nu = \nu$, so that ν is critical, and hence h is strictly big.

(b) For the given helpful function h and ordinal ζ let

$$f\alpha = h^\alpha\zeta$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$.

By construction for each ordinal α we have

$$f(\alpha+1) = h(f\alpha) > f\alpha$$

since h is strictly inflationary. Also by construction we have

$$f\lambda = \bigvee\{f\alpha \mid \alpha < \lambda\}$$

for each limit ordinal λ. But if $\alpha < \lambda$ then $\alpha < \alpha + 1 < \lambda$ so that $f\alpha < f(\alpha+1) \le f\lambda$ which is enough to show that f is strictly monotone.

A simple argument now shows that f is continuous.

For non-zero α the value $f\alpha$ is either a value of h or a supremum of such values. Thus $f\alpha$ is critical. But $\alpha \le f\alpha$ and hence $\omega^\alpha \le \omega^{f\alpha} = f\alpha$ to show that f is big. ■

By part (b) of this result, for each helpful function h and ordinal ζ the function $\alpha \mapsto h^\alpha\zeta$ is fruitful (in fact, normal) and provides an enumeration of a set of critical ordinals. This will be useful if only we can find some helpful functions. That is where part (a) comes into play.

The smallest fruitful function (we are interested in) is $\omega^\bullet$, exponentiation to base ω.

3.10 DEFINITION. Let ***Next*** $=$ ***Fix*** $\omega^\bullet$. ■

By Lemma 3.9(a) the function $\boldsymbol{Next}$ is helpful. Let h be any helpful function, and let $f = \omega^{\bullet}$. For each $\zeta \in \mathbb{O}\mathrm{rd}$ we have $\zeta{+}1 \leq h\zeta$ and $h\zeta$ is critical, so that $f(\zeta{+}1) \leq \omega^{h\zeta} = h\zeta$. An easy induction gives $f^r(\zeta+1) \leq h\zeta$ for all $r \leq \omega$, and hence $\boldsymbol{Next}\zeta \leq h\zeta$, to show the following

3.11 LEMMA. *The function* $\boldsymbol{Next}$ *is the smallest helpful function.*

A simple argument shows that $\boldsymbol{Next}\zeta$ is the next critical ordinal strictly beyond ζ. In particular

$$\epsilon_\alpha = \boldsymbol{Next}^\alpha \epsilon_0 = \boldsymbol{Next}^{1+\alpha}\omega$$

is a long list of critical ordinals. This will run out of steam at

$$\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot}}}}$$

the least ordinal ν with $\epsilon_\nu = \nu$. To generate larger critical ordinals we need more powerful helpful functions. We show how to produce these in the next section. To conclude this section we obtain a couple of properties of an arbitrary helpful function.

3.12 LEMMA. *Suppose* $h \in \mathbb{H}\mathrm{elp}$. *Then*

(a) $\zeta + \alpha \leq h^\alpha \zeta$.

(b) $h^\lambda \zeta = h^\lambda 0$

(c) $(\zeta < \nu = h^\nu 0) \Longleftrightarrow (0 < \nu = h^\nu \zeta)$

hold for all ordinals α, ν, ζ *and (additively) critical ordinal* λ *with* $\zeta < \lambda$.

Proof. (a) We prove this by induction on α.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, since h is strictly inflationary we have

$$h^{\alpha+1}\zeta = h(h^\alpha\zeta) \geq h^\alpha\zeta + 1 \geq \zeta + \alpha + 1$$

using the induction hypothesis.

For the induction leap to a limit ordinal λ we have

$$h^\lambda\zeta = \bigvee\{h^\alpha\zeta \mid \alpha < \lambda\} \geq \bigvee\{\zeta + \alpha \mid \alpha < \lambda\} = \zeta + \bigvee\{\alpha \mid \alpha < \lambda\} = \zeta + \lambda$$

as required.

(b) The iterate h^λ is helpful, and hence monotone, so that $h^\lambda\zeta \geq h^\lambda 0$. For the converse we have $\zeta \leq h^\zeta 0$ (by part (a)) and hence $h^\lambda\zeta \leq h^\lambda(h^\zeta 0) = h^{\zeta+\lambda}0$ by Lemma 3.3. But $\zeta < \lambda$ and λ is (additively) critical so that $\zeta + \lambda = \lambda$, to give the required result. ■

(c) Let

$$\mu = h^\nu\zeta$$

for any pair ν, ζ of ordinals with $\nu \neq 0$. If $\nu = \alpha + 1$ then

$$\mu = h(h^\alpha\zeta)$$

so that μ is critical since it is a value of the helpful function h. If ν is a limit ordinal, then μ is a supremum of values of h, and so is again critical. In other words, μ is critical for all $\nu > 0$. Note also that $\zeta < \mu$, so that two uses of part (b) gives

$$\mu = h^\nu \eta$$

for each $\eta < \nu$. Further uses of part (b) give the two required implications. ■

A standard development of ordinal notations would make much use of normal functions. We will see that fruitful functions are a more amenable way of doing this. The helpful functions provide a canonical way of generating such functions. However, a more important benefit of the notion of helpfulness is that it lifts to higher levels.

4 Helpful functions

By Section 3 we know that for each helpful function $h \in \mathbb{H}\mathrm{elp}$ the fruitful function

$$\alpha \mapsto h^{1+\alpha}0$$

generates a sequence of critical ordinals. (The '1+' can be avoided here if we are prepared to start from a known critical ordinal.) Furthermore, for each fruitful function $f \in \mathbb{F}\mathrm{ruit}$, the function $\boldsymbol{Fix} f$ is helpful, and

$$\alpha \mapsto (\boldsymbol{Fix} f)^{1+\alpha}0$$

enumerates the fixed points of f. Thus the higher level function $\boldsymbol{Fix} : \mathbb{O}\mathrm{rd}''$ converts a fruitful function into a helpful function.

In this section we decribe a method of producing helpful functions which doesn't require a given fruitful function. The technique makes use of functions $g : \mathbb{O}\mathrm{rd}^{(l+1)}$ on all levels.

Recall that by decomposing the space

$$\mathbb{O}\mathrm{rd}^{(l+2)} = \mathbb{O}\mathrm{rd}^{(l+1)} \to \mathbb{O}\mathrm{rd}^{(l)} \to \cdots \to \mathbb{O}\mathrm{rd}' \to \mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd}$$

we see that each function $H : \mathbb{O}\mathrm{rd}^{(l+2)}$ must receive successive inputs

$$h : \mathbb{O}\mathrm{rd}^{(l+1)},\ h_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ h_1 : \mathbb{O}\mathrm{rd}',\ \zeta : \mathbb{O}\mathrm{rd}$$

to return its eventual output $Hhh_l \cdots h_1\zeta$. Often these central inputs $h_l, \ldots, h_1$ play only a passive role, so we abbreviate the list $h_l \cdots h_1$ to $\boldsymbol{h}$ and write $Hh\boldsymbol{h}\zeta$ for the eventual output. We do not use this abbreviation in the following definition, but we will in the subsequent analysis.

4.1 DEFINITION. (Base) Let $\mathbb{H}^{(1)} = \mathbb{H}\mathrm{elp}$, the class of helpful functions on level 1.
(Step) For each $l < \omega$ a function $H : \mathbb{O}\mathrm{rd}^{(l+2)}$ is helpful on level $l + 2$ if

$$\begin{array}{ll} \text{(Help1)} & Hh \text{ is helpful on level } l+1 \\ \text{(Help2)} & h^2 h_l \cdots h_1 \le Hhh_l \cdots h_1 \\ \text{(Help3)} & Hhh_l \cdots h_2 f \le Hhh_l \cdots h_2 g \end{array}$$

for all $h : \mathbb{H}^{(l+1)}$, $h_l : \mathbb{H}^{(l)}$, $\ldots$, $h_1 : \mathbb{H}^{(1)}$, and $f, g : \mathbb{H}^{(1)}$ with $f \le g$.
Let $\mathbb{H}^{(l+2)}$ be the class of helpful functions on level $l + 2$. ■

This is a construction by recursion on the level l to produce $\mathbb{H}^{(l+1)} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$. The comparison in (Help2, Help3) takes place in $\mathbb{O}\mathrm{rd}'$. Since the functions involved are in $\mathbb{IM}$, this doesn't lead to difficulties. Notice that the first two defining clauses of $\mathbb{H}^{(l+2)}$ can be written

$$\text{(Help1)} \quad Hh \in \mathbb{H}^{(l+1)} \qquad \text{(Help2)} \quad h^2\boldsymbol{h} \leq Hh\boldsymbol{h}$$

using the abbreviation $\boldsymbol{h}$ explained above. The third clause is not so straight forward. In particular, for the case $l = 0$ you should read (Help3) with some care, because the sequence $h, h_l, \ldots, h_2$ is empty. A function $H : \mathbb{O}\mathrm{rd}''$ is in $\mathbb{H}^{(2)}$ precisely when

$$(1) \quad Hh \in \mathbb{H}^{(1)} \qquad (2) \quad h^2 \leq Hh \qquad (3) \quad Hf \leq Hg$$

for all $f, g, h : \mathbb{H}^{(1)}$ with $f \leq g$.

The squaring property (Help2) is quite powerful, especially when used at higher levels. This will be a crucial component of the proof of several results.

4.2 LEMMA. *For each $H \in \mathbb{H}^{(l+2)}$, $h \in \mathbb{H}^{(l+1)}$, $h_l \in \mathbb{H}^{(l)}$, $\ldots$, $h_1 \in \mathbb{H}^{(1)}$ we have*

$$(h\boldsymbol{h})^2 \leq Hh\boldsymbol{h}$$

(where $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$).

Proof. We proceed by induction on the level l. For the base case, $l = 0$, the parameter sequence $\boldsymbol{h}$ is empty, and the required comparison $h^2 \leq Hh$ is just (Help2). For the induction step, $l \mapsto l+1$, consider a helpful $K : \mathbb{O}\mathrm{rd}^{(l+3)}$, as well as the helpful $H, h, \boldsymbol{h}$. By (Help1) we know that Hh is helpful. Thus, using (Help2) for K and the induction hypothesis, we have

$$KHh\boldsymbol{h} \geq H^2h\boldsymbol{h} = H(Hh)\boldsymbol{h} \geq (Hh\boldsymbol{h})^2$$

as required. ■

In Lemma 3.6 we saw that the class $\mathbb{H}^{(1)} = \mathbb{H}\mathrm{elp}$ is smooth. We now generalize this.

4.3 LEMMA. *For each $l < \omega$, the class $\mathbb{H}^{(l+1)}$ is smooth.*

Proof. Lemma 3.6 gives the result for $\mathbb{H}^{(1)}$. We look at $\mathbb{H}^{(l+2)}$ for arbitrary $l < \omega$.

We show first that $\mathbb{H}^{(l+2)}$ is closed under composition. To this end consider any $G, H \in \mathbb{H}^{(l+2)}$. To show $G \circ H \in \mathbb{H}^{(l+2)}$ we look at (Help1, Help2, Help3) in turn.

For each $h \in \mathbb{H}^{(l+1)}$ we have $Hh \in \mathbb{H}^{(l+1)}$ and hence $G(Hh) \in \mathbb{H}^{(l+1)}$ to verify (Help1).

To verify (Help2) consider any compatible family $h, \boldsymbol{h}$ of helpful functions. Then

$$(G \circ H)h\boldsymbol{h} = G(Hh)\boldsymbol{h} \geq (Hh\boldsymbol{h})^2 \geq Hh\boldsymbol{h} \geq h^2\boldsymbol{h}$$

as required. Here the first comparison follows by Lemma 4.2, the second follows since $Hh\boldsymbol{h}$ is inflationary, and the third uses (Help2) for H.

To verify (Help3) observe that

$$(G \circ H)hh_l \cdots h_2 g = G(Hh)h_l \cdots h_2 g \qquad (G \circ H)hh_l \cdots h_2 f = G(Hh)h_l \cdots h_2 f$$

$$\begin{aligned} (G \circ H)hh_l \cdots h_2 g &= G(Hh)h_l \cdots h_2 g \\ (G \circ H)hh_l \cdots h_2 f &= G(Hh)h_l \cdots h_2 f \end{aligned}$$

so the known monotone property of G gives the required result. (Strictly speaking, this is the argument for $l \neq 0$. A slight variant is needed for $l = 0$.)

To show that $\mathbb{H}^{(l+2)}$ is closed under pointwise suprema, consider a non-empty subset $\mathcal{H}$ of $\mathbb{H}^{(l+2)}$. We show that $\bigvee \mathcal{H}$ is helpful. We look at (Help1, Help2, Help3) in turn.

For each $h \in \mathbb{H}^{(l+1)}$ we have

$$(\bigvee \mathcal{H})h = \bigvee\{Hh \mid H \in \mathcal{H}\}$$

so the known closure property of $\mathbb{H}^{(l+1)}$ gives the required property of $\mathbb{H}^{(l+2)}$. (Strictly speaking, this is a proof by induction on l.) This verifies (Help1).

To verify (Help2) consider any compatible family $h, \boldsymbol{h}$ family of helpful functions. Since $\mathcal{H}$ is non-empty we have

$$(\bigvee \mathcal{H})h\boldsymbol{h} \geq Hh\boldsymbol{h} \geq h^2\boldsymbol{h}$$

for any selected any member H of $\mathcal{H}$.

Property (Help3) follows in the same way. ■

For each $l < \omega$ the class $\mathbb{H}^{(l+2)}$ is smooth, and hence is closed under ordinal iteration. This has a useful consequence.

4.4 COROLLARY. *For each $l < \omega, H \in \mathbb{H}^{(l+2)}, h \in \mathbb{H}^{(l+1)}$ we have $H^\alpha h \in \mathbb{H}^{(l+1)}$ for each $\alpha \in \mathbb{O}\mathrm{rd}$.*

Lemmas 3.9(b) and 3.12 give us some crucial properties of helpful functions $h \in \mathbb{H}^{(1)}$. These properties lift to higher levels.

4.5 LEMMA. *Let $H : \mathbb{H}^{(l+2)}$, $h : \mathbb{H}^{(l+1)}$, $h_l : \mathbb{H}^{(l)}$, $\ldots$, $h_1 : \mathbb{H}^{(1)}$. The the function*

$$\alpha \mapsto H^\alpha h\boldsymbol{h}\zeta$$

is normal, and

(a) $H^\alpha h\boldsymbol{h}\alpha \leq H^{\alpha+1}h\boldsymbol{h}0$

(b) $H^\lambda h\boldsymbol{h}\zeta = H^\lambda h\boldsymbol{h}0$

(c) $(\zeta < \nu = H^\nu h\boldsymbol{h}0) \iff (0 < \nu = H^\nu h\boldsymbol{h}\zeta)$

hold for all ordinals α, ν, ζ and (additively) critical ordinal λ with $\zeta < \lambda$. Here $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$.

Proof. Let f be this function, that is

$$f\alpha = H^\alpha h\boldsymbol{h}\zeta$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$.

The function $h\boldsymbol{h}$ is helpful, hence strictly inflationary, so that

$$Hh\boldsymbol{h}\zeta \geq (h\boldsymbol{h})^2\zeta = h\boldsymbol{h}(h\boldsymbol{h}\zeta) > h\boldsymbol{h}\zeta$$

by Lemma 4.2. In particular

$$f(\alpha+1) = H(H^\alpha h)\boldsymbol{h}\zeta > H^\alpha h\boldsymbol{h}\zeta = f\alpha$$

(using $H^\alpha h$ in place of h).

For each limit ordinal λ and ordinal $\alpha < \lambda$, we have

$$\alpha + 1 < \lambda$$

and so (by the definition of $f\lambda$) we have $f\alpha < f(\alpha+1) \leq f\lambda$ using the previous observation.

This shows that f is strictly monotone.

By construction the function f is continuous.

Finally, for each α the function $H^\alpha h\boldsymbol{h}$ is helpful, and so takes only critical values. But $\alpha \leq f\alpha$, so that $\omega^\alpha \leq \omega^{f\alpha} = f\alpha$, as required to show that f is normal.

(a) Using Lemma 4.2 we have

$$H^{\alpha+1}h\boldsymbol{h}0 = H(H^\alpha h)\boldsymbol{h}0 \geq (H^\alpha h\boldsymbol{h})^2 0 = H^\alpha h\boldsymbol{h}(H^\alpha h\boldsymbol{h}0) \geq H^\alpha h\boldsymbol{h}\alpha$$

where last comparison holds since each helpful function is inflationary.

(b) The comparison $H^\lambda h\boldsymbol{h}\zeta \geq H^\lambda h\boldsymbol{h}0$ is immediate.

For the converse consider any ordinal α with $\zeta < \alpha < \lambda$. Then, using part (a) we have

$$H^\alpha h\boldsymbol{h}\zeta \leq H^\alpha h\boldsymbol{h}\alpha \leq H^{\alpha+1}h\boldsymbol{h}0 \leq H^\lambda h\boldsymbol{h}0$$

where the last comparison holds by the construction of H^λ. Thus taking the supremum over all $\alpha < \lambda$ gives the required result.

(c) For the given $H, h, \boldsymbol{h}$ let

$$\mu = H^\nu h\boldsymbol{h}\zeta$$

for any pair ν, ζ of ordinals with $\nu \neq 0$. If $\nu = \alpha + 1$ then

$$\mu = H(H^\alpha h)\boldsymbol{h}\zeta$$

so that μ is citical since it is a value of the helpful function $(H^\alpha h)\boldsymbol{h}$. If ν is a limit ordinal, then μ is a supremum of values of $(H^\alpha h)\boldsymbol{h}$, and so is again critical. In other words, μ is critical for all $\nu > 0$. Note also that $\zeta < \mu$, so that two uses of part (b) gives

$$\mu = H^\nu h\boldsymbol{h}\eta$$

for each $\eta < \nu$. Further uses of part (b) give the two required implications. ■

Lemma 3.9(a) gives us $\boldsymbol{Fix} f \in \mathbb{H}^{(1)}$ for each $f \in \mathbb{F}\mathrm{ruit}$. By modifying that idea we can generate helpful functions at all levels.

4.6 DEFINITION. For each level l let $[\imath] : \mathbb{O}\mathrm{rd}^{(l+2)}$ be the function given by

$$[\imath]h\boldsymbol{h} = \boldsymbol{Fix} f \qquad \text{where } f\alpha = h^\alpha \boldsymbol{h}0 \text{ (for } \alpha \in \mathbb{O}\mathrm{rd})$$

for each compatible family $h, \boldsymbol{h}$ of (helpful) functions. ■

It is important to understand what these functions do, so let's take a look at $[\,0\,]$.

Consider any helpful function $h : \mathbb{O}\mathrm{rd}'$. By Lemma 3.9(b) we have a normal function $f : \mathbb{O}\mathrm{rd}'$ given by $f\alpha = h^\alpha 0$ (for $\alpha \in \mathbb{O}\mathrm{rd}$). By Lemmas 3.8 and 4.5(c), we have

$$[\,0\,]h\zeta = \big(\text{the least } \nu \text{ with } \zeta < \nu = h^\nu 0\big) = \big(\text{the least } \nu \text{ with } 0 < \nu = h^\nu \zeta\big)$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. By definition, this ν is non-zero. Hence $\nu = h^\nu 0$ is either a value of h (if ν is a successor) or is a supremum of such values. Thus ν is critical.

By construction, $[\,0\,]h$ is strictly inflationary, and a simple argument shows that it is monotone. We have just seen that $[\,0\,]h$ takes only critical values, and hence $[\,0\,]h$ is helpful. In particular, $[\,0\,]$ satisfies (Help1).

Using the ζ insensitivity we have

$$[\,0\,]h\zeta = \nu = h^\nu 0 = h^\nu \zeta > h^2\zeta$$

and hence $h^2 \le [\,0\,]h$. This shows that $[\,0\,]$ satisfies (Help2). A similar argument shows that is also satisfies (Help3). Thus $[\,0\,] : \mathbb{O}\mathrm{rd}''$ is helpful.

4.7 THEOREM. *For each $l < \omega$ the function $[\,l\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$ is helpful.*

Proof. We have just seen the proof for $l = 0$, so let's look at the non-zero case. Thus we show that $[\,l{+}1\,] : \mathbb{O}\mathrm{rd}^{(l+3)}$ is helpful.

Consider a family $H : \mathbb{O}\mathrm{rd}^{(l+2)}$, $h : \mathbb{O}\mathrm{rd}(l+1)$, $h_l : \mathbb{O}\mathrm{rd}^{(l)}$, $\ldots$, $h_1 : \mathbb{O}\mathrm{rd}'$ of helpful functions. By Lemma 4.5(a) the function $f : \mathbb{O}\mathrm{rd}'$ given by

$$f\alpha = H^\alpha h\boldsymbol{h}0$$

(for $\alpha \in \mathbb{O}\mathrm{rd}$) is normal. By Lemma 3.9(a), the function

$$[\,l{+}1\,]Hh\boldsymbol{h} = \boldsymbol{Fix}\, f$$

is helpful. Since $h_1, \ldots, h_l, h$ are arbitrary, this shows that $[\,l{+}1\,]H$ is helpful, and hence $[\,l{+}1\,]$ satisfies (Help1).

The value $[\,l{+}1\,]Hh\boldsymbol{h}\zeta$ is the least ordinal ν such that

$$\zeta < \nu = H^\nu h\boldsymbol{h}0$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. By Lemma 4.3 the function $H^\nu h\boldsymbol{h}$ is helpful, and so takes only critical values. In particular, ν is critical, and hence using Lemma 4.5(c) we have

$$[\,l{+}1\,]Hh\boldsymbol{h}\zeta = H^\nu h\boldsymbol{h}0 = H^\nu h\boldsymbol{h}\zeta > H^2 h\boldsymbol{h}\zeta$$

to show that $[\,l{+}1\,]$ satisfies (Help2).

Finally, to verify (Help3) consider helpful $f, g : \mathbb{O}\mathrm{rd}'$ with $f \le g$. (Of course, this 'f' is not the same as before.) For $\zeta \in \mathbb{O}\mathrm{rd}$ let

$$\mu = [\,l{+}1\,]Hhh_l\cdots h_2 f\zeta \qquad \nu = [\,l{+}1\,]Hhh_l\cdots h_2 g\zeta$$

so we require $\mu \le \nu$. But

$$\mu = H^\mu hh_l\cdots h_2 f\zeta \qquad \nu = H^\nu hh_l\cdots h_2 g\zeta$$

with a certain minimality. Also, using Lemma 4.5(a), we have

$$\zeta < \nu \leq H^{\nu}hh_l \cdots h_2 f0 \leq H^{\mu}hh_l \cdots h_2 g0 = \nu$$

so that $\mu \leq \nu$, as required. ∎

This fills the gaps of [7].

There is another possible development of this material.

The first observation is that a helpful function on some level is applied only to a helpful input at the next level down (where each ordinal is viewed as helpful). We may set $\mathbb{H}^{(0)} = \mathbb{O}\mathrm{rd}$. In fact, we are not interested in the behaviour of a helpful function outside the helpful inputs. The next observation is that a helpful function applied to a helpful input returns a helpful output. This is the condition (Help1). Thus we could define

$$\mathbb{H}^{(l+1)} = \text{ those functions of type } \mathbb{H}^{(l)} \longrightarrow \mathbb{H}^{(l)}$$

which satisfy certain restriction. At the same time we can partially order each class $\mathbb{H}^{(l)}$. Since $\mathbb{H}^{(0)} = \mathbb{O}\mathrm{rd}$ we have the linear comparison on $\mathbb{H}^{(0)}$, and since $\mathbb{H}^{(1)} \subseteq \mathbb{IM}$ we have the pointwise comparison on $\mathbb{H}^{(1)}$. This idea lifts all the way up the levels. Thus for $H, K \in \mathbb{H}^{(l+2)}$ we use

$$H \leq K \iff (\forall h \in \mathbb{H}^{(l)})[Hh \leq Kh]$$

to produce a pointwise comparison on $\mathbb{H}^{(l+2)}$. With this we find that

$$h^2 \leq Hh \qquad H \text{ is monotone}$$

are rephrasings of (Help2) and (Help3). Furthermore, the pointwise supremum on $\mathbb{H}^{(l)}$ is the actual supremum. As a consequence of this we obtain higher level analogues of Lemma 3.3. Thus

$$H^{\alpha} \circ H^{\beta} = H^{\beta \circ \alpha} \qquad (H^{\beta})^{\alpha} = H^{\beta \times \alpha}$$

holds for each $H \in \mathbb{H}^{(l+2)}$ and ordinals α, β.

The problem with this approach is that it requires the definitions and properties to be developed in parallel, which can be a bit messy. Thus I chose to present it as above.

5 The Eastwood hierarchy

In this final section I will indicate how the use of helpful functions relates to other methods of generating ordinal notations. Of course, [7] is concerned with the relationship with the Bachmann method, but the earlier methods also fit into the same picture.

It is convenient to take an historical perspective. This enables us to produce a sequence

$$\Delta[0], \Delta[1], \Delta[2], \Delta[3], \ldots$$

of larger and larger ordinals the first few of which are milestones along the journey.

God created the natural numbers

$$0, 1, 2, \ldots$$

but, by design or oversight, forgot to tell us the limit point

$$\Delta[0] = \omega$$

of this sequence. This is the zeroth ordinal in our Δ-sequence. It was left to Cantor to discover ω and peer beyond. (Some people have tried to convince me that Cantor invented ω, not just discover it.)

It is fair to say that the Cantor normal form to base ω gives the first system of ordinal notations. This uses $\omega^\bullet$, exponentiation to base ω, and is good enough to name all the ordinals below ϵ_0, the least ordinal ϵ with $\omega^\epsilon = \epsilon$. Thus this system closes off at

$$\Delta[1] = \boldsymbol{Next}\omega = \epsilon_0$$

the first ordinal in our sequence. Notice how this uses $\boldsymbol{Next}$, the simplest helpful function.

To go beyond ϵ_0 we must name larger critical ordinals. Since

$$\epsilon_\alpha = \boldsymbol{Next}^{1+\alpha}\omega$$

we can do this by iterating $\boldsymbol{Next}$. With these ordinals we can extend the Cantor normal form for quite a bit further. This extended system closes off at the least ordinal ν with $\nu = \boldsymbol{Next}^\nu\omega$, which is

$$\Delta[2] = [\,0\,]\boldsymbol{Next}\omega = \epsilon_{\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot_{\cdot}}}}}}$$

the second ordinal in our sequence. This ordinal rarely gets a mention. I don't know what it has done to deserve that. I hope it is not some cardinal sin.

The next extension was made by Veblen. In [8] he constructed what we now call the Veblen hierarchy. Starting from any normal function $f : \mathbb{O}\mathrm{rd}'$ we set

$$\begin{aligned}
\phi_f 0 &= f \\
\phi_f(\alpha+1) &= \text{enumeration of fixed points of } \phi_f\alpha \\
\phi_f\lambda &= \text{enumeration of common fixed points of } \phi_f\alpha \text{ for all } \alpha < \lambda
\end{aligned}$$

for each ordinal α and limit ordinal λ. This gives us a whole hierarchy of normal functions with a substantial harvest of fixed points. With hindsight we see that we may generate such a hierarchy ϕ_f on any fruitful function f, and this does have some simplifying consequences.

This construction can be rephrased as a use of iteration.

Let $\boldsymbol{Veb} : \mathbb{O}\mathrm{rd}''$ be function given by

$$\boldsymbol{Veb}f\zeta = h^{1+\zeta}0 \quad \text{where } h = \boldsymbol{Fix}f$$

for $f : \mathbb{O}\mathrm{rd}'$ and $\zeta \in \mathbb{O}\mathrm{rd}$. Thus for each $f \in \mathbb{F}\mathrm{ruit}$ the function $\boldsymbol{Veb}f$ enumerates the fixed points of a normal (or fruitful) function f. The Veblen hierarchy ϕ_f on f is obtained by iterating $\boldsymbol{Veb}$. For historical reasons the limit levels are usually omitted, and we have

$$\phi_f(1+\alpha,\zeta) = \boldsymbol{Veb}^{\alpha+1}f\zeta$$

for each $\alpha, \zeta \in \mathbb{O}\mathrm{rd}$. Using the base function $f = \omega^\bullet$ this system of notations closes off at the least ordinal ν with $\nu = \boldsymbol{Veb}^\nu f0$, which we now call at Γ_0, the least strongly critical ordinal.

It is not hard to see that for each $f \in \mathbb{F}\mathrm{ruit}$ we have

$$(\boldsymbol{Fix} \circ \boldsymbol{Veb})f = \big([\,0\,] \circ \boldsymbol{Fix}\big)f$$

using the helpful function $[\,0\,] : \mathbb{H}^{(2)}$. Thus

$$(\boldsymbol{Fix} \circ \boldsymbol{Veb}^{\alpha})f = [\,0\,]^{\alpha}h \quad \text{where } h = \boldsymbol{Fix}\,f$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$, so that

$$\phi_f(1+\alpha,\zeta) = \boldsymbol{Veb}^{\alpha+1}f\zeta = (\boldsymbol{Veb} \circ \boldsymbol{Veb}^{\alpha})f\zeta = ((\boldsymbol{Fix} \circ \boldsymbol{Veb}^{\alpha})f)^{1+\zeta}0 = \big([\,0\,]^{\alpha}h\big)^{1+\zeta}0$$

which, for us, is a more convenient description of ϕ_f. (This calculation is a simple example of the shuffle technique which will be described in more detail elsewhere.)

For $f = \omega^{\bullet}$ this first Veblen system closes off at the least ν with $\nu = [\,0\,]^{\nu}\boldsymbol{Next}\omega$, which is

$$\Delta[3] = [\,1\,][\,0\,]\boldsymbol{Next}\omega = \Gamma_0$$

the third ordinal in our sequence.

How can we get beyond Γ_0? With hindsight we may use the battery of critical ordinals

$$\big(([\,1\,]^{\alpha}[\,0\,])^{\beta}\boldsymbol{Next}\big)^{1+\gamma}\omega$$

for previously generated ordinals α, β, γ. In particular, it can be checked that

$$\Gamma_{\alpha} = \big([\,1\,][\,0\,]\boldsymbol{Next}\big)^{1+\alpha}\omega$$

generates the sequence of strongly critical ordinals. This system closes off at the least solution of $\nu = [\,1\,]^{\nu}[\,0\,]\boldsymbol{Next}\omega$, which is

$$\Delta[4] = [\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega$$

the next ordinal in our sequence. After that it is obvious what to do, we use more and more of the higher order fixed point extractors $[\,\imath\,]$.

Veblen didn't do this. He set

$$f^{+}\zeta = \phi_f(1+\zeta, 0) = \boldsymbol{Veb}^{\zeta+1}f0 = [\,0\,]^{\zeta}h0$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. He proved directly from the properties of ϕ_f that f^{+} is a normal function which is much faster that f. He then generated a new hierarchy $\phi_{f^{+}}$ using f^{+} as the new base function. After that Veblen iterated this process and described an intricate system of indexing the whole family of hierarchies. I think it is fair to say that by that stage Veblen's description is not exactly crystal clear.

In [6] Schütte reorganized and extended Veblen's method to produce some quite large ordinals. A Schütte bracket is an array of ordinals

$$\begin{pmatrix} \zeta & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ r & 1+i(1) & \cdots & 1+i(s) \end{pmatrix}$$

where r is finite. (There are various other restrictions on the occurring ordinals, but we don't need to worry about those here.) Schütte thought of each bracket as an 'input' to a normal function f, which then produces an ordinal 'output'

$$f\begin{pmatrix} \zeta & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ r & 1+i(1) & \cdots & 1+i(s) \end{pmatrix}$$

via a rather intricate multi-recursion. In fact, it is better to think of the bracket as an operator which converts the function f into an ordinal, that is we think of the composite as

$$\begin{pmatrix} \zeta & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ r & 1+i(1) & \cdots & 1+i(s) \end{pmatrix} f$$

with the bracket acting on f rather than the other way round.

After some rather finicky analysis it can be shown that

$$\begin{pmatrix} \zeta & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ 0 & 1+i(1) & \cdots & 1+i(s) \end{pmatrix} f = \left(\left(\nabla \begin{bmatrix} \alpha(1)+1 \\ i(1)+1 \end{bmatrix} \circ [\,0\,] \circ \cdots \circ \nabla \begin{bmatrix} \alpha(s)+1 \\ i(s)+1 \end{bmatrix} \circ [\,0\,]\right) h\right)^{1+\zeta} 0$$

where

$$h = \boldsymbol{Fix}\, f$$

and

$$\nabla \begin{bmatrix} \alpha+1 \\ i+1 \end{bmatrix} = \left([\,1\,]^{i}[\,0\,]\right)^{1+\alpha}$$

for each pair of ordinal i, α. Actually, this last definition is not quite right when $i = 0$ and α is finite. Also, in most cases the somewhat odd occurrences of $[\,0\,]$ in the expansion of the bracket can be absorbed by the other components. This doesn't matter here, for the important message is that each such bracket is essentially a helpful function in $\mathbb{H}^{(2)}$ built up from certain legal combinations of $[\,0\,] \in \mathbb{H}^{(2)}$ and $[\,1\,] \in \mathbb{H}^{(3)}$. The appropriate combination for a bracket is determined by the ordinals occurring in the bracket. This is also true when $r \neq 0$.

As you can perhaps imagine, the proofs of these assertions are not entirely straight forward. I won't give any of the details here, but will give a full account elsewhere. The proofs depend on an extension of the shuffle technique mentioned above.

The Schütte brackets can be seen as a somewhat recondite way of producing helpful functions using only $[\,0\,]$ and $[\,1\,]$. In particular,

$$\Delta[4] = [\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega$$

is an upper bound for the ordinals generated by this method. I have been told that this is sometimes known as the Ackermann ordinal.

The next few ordinals in our sequence are

$$\begin{aligned} \Delta[5] &= [\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega = \text{least } \nu \text{ with } \nu = [\,2\,]^{\nu}[\,1\,][\,0\,]\boldsymbol{Next}\omega \\ \Delta[6] &= [\,4\,][\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega = \text{least } \nu \text{ with } \nu = [\,3\,]^{\nu}[\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega \\ \Delta[7] &= [\,5\,][\,4\,][\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega = \text{least } \nu \text{ with } \nu = [\,4\,]^{\nu}[\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\omega \end{aligned}$$

and so on. As far as I know these have not appeared explicitly in any method of generating ordinals.

The Bachmann method seems to be quite different. It uses an appropriate collapsing function

$$\psi : [0, \Omega^{+}) \longrightarrow [0, \Omega)$$

which enumerates the critical ordinals. Of course, such a function must be constant for long stretches. The precise details of ψ are not needed here. Using an iteration of the exponentiation function to base Ω, for each $l < \omega$ let

$$\nabla[l] = \psi\big((\Omega^{\bullet})^{l}0\big)$$

to obtain a fundamental sequence $\nabla[\cdot]$ for the Howard ordinal. For instance

$$\nabla[0] = \psi 0 \quad \nabla[1] = \psi 1 \quad \nabla[2] = \psi\Omega \quad \nabla[3] = \psi(\Omega^{\Omega}) \quad \nabla[4] = \psi(\Omega^{\Omega^{\Omega}})$$

and so on. As in [7], with a bit of effort it can be shown that $\nabla[\cdot]$ and $\Delta[\cdot]$ are essentially the same sequence, that is

$$\Delta[0] = \omega < \epsilon_0 = \nabla[0] \quad \Delta[1] = \epsilon_0 < \epsilon_1 = \nabla[1] \quad \Delta[l+2] = \nabla[l+2]$$

for each $l < \omega$. As can be seen from the analysis in [7], it is not so much the size of the ξ that determines the output $\psi\xi$, but the type structure hidden in a canonical expansion of ξ to base Ω. The method of generating ordinals 'from below' simply makes that type structure and associated gadgetry more explicit.

Let me conclude with some remarks on the work of Setzer as described in [4].

Here and in [7] I used the phrase 'from below' to describe the method of naming ordinals based on iterates of the helpful functions $[\,\imath\,]$. This is merely a convenient way of distinguishing that method from the Bachmann method, which I described as 'from above'. Nevertheless, there is clearly a more fundamental difference between the two methods. In [4] Setzer puts more meat on the skeletal phrase 'from below'. That work is clearly an important step towards making the difference between the two methods quite precise.

As part of the analysis he uses extended Schütte brackets in which weaker version are nested to produce more powerful versions. We have seen here (but not proved) that the standard Schütte brackets are essentially those helpful functions that can be built in a certain way from $[\,0\,]$ and $[\,1\,]$. Some of the calculations in [4] seem to suggest that the extended Schütte brackets can be built in a similar way from $[\,2\,], [\,3\,], [\,4\,], \ldots$ and so on. It would be interesting to see the details worked out.

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