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The coverage technique for enriched posets

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For the purpose of these notes the coverage technique is a method of taking a poset, perhaps furnished with some extra structure, and converting it into a complete poset with some required properties and perhaps some extra structure. There is also a more general method which is concerned with Grothendieck topologies and Gabriel topologies on appropriate categories. That generalization is not dealt with here, but will be in a companion set of notes. However, I will give some historical remarks concerning the connection between these various versions in section 9. *[Not yet written]*

The poset based version of the technique was first used in earnest in [2, 3, 4], and was developed into a general method in [6]. Since then the technique has been used on many occasions and, in fact, is one of the routine methods employed by workers in the field.

Unfortunately, in recent years there has been a lot of rubbish published by people who have just discovered hot water but don't know what to do with it. It is sometimes clear that they are unaware that the technique has been used in this simplified form for well over twenty years.

These notes have been around in various formats for less time, but perhaps not as widely circulated as they should have been. After some prompting I have got round to revising the older versions to produce what I hope is a useful description of the method and some applications.

The notes split into three parts. Sections 1–4 gather together various background facts about posets, complete poset, frames, quantales, and the like. I try not to stray too far from the central aim, so I consider only those facts that are needed later. In sections 5,6,7 I discuss the coverage technique proper. In the rather long section 8 I give a selection of examples of the use of the technique. These are not unknown, but I do set them in a general context to show there is some method behind the madness.

I don't claim any originality for this material. My aim is merely to set down the basic technique with a few examples. I doubt if people who know what they are doing will find much novelty here. However, if the notes prevent some newcomers being lead astray, then I will have achieved my purpose.

FOR HORATIO

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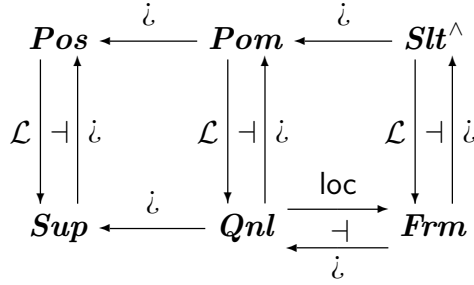


Table 1: The participating categories

1 The participating categories

We need to look at six different categories and the connections between them, as indicated in Table 1. In this section I will describe these categories and set down various relevant details. All of this stuff is well known, at least to any half competent practitioner, so we don't need to spend much time on the material.

The category *Pos*

This is the category of posets and monotone maps, and hardly needs an introduction, but let's set down the definitions just to be sure.

1.1 DEFINITION. A **poset** is a set A furnished with a comparison \leq , which is reflexive, antisymmetric, and transitive. That is, the comparison is a partial ordering of A .

Two posets A, B are compared

$$A \xrightarrow{f} B$$

using a **monotone map**, that is a function $f : A \longrightarrow B$ which satisfies

$$x \leq y \implies f(x) \leq f(y)$$

for all $x, y \in A$.

These give the objects and arrows of the category *Pos*. ■

This is the category inside which almost everything will happen. (Actually, there is an argument for working with the more general notion of a preset – a pre-ordered set. We won't do that here since it doesn't give any immediate benefits. However, at a later stage you may find that presets rather than posets are the appropriate vehicle.)

In general a poset need not have any completeness properties, or may have some but not other such properties. Given a subset X of a poset A we write

$$\bigvee X \quad \bigwedge X$$

for, respectively, the supremum and the infimum of X in A . Of course, for a particular poset A and subset X each of these elements may or may not exist, and one may exist without the other. When X is a 2-element set

$$X = \{x, y\}$$

we write

$$x \vee y = \bigvee X \quad x \wedge y = \bigwedge X$$

and refer to these as the join and meet of the pair, respectively. This is a convenient linguistic trick to distinguish between the finitary and the infinitary. We write

$$\perp \quad \top$$

for, respectively, the bottom and top of the poset A , assuming these exists. Each can exist without the other, but

$$\bigvee \emptyset = \bigwedge A = \perp \quad \bigwedge \emptyset = \bigvee A = \top$$

when they do exist.

The category **Sup**

A poset A is complete if

$$\bigvee X \quad \bigwedge X$$

exist for each subset $X \in \mathcal{P}A$. If A has all suprema then A has all infima. These complete poset form the objects of several different categories. We use one of these.

1.2 DEFINITION. A \bigvee -semilattice (or a sup-semilattice) is a structure

$$(A, \leq, \bigvee, \perp)$$

where (A, \leq) is a complete poset with supremum operation \bigvee and bottom $\perp = \bigvee \emptyset$.

Two such structures are compared

$$A \xrightarrow{f} B$$

using a \bigvee -continuous function, that is a monotone function f which also satisfies

$$f\left(\bigvee X\right) = \bigvee f[X]$$

for all subsets X of A . In particular $f(\perp) = \perp$.

This gives us the category **Sup**. ■

The two categories **Pos**, **Sup** are intimately connected. There is a forgetful functor

$$\mathbf{Pos} \xleftarrow{i} \mathbf{Sup}$$

obtained by simply viewing each object and arrow of **Sup** as gadgets of **Pos**. It is not hard to see that this functor has a left adjoint \mathcal{L} , and we produce an explicit description of this in section 2. The nature of this adjoint drives almost everything we do.

As well as this and other functorial adjunctions we need some miniature examples.

1.3 DEFINITION. An **adjunction** between two posets A, B is a pair

$$A \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_*} \end{array} B$$

of monotone maps such that

$$f^*a \leq b \iff a \leq f_*b$$

for each $a \in A$ and $b \in B$. ■

By convention we call f^* the left adjoint, and f_* the right adjoint. A simple argument shows that each of these monotone maps f^* and f_* determines the other. In other words, each monotone map can have at most one left adjoint and at most one right adjoint. Furthermore, it can have one without the other, and it can have neither. It is even possible that a monotone map has both a left and a right adjoint and these are different. The existence of an adjoint (on some side) is concerned with completeness properties of the two posets involved.

1.4 LEMMA. Each **Sup**-arrow

$$A \xrightarrow{f = f^*} B$$

has a (unique) right adjoint

$$A \xleftarrow{g = f_*} B$$

given by

$$gb = \bigvee \{a \in A \mid fa \leq b\}$$

for each $b \in B$.

Here both A and B are \bigvee -semilattices, and the function f is a **Sup**-arrow, as indicated. The right adjoint g is a monotone map, but need *not* be a **Sup**-arrow.

The category \mathbf{Slt}^\wedge

We now move to the other end of the top row in Table 1

1.5 DEFINITION. A \wedge -semilattice (or a meet semilattice) is a structure

$$(A, \leq, \wedge, \top)$$

where (A, \leq) is a poset with binary meet operation \wedge and top \top .

These structures are compared

$$A \xrightarrow{f} B$$

using monotone functions which preserve the distinguished attributes, that is

$$f(x \wedge y) = fx \wedge fy \quad f\top = \top$$

for all $x, y \in A$.

This gives the category \mathbf{Slt}^\wedge of \wedge -semilattices. ■

Although these structures play a central role in the coverage technique, there is little else we need to know about them.

The category **Frm**

In the previous two subsections we have used two different kinds of completeness properties of a poset. We now combine these and make them interact.

1.6 DEFINITION. A frame is a structure

$$(A, \leq, \bigvee, \perp, \wedge, \top)$$

where

- $(A, \leq, \bigvee, \perp)$ is a \bigvee -semilattice
- (A, \wedge, \top) is a \wedge -semilattice
- the frame distributive law

$$a \wedge \bigvee X = \bigvee \left\{ a \wedge x \mid x \in X \right\}$$

holds for all $a \in A$ and $X \subseteq A$.

Frames are compared

$$A \xrightarrow{f} B$$

using functions f which are both **Sup**-arrows and $\{\wedge, \top\}$ -preserving.

This gives the category **Frm** of frames. ■

By construction there are two forgetful functors

$$\mathbf{Slt}^\wedge \xleftarrow{i} \mathbf{Frm} \qquad \mathbf{Sup} \xleftarrow{i} \mathbf{Frm}$$

each forgetting a part of the structure on a frame. In section 2 we locate the left adjoint \mathcal{L} to the first of these.

The most immediate examples of frames arise by taking the topology $\mathcal{O}S$ of open sets of a topological space. Each continuous map

$$T \xrightarrow{\phi} S$$

between topological spaces gives a frame morphism

$$\mathcal{O}S \xrightarrow{\phi^\leftarrow} \mathcal{O}T$$

by taking the inverse image map. This construction gives a contravariant functor

$$\mathbf{Frm} \longleftarrow \mathbf{Top}$$

from the category **Top** of topological spaces to **Frm**. This is one half of a contravariant adjunction. The coverage technique is often used to produce a frame from which we can obtain a space via the missing adjoint functor.

Each frame is a \bigvee -semilattice with some extra structure and, crucially, an extra property in the form of the frame distributive law. What is the content of this? A \bigvee -semilattice A has an implication if there is a 2-placed operation $(\cdot \supset \cdot)$ on A where

$$x \leq (a \supset b) \iff a \wedge x \leq b$$

holds for all $a, b, x \in A$. Tivially, there is at most one such operation on A .

1.7 LEMMA. A \bigvee -semilattice A is a frame precisely when it has an implication.

Proof. Suppose that A is a frame. For $a, b \in A$ let

$$(a \supset b) = \bigvee X$$

where the set X is defined by

$$x \in X \iff a \wedge x \leq b$$

(for $x \in A$). The frame distributive law gives

$$a \wedge (a \supset b) = \bigvee \{a \wedge x \mid x \in X\} \leq b$$

from which we see that $(\cdot \supset \cdot)$ is the required implication operation.

Conversely, suppose A has an implication operation $(\cdot \supset \cdot)$. For $X \in \mathcal{P}A$ let

$$b = \bigvee \{a \wedge x \mid x \in X\}$$

so that a comparison

$$a \wedge \bigvee X \leq b$$

will give the required result. For each $x \in X$ we have $a \wedge x \leq b$ so that $x \leq (a \supset b)$ and hence

$$\bigvee X \leq (a \supset b)$$

which leads to the required result. ■

Later in this section we obtain a generalization of this result.

The category *Pom*

Recall that a monoid is a structure

$$(A, \bullet, 1)$$

where A is a set furnished with an associative binary operation ' \bullet ' and a neutral element 1 for that operation. We indicate values of the operation by concatenation and write

$$ab \text{ for } a \bullet b$$

(for $a, b \in A$). The operation is associative, so we may omit certain brackets and write

$$abc$$

for the two punctuated versions.

A monoid morphism

$$A \xrightarrow{f} B$$

is a function between the two carrying sets which preserves the structures, that is

$$f1 = 1 \quad f(xy) = (fx)(fy)$$

for $x, y \in A$. This gives the category of monoids. We need an enriched version of this.

1.8 DEFINITION. A pom (or, in full, a partially ordered monoid) is a structure

$$(A, \leq, \bullet, 1)$$

where

$$(A, \leq) \text{ is a poset} \quad (A, \bullet, 1) \text{ is a monoid}$$

and where

$$\left. \begin{array}{l} x \leq a \\ y \leq b \end{array} \right\} \implies xy \leq ab$$

holds for all $a, b, x, y \in A$.

Poms are compared

$$A \xrightarrow{f} B$$

using monotone monoid morphisms.

This gives the category **Pom** of poms. ■

Just as a poset can be generalized to a pre-ordered set, so the notion of a pom can be generalized to a similar structure carried by a preset. In the final analysis the coverage technique ought to be developed on such structures. I won't describe that more general version, but an indication of what happens is given in section 8.

Each \wedge -semilattice is a pom of a special kind. A poset can be a \wedge -semilattice in at most one way, but it can carry many different monoids structures to become a pom.

Poms have not had the grilling that other less interesting algebras have enjoyed. At the very least there is a couple of half-baked theses on these creatures that could be cooked up.

1.9 DEFINITION. A pom A is residuated if it carries a 3-placed operation

$$a, b, c \longmapsto (a, b) \multimap c$$

characterized by

$$x \leq (a, b) \multimap c \iff axb \leq c$$

(for $x, a, b, c \in A$. The two 2-placed operations

$$(1, \cdot) \multimap \cdot \quad (\cdot, 1) \multimap \cdot$$

give 1-sided residuations. ■

When the pom is commutative (or symmetric) the 3-placed operation and the two 2-placed operations coincide. In this terminology Lemma 1.7 says that a \vee -semilattice is frame precisely when viewed as a pom the underlying \wedge -semilattice is residuated.

The category **Qnl**

There is a sense in which a frame is just a \wedge -semilattice viewed as an object of **Sup**. In the same way we may view certain poms as objects of **Sup**.

1.10 DEFINITION. A quantale is a structure

$$(A, \leq, \bigvee, \perp, \bullet, 1)$$

where

- $(A, \leq, \bigvee, \perp)$ is a \bigvee -semilattice
- $(A, \bullet, 1)$ is a monoid
- the quantale distributive law

$$l(\bigvee X)r = \bigvee \{lxr \mid x \in X\}$$

holds for all $l, r \in A$ and $X \subseteq A$.

Quantales are compared

$$A \xrightarrow{f} B$$

using functions f which are both **Sup**-arrows and $\{\bullet, 1\}$ -preserving.

This gives the category **Qnl** of quantales. ■

Each frame is a quantale of a very special kind. It is commutative and the associated pom is a \wedge -semilattice. These quantales have a nice characterization.

1.11 LEMMA. *A quantale is a frame exactly when $1 = \top$ and each element is idempotent.*

The family of ideals of a commutative ring is a quantale that need not be a frame.

1.12 LEMMA. *Suppose the underlying poset of a pom A is a \bigvee -semilattice. Then A is a quantale if and only if A is residuated.*

The proof of this follows the same pattern as that of Lemma 1.7.

2 Free completions

From Table 1 let **Fin** be any one of the three categories

$$\mathbf{Pos} \quad \mathbf{Pom} \quad \mathbf{Sl}^\wedge$$

of finitary algebras, and let **Inf** be the corresponding category

$$\mathbf{Sup} \quad \mathbf{Qnl} \quad \mathbf{Frm}$$

of infinitary algebras. Each of the three pairs of vertical arrows of Table 1 gives an adjunction

$$\mathbf{Fin} \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \dashv \\ \xleftarrow{\mathcal{I}} \end{array} \mathbf{Inf}$$

where ι is the forgetful functor and \mathcal{L} is the left adjoint that we describe. As the notation suggests, we will see that the three left adjoints are obtained by essentially the same construction. Furthermore, the construction is a completion process.

For each object A of **Fin** we construct an object $\mathcal{L}A$ of **Inf**. In each of the three cases the carrier of $\mathcal{L}A$ is produced in the same way; only the furnishings depend on the case. We also construct a certain arrow of **Fin**

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

carried by the same function $\eta_A : A \longrightarrow \mathcal{L}A$ in each case. (Of course, we now view $\mathcal{L}A$ as an object of **Fin**, so strictly speaking we should write

$$A \xrightarrow{\eta_A} (\iota \circ \mathcal{L})A$$

for the arrow, but we don't need to be so pernickity.) It is this arrow that we call the free completion of A , because of the following property.

2.1 FREE COMPLETION. For each **Fin**-object A and **Fin**-arrow

$$A \xrightarrow{f} B$$

to a **Inf**-object B , there is a unique **Inf**-arrow

$$\mathcal{L}A \xrightarrow{f^\#} B$$

such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \eta_A & \nearrow f^\# \\ & \mathcal{L}A & \end{array}$$

commutes. ■

We can take this as the definition of free completion. Our problem here is to show that each **Fin**-object A does have one of these. We know, on general grounds, that the existence of free completions ensures that the construction \mathcal{L} is left adjoint to the forgetful functor. However, here the adjointness properties are not important. Indeed, the coverage technique gives a method of attacking a single finitary object to produce an infinitary object which is complete *and* satisfies other conditions. This technique is built on top of the free completion method described in this section.

Free \vee -semilattices

The forgetful functor

$$\mathbf{Pos} \longleftarrow \mathbf{Sup}$$

forgets in two ways. It forgets the furnishings \vee and \perp , and it forgets that the poset is complete. Our problem is to go the other way. The construction we use drives almost everything we do, so it is worth looking at it internal workings in some detail.

A lower section or an initial section of a poset A is a subset X of A such that

$$x \in X \implies y \in X$$

holds for all elements x, y of A with $y \leq x$. In particular, the extremes \emptyset and A are lower sections. For each subset H of A we construct the subset $\downarrow H$ by

$$x \in \downarrow H \iff (\exists y : A)[x \leq y \in H]$$

to obtain the lower section generated by H , that is the smallest lower section which includes H . For each $a \in A$ we set

$$\downarrow a = \downarrow \{a\}$$

to obtain the principal lower section generated by a .

2.2 DEFINITION. (a) For each poset A let $\mathcal{L}A$ be the poset of lower sections of A (under inclusion). This is closed under arbitrary unions and hence is a complete poset.

(b) For each monotone map

$$A \xrightarrow{f} B$$

between posets A and B , let

$$\mathcal{L}A \xrightarrow{\mathcal{L}(f)} \mathcal{L}B$$

be the monotone map given by

$$\mathcal{L}(f)(X) = \downarrow f[X]$$

for each $X \in \mathcal{L}(A)$. ■

Here

$$f[X] = \{fx \mid x \in X\}$$

is the direct image of X across f .

For each poset A the poset $\mathcal{L}A$ is complete, and hence we obtain a \vee -semilattice

$$(\mathcal{L}A, \subseteq, \bigcup, \emptyset)$$

with set theoretic furnishings. Thus we have an object assignment $\mathbf{Pos} \longrightarrow \mathbf{Sup}$.

Each \mathbf{Pos} -arrow

$$A \xrightarrow{f} B$$

produces a monotone map

$$\mathcal{L}A \xrightarrow{\mathcal{L}(f)} \mathcal{L}B$$

by taking direct images. By a further simple calculation we find that

$$\mathcal{L}(f) \left(\bigcup \mathcal{X} \right) = \bigcup \{ \mathcal{L}(f)(X) \mid X \in \mathcal{X} \}$$

for each subset \mathcal{X} of $\mathcal{L}A$. From this we see that $\mathcal{L}(f)$ is a **Sup**-arrow.

Finally we check that for each composable pair

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of monotone maps the equality

$$\mathcal{L}(g \circ f) = (\mathcal{L}g) \circ (\mathcal{L}f)$$

holds. Thus we have a functor

$$\mathbf{Pos} \xrightarrow{\mathcal{L}} \mathbf{Sup}$$

which we will show is the required left adjoint. The ‘ \mathcal{L} ’ reminds us of ‘lower’ and ‘left’.

We have indicated a direct verification that \mathcal{L} is a functor. Strictly speaking, this is not necessary, for functoriality follows from the free completion property.

For each poset A and each $a \in A$ let

$$\eta_A(a) = \downarrow a$$

to produce a monotone map

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

which is natural for variation of A . More precisely, η_\bullet is a natural transformation

$$Id_{\mathbf{Pos}} \xrightarrow{\eta} (i \circ \mathcal{L})$$

where $Id_{\mathbf{Pos}}$ is the identity endofunctor of \mathbf{Pos} and ‘ i ’ is the forgetful functor. Usually we will not be so pedantic as this. Trivially, η_A is an injective monotone map. It is not epic (in \mathbf{Pos}) but it does have an epic-like property.

2.3 LEMMA. *For each poset A , the monotone map η_A is ‘**Sup**-epic’ in the sense that*

$$g \circ \eta_A = h \circ \eta_A \implies g = h$$

holds for each parallel pair

$$\mathcal{L}A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

*of **Sup**-arrows.*

Proof. For each $X \in \mathcal{L}A$ we have

$$X = \bigcup \{ \downarrow x \mid x \in X \} = \bigcup \eta[X]$$

from which the result follows easily. ■

To verify that η_A is the free completion of A we must show that each arrow f of a certain kind factors uniquely through η_A . Lemma 2.3 gives us the uniqueness.

2.4 THEOREM. *For each poset A the arrow*

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

is the free completion of A .

Proof. Consider any monotone map

$$A \xrightarrow{f} B$$

to a **Sup**-object B . For each $X \in \mathcal{L}A$ set

$$f^\sharp(X) = \bigvee f[X]$$

to obtain a monotone map. A few calculations give the required preservation and lifting properties. The required uniqueness follows since η_A is **Sup**-epic. ■

In the next two subsections we refine this construction to produce the free completion of a pom and a \wedge -semilattice.

Free quantales

For each poset A we have a monotone map

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

to a sup-semilattice $\mathcal{L}A$. When A is a pom we can use its furnishings to impose extra structure on $\mathcal{L}A$. For $X, Y \in \mathcal{L}A$ let

$$XY = \downarrow \{xy \mid x \in X, y \in Y\}$$

to obtain a binary operation on \mathcal{L} . In the same way let $\mathbf{1} = \downarrow 1$ to obtain a particular member of $\mathcal{L}A$. We can now consider the structure

$$(\mathcal{L}A, \subseteq, \bigcup, \emptyset, \cdot, \mathbf{1})$$

formed by furnishing the free \bigvee -semilattice $\mathcal{L}A$ with the two extra attributes.

2.5 THEOREM. *For each pom A the furnished $\mathcal{L}A$ is a quantale and*

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

*is a **Pom**-arrow which is the free completion of A (to a quantale).*

Proof. The proof is a series of simple observations. Let's see what has to be done.

(1) The operation on $\mathcal{L}A$ is associative, that is

$$(XY)Z = X(YZ)$$

holds for each $X, Y, Z \in \mathcal{L}A$. Each side becomes

$$\downarrow\{xyz \mid x \in X, y \in Y, z \in Z\}$$

by a simple unravelling.

(2) The element $\mathbf{1}$ is neutral for the operation.

(3) The distributive law

$$X\left(\bigcup \mathcal{Y}\right)Z = \bigcup \left\{XYZ \mid Y \in \mathcal{Y}\right\}$$

holds for each $X, Z \in \mathcal{L}A$ and $\mathcal{Y} \subseteq \mathcal{L}A$. Each side is the lower section generated by all xyz for $x \in X, y \in Y \in \mathcal{Y}, z \in Z$.

This shows that $\mathcal{L}A$ is a quantale.

(4) For each $a, b \in A$

$$\eta_A(ab) = \downarrow ab = (\downarrow a)(\downarrow b) = (\eta_A a)(\eta_A b)$$

to show that η_A is a **Pom**-arrow.

(5) For each parallel pair

$$\mathcal{L}A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

of **Qnl**-arrows, the implication

$$g \circ \eta_A = h \circ \eta_A \implies g = h$$

holds, to show that η_A is **Qnl**-epic. This is a consequence of Lemma 2.3.

(6) Finally, consider any **Pom**-arrow

$$A \xrightarrow{f} B$$

to a **Qnl**-object B . By Theorem 2.4 we know there is a unique **Sup**-arrow

$$\mathcal{L}A \xrightarrow{f^\#} B$$

such that $f = f^\# \circ \eta_A$. It suffices to show that $f^\#$ is a **Qnl** arrow. To this end consider any $X, Y \in \mathcal{L}(A)$ and let

$$Z = \{xy \mid x \in X, y \in Y\}$$

so that $XY = \downarrow Z$. Since f is a **Pom**-arrow, we have

$$f^\#(XY) = f^\#(\downarrow Z) = \bigvee f[Z] = \left(\bigvee f[X]\right)\left(\bigvee f[Y]\right) = (f^\#X)(f^\#Y)$$

as required.

There is nothing very sophisticated going on in this proof. ■

By Lemma 1.12 every quantale is residuated. In particular the completion $\mathcal{L}A$ is residuated, and it doesn't take too long to show that

$$a \in (L, R) \multimap C \iff LaR \subseteq C$$

for each $L, R, C \in \mathcal{L}(A)$ (and $a \in A$).

The parent pom A may itself be residuated, but this doesn't lead to any confusion.

2.6 THEOREM. *If the pom A is a residuated then*

$$\eta_A((l, r) \multimap c) = (\eta_A(l), \eta_A(r)) \multimap \eta_A(c)$$

holds for all $r, l, c \in A$

Proof. A calculation shows that each side consists of those $a \in A$ with $lar \leq c$. ■

The pom A may have other properties, and some of these transfer to $\mathcal{L}A$. For instance, if A is commutative then so is $\mathcal{L}A$.

Free frames

A \wedge -semilattice is a pom of a particular kind, and a frame is a quantale of a particular kind. In both cases it is the same kind.

2.7 THEOREM. *Let A be a \wedge -semilattice. Then the free quantale $\mathcal{L}A$ is a frame, and*

$$A \xrightarrow{\eta_A} \mathcal{L}A$$

*is a **Stt**[^]-arrow which is the free completion of A (to a frame).*

The frame in question is just

$$(\mathcal{L}A, \subseteq, \bigcup, \emptyset, \cap, A)$$

with set theoretic furnishings. In fact, this is a frame for *any* poset A . The extra content of Theorem 2.7 is that when A is a \wedge -semilattice the function η is \wedge -preserving.

3 Some universal algebra

Let **Inf** be one of the categories **Sup**, **Qnl**, or **Frm**. A quotient in **Inf** is an arrow

$$A \xrightarrow{f} B$$

which is surjective as a function. Each arrow (surjective or not) determines an equivalence relation on its source which is a congruence appropriate for the algebras under consideration. For a quotient that congruence determines the target algebra (up to isomorphism), and this can be viewed as carried by the blocks of the congruence. In this section we describe the construction of that quotient in more amenable terms.

Quotients in *Sup*

For many algebras (such as groups or rings) the blocks of a quotient are determined by one special block (by ‘translation’). For partially ordered structures that may not happen, and does not happen with the three classes considered here. However, because each morphism is at least \vee -preserving, each block contains a unique maximum member. We use that special member as the representative of the block.

3.1 DEFINITION. Let A be a \vee -semilattice. A closure operation on a A is a function $j : A \longrightarrow A$ which is

inflationary monotone idempotent

that is

$$x \leq jx \quad y \leq x \implies jy \leq jx \quad j^2x = jx$$

hold for each $x, y \in A$. ■

Consider any *Sup*-arrow

$$A \xrightarrow{f} B$$

(which need not be surjective). By Lemma 1.4 we know f has a right adjoint

$$A \xrightleftharpoons[f_*]{f^* = f} B$$

which need not be a *Sup*-arrow, but is certainly monotone. By definition, we have

$$f^*a \leq b \iff a \leq f_*b$$

for all $a \in A, b \in B$. The composite $f_* \circ f^*$ is a closure operation on A .

3.2 DEFINITION. For each *Sup*-arrow f , as above, the closure operation

$$\ker(f) = f_* \circ f^*$$

is the kernel of f . ■

It doesn’t take to long to see that $k = \ker(f)$ is characterized by

$$x \leq ka \iff fx \leq fa$$

for $a, x \in A$.

We will verify two observations. Each closure operation j on a \vee -semilattice A determines a quotient

$$A \xrightarrow{j^*} A_j$$

where the kernel is j . Each quotient

$$A \xrightarrow{f} B$$

is canonically isomorphic to the quotient j^* determined by the kernel $j = \ker(f)$. The meaning of ‘canonical’ will become clear.

3.3 DEFINITION. For a closure operation j on a \vee -semilattice A let

$$A_j = j[A] = \{a \in A \mid ja = a\}$$

and view A_j as a poset (under the restriction of the comparison from A). ■

Thus A_j is just the set of elements of A that are fixed by j . Since j is inflationary and idempotent this is just the range, set of values, of j . Since A_j is a subset of A it is a poset under the comparison inherited from A . We want A_j to be a \vee -semilattice.

Consider any subset $X \subset A_j$. This has a supremum $\bigvee X$ in A , but this element may not be in A_j . However

$$\bigvee^j X = j\left(\bigvee X\right)$$

is in A_j and is certainly an upper bound for X . Consider any upper bound a of X in A_j . Then $x \leq a$ for each $x \in X$, so that $\bigvee X \leq a$ in A , and hence

$$\bigvee^j X = j\left(\bigvee X\right) \leq ja = a$$

since $a \in A_j$. This shows that A_j is a \vee -semilattice. In fact, we have the following.

3.4 LEMMA. For each closure operation j on the \vee -semilattice A , the poset A_j is a \vee -semilattice where

$$\bigvee^j X = j\left(\bigvee X\right)$$

gives the supremum of $X \subseteq A_j$. Furthermore, the assignment

$$\begin{array}{ccc} A & \xrightarrow{j^*} & A_j \\ a & \longmapsto & ja \end{array}$$

is a quotient with j as its kernel.

Proof. We have proved above that A_j is a \vee -semilattice.

Trivially, the assignment j^* is surjective and monotone. We need to show that

$$j^*\left(\bigvee X\right) = \bigvee^j j^*[X]$$

holds for each $X \subseteq A$. But these two compounds unravel to

$$j\left(\bigvee X\right) \quad j\left(\bigvee j[X]\right)$$

and a one line calculations shows these are equal.

Finally we check that the right adjoint

$$\begin{array}{ccc} A & \xrightarrow{j^*} & A_j \\ & \xleftarrow{j_*} & \end{array}$$

j_* of j^* is just the insertion

$$A \xleftarrow{j_*} \supset A_j$$

and hence $\ker(j^*) = j_* \circ j^* = j$, to complete the proof. ■

This shows how each closure operation on a \vee -semilattice produces a quotient. Our second job is to show that this is essentially the only way of producing quotients.

For a \vee -semilattice A we may compare two closure operations j, k on A by

$$j \leq k \iff (\forall x \in A)[jx \leq kx]$$

which is the pointwise comparison. This imposes a partial ordering on the set of all closure operations carried by A .

3.5 THEOREM. *Let j be a closure operation on the **Sup**-object A . Let*

$$A \xrightarrow{f} B$$

*be a **Sup**-arrow with kernel $k = \ker(f)$ where $j \leq k$. Then there is a unique **Sup**-arrow f_j such that the **Sup**-triangle*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow j^* & \nearrow f_j \\ & A_j & \end{array}$$

commutes.

Proof. By definition of k we have

$$x \leq ka \iff fx \leq fa$$

for all $x, a \in A$. Also $ja \leq ka$ for each $a \in A$, so that

$$fa \leq f(ja) \leq fa$$

and hence $f \circ j = f$.

For each $a \in A_j$ set

$$f_j a = fa$$

to obtain a function $f_j : A \longrightarrow B$. For $a \in A$ we have

$$(f_j \circ j^*)(a) = f_j(ja) = f(ja) = fa$$

and hence $f_j \circ j^* = f$.

If a function $g : A_j \longrightarrow B$ satisfies $g \circ j^* = f$ then for each $a \in A_j$ we have

$$ga = g(ja) = fa = f_j a$$

to give so that $g = f_j$.

It remains to show that f_j is \bigvee -preserving. Consider any $X \subseteq A_j$. With

$$a = \bigvee^j X = j\left(\bigvee X\right)$$

we have

$$f_j a = f a = f\left(j\left(\bigvee X\right)\right) = f\left(\bigvee X\right) = \bigvee f[X] = \bigvee f_j[X]$$

to give the required result. ■

We develop a method of producing quotients of the infinitary algebras with certain predetermined properties. The results of this section show that this is equivalent to finding a certain closure operation on the algebra. When the algebra is a \bigvee -semilattice that is all we need. When the algebra is a quantale or frame then we need more.

Quotients in **Qnl**

Consider a quotient

$$A \xrightarrow{f} B$$

in **Qnl**. This is at least a quotient in **Sup** and so is determined by the closure operation $k = \ker(f)$ which, as before, is characterized by

$$x \leq ka \iff fx \leq fa$$

for $a, x \in A$. In particular

$$f(ka) = fa$$

for each $a \in A$. By the previous subsection each closure operation j on A will give a quotient of A in **Sup**, but that may not produce a quantale. We need to isolate a special class of closure operations. Here is the definition analogous to Definition 3.1.

3.6 DEFINITION. Let A be a quantale. A **quantic nucleus** on A is a closure operation $j : A \longrightarrow A$ for which

$$(jx)(jy) \leq j(xy)$$

holds for all $x, y \in A$. ■

A quantic nucleus is a closure operation which respects the associative operation.

3.7 LEMMA. For each **Qnl**-arrow

$$A \xrightarrow{f} B$$

the kernel $\ker(f)$ is a quantic nucleus on A .

Proof. We know that $k = \ker(f)$ is a closure operation. For $x, y \in A$ the various properties of f and k give

$$f((kx)(ky)) = (f(kx))(f(ky)) = (fx)(fy) = f(xy)$$

and hence

$$(kx)(ky) \leq k(xy)$$

by the construction of the kernel k . ■

For an analogue of Lemma 3.4 consider a quantic nucleus j on a quantale A . We have a **Sup**-quotient

$$A \xrightarrow{j^*} A_j$$

and our job is to show that A_j carries a monoid structure to become a quantale and such that j^* is a quantale morphism.

For $a, b \in A_j$ set

$$a \bullet b = j(ab)$$

to obtain a binary operation on A_j . To show that this is associative consider $a, b, c \in A_j$ and let $x = a \bullet b$. Then $ab \leq x$ so that

$$abc \leq xc \leq (a \bullet b) \bullet c$$

and hence

$$j(abc) \leq (a \bullet b) \bullet c$$

holds. Also

$$xc \leq (j(ab))(jc) \leq j(abc)$$

so that

$$(a \bullet b) \bullet c = j(xc) \leq j(abc)$$

and hence

$$(a \bullet b) \bullet c = j(abc)$$

holds. This combined with a similar argument shows that

$$(a \bullet b) \bullet c = j(abc) = a \bullet (b \bullet c)$$

and hence the defined operation is associative. A similarly simple argument shows that $1_j = j(1)$ is a unit for this operation, so we do have a monoid.

3.8 LEMMA. *Let j be a quantic nucleus on the quantale A . Then the **Sup**-object A_j carries a monoid structure making A_j a quantale such that the **Sup**-arrow*

$$A \xrightarrow{j^*} A_j$$

is a quantale morphism.

Proof. We have described above the monoid structure imposed on A_j . Furthermore, for $a, b \in A$ we have

$$(j^*a) \bullet (j^*b) = j((ja)(jb)) = j(ab) = j^*(ab)$$

which more or less shows that j^* is a monoid morphism.

To complete the proof it suffices to show that A_j is a quantale, that is both

$$a \bullet \left(\bigvee^j X \right) = \bigvee^j \{a \bullet x \mid x \in X\} \quad \left(\bigvee^j X \right) \bullet a = \bigvee^j \{x \bullet a \mid x \in X\}$$

hold for all $a \in A_j$ and $X \subseteq A_j$. For the left hand equality we have

$$l = a \bullet \left(\bigvee^j X \right) = j \left(aj \left(\bigvee^j X \right) \right) \quad r = \bigvee^j \{a \bullet x \mid x \in X\} = j \left(\bigvee \{j(ax) \mid x \in X\} \right)$$

so we show $l = r$ via two comparisons. Firstly we have

$$aj \left(\bigvee^j X \right) \leq (ja)j \left(\bigvee^j X \right) \leq j \left(a \left(\bigvee^j X \right) \right) \leq j \left(\bigvee \{ax \mid x \in X\} \right) \leq r$$

and hence $l \leq jr = r$. Secondly, for each $x \in X$ we have

$$x \leq \bigvee^j X \leq j \left(\bigvee^j X \right)$$

so that

$$ax \leq aj \left(\bigvee^j X \right)$$

and hence

$$j(ax) \leq l$$

holds. Since x is arbitrary this gives

$$\bigvee \{j(ax) \mid x \in X\} \leq l$$

and hence $r \leq jl = l$. The right hand equality is proved in then same way. \blacksquare

This with a couple more observations gives the quantale version of Theorem 3.5.

3.9 THEOREM. *Let j be a quantic nucleus on the \mathbf{Qnl} -object A . Let*

$$A \xrightarrow{f} B$$

be a \mathbf{Qnl} -arrow with kernel $k = \ker(f)$ where $j \leq k$. Then there is a unique \mathbf{Qnl} -arrow f_j such that the \mathbf{Qnl} -triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow j^* & \nearrow f_j \\ & & A_j \end{array}$$

commutes.

Proof. By Theorem 3.5 we know there is precisely one **Sup**-arrow f_j which make the triangle commute. Thus it suffices to show that this function f_j is a monoid morphism, and hence is a \mathbf{Qnl} -arrow. But, for $a, b \in A_j$ we have

$$f_j(a \bullet b) = f(j(ab)) = f(ab) = (fa)(fb) = (f_j a)(f_j b)$$

as required. \blacksquare

This result shows the importance of the quantic nuclei on a quantale.

Quotients in **Frm**

Each frame is a special kind of quantale. Each quotient

$$A \xrightarrow{f} B$$

in **Frm** is a special kind of quantale quotient. Thus, by Theorem 3.9 it is characterized by its kernel and this is a special kind of quantic nucleus.

3.10 DEFINITION. Let A be a frame. A nucleus on A is a closure operation $j : A \longrightarrow A$ for which

$$(jx) \wedge (jy) = j(x \wedge y)$$

holds for all $x, y \in A$. ■

You might wonder why this notion has an equality whereas the more general notion of Definition 3.6 has a comparison. If so, you should.

The analogue of Lemma 3.7 does hold, and the proof is just the same.

3.11 LEMMA. For each **Frm**-arrow

$$A \xrightarrow{f} B$$

the kernel $\ker(f)$ is a nucleus on A .

Each nucleus j on a frame A is a quantic nucleus on A viewed as a quantale. Thus the fixed set A_j carries a quantale structure. As usual suprema in A_j are just j -modified suprema in A . Remembering that the monoid operation on A is just meet we see that the imposed monoid operation on A_j is give by

$$a \bullet b = j(a \wedge b) = (ja) \wedge (jb) = a \wedge b$$

(since both a, b are fixed by j). In other words, the monoid operation on A_j is just meet. This gives the following refinement of Lemma 3.8.

3.12 LEMMA. Let j be a nucleus on the frame A . Then the **Sup**-object A_j carries is a \wedge -semilattice making A_j a frame such that the **Sup**-arrow

$$A \xrightarrow{j^*} A_j$$

is a frame morphism.

Finally, we have the analogue of theorem 3.9.

3.13 THEOREM. Let j be a nucleus on the **Frm**-object A . Let

$$A \xrightarrow{f} B$$

be a **Frm**-arrow with kernel $k = \ker(f)$ where $j \leq k$. Then there is a unique **Frm**-arrow f_j such that the **Frm**-triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow j^* & \nearrow f_j \\ & & A_j \end{array}$$

commutes.

The results of the subsection can be proved directly with reference to quantales.

From *Qnl* to *Frm*

To conclude this section let's look at a result which may not seem directly relevant but which does have an impact on some uses (and mis-uses) of the coverage technique.

Let A be a quantale. There is at least one quotient of A which is a frame (for we may collapse everything to the 1-element frame). Thus there are quantic nuclei j on A for which the quotient A_j is a frame. There is a universal example of such a nucleus.

3.14 THEOREM. *For each quantale A there is a unique quantic nucleus ℓ such that for each quantic nucleus j on A , the quotient A_j is a frame precisely when $\ell \leq j$.*

This special nucleus ℓ is a kind of 'radical' operation on A , and it provides a reflection of *Qnl* into *Frm*. An analysis of this and related properties is given in [7].

4 How to generate nuclei

Each quotient of a

\bigvee -semilattice frame quantale

is determined by a

closure operation nucleus quantic nucleus

respectively. Some of these can be quite complicated, so we will develop a method of producing such a gadget from a simpler function. The coverage technique, which is developed in the following sections, takes this method one step further to show how these simple functions can be generated from more primitive information.

In this section we first develop a method for producing closure operations on a \bigvee -semilattice. We then modify this method to produce quantic nuclei on a quantale. Finally, we look at the particular case where the quantale is a frame. In fact, for these notes it is the frame version that is important, but the more general quantale version should not be forgotten. If you want to, at a first reading you may assume that each quantale we meet is a frame

4.1 DEFINITION. An inflator on a \bigvee -semilattice A is a function $d : A \longrightarrow A$ which is both inflationary and monotone, that is

$$x \leq dx \quad x \leq y \implies dx \leq dy$$

hold for all $x, y \in A$.

Let $\mathcal{I}A$ be the set of inflators on A . ■

A closure operation on A is an inflator j which is also idempotent, that is $j^2 = j$. We will see that inflators are easier than closure operations to deal with. For instance, the composite of two closure operations need not be a closure operation, but the composite of two inflators is an inflator.

When dealing with a frame or quantale we use a special class of closure operations, the nuclei or quantic nuclei. Here we won't come to any harm if we use 'nucleus' to mean either nucleus (on the parent frame) or quantic nucleus (on the parent quantale). It is always clear what the parent algebra is.

There are several special classes of inflators.

4.2 DEFINITION. Let A be a frame or quantale. A **pre-nucleus** on A is an inflator d such that, respectively,

$$(dx) \wedge (dy) = d(x \wedge y) \quad (dx)(dy) \leq d(xy)$$

holds for all $x, y \in A$. ■

Thus a nucleus is just a pre-nucleus that is idempotent. Notice that the composite of two pre-nuclei is itself a pre-nucleus.

4.3 DEFINITION. For a \vee -semilattice A the **pointwise comparison** on $\mathcal{I}A$ is given by

$$d \leq e \iff (\forall x \in A)[dx \leq ex]$$

for inflators d, e . ■

This is a partial ordering of $\mathcal{I}A$. Furthermore, it includes the poset of closure operations or nuclei on A . Note that the identity function on A is the bottom of $\mathcal{I}A$.

The assumed completeness of A gives some completeness properties of $\mathcal{I}A$.

4.4 LEMMA. For each subset $D \subseteq \mathcal{I}A$ the pointwise infimum $\bigwedge D$ given by

$$\left(\bigwedge D \right) x = \bigwedge \{ dx \mid d \in D \}$$

is an inflator, and is the infimum of D in $\mathcal{I}A$.

This shows that the poset $\mathcal{I}A$ has all infima, and hence it has all suprema. A simple exercise (which you should do) shows that if D is a set of closure operations or pre-nuclei or nuclei, then $\bigwedge D$ is a closure operation or pre-nucleus or nucleus, respectively. Thus these various subclasses of inflators also form complete posets. However, there is a subtlety here.

Suppose, for instance, D is a set of nuclei. Then we may look at $\bigvee D$ as a nucleus, as a pre-nucleus, or even as an inflator. However, these different suprema may not be the same. We need to analyse this problem.

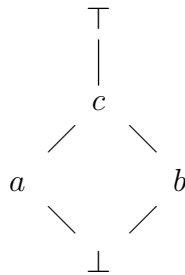
4.5 LEMMA. For each subset $D \subseteq \mathcal{I}A$ the pointwise supremum $\bigvee D$ given by

$$\left(\bigvee D \right) x = \bigvee \{ dx \mid d \in D \}$$

is an inflator, and is the supremum of D in $\mathcal{I}A$.

We need similar way of calculating the suprema of a set of nuclei as a nucleus. This is not so straight forward.

4.6 EXAMPLE. For the 5-element frame



set

$$jx = (a \supset x) \quad kx = (b \supset x)$$

(for each element x) to obtain a pair of nuclei. We have

$$ja = \top \quad j\perp = b \quad kb = \top \quad k\perp = a$$

to give

$$(j \dot{\vee} k)a = \top \quad (j \dot{\vee} k)b = \top$$

whereas

$$(j \dot{\vee} k)(a \wedge b) = (j \dot{\vee} k)\perp = j\perp \vee k\perp = c$$

and hence $j \dot{\vee} k$ is not a pre-nucleus. ■

Pointwise suprema are certainly useful, but we need some other tricks.

Each inflator d has a family of finite iterates d^m (for $m < \omega$) each of which is an inflator. Lemma 4.5 enables us to extend this chain into the transfinite.

4.7 DEFINITION. For each inflator $d \in \mathcal{IA}$, the family d^\bullet of **ordinal iterates** of d is generated by

$$d^0 = id_A \quad d^{\alpha+1} = d \circ d^\alpha \quad d^\lambda = \dot{\bigvee} \{d^\alpha \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ . ■

It is easy to check that d^\bullet is an ascending chain of inflators. On cardinality grounds there is some ordinal ∞ (which depends on A and d) such that

$$d^{\infty+1} = d^\infty$$

and then d^∞ is a closure operation. The size of this ordinal ∞ is an important measure of the complexity of the situation, so it shouldn't be buried under linguistic nonsense.

4.8 LEMMA. *For each $d \in \mathcal{IA}$, the inflator d^∞ is the least closure operation above d . Furthermore,*

$$dx = a \iff d^\infty x = x$$

holds for each $x \in A$.

Suppose we wish to construct a quotient A_j of A . This may be quite tricky if j is complicated. However, if we can find an inflator d with $d^\infty = j$, then we can use the set of fixed elements of d . Sometimes there is a d that is much simpler than j .

When A is a frame or quantale we probably want the closure operation j to be a nucleus. To express this in the form $j = d^\infty$ we need a special kind of inflator.

Remember that a subset $D \subseteq \mathcal{IA}$ is **directed** if it is non-empty and for each $d, e \in D$ there is some $f \in D$ with $d, e \leq f$. For instance, the family

$$\{d^\alpha \mid \alpha \leq \infty\}$$

of ordinal iterates of an inflator d is directed.

4.9 LEMMA. *For each directed set D of pre-nuclei on a quantale A , the pointwise supremum $\dot{\bigvee} D$ is a pre-nucleus.*

Proof. Consider any $x, y \in A$. By two uses of the distributive law we have

$$\left(\left(\dot{\bigvee} D\right)_x\right)\left(\left(\dot{\bigvee} D\right)_y\right) = \bigvee\{(dx)(ey) \mid d, e \in D\}$$

where, in the first instance, d and e may be different. However, for each such pair there is some $f \in D$ with $d, e \leq f$, and then

$$\begin{aligned} \left(\left(\dot{\bigvee} D\right)_x\right)\left(\left(\dot{\bigvee} D\right)_y\right) &\leq \bigvee\{(fx)(fy) \mid f \in D\} \\ &\leq \bigvee\{f(xy) \mid f \in D\} = \left(\dot{\bigvee} D\right)(xy) \end{aligned}$$

as required. ■

This result with the following consequence is one of the reasons why pre-nuclei are useful.

4.10 LEMMA. *For each pre-nucleus d on a quantale A , the ordinal iterates d^\bullet are pre-nuclei. In particular d^∞ is a nucleus, and is the least nucleus above d .*

Proof. The fact that each d^α is a pre-nucleus follows by a simple induction over α using Lemma 4.9 at the limit leaps. In particular, d^∞ is a pre-nucleus and a closure operation, and therefore is a nucleus. Finally, consider any nucleus j with $d \leq j$. Then

$$d^2 \leq j^2 = j$$

and, more generally,

$$d^\alpha \leq j$$

follows by induction over α . In particular, $d^\infty \leq j$, as required. ■

The result gives us a nice way of getting at a quotient A_j of a quantale when the controlling nucleus j is quite complicated. We look for a simpler pre-nucleus d with $d^\infty = j$ and then work with d . This method has been used on many occasions, and we will see examples of it in section 8. However, things can get even better.

It can happen that the idempotent closure of an inflator d is a nucleus even though d is not a pre-nucleus. All that we need is that some iterate d^ν of d is a pre-nucleus. Suppose this happen and consider the pre-nucleus $e = d^\nu$. The ordinal iterates of e are cofinal in the ordinal iterates of d , so that d and e have the same idempotent closure, and hence $d^\infty = e^\infty$ is a nucleus.

There is a useful condition which ensures this does happen with $\nu = \omega$.

4.11 DEFINITION. An inflator d on a quantale A is **stable** if both the comparisons

$$l(dx) \leq d(lx) \quad (dx)r \leq d(xr)$$

holds for all $x, l, r \in A$. ■

You may question the use of the word ‘stable’ for this notion, for it doesn’t seem to have much to do with stability. You are probably right. However, the term is used here because, as we will see in section 7, the notion is related to another property of a different kind of gadget which has already been termed stability (and this does have a superficial connection with stability).

4.12 THEOREM. *Suppose the inflator d on the quantale A is stable. Then*

$$(dx)(dy) \leq d^2(xy)$$

for each $x, y \in A$; the iterate d^ω is a pre-nucleus; and the idempotent closure d^∞ is a nucleus.

Proof. Consider $x, y \in A$. The right hand stability condition gives

$$(dx)(dy) \leq d(x(dy))$$

using $r = dy$. With a different l, x the left hand condition gives

$$x(dy) \leq d(xy)$$

and hence

$$(dx)(dy) \leq d(x(dy)) \leq d^2(xy)$$

as required for the first part.

Consider now the following property of A depending on a variable $m \in \mathbb{N}$.

$$[m] \quad (\forall x, y \in A)(\exists n \in \mathbb{N})[(d^m x)(d^m y) \leq d^n(xy)]$$

The property [0] always holds (since we can take $n = 0$), and we have just seen that for a stable inflator we can take $n = 2$ to verify [1].

We now verify

$$[1] \implies (\forall m \in \mathbb{N})[m]$$

by a simple induction. Thus, assuming [m] we have

$$(d^{m+1}x)(d^{m+1}y) = (d(d^m x))(d(d^m y)) \leq d^k((d^m x)(d^m y)) \leq d^k(d^n(xy)) = d^{n+k}(xy)$$

where second step uses [1] (to select an appropriate k) and the third step uses [m] (to select an appropriate n).

Observe that there is quite a lot of leeway in the induction step.

We use the conditions [m] to show that d^ω is a pre-nucleus. For $x, y \in A$ we have

$$\begin{aligned} (d^\omega x)(d^\omega y) &= \left(\dot{\bigvee} \{d^l x \mid l < \omega\} \right) \left(\dot{\bigvee} \{d^r y \mid r < \omega\} \right) \\ &\leq \dot{\bigvee} \{(d^l x)(d^r y) \mid l, r < \omega\} \\ &= \dot{\bigvee} \{(d^m x)(d^m y) \mid m < \omega\} \\ &\leq \dot{\bigvee} \{d^n(xy) \mid n < \omega\} = d^\omega(xy) \end{aligned}$$

to give the required result. The first step uses the construction of d^ω , the second step uses the distributive law, the third merely bounds the exponents l, r , the fourth invokes the family of properties [m], and the fifth again use the construction of d^ω .

Finally, the remarks preceding Definition 4.11 show that d^∞ is a nucleus. ■

There is quite a lot of slack in this notion of stability and the proof of the result. Consider the property of an inflator

$$\langle \theta \rangle \quad (\forall x, y \in A)(\exists \alpha \leq \theta)[(dx)(dy) \leq d^\alpha(xy)]$$

which depends on an ordinal θ (where the quantified α ranges over smaller ordinals). Only the trivial inflator $d = id_A$ satisfies $\langle 0 \rangle$. An inflator satisfies $\langle 1 \rangle$ precisely when it is a pre-nucleus. We have just seen that a stable inflator satisfies $\langle 2 \rangle$. A slight modification of the proof of the Theorem shows that if an inflator d satisfies $\langle \theta \rangle$ for some ordinal θ , then its idempotent closure d^∞ is a nucleus, since some iterate d^ν will be a pre-nucleus. However, depending on the size of θ , the appropriate nice ordinal ν might be bigger than ω .

I do not know any applications of these weaker properties, but that does not mean that there aren't any (for the uses of coverages as hardly taken off yet). What it does mean is that at this stage we shouldn't worry the properties to death by trying to formulate what we think might be the most useful versions.

The two stability comparisons can be replaced by the single comparison

$$l(dx)r \leq d(lxr)$$

which is sometimes useful.

4.13 LEMMA. *For each set D of stable inflators on a quantale A , the pointwise supremum $\dot{\bigvee} D$ is a stable.*

Proof. By Lemma 4.5 we know that $\dot{\bigvee} D$ is an inflator, so it suffices to show that

$$l\left(\left(\dot{\bigvee} D\right)x\right)r \leq \left(\dot{\bigvee} D\right)(lxr)$$

for all $l, r, x \in A$. The quantale distributive law and the stability of each $d \in D$ gives

$$\begin{aligned} l\left(\left(\dot{\bigvee} D\right)x\right)r &= l(\bigvee\{dx \mid d \in D\})r \\ &= \bigvee\{l(dx)r \mid d \in D\} \\ &\leq \bigvee\{d(lxr) \mid d \in D\} = \left(\dot{\bigvee} D\right)(lxr) \end{aligned}$$

as required. ■

Suppose J is a family of nuclei on a quantale A where each $j \in J$ is given as the closure d^∞ of some stable inflator. Let D be the family of all these d . Then, by Lemma 4.13, the pointwise supremum $\dot{\bigvee} D$ is a stable inflator and, by Theorem 4.12 the closure

$$\left(\dot{\bigvee} D\right)^\infty$$

is a nucleus. This is the supremum of J as a family of nuclei.

Our main interest here is the construction method for frames, so we will use a particular case of this material. Let's state the relevant property and result separately.

4.14 DEFINITION. An inflator d on a frame A is **stable** if the comparison

$$(dx) \wedge y \leq d(x \wedge y)$$

holds for all $x, y \in A$. ■

Of course, a frame is an example of a commutative quantale, which is why stability can be replaced by a single comparison in this way.

4.15 THEOREM. *Suppose the inflator d on the frame A is stable. Then*

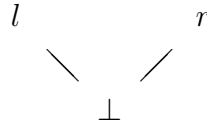
$$(dx) \wedge (dy) \leq d^2(x \wedge y)$$

for each $x, y \in A$; the iterate d^ω is a pre-nucleus; and the closure d^∞ is a nucleus.

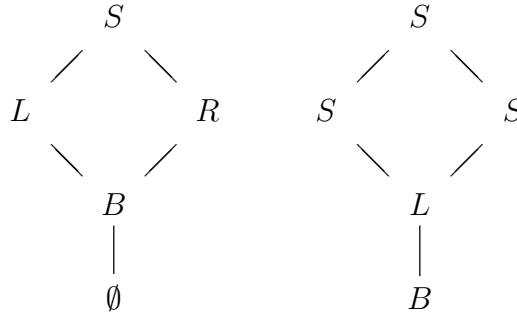
For each set D of stable inflators on A , the pointwise supremum $\bigvee D$ is a stable.

You may wonder if the comparison in this result is a characterization of stability. It is not, and a simple example show why.

4.16 EXAMPLE. Let S the the 3-element poset



as shown. Let A be the finite frame of lower sections of S . Thus A is the 5-element frame as shown on the left



where

$$L = \{\perp, l\} \quad B = \{\perp\} \quad R = \{\perp, r\}$$

are the three component lower sections. Consider the inflator d on A with values as shown on the right. In particular

$$dB = L$$

is the crucial value.

We first check that

$$(dX) \cap (dY) \subseteq d^2(X \cap Y)$$

holds for all $X, Y \in A$. Since $d^2B = S$ the only possible problem is when $X \cap Y = \emptyset$. But then at least one of X, Y is empty (for otherwise $\perp \in X \cap Y$) so that

$$(dX) \cap (dY) \subseteq d\emptyset = B \subseteq L = d^2\emptyset = d^2(X \cap Y)$$

as required.

Finally

$$(dL) \cap R = R \quad d(L \cap R) = dB = L$$

so that d is not stable. ■

This concludes the required background material, and we can now begin the study of the coverage technique proper.

5 The basic technique

In this section I will describe the coverage technique in its most basic form, that is for converting a poset into a \bigvee -semilattice. In the following sections we look at refinements of the technique where the target \bigvee -semilattice is required to have extra structure to make it a frame or quantale.

The coverage technique is a way of finding the universal solution to a certain kind of problem. To describe this kind we need a couple of preliminary notions.

5.1 DEFINITION. Let A be a poset. A **poser** on A is a relation

$$a \Vdash X$$

between elements $a \in A$ and subsets $X \in \mathcal{P}A$.

A **site** is a poset furnished with a particular poser. ■

There are no restrictions whatsoever on a poser except that it must compare elements with subsets. So why are these things useful, and where does the name come from? It is because each such relation poses a problem of which we want the universal solution. The site is the place where we build this solution.

5.2 DEFINITION. Let (A, \Vdash) be a site. A (not necessarily universal) solution to the associated problem is a monotone function

$$A \xrightarrow{f} B$$

from A to a \bigvee -semilattice B such that

$$a \Vdash X \implies fa \leq \bigvee f[X]$$

holds for all $a \in A, X \in \mathcal{P}A$.

We say such a function f **solves** the site (A, \Vdash) . ■

Trivially, for each site there is at least one solution, for we can take the 1-element poset for B . What we want is a solution at the other end, a solution through which each other solution factors via a unique mediating arrow. This is the universal solution, and the coverage technique gives us a way of generating this.

A poser on a poset is a completely free spirit, and need not satisfy any conventions whatsoever. However, for a more civilized account of the various properties it is convenient to impose a couple of restrictions.

5.3 DEFINITION. Let A be a poset. A **cover** on A is a relation

$$a \dashv X$$

between elements $a \in A$ and lower section $X \in \mathcal{L}A$ satisfying

$$\frac{a \in X}{a \dashv X} \text{ (infl)} \qquad \frac{a \leq b \quad b \dashv Y \quad Y \subseteq X}{a \dashv X} \text{ (mono)}$$

for all $a, b \in A$ and $X, Y \in \mathcal{L}A$. ■

A crucial difference between a poset and a cover is that while a poset may handle arbitrary subsets of the poset, a cover only handles lower sections.

The relation $a \vdash X$ should be read ‘ a is covered by X ’ (in line with the name of the notion). The two restrictions on a cover have been written as derivation rules for this can be a useful way of thinking of covers. However, these restrictions are nothing more than implications from numerator (the top) to denominator (the bottom).

A word about the terminology is in order here. The term ‘coverage’ is used in the literature (and between people who know about these things) for a notion that is not quite a poset nor a cover. In fact, there are several slight variants of this coverage idea. I have chosen the terms ‘poset’ and ‘cover’ so as not to confuse the issue. In speech I would normally say ‘coverage’ to refer to these and the other coverage notions. In fact, when we get to the examples in section 8 we will use ‘coverage’ in this looser sense.

Of course, each cover is a poset but, because of the extra conditions, we might expect that a cover has more special properties. Well, yes and no. Each poset can be converted into a cover in a canonical way, and then we find that the two relations are more or less equivalent (in a sense that is made precise by Lemma 5.6 below).

For a poset \Vdash and a cover \vdash on a poset A we say \vdash refines \Vdash or \Vdash is refined by \vdash if

$$a \Vdash X \implies a \vdash_2 \downarrow X$$

holds for all $a \in A$ and $X \in \mathcal{P}A$.

Given two covers \vdash_1, \vdash_2 of a poset A we write

$$\vdash_1 \leq \vdash_2$$

and say ‘ \vdash_1 is smaller than \vdash_2 ’ or ‘ \vdash_2 is larger than \vdash_1 ’ or some such similar phrase if

$$a \vdash_1 X \implies a \vdash_2 X$$

holds for all $a \in A$ and $X \in \mathcal{L}A$.

We look for the smallest cover which refines a poset.

5.4 DEFINITION. For a poset \Vdash on a poset A let \vdash be the relation generated by the two rules

$$\frac{a \in X}{a \vdash X} \quad \frac{a \leq c \quad c \Vdash Z \quad Z \subseteq X}{a \vdash X}$$

for $a, c \in A$ and $X \in \mathcal{L}A, Z \in \mathcal{P}A$. ■

In other words, an instance

$$a \vdash X$$

of the generated relation \vdash can be witnessed in one of two ways. Either $a \in X$ or there is an instance

$$c \Vdash Z$$

of the given poset where $a \leq c$ and $Z \subseteq X$.

5.5 LEMMA. For each poset \Vdash on a poset A the relation \vdash generated by Definition 5.4 is the smallest cover that refines \Vdash .

Proof. We go through a list of simple properties.

We must verify that the generated relation \vdash is a cover. The left hand property of Definition 5.3 is immediate. For the right hand property suppose

$$a \leq b \quad b \vdash Y \quad Y \subseteq X$$

for $a, b \in A$ and $X, Y \in \mathcal{L}A$. The middle condition gives

$$b \leq c \quad c \Vdash Z \quad Z \subseteq Y$$

for some $c \in A$ and $Z \in \mathcal{P}A$. But now

$$a \leq c \quad c \Vdash Z \quad Z \subseteq X$$

and hence $a \vdash X$, as required.

The fact that \vdash refines \Vdash is more or less trivial.

Finally, suppose \vdash' is any cover which refines \Vdash , and suppose

$$a \vdash X$$

for some $a \in A, X \in \mathcal{L}A$. We have

$$a \leq c \quad c \Vdash Z \quad Z \subseteq X$$

for some $c \in A$ and $Z \in \mathcal{P}A$. But now

$$a \leq c \quad c \vdash' \downarrow Z \quad \downarrow Z \subseteq X$$

(since \vdash' extends \Vdash), and hence

$$a \vdash' X$$

(since \vdash' is a cover). Thus \vdash is smaller than \vdash' , as required. ■

Each cover \vdash on a poset A is a poser, and so gives a site (A, \vdash) . We refer to such a gadget as a **covered site**. By Lemma 5.5 each site (A, \Vdash) gives a covered site (A, \vdash) , and both of these sites have solutions. How are these related?

5.6 LEMMA. *A site (A, \Vdash) and its associated covered site (A, \vdash) have exactly the same solutions.*

Proof. Consider a potential solution

$$A \xrightarrow{f} B$$

from the parent poset A to a \vee -semilattice B .

Suppose that f solves (A, \vdash) and that

$$a \Vdash X$$

for $a \in A, X \in \mathcal{P}A$. Then

$$a \vdash \downarrow X$$

(since $X \subseteq \downarrow X \in \mathcal{L}A$), and

$$\bigvee f[\downarrow X] = \bigvee f[X]$$

(since f is monotone), to give the required

$$fa \leq \bigvee f[\downarrow X] = \bigvee f[X]$$

(since f solves (A, \vdash)).

Conversely, suppose that f solves (A, \vdash) and that

$$a \vdash X$$

for $a \in A, X \in \mathcal{L}A$. Then

$$a \leq c \Vdash Z \subseteq X$$

for some $c \in A, Z \subseteq \mathcal{P}A$, to give

$$fa \leq fc \leq \bigvee f[Z] \leq \bigvee f[X]$$

as required to show that f solves (A, \vdash) . ■

Since posers and covers have the same solutions, why bother with the two notions?

As we will see, the general mechanics and analysis of covers is rather smooth. (In fact, we have already done most of it.) We could, of course, carry out the whole development using posers, but then we find that certain quantifications insists on being seen. (These are the quantifiers in the construction of Definition 5.4.) This becomes a bit messy, and can obscure what is going on. I know that some people like this kind of thing, but I prefer an uncluttered presentation, and covers are just the job for this.

That's all very well for the generalities, but consider what might happen when we try to solve a particular problem on a particular poset. We have a certain cover in mind, and we know that it can be generated by at least one poser (the cover itself). However, it can happen that there are rather simpler posers which generate the cover, and some of these may be easier to calculate with. In other words, posers are calculation devices suitable for particular jobs, whereas covers fit more neatly into the general scheme of things. We will see some examples of this difference in section 8.

5.7 LEMMA. *For each poset A there is a bijective correspondence between the covers \vdash on A and the inflators d on $\mathcal{L}A$ given by*

$$a \vdash X \iff a \in dX$$

for $a \in A, X \in \mathcal{L}A$.

Proof. We must verify a list of rather trivial properties.

Suppose that \vdash is a cover on A and consider the function

$$d : \mathcal{L}A \longrightarrow \mathcal{P}A$$

given by the equivalence. A use of the rule (mono) show that $dX \in \mathcal{L}A$ for each lower section $X \in \mathcal{L}A$, and hence d is an operation on $\mathcal{L}A$. The rule (infl) ensures that d is inflationary, and a different use of (mono) ensures that d is monotone.

Conversely, suppose that d is an inflator on $\mathcal{L}A$ and consider the relation given by the equivalence. Since d is inflationary we have

$$a \in X \implies a \in dX \implies a \vdash X$$

to verify the rule (infl). To verify the rule (mono) suppose

$$a \leq b \vdash Y \subseteq X$$

for $a, b \in A$ and $X, Y \in \mathcal{L}A$. Then, since d is monotone, we have

$$a \leq b \in dY \subseteq dX$$

to show that $a \vdash X$, as required.

The required bijective properties are immediate. ■

This result shows that the difference between covers and inflators is superficial; nothing more than notation. Once this is understood much of what we do with covers is routine. Unfortunately, for some people understanding even the simplest things takes a long time, if it ever happens at all.

For each site (A, \vdash) we are interested in certain monotone maps

$$A \xrightarrow{f} B$$

to some \vee -semilattice B . By section 2 this map f must factor uniquely as

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \eta & \nearrow f^\# \\ & \mathcal{L}A & \end{array}$$

for some \vee -morphism $f^\#$. Here η is the canonical unit for A . This morphism $f^\#$ has a kernel $\ker(f^\#)$ which is a closure operation on $\mathcal{L}A$. It is not too hard to see that when f solves (A, \vdash) then so does the composite

$$A \xrightarrow{\eta} \mathcal{L}A \longrightarrow (\mathcal{L}A)_k$$

where $k = \ker(f^\#)$. (A proof of this is a simple consequence of our next result.) In other words, to solve (A, \vdash) we need to select certain closure operations on $\mathcal{L}A$, and the universal solution (if this exists) corresponds to a particular closure operation.

There is a canonical way of obtaining such a closure operation. By Lemma 5.6 the site (A, \vdash) may be converted into a covered site (A, \vdash) with exactly the same solutions, and by Lemma 5.7 this is equivalent to an inflator d on $\mathcal{L}A$. The idempotent closure $j = d^\infty$ is a closure operation on $\mathcal{L}A$.

5.8 THEOREM. *Consider the situation describe above for a parent poset A . Then the morphism f solves (A, \vdash) if and only if $d \leq \ker(f^\#)$.*

Proof. Suppose first that $d \leq k = \ker(f^\sharp)$. Then $j = d^\infty \leq k$ and we have a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & (\mathcal{L}A) \\ & \searrow g & \downarrow & \searrow f^\sharp \\ & & (\mathcal{L}A)_j & \xrightarrow{h} & B \end{array}$$

where

$$f = f^\sharp \circ \eta_A = h \circ g$$

is the given morphism, $(\mathcal{L}A)_j$ is the quotient of $\mathcal{L}A$ produced by $j = d^\infty$, and the unnamed vertical arrow is the morphism of this quotient. Note that g is merely a monotone map, but h is a \vee -arrow. By Lemma 5.6 it suffices to show that f solves the problem for the cover \vdash associated with the given poset \Vdash .

For each $x \in A$ we have $gx = j(\downarrow x)$. For $X \in \mathcal{L}A$ we have

$$X = \bigcup \{\downarrow x \mid x \in X\} \subseteq \bigcup \{j(\downarrow x) \mid x \in X\} = \bigcup g[X]$$

so that

$$jX \subseteq j\left(\bigcup g[X]\right) = \bigvee^j g[X] = G \quad (\text{say})$$

where this supremum is computed in $(\mathcal{L}A)_j$, as indicated. With this we have

$$\begin{aligned} a \vdash X &\implies a \in dX \subseteq jX \subseteq G \\ &\implies \downarrow a \subseteq G \\ &\implies ga = j(\downarrow a) \subseteq jG = G \\ &\implies fa = h(ga) \leq h(G) \implies fa \leq \bigvee (h \circ g)[X] = \bigvee f[X] \end{aligned}$$

to show that f solves (A, \vdash) . The last steps hold since h preserves suprema, and the final supremum is computed in B

Conversely, suppose that f solves (A, \Vdash) , and hence solves (A, \vdash) . For each $X \in \mathcal{L}A$ we have

$$f^\sharp X = \bigvee f[X]$$

so that, for each $a \in A$ we have

$$\begin{aligned} a \in dX &\implies a \vdash X \\ &\implies fa \leq \bigvee f[X] \\ &\implies f^\sharp(\downarrow a) \leq f^\sharp X \\ &\implies \downarrow a \subseteq kX \implies a \in kX \end{aligned}$$

to give $dX \subseteq \ker(f^\sharp)X$, as required. The penultimate step uses the construction of $k = \ker(f^\sharp)$. ■

This result tells us that to solve (A, \Vdash) it is sufficient to find a closure operation above the associated inflator d . Furthermore, the universal solution is provided by the

smallest such closure operation. But that is just d^∞ , and hence we have a canonical description of that universal solution. We write

$$(\mathcal{L}A)_{\Vdash}$$

for that universal solution, that is $(\mathcal{L}A)_j$ where $j = d^\infty$ where d is obtained from \Vdash via the refining cover.

This \vee -semilattice $(\mathcal{L}A)_{\Vdash}$ is carried by those $X \in \mathcal{L}A$ which are fixed by $j = d^\infty$, or equivalently by d . These are just those $X \in \mathcal{L}A$ for which

$$a \vdash X \implies a \in X$$

holds for all $a \in A$. Remembering how \vdash arises from \Vdash we see that $(\mathcal{L}A)_{\Vdash}$ is carried by those $X \in \mathcal{L}A$ such that

$$a \Vdash Z \subseteq X \implies a \in X$$

for all $a \in A$ and $Z \in \mathcal{P}A$. The comparison on $(\mathcal{L}A)_{\Vdash}$ is just inclusion, and suprema are given as j -modified unions.

The construction of $(\mathcal{L}A)_{\Vdash}$ can be carried out directly from the characterization in terms of \Vdash of the relevant $X \in \mathcal{L}A$. This has been done several times in the literature. And a right mess it looks as well.

To solve a site (A, \Vdash) we first pass to the associated cover \vdash , then to the associated inflator d , and then to the associated closure operation $j = d^\infty$. This j is itself an inflator, and so corresponds to a cover on A . Such a cover must have some special properties. What are these? How is the special cover related to the original cover? Both of these questions are easy to solve.

A cover on a poset A is a relation

$$a \vdash X$$

between elements a and certain subsets X . To help with our analysis we let

$$Y \vdash X \quad \text{abbreviates} \quad (\forall y \in Y)[y \vdash X]$$

for subsets Y, X of A . Notice that this hides a quantifier and so, at times, must be used with some care.

5.9 LEMMA. *Let \vdash be a cover on a poset A and let d be the associated inflator on $\mathcal{L}A$. Then d is idempotent if and only if*

$$\frac{a \vdash Y \quad Y \vdash X}{a \vdash X} \quad (\text{idem})$$

holds for each $a \in A$ and $X, Y \in \mathcal{L}A$.

Proof. We observe that

$$Y \vdash X \iff Y \subseteq dX$$

and hence the rule can be rephrased as

$$a \in dY \text{ and } Y \subseteq dX \implies a \in dX$$

which is just the idempotency of d . ■

The rule displayed in this result is often referred to as transitivity, a terminology that goes back to an analogous property of Grothendieck topologies (where it enables certain properties to be transferred along arrows).

Each cover on a poset A corresponds to an inflator on $\mathcal{L}A$ which generates a closure operation and this corresponds to a second cover on A . How are the first and second covers related?

5.10 DEFINITION. For a cover \vdash on a poset A , let \models be the relation generated by the derivation rules

$$\frac{a \in X}{a \models X} \quad \frac{a \vdash Y \quad Y \models X}{a \models X}$$

for $a \in A$ and $X, Y \in \mathcal{L}A$. ■

Let's work through this idea more slowly.

Given the covered site (A, \vdash) we take \models to be the smallest relation (between elements $a \in A$ and lower sections $X \in \mathcal{L}A$) which satisfies the two displayed conditions. Thus each instance

$$x \models X$$

of \models can be witnessed by a tree of 'earlier' instances. If $x \in X$ then we have $x \models X$ without further ado. Otherwise we require some $Y \in \mathcal{L}A$ such that

$$x \vdash Y \quad Y \models X$$

hold. The right hand condition is an abbreviation for a whole family

$$y \models X$$

of instances of \models for $y \in Y$. Each of these must be witnessed, either trivially ($y \in X$) or by 'even earlier' instances of \models .

In this way we can unravel the witnessing process

$$\frac{x \vdash Y \quad \dots \quad \frac{y \vdash Z \quad \dots \quad \frac{\vdots}{z \models X} \quad \dots}{y \models X} \quad \dots}{x \models X}$$

to produce a tree of instances of \models . By definition, \models is that relation such that each branch of such a tree eventually closes off with a trivial instance

$$\frac{w \in X}{w \models X}$$

of \models .

Of course, since a collection of instances

$$Y \models X$$

may require an infinite splitting, the whole tree may be infinite with infinite height. However, each branch must be finite.

The easiest way to handle such a splitting is to employ ordinals.

5.11 DEFINITION. For a cover \vdash on a poset A , the ordinal indexed family of relations

$$\vdash^\alpha$$

is generated by the derivation rules

$$\frac{a \in X}{a \vdash^\alpha X} \quad \frac{a \vdash Y \quad Y \vdash^\beta X}{a \vdash^\alpha X}$$

for $a \in A$ and $X, Y \in \mathcal{L}A$, and all ordinals $\beta < \alpha$. ■

In other words the indexing ordinal gives an upper bound for the height of the derivation. Note that the relations are cumulative, that is

$$a \vdash^\beta X \implies a \vdash^\alpha X$$

has been arranged for ordinals $\beta \leq \alpha$.

5.12 LEMMA. For a cover \vdash on a poset A with associated inflator d , the equivalence

$$a \vdash^\alpha X \iff a \in d^\alpha X$$

holds for each $a \in A$ and $X, Y \in \mathcal{L}A$, and each ordinal α .

Proof. We can prove the two implications separately or together. Either way we proceed by a progressive induction over the ordinals α .

Suppose $a \vdash^\alpha X$. Then either $a \in X$ in which case $a \in d^\alpha X$, or

$$a \vdash Y \quad Y \vdash^\beta X$$

for some $Y \in \mathcal{L}A$ and ordinal $\beta < \alpha$. These give

$$a \in dY \quad Y \subseteq d^\beta X$$

where the second uses the induction hypothesis. Thus

$$a \in d^{\beta+1}X \subseteq d^\alpha X$$

as required.

Conversely, suppose $a \in d^\alpha X$. Then either $a \in X$ in which case $a \vdash^\alpha X$, or there is some ordinal $\beta < \alpha$ with $a \in d^{\beta+1}X$. In the second case we have

$$a \in dY \quad \text{where } Y = d^\beta X$$

and hence

$$a \vdash Y \quad Y \vdash^\beta X$$

where the second uses the induction hypothesis. These give

$$a \vdash^\alpha X$$

as required. ■

Definition 5.10 says that \models is the cumulative union of all the relations \vdash^α , that is

$$a \models X \iff (\exists \alpha)[a \vdash^\alpha X]$$

holds for $a \in A, X \in \mathcal{L}A$. Thus, using Lemma 5.12 we have

$$a \in d^\infty X \iff (\exists \alpha)[a \in d^\alpha X] \iff (\exists \alpha)[a \models^\alpha X] \iff a \models X$$

to give the following.

5.13 COROLLARY. *For each cover \vdash on a poset A , the generated relation \models is the idempotent closure of \vdash .*

This basic method converts a poset into a \vee -semilattice with some required completeness properties. Usually, in practice, the target \vee -semilattice is required to be a frame or sometimes a quantale of a special kind. To achieve this we need to build more information into the parent poset. Sometimes this can be done by having extra structure on the poset, and sometimes it can be done by selecting the appropriate kind of poset. We look at these refinements in the next two sections.

Let me outline how the coverage technique can be used, indicate some fertile regions which may benefit from the technique, and mention some quite futile uses.

Suppose we have in mind a quotient of a free \vee -semilattice $\mathcal{L}A$ on a poset A . In some cases we may want to view $\mathcal{L}A$ as a quantale obtained from a pom carried by A . In other cases we may want to use the canonical frame structure on $\mathcal{L}A$, perhaps related to a \wedge -semilattice carried by A . In all case the quotient is given by a closure operation j on $\mathcal{L}A$, and perhaps we require this to be a nucleus. The quotient is carried by the family of j -fixed sets.

We will have some idea of what these fixed sets are. In the simplest case we will have a complete description of the closure operation j . In such circumstances we can move directly to $(\mathcal{L}S)_j$ via the standard construction.

We could, of course, rephrase the description of j as a poset on A . This is futile since the coverage approach tells us nothing new whatsoever. The purpose of the coverage technique is to get at complicated nuclei via simpler gadgetry.

In some cases we may have some idea of the nature of the j -fixed sets without knowing too much about j itself. In some cases we may be able to show that j is the closure d^∞ of some inflator d . This is easier to understand. When the closure ordinal ∞ is reasonably large (and perhaps unknown) it is still possible to have a complete description of d . As we will see in section 8, it can happen that d is simple but d^∞ is complicated partly because ∞ is at least ω .

It can happen that the required closure operation has to do several different jobs, and seems to be a composite of a few different closure operation. The coverage technique gives us a way of organizing this multiplicity.

5.14 DEFINITION. The syntactic join of a pair \Vdash_1, \Vdash_2 of posers on a poset A is the poset \Vdash given by

$$a \Vdash X \iff a \Vdash_1 X \text{ or } a \Vdash_2 X$$

for $a \in A$ and $X \in \mathcal{P}A$. ■

Because there are no restrictions on posers we can put them together in this way. Suppose we have such a pair of posers and their syntactic join.

$$\Vdash_1 \quad \Vdash_2 \quad \Vdash$$

Each of these generates an inflator

$$d_1 \quad d_2 \quad d$$

each of which has a closure

$$d_1^\infty \quad d_2^\infty \quad d^\infty$$

perhaps with different closure ordinals.

5.15 THEOREM. *The syntactic join of a pair of posers generates the join of the two closure operations produced by the posers. That is*

$$d_1^\infty \vee d_2^\infty = d^\infty$$

in the notation above.

The join of the two closure operations (as closure operations) can be obtained as the closure of the pointwise join of the two. Unfortunately this join can be far more complicated than the two components. The coverage technique provides a method of putting this complexity to one side, perhaps for later analysis.

6 Property induced stability

Each poser \Vdash on a poset A generates a closure operation j on $\mathcal{L}A$, and so leads to a \vee -semilattice $(\mathcal{L}A)_{\Vdash}$ which is the universal solution to the problem posed by (A, \Vdash) . However, $\mathcal{L}A$ has more structure, in particular it is a frame, and we can ask whether this structure can be transferred to $(\mathcal{L}A)_{\Vdash}$. For this to happen we need the closure operation j to be a nucleus. We consider some ways that this can be arranged.

The poset \Vdash refines to a cover \vdash which is equivalent to an inflator d on $\mathcal{L}A$ and j is the idempotent closure d^∞ . From section 4 we know various conditions on d which ensure that $j = d^\infty$ is a nucleus. Our problem is to find useful restrictions on \Vdash or \vdash which ensure one of these conditions.

For many uses of the coverage technique the poset A is, in fact, a \wedge -semilattice. In those circumstances we can use the carried meet to impose extra conditions on the poset under consideration. This is the trick used in the original account of the technique, [6]. We look at this in the next section. In this section we develop a more general method which works without any extra structure on A .

In Definition 4.13 we introduced the notion of a stable inflator. That property is easily transferred to the corresponding cover.

6.1 LEMMA. *Let A be a poset and let \vdash be a cover on A with equivalent inflator d on $\mathcal{L}A$. Then the three conditions*

(i) *The rule*

$$\frac{a \vdash X}{a \vdash X \cap \downarrow a}$$

holds for all $a \in A, X \in \mathcal{L}A$.

(ii) *The inflator d is stable.*

(iii) *The rule*

$$\frac{b \leq a \vdash X}{b \vdash X \cap \downarrow b}$$

holds for all $a, b \in A, X \in \mathcal{L}A$.

are equivalent.

Proof. (i) \Rightarrow (ii). Assuming (i) we must show that

$$(dX) \cap Y \subseteq d(X \cap Y)$$

holds for all $X, Y \in \mathcal{L}A$. To do this consider any

$$a \in (dX) \cap Y$$

for such X, Y . Then

$$a \vdash X \quad a \in Y$$

and hence, by (i), we have

$$a \vdash X \cap \downarrow a \subseteq X \cap Y$$

to give

$$a \vdash X \cap Y$$

as required.

(ii) \Rightarrow (iii). Assuming (ii) suppose

$$b \leq a \vdash X$$

for $a, b \in A, X \in \mathcal{L}A$. Then

$$b \vdash X \quad b \in \downarrow b$$

to give

$$b \in (dX) \cap \downarrow b$$

and hence, by (ii), we have

$$b \in d(X \cap \downarrow b)$$

as required.

(iii) \Rightarrow (i). This is immediate by considering the case $b = a$. ■

The triviality of this proof makes it clear that the three conditions (i, ii, iii) are little more than rephrasings. Condition (iii) is included here merely because similar looking properties have been used in the literature.

From the analysis of section 4 we have the following.

6.2 COROLLARY. *If the covered site (A, \vdash) is stable then the generated closure operation is a nucleus and the universal solution $(\mathcal{L}A)_{\vdash}$ is a frame.*

We know that each pre-nucleus on $\mathcal{L}A$ is a stable inflator (but not conversely). The pre-nucleus property of an inflator can be rephrased as a derivation rule. A simple calculation shows that the rule

$$\frac{a \vdash X \quad a \vdash Y}{a \vdash X \cap Y}$$

is equivalent to the corresponding inflator being a pre-nucleus.

The following notion occur in the work of N. Gambino, [1], hence the terminology.

6.3 DEFINITION. (a) A poset \Vdash on a poset A is **bounded** if

$$a \Vdash X \implies X \subseteq \downarrow a$$

holds for all $a \in A, X \in \mathcal{P}A$.

(b) A poset \Vdash on a poset A is **NG-stable** if

$$b \leq a \Vdash X \implies (\exists Y \subseteq \downarrow X)[b \Vdash Y]$$

for all $a \in A, X \in \mathcal{P}A$.

An **NG-poset** is a bounded poset which is NG-stable.

An **NG-site** is a site (A, \Vdash) where \Vdash is an NG-poset. ■

These various notions fit together as follows.

6.4 LEMMA. *If \Vdash is an NG-poset on a poset A , then the associated cover \vdash is stable.*

Proof. Consider any instance

$$a \vdash X$$

of the associated cover. Thus

$$a \leq c \vdash Z \subseteq X$$

for some $c \in A, Z \in \mathcal{P}A$. The NG-stability gives some $Y \in \mathcal{P}A$ with

$$a \vdash Y \subseteq \downarrow Z \subseteq X$$

and then

$$Y \subseteq \downarrow a$$

since \Vdash is bounded. Thus we have

$$a \vdash Y \subseteq X \cap \downarrow a$$

and hence

$$a \vdash X \cap \downarrow a$$

as required. ■

Combining Lemma 6.4 and Corollary 6.2 we obtain the following basic result.

6.5 THEOREM. *The universal solution of an NG-site is a frame.*

In [1] this result and technique is used to construct various examples of frames which, given sufficient choice principles, have enough points.

Stability is not the only way to ensure that the closure of an inflator is a nucleus. As in section 4 there are several weaker properties we could use. As there, I do not know of any applications of these weakenings, but that does not mean that there aren't any waiting to be discovered. However, it does mean that we should not try to sort out the generalities of the possible weakenings before a particular use comes along.

7 Structure induced stability

The first significant use of the coverage technique to produce a frame is by P. T. Johnstone in [3], and a general account is given in [6]. For those purposes Johnstone assumes that the carrier is a \wedge -semilattice, not just a poset. In this section we first look at that method, and then describe a generalization which produces a quantale.

(As explained in [6], this poset based method is derived from a method used with Grothendieck topologies, and is related to a similar method used with Gabriel topologies. These more sophisticated versions are discussed elsewhere.)

The following notion should be compared with Definition 6.3.

7.1 DEFINITION. A poset \Vdash on a \wedge -semilattice A is **PTJ-stable** if

$$b \leq a \Vdash X \implies b \Vdash Y \quad \text{where } Y = \{x \wedge b \mid x \in X\}$$

for all $a \in A, X \in \mathcal{P}A$.

A **PTJ-poset** is a bounded poset which is PTJ-stable.

A **PTJ-site** is a site (A, \Vdash) where A is a \wedge -semilattice and \Vdash is a PTJ-poset. ■

In other words, a PTJ-poset is nothing more than a coverage in the sense of [6], and a PTJ-site is nothing more than a site in the sense of [6].

Clearly, the two notions of NG-stability and PTJ-stability are closely related. Each is a condition

$$b \leq a \Vdash X \implies b \Vdash Y$$

where the first requires the existence of some subset Y whereas the second says which subset Y should be used. Thus we have the following.

7.2 LEMMA. *Let A be a \wedge -semilattice.*

Each poset on A which is PTJ-stable is also NG-stable.

Each PTJ-site carried by A is an NG-site carried by A .

This observation with Theorem 6.5 gives a second basic result.

7.3 THEOREM. *The universal solution of a PTJ-site is a frame.*

Each \wedge -semilattice is a particularly nice pom, and each pom reflects into a quantale which is a frame when the parent pom is a \wedge -semilattice. Remembering this we can extend the coverage technique to produce quantales.

Thus we assume that the carrying poset A is furnished as a monoid to become a pom. We could attempt to work with suitable posets on A , but at the moment there is little virtue in such generality. Thus we consider only suitable covers.

We use a generalization of the notion of a stable cover as isolated in Lemma 6.1. Consider $l, r \in A$ and $X \in \mathcal{L}A$ where A is a pom A . We set

$$lXr = \downarrow\{lrx \mid x \in X\}$$

to produce some member of $\mathcal{L}A$. Notice that if A is a \wedge -semilattice then

$$lXr = X \cap \downarrow(l \wedge r)$$

and hence we have

$$\top X a = a X \top = X \cap \downarrow a$$

as a particular case. With this we have the required generalization.

7.4 DEFINITION. A cover \vdash on a pom A is pom-stable if

$$\frac{a \vdash X}{lar \vdash lXr}$$

holds for all $l, r \in A, X \in \mathcal{L}A$. ■

The following result should not come as a surprise.

7.5 LEMMA. Let \vdash be a cover on the pom A with equivalent inflator d on $\mathcal{L}A$. Then

- (i) The cover \vdash is pom-stable. (ii) The inflator d is stable.

are equivalent.

Proof. (i) \Rightarrow (ii). Assuming (i) we must show that

$$L(dX)R \subseteq d(LXR)$$

holds for all $L, X, R \in \mathcal{L}A$. To this end consider any $b \in L(dX)R$ so that $b \leq lar$ for some $l \in L, r \in R$ and $a \vdash X$. But now (i) gives

$$b \leq lar \vdash lXr \subseteq LXR$$

and hence $b \vdash LXR$ as required.

(ii) \Rightarrow (i). Assuming (ii) suppose $a \vdash X$ for $a \in A, X \in \mathcal{L}A$ and consider any $l, r \in A$. Then $a \in dX$ so that

$$lar \in l(dX)r \subseteq L(dX)R \subseteq d(LXR)$$

where $L = \downarrow l, R = \downarrow r$ and (ii) gives the final inclusion. But, by a simple calculation, we have $LXR = lXr$ so that $lar \vdash lXr$ as required. ■

Each pom-stable site gives a stable inflator on the corresponding quantale of lower section. This inflator closes off to a quantic nucleus which provides the universal solution to the original site. Thus we have the following.

7.6 THEOREM. The universal solution of a pom-stable site is a quantale.

We know that a quantale nucleus on a quantale can produce a frame. By Theorem 3.14 each quantale carries a universal such nucleus ℓ . This has a coverage version.

7.7 THEOREM. Let \vdash be a cover on a pom A which is pom stable, idempotent, and satisfies

$$\frac{a \vdash X \quad b \vdash Y}{a \vdash XY}$$

for $a \in A$ and $X, Y \in \mathcal{L}A$. Then the associated inflator is a quantic nucleus and $(\mathcal{L}A)_{\vdash}$ is a frame.

This rather futile use of the coverage technique can be found in the literature. However, there is a related question which does have some content. Given a pom A find a poser \Vdash on A which generates the nucleus ℓ . In section 8 we will see a solution of this for a special kind of pom.

8 Examples

In this section we look at some particular examples which illustrate various aspects of the coverage technique. In the later part of the section we look at a use of quantales, but in the earlier part we concentrate on frames. The examples are not unknown, but I will set them in a general context which shows they are not just one-off constructions. This setting also suggests some aspects of the technique that have not yet been explored.

The section is rather long, so I have split it into seven subsections.

The basic idea

We wish to produce some interesting topological spaces. The idea is to produce each example as the point space $\mathbf{pt}(\Omega)$ of some frame Ω , or perhaps as a suitable subspace of $\mathbf{pt}(\Omega)$. To do that we must first produce the frame Ω , and this must be done without recourse to the target space. How might we do that?

Let S be some topological space that we are happy to use. In the particular examples this will be a rather pathetic space in the sense that it hardly needs topological methods to analyse it. (In fact, it will be an Alexandroff space, that is a partially ordered set in drag). However, for the time being let's consider an arbitrary space S .

The space S has a topology $\mathcal{O}S$ of open sets, which is a frame, and a family $\mathcal{C}S$ of closed sets. Suppose we have in mind a nucleus j on $\mathcal{O}S$. This gives a quotient

$$\mathcal{O}S \longrightarrow (\mathcal{O}S)_j = \Omega$$

where the target Ω need not be spatial. Let

$$T = \mathbf{pt}(\Omega)$$

so we have a composite quotient

$$\mathcal{O}S \longrightarrow \Omega \longrightarrow \mathcal{O}T$$

obtained in a canonical fashion. It is this quotient we wish to describe.

Each space S carries a specialization order \sqsubseteq . This is a pre-order, and is a partial order precisely when S is T_0 . Each open set $U \in \mathcal{O}S$ is a \sqsubseteq -upper section, but in an interesting space there are many such upper section which are not open. However, starting from any poset (S, \sqsubseteq) the family of all \sqsubseteq -upper sections forms a topology on S for which \sqsubseteq is the specialization order. We will always start from such a space.

We have a nucleus on $\mathcal{O}S$ in mind. This may be quite complicated, so we try to generate it from a poset carried by S . To do this we need a bit of a twist.

Let \leq be the opposite of the specialization order. The trick is to furnish (S, \leq) with a poset. Notice that in the pathetic case, the topology $\mathcal{O}S$ is the family of all \sqsubseteq -upper sections, and this is the family of all \leq -lower section. Thus the poset eventually produces a quotient of $\mathcal{O}S$. We will see that a simple poset can generate a complicated nucleus. One of the reasons for this is that the closure ordinal of the associated inflator can be quite large. (In particular, there is little point in using the coverage technique if only idempotent coverages are considered.)

That is the bare bones of the idea. Before we flesh it out let's look at a slight variant which looks rather silly but turns out to be a crucial component.

For the space S , suppose the nucleus we have in mind is the identity, so we obtain the point space $\text{pt}(\mathcal{O}S)$. On general grounds there is a canonical continuous map

$$S \longrightarrow \text{pt}(\mathcal{O}S)$$

which is a topological embedding when S is T_0 (as it always will be for us). In such a case we may view S as a subspace of $\text{pt}(\mathcal{O}S)$. The residue

$$\widehat{S} = \text{pt}(\mathcal{O}S) - S$$

can be non-empty, and can produce a space which is far more interesting than S . This residue space gives a quotient

$$\mathcal{O}S \longrightarrow \mathcal{O}\widehat{S}$$

which is determined by some nucleus on $\mathcal{O}S$. What can this be?

For every space the topology $\mathcal{O}S$ carries a canonical nucleus which, it is reasonable to claim, was one of the reasons why topology was developed. It turns out that in the pathetic case this is precisely the nucleus which gives the residue space. Can you guess what it is? The answer is given in the next subsection.

To conclude this subsection let's return to the general situation of a quotient frame $(\mathcal{O}S)_j$ of a topology determined by a nucleus. We describe the points of the quotient in more concrete terms as certain closed subsets of S .

The points of any frame Ω can be characterized in several ways: as frame morphisms

$$\Omega \longrightarrow 2$$

to the 2-element frame; as completely prime filters on Ω ; or (and this is often the most useful characterization) as the \wedge -irreducible elements of Ω , those $p \in \Omega - \{\top\}$ with

$$a \wedge b \leq p \implies a \leq p \text{ or } b \leq p$$

for all $a, b \in \Omega$. When the frame is a topology, $\Omega = \mathcal{O}S$, there is a fourth characterization, for we may take those closed sets whose complement is \wedge -irreducible in $\mathcal{O}S$. These are the closed irreducible sets of S . (Curiously, it is in this guise that the points of a topology, as opposed to the points of the parent space, were first considered.)

A closed set $\pi \in \mathcal{C}S$ is irreducible if it is non-empty and such that if it meets each of a pair of open sets then it meets the intersection of that pair. We will write ' π ' for a typical closed irreducible set (for it will be a point of some space or other).

Consider a quotient

$$\mathcal{O}S \longrightarrow (\mathcal{O}S)_j = \Omega$$

as above. What is a point of Ω ? Each such point is given by a morphism

$$\mathcal{O}S \longrightarrow \Omega \longrightarrow 2$$

and hence a point of $\mathcal{O}S$. We are looking for certain closed irreducible subsets of S .

8.1 LEMMA. *For a nucleus j on a topology $\mathcal{O}S$ the points of $(\mathcal{O}S)_j$ are in bijective correspondence with those $\pi \in \mathcal{C}S$ for which both*

$$j(\pi') \subseteq \pi' \quad \pi \text{ is closed irreducible in } S$$

hold.

Proof. We certainly want to restrict to those $\pi \in \mathcal{CS}$ with $\pi' \in (\mathcal{OS})_j$, that is with $j(\pi') = \pi'$. Within this family we want those π for which π' is \wedge -irreducible in $(\mathcal{OS})_j$.

Consider any $\pi \in \mathcal{OS}$ for which π' is \wedge -irreducible in $(\mathcal{OS})_j$. We show that π' is \wedge -irreducible in \mathcal{OS} (and hence π is closed irreducible in S). Since $\pi' = j(\pi') \neq S$, we have $\pi \neq \emptyset$. Suppose

$$U \cap V \subseteq \pi'$$

for $U, V \in \mathcal{OS}$. Then

$$j(U) \cap j(V) = j(U \cap V) \subseteq j(\pi') = \pi'$$

and hence

$$U \subseteq j(U) \subseteq \pi'$$

(say). Thus π is closed irreducible in S .

The required converse is even easier. ■

This gives us a concrete representation of the members of $\mathbf{pt}((\mathcal{OS})_j)$. Let \check{S} be this subfamily of \mathcal{CS} . The composite morphism

$$\mathcal{OS} \longrightarrow \Omega \longrightarrow \mathcal{OT}$$

for $T = \mathbf{pt}(\Omega)$ can be rephrased as an assignment

$$\begin{array}{ccc} \mathcal{OS} & \longrightarrow & \mathcal{O}\check{S} \\ U & \longmapsto & \check{U} \end{array}$$

given by

$$\pi \in \check{U} \iff \pi \text{ meets } U$$

for $U \in \mathcal{OS}$ and $\pi \in \check{S}$. In other words, we use the opens of S to index the opens of \check{S} , and the indexing assignment (\cdot) is a frame morphism. (It is an instructive exercise to verify directly that, indeed, this is a frame morphism.) In a similar way we find that

$$\pi \in \check{X} \iff \pi \subseteq X$$

for $X \in \mathcal{CS}$ and $\pi \in \check{S}$ indexes the closed sets of \check{S} . (Of course, this second assignment is not a frame morphism. In fact, that doesn't even make sense.) Notice that

$$(\check{X})' = \check{U}$$

when $X = U'$. For some people this construction will ring some bells, but perhaps not the most tuneful ones. It is an example of the ubiquitous hull-kernel construction.

A general context

We set up the general framework in which we will work. We describe a family of pathetic spaces and some associated gadgetry. In the later subsections we look at particular cases of this general set-up. We begin by isolating a topological gadget which is the root cause of much of the problems in the subject.

For frames the pivotal notion is that of nuclei. Some of these are quite complicated, so we make use of inflators and a pre-nuclei. These certainly help to make the general development smoother, but are they necessary when we look at particular examples?

Each pre-nucleus d on a frame has an closure d^∞ which is a nucleus. In general the closure ordinal ∞ can be quite large. Every nucleus j can be generated in this way, for we can take $d = j$ (and so get $\infty = 1$). Perhaps the use of large ordinals is unnecessary. To nail this idea let's look at the situation for which ordinals were invented.

Let S be a topological space with topology \mathcal{OS} of open sets and family \mathcal{CS} of closed sets. Consider the Cantor-Bendixson process on S . Thus, in the first instance we attach to each $X \in \mathcal{CS}$ the subset

$$\mathbf{cb}(X) \subseteq X$$

of all non-isolated points of X (that is, those points in X which are not isolated in X). It is routine to check that $\mathbf{cb}(X) \in \mathcal{CS}$ (for $X \in \mathcal{CS}$), so that \mathbf{cb} is an operation on \mathcal{CS} . This operation is deflationary (as indicated above), monotone, and satisfies

$$\mathbf{cb}(X) \cup \mathbf{cb}(Y) = \mathbf{cb}(X \cup Y)$$

for $X, Y \in \mathcal{CS}$. Thus, by taking the dual complement, we may set

$$\mathbf{der}(U) = \mathbf{cb}(U)'$$

for $U \in \mathcal{OS}$ to obtain a pre-nucleus on \mathcal{OS} . This is the **CB-derivative** on \mathcal{OS} .

(This derivative can be set up on any frame by entirely algebraic means. It has an important place in the study of frames. However, that is not directly relevant here.)

A closed set $X \in \mathcal{CS}$ is perfect if it has no isolated points, that is if $X = \mathbf{cb}(X)$. The perfect part of a closed set can be obtained by iterating the CB-process. Thus

$$\mathbf{per}(X) = \mathbf{cb}^\infty(X)$$

is the perfect part of $X \in \mathcal{CS}$. The ordinal ∞ required here is the global Cantor-Bendixson rank of the space. It is easy to produce examples where this rank is as large as we like. Furthermore, it is known that the perfect part operation \mathbf{per} can be very messy, whereas the 1-step process \mathbf{cb} can be rather simple.

By taking the dual complement we have

$$\mathbf{der}^\infty(U) = \mathbf{per}(U)'$$

for each $U \in \mathcal{OS}$. This gives the **CB-nucleus** $\delta = \mathbf{der}^\infty$ on \mathcal{OS} .

This shows that the closure ordinal of a pre-nucleus can be large and the resulting nucleus can be much more complicated than the generating pre-nucleus. Remember this example, it will jump out from behind the covers a little later.

We need a family of pathetic spaces which we can mould into more interesting stuff.

8.2 DEFINITION. Let (S, \sqsubseteq) be a poset. The family \mathcal{OS} of \sqsubseteq -upper sections of S is a topology on S . The family \sqsubseteq -lower sections is the corresponding family \mathcal{CS} of closed sets. We refer to this furnished poset as a **pathetic space**. ■

What could be more pathetic than such a space? (Actually, it is an Alexandroff space, and these have some surprising applications.) From now on we work with an arbitrary poset S furnished in this way. Later, when we start to produce posers, we will take the opposite \leq of \sqsubseteq , so that \mathcal{OS} will be the family of \leq -lower sections.

What are the points of \mathcal{OS} ? For this we use Lemma 8.1 where the nucleus j is the identity. Thus we want those non-empty \sqsubseteq -lower sections that are closed irreducible. Recall that a subset of S is \sqsubseteq -directed if it is non-empty and for each pair r, s of its members there is a member t of the subset with $r, s \leq t$.

8.3 LEMMA. For a pathetic space S the points of \mathcal{OS} are the directed lower sections.

Proof. We must show that $\pi \in \mathcal{CS}$ is irreducible precisely when it is directed. The trick is that the principal \sqsubseteq -upper section generated by each $s \in S$ is open in S . ■

Each $s \in S$ gives a point of \mathcal{OS} , namely the principal \sqsubseteq -lower section generated by s . These are not the interesting points, its the others that will give us hours of fun.

8.4 DEFINITION. For a pathetic space S let \widehat{S} be the set of non-principal points of \mathcal{OS} , That is, the set of those \sqsubseteq -lower section π which are \sqsubseteq -directed and have no maximal elements. ■

We topologize \widehat{S} as a subspace of $\mathbf{pt}(\mathcal{OS})$. For each $U \in \mathcal{OS}$ and $X \in \mathcal{CS}$ we use

$$\pi \in \widehat{U} \iff \pi \text{ meets } U \quad \pi \in \widehat{X} \iff \pi \subseteq U$$

(for $\pi \in \widehat{S}$) to produce subsets of \widehat{S} . We find that

$$\mathcal{OS}^\widehat{S} = \{\widehat{U} \mid U \in \mathcal{OS}\} \quad \mathcal{CS}^\widehat{S} = \{\widehat{X} \mid X \in \mathcal{CS}\}$$

and

$$\begin{array}{ccc} \mathcal{OS} & \longrightarrow & \mathcal{OS}^\widehat{S} \\ U & \longmapsto & \widehat{U} \end{array}$$

is a frame morphism. Put another way, for each $s \in S$ let

$$\pi \in B_s \iff s \in \pi$$

(for $\pi \in \widehat{S}$) to obtain a subset $B_s \subseteq \widehat{S}$. Since each $\pi \in \widehat{S}$ is directed the family

$$\mathcal{B} = \{B_s \mid s \in S\}$$

is a base for $\mathcal{OS}^\widehat{S}$. We call \widehat{S} the **residue space** of S .

What is the kernel of this residue quotient? If you can't guess what it is, then you haven't been paying attention.

8.5 LEMMA. Let S be a pathetic space S . Then

$$\mathbf{per}(X) = \bigcup \{\pi \in \widehat{S} \mid \pi \subseteq X\}$$

for each $X \in \mathcal{CS}$.

Proof. Given a $X \in \mathcal{CS}$ let

$$Y = \bigcup \{\pi \in \widehat{S} \mid \pi \subseteq X\} \quad Z = \mathbf{per}(X)$$

so that $Y = Z$ is required.

We first show that Y is perfect, and hence $Y \subseteq Z$. Consider any $s \in Y$, Then

$$s \in \pi \subseteq X$$

for some $\pi \in \widehat{S}$. But π has no maximal elements and hence

$$s \sqsubseteq t \in \pi \subseteq X$$

for some $t \in S$ with $t \neq s$. This shows that s is not maximal in Y .

Conversely we show that $Z \subseteq Y$. We use the fact that Z is perfect. Consider any $s \in Z$. Then s is not a maximal element of Z and hence there is some $s \sqsubseteq t \in Z$ with $t \neq s$. By iterating this process we obtain a strictly ascending chain

$$s = s_0 \sqsubseteq s_1 \sqsubseteq s_2 \sqsubseteq \cdots \sqsubseteq s_i \sqsubseteq \cdots \quad (i < \omega)$$

of elements of Z . Using this chain we see that

$$r \in \pi \iff (\exists i < \omega)[r \sqsubseteq s_i]$$

gives some $\pi \in \widehat{S}$ with

$$s \in \pi \subseteq Z \subseteq X$$

and hence $s \in Y$, as required. ■

The more perceptive amongst you, or perhaps the acutely neurotic, will have noticed that a choice principle is used in this proof. (It is needed to generate the chain of elements s_i from which we obtain π .) This is not surprising. The existence of points of frames is intimately connected with choice principles, and working with frames is one way of avoiding such decadence. In fact, this is half the point of the original application [3] of the coverage technique [joke intended].

8.6 THEOREM. *Let S be a pathetic space S , and let*

$$\mathcal{O}S \longrightarrow \mathcal{O}\widehat{S}$$

be its residue quotient. Then the kernel of this quotient is the CB-nucleus.

Proof. We show

$$\widehat{V} \subseteq \widehat{U} \iff \mathbf{per}(U') \subseteq \mathbf{per}(V')$$

for each $U, V \in \mathcal{O}S$. This equivalence follows by Lemma 8.5 and a few negative somersaults. For instance, suppose $\widehat{V} \subseteq \widehat{U}$ and consider any $s \in \mathbf{per}(U')$. By Lemma 8.5 we have

$$s \in \pi \subseteq U'$$

for some $\pi \in \widehat{S}$. But now π does not meet U and hence $\pi \notin \widehat{U}$. In particular, we have $\pi \notin \widehat{V}$ so that

$$s \in \pi \subseteq V'$$

to give $s \in \mathbf{per}(V')$, as required. The converse implication follows in the same way. ■

This is the basic set-up. What we must do now is to find some very simple posets S for which the residue \widehat{S} is interesting, and can be obtained from a coverage. After that we will look at a more complicated poset for which the residue contains a very interesting space and can be got at by a coverage.

Some tree spaces

We look at a family of very simple but non-trivial examples of pathetic spaces. Each of the carrying posets is a tree of height ω . For each of these the residue space is moderately interesting and, in particular, both Cantor space and Baire space can be obtained in this way. For these examples we use a poset to generate the CB-derivative.

8.7 DEFINITION. Let I be any non-empty set of tags, let S be the set of all list taken from I , including the empty list \perp . We let $r, s, t \dots$ range over S , and write

$$si$$

for the list s extended by adding the tag i to the end. Thus each list has the form

$$\perp i_1 i_2 \cdots i_l$$

for tags i_1, i_2, \dots, i_l . This list has length l , so the empty list has length 0.

Let \sqsubseteq be the extension comparison, that is $s \sqsubseteq t$ holds precisely when

$$si_1 i_2 \cdots i_l = t$$

for some tags i_1, i_2, \dots, i_l (with $l = 0$ allowed). This converts S into a poset with the canonical topology $\mathcal{O}S$ of upper sections. This poset is a tree, that is if $s, t \in S$ have a common extension then they are comparable.

Consider the set \widehat{S} of residue points, that is the set of lower sections which are directed and have no maximal elements. Since S is a tree such a directed set π is a linear lower section of S . Since π it has no maximal element it is just a branch of S . Thus we may view π as a function

$$\pi(\cdot) : \mathbb{N} \longrightarrow I$$

where

$$\perp, \perp\pi(0), \perp\pi(0)\pi(1), \dots, \perp\pi(0)\pi(1)\cdots\pi(r), \dots$$

are the initial sections of length $0, 1, 2, \dots, r + 1, \dots$ in the branch π . ■

There are two particular cases of this construction worth mentioning. When I is a 2-element set we see that S is the full binary splitting tree, and \widehat{S} is Cantor space which is homeomorphic to the Cantor middle third set. When $I = \mathbb{N}$ we find that S is the Baire tree and \widehat{S} is Baire space, which is homeomorphic to the irrationals. In these cases S is hardly worth calling a space, but \widehat{S} is much more interesting.

We want to generate the CB-nucleus on $\mathcal{O}S$ using a poset. To do this it is convenient to turn the tree upside-down. This is why we used ' \sqsubseteq ' for the extension comparison.

8.8 DEFINITION. For the space S of Definition 8.7 let \leq be the opposite of \sqsubseteq , that is

$$t \leq s \iff s \sqsubseteq t$$

for $s, t \in S$. For each $s \in S$ let

$$sI = \{si \mid i \in I\}$$

the set of immediate extensions of s . Let \Vdash be the relation whose only instances are

$$s \Vdash sI$$

for $s \in S$. This gives us a site carried by the poset (S, \leq) . ■

The poset \Vdash induces, via the usual sequence of steps a closure operation on the poset $\mathcal{L}(S, \leq)$ of \leq -lower section of S . This is just the poset of \sqsubseteq -upper sections of S , in other words it is the topology $\mathcal{O}S$.

8.9 LEMMA. *Definition 8.8 produces an NG-poset \Vdash on the poset (S, \leq) .*

Proof. To show the poset is bounded consider any instance

$$s \Vdash sI$$

of the relation. By definition $\downarrow s$ is the \leq -lower section generated by s , and this is just the set of extensions of s . But each si is an extension of s , so that $sI \subseteq \downarrow s$, as required.

To show the poset is NG-stable we must show that

$$t \leq s \implies tI \subseteq \downarrow(sI)$$

holds for $s, t \in S$. To this end consider any $t \leq s$. If $t = s$ then $tI = sI$ and we are done. If $t \neq s$ then t is a proper extension of s , and hence $t \leq si < s$ for some tag i . But now for each tag j we have

$$tj < t \leq si \in sI$$

to give $tj \in \downarrow(sI)$, as required. ■

This shows that the closure operation induced on $\mathcal{O}S$ by the poset is a nucleus. To isolate this nucleus we first described the refining cover. This is a relation

$$s \vdash U$$

between elements $s \in S$ and \leq -lower section U or, equivalently, members of $\mathcal{O}S$.

8.10 LEMMA. *The refining cover of the poset of Definition 8.8 is given by*

$$s \vdash U \iff sI \subseteq U$$

for $s \in S$ and $U \in \mathcal{O}S$.

Proof. Consider first an instance $s \vdash U$ of the cover. By construction of \vdash we have

$$s \leq t \quad tI \subseteq U$$

for some $t \in S$. If $t = s$ then $sI = tI \subseteq U$, and we are done. If $t \neq s$ then s is a proper extension of t and hence $s \leq tj$ for some tag j . But then $tj \in tI \subseteq U$ and hence $s \in U$ (since U is a \leq -lower section of S). Finally, for each tag i we have

$$si \leq s \in U$$

so that $si \in U$ (since U is still a \leq -lower section of S).

Conversely, if $sI \subseteq U$ then

$$s \Vdash sI \subseteq U$$

to give $s \vdash U$. ■

To complete this collection of examples we pull the various constructions together.

8.11 THEOREM. *The refined version of the poser \Vdash of Definition 8.8 is just the CB-derivative \mathbf{der} on the topology \mathcal{OS} .*

Proof. For each $s \in S$ and $X \in \mathcal{CS}$ we see that X meets sI precisely when $si \in X$ for some tag i . Since X is a \sqsubseteq -lower section, there is such a tag precisely when s is a non-maximal member of X . In other words, we have

$$X \text{ meets } sI \iff s \in \mathbf{cb}(X)$$

for each $s \in S$ and $X \in \mathcal{CS}$. Thus, for $s \in S$ and $U \in \mathcal{OS}$ we have

$$s \in \mathbf{der}(U) \iff s \notin \mathbf{cb}(U') \iff U' \text{ does not meet } sI \iff sI \subseteq U$$

which, by Lemma 8.10, gives the required result. ■

As indicated above, using this construction with $I = \{0, 1\}$ or $I = \mathbb{N}$ produces either Cantor space or Baire space. However, in the proof of Lemma 8.5 a choice principle poked its head above the parapet. In [1] there is an analysis of precisely which principle is needed to obtain the description of either space. The more general construction of the previous subsection is used in [8] for what may seem to be a different purpose.

To conclude this subsection we recall a significant characteristic of the tree space S which is determined by the size of the tag set I . This is concerned with the closure ordinal of \mathbf{der} of, put differently, the global CB-rank of the space S .

8.12 LEMMA. *When the tag set I is finite the iterate \mathbf{der}^ω is a nucleus.*

Proof. It suffices to show that \mathbf{der}^ω is idempotent. To this end consider

$$s \in \mathbf{der}^{\omega+1}(U)$$

for some $U \in \mathcal{OS}$. Then

$$sI \subseteq \mathbf{der}^\omega(U)$$

and hence for each $i \in I$ there is some finite exponent $r(i)$ such that

$$si \in \mathbf{der}^{r(i)}(U)$$

holds. Since I is finite we can take a finite exponent r larger than each $r(i)$. But then

$$sI \subseteq \mathbf{der}^r(U)$$

to show

$$s \in \mathbf{der}^{r+1}(U) \subseteq \mathbf{der}^\omega(U)$$

as required. ■

Clearly, when I is infinite this argument doesn't work. Indeed the case of Baire space where $I = \mathbb{N}$ shows what can happen. It is easy to attach to each countable ordinal α a set $\nabla(\alpha) \in \mathcal{CS}$ such that

$$\mathbf{cb}^{\alpha+1}(\nabla(\alpha)) = \emptyset \quad \mathbf{cb}^\alpha(\nabla(\alpha)) \neq \emptyset$$

and hence the closure ordinal of \mathbf{cb} and \mathbf{der} is large.

A non-tree space

In this subsection we set up a particular pathetic space which eventually will lead to a construction of the the reals \mathbb{R} and the irrationals (with the metric topology). The construction is based on that given in section IV.12 of [6]. However, with the benefit of hindsight we can give a more detailed analysis, and locate several posers, inflators, and nuclei all of which have some special relationship with \mathbb{R} . The construction is also related to a common domain-theoretic characterization of the reals, but better done.

We produce a pathetic space \mathbb{S} (which is not a tree) and a chain of quotients

$$\begin{array}{ccccccc}
 \mathcal{O}\mathbb{S} & \longrightarrow & (\mathcal{O}\mathbb{S})_\delta & \longrightarrow & (\mathcal{O}\mathbb{S})_{??} & \longrightarrow & (\mathcal{O}\mathbb{S})_\rho & \longrightarrow & (\mathcal{O}\mathbb{S})_\zeta \\
 U & \longmapsto & \dot{U} & & & & & & \\
 U & \longmapsto & & \longrightarrow & \ddot{U} & & & & \\
 U & \longmapsto & & & & \longrightarrow & \ddot{\ddot{U}} & & \\
 U & \longmapsto & & & & & & \longrightarrow & \ddot{\ddot{\ddot{U}}}
 \end{array}$$

where the first gives the residue space (via the CB-nucleus) and the last two gives the reals and the irrationals. It is not absolutely clear what $(\mathcal{O}\mathbb{S})_{??}$ is, but the nucleus involved is obviously important. The notation for the frame morphisms

$$(\dot{\cdot}) \quad (\ddot{\cdot}) \quad (\ddot{\ddot{\cdot}}) \quad (\ddot{\ddot{\ddot{\cdot}}})$$

is intended to give the impression of a chain of important derivatives finishing with the irrational notation $(\ddot{\ddot{\ddot{\cdot}}})$. (In fact, the first two will not be used in the development.)

Rather than construct the reals \mathbb{R} we will construct

$$\mathbb{I} = (0, 1)$$

as a real interval. We do this so that we don't have to mess around with points at infinity, we can use the actual reals 0 and 1. However, the whole development could be rephrased using \mathbb{R} with $-\infty$ for 0 and $+\infty$ for 1.

Our first job is to construct the pathetic space \mathbb{S} . To do that we use the rationals

$$\mathbb{H} = \mathbb{Q} \cap [0, 1]$$

which we think of as markers of some kind.

8.13 DEFINITION. Let \mathbb{S} be the set of all ordered pairs

$$(l, r)$$

of rations numbers $0 \leq l < r \leq 1$. We compare such pairs by

$$(a, b) \leq (l, r) \iff l \leq a < b \leq r$$

to obtain a poset. (Here, of course, the comparisons on the right are those on \mathbb{H} .) ■

We may think of such a pair (l, r) as the real interval strictly between l and r . Thus we make use of the coincidence between the ordered pair notation and the open interval notation. With this interpretation the comparison \leq is just inclusion. Thus we will

always write ‘ \subseteq ’ for this comparison. That allows us to reserve ‘ \leq ’ for the standard comparison on \mathbb{R} . It is this comparison \subseteq which is used in the coverage relations.

As usual the topology on \mathbb{S} is given by the opposite \sqsubseteq of \subseteq , that is

$$t \sqsubseteq s \iff s \subseteq t$$

for $s, t \in \mathbb{S}$. However, except for the proof of the next result, for what we do here it is more convenient to work with \subseteq . Thus $\mathcal{O}\mathbb{S}$ is the family of \subseteq -lower sections and $\mathcal{C}\mathbb{S}$ is the family of \subseteq -upper sections of \mathbb{S} .

What are the residue points in $\widehat{\mathbb{S}}$?

The poset (\mathbb{S}, \subseteq) is not complete, but it is directed complete. It is not a \wedge -semilattice (since the empty interval is not in \mathbb{S}). Nevertheless, the notion of a filter still makes sense. Thus a filter is a non-empty set ∇ of members of \mathbb{S} such that both

$$\left. \begin{array}{l} s \in \nabla \\ s \subseteq t \end{array} \right\} \implies t \in \nabla \quad s, t \in \nabla \implies s \cap t \in \nabla$$

hold for all $s, t \in \mathbb{S}$. Examples of such filters are easy to find. For each $s \in \mathbb{S}$ the set

$$\nabla_s = \{t \in \mathbb{S} \mid s \subseteq t\}$$

is the principal filter generated by s . As usual, these are not the ones we want.

8.14 LEMMA. *The members of $\widehat{\mathbb{S}}$ are precisely the non-principal filters of (\mathbb{S}, \subseteq) .*

Proof. We need to remember the general description of $\pi \in \widehat{\mathbb{S}}$. Thus, for this proof, it is convenient to use the opposite \sqsubseteq of \subseteq .

Suppose first that $\pi \in \widehat{\mathbb{S}}$. Then π is certainly non-empty, and a \sqsubseteq -lower section and hence a \subseteq -upper section of \mathbb{S} . Consider any $s, t \in \pi$. Since π is \sqsubseteq -directed there is some $r \in \mathbb{S}$ with

$$s, t \sqsubseteq r \in \pi$$

which gives

$$r \in \pi \quad r \subseteq s \cap t$$

and hence $s \cap t \in \pi$. This shows that π is a filter of (\mathbb{S}, \subseteq) . If π is principal then it has a \subseteq -least member, which is a \sqsubseteq -maximal member, which a member of $\widehat{\mathbb{S}}$ can not have.

Conversely, suppose π is a non-principal filter of (\mathbb{S}, \subseteq) . Then, by a similar argument, we see that $\pi \in \widehat{\mathbb{S}}$. ■

This is the last time we use \sqsubseteq . From now on we work entirely with \subseteq .

It is useful to have a description of these residue points in terms of real numbers.

8.15 DEFINITION. For each pair of reals

$$0 \leq m \leq n \leq 1$$

from the interval $[0,1]$ consider the subsets of \mathbb{S}

$$\begin{array}{ll} \llbracket m, n \rrbracket & \text{given by } (l, r) \in \llbracket m, n \rrbracket \iff l \leq m \leq n \leq r \\ \llbracket m, n \rangle & \text{given by } (l, r) \in \llbracket m, n \rangle \iff l \leq m \leq n < r \\ \langle\langle m, n \rrbracket & \text{given by } (l, r) \in \langle\langle m, n \rrbracket \iff l < m \leq n \leq r \\ \langle\langle m, n \rangle & \text{given by } (l, r) \in \langle\langle m, n \rangle \iff l < m \leq n < r \end{array}$$

for $(l, r) \in \mathbb{S}$. ■

Each of these is in $\mathcal{C}\mathbb{S}$, and most of them are filters, but a few are empty. Thus

$$\llbracket m, 1 \rrbracket = ((m, 1)) = \emptyset = ((0, n)) = \llbracket 0, n \rrbracket$$

but all other are non-empty, and hence are filters.

A simple calculation shows that for $0 < m \leq n < 1$ each of

$$\llbracket m, n \rrbracket \quad ((m, n)) \quad \llbracket m, n \rrbracket$$

is in $\widehat{\mathbb{S}}$, and we find that

$$((m, n)) = \llbracket m, n \rrbracket \cap \llbracket m, n \rrbracket$$

holds. These three filters need not be distinct. For instance, we have

$$\llbracket m, n \rrbracket = ((m, n)) \quad ((m, n)) = \llbracket m, n \rrbracket$$

when m is irrational or n is irrational, respectively. When both m, n are rational the filter $\llbracket m, n \rrbracket$ is principal, and so not in $\widehat{\mathbb{S}}$. When both m, n are irrational we have $\llbracket m, n \rrbracket = ((m, n))$, and this is in $\widehat{\mathbb{S}}$.

8.16 THEOREM. *For each $\pi \in \widehat{\mathbb{S}}$ there are unique reals*

$$0 \leq m \leq n \leq 1$$

such that one of

$$\pi = \llbracket m, n \rrbracket \quad \pi = ((m, n)) \quad \pi = \llbracket m, n \rrbracket$$

holds.

Proof. Consider any $\pi \in \widehat{\mathbb{S}}$. Each interval $s = (l, r) \in \pi$ has a closure

$$s^- = [l, r]$$

and, using the filter properties, we find that the family

$$\{s^- \mid s \in \pi\}$$

has the finite intersection property. By the appropriate compactness result

$$\bigcap \{s^- \mid s \in \pi\}$$

is a non-empty closed interval. Let $[m, n]$ be this interval. In other words

$$m = \bigvee \{l \mid (\exists r)[(l, r) \in \pi] \quad n = \bigwedge \{r \mid (\exists l)[(l, r) \in \pi]$$

are the two reals in question.

Consider any $(l, r) \in \pi$. We have

$$l \leq m \leq n \leq r$$

by the construction of m and n . Also $(m, n) \notin \pi$, for otherwise both m and n are rational and π is the the principal filter generated by (m, n) , which is not so. Thus either $l \neq m$ or $n \neq r$ or both. We consider the various possibilities.

Suppose we have $(m, k) \in \pi$ for at least one k . If $(l, n) \in \pi$ for some l , then

$$(m, n) = (m, k) \cap (l, n) \in \pi$$

which is not so. Thus

$$(l, r) \in \pi \implies l \leq m \leq n < r$$

and hence $\pi \subseteq \llbracket m, n \rrbracket$. Consider any $(l, r) \in \llbracket m, n \rrbracket$. Since $n < r$, by the construction of n we have some $n < k < r$ with $(m, k) \in \pi$. But now

$$l \leq m \leq n < k < r$$

so that

$$(m, k) \subseteq (l, r)$$

and hence $(l, r) \in \pi$. This show that $\pi = \llbracket m, n \rrbracket$.

Suppose we have $(k, n) \in \pi$ for at least one k . A similar argument gives $\pi = \llbracket m, n \rrbracket$.

Finally, we may exclude the two previous cases to get

$$(l, r) \in \pi \implies l < m \leq n < r$$

and hence $\pi \subseteq \llbracket m, n \rrbracket$. As above, the limiting properties of m, n give $\pi = \llbracket m, n \rrbracket$.

This gives the existence of the required reals m, n . To verify the uniqueness we simply consider all possible cases of different representations. In each case we use the fact that for any two distinct reals there is a rational strictly between them. ■

The parent space \mathbb{S} is not quite as pathetic as the previous examples, but it is still an Alexandroff space (turned upside down). Its residue space $\widehat{\mathbb{S}}$ is quite complicated. It can be checked that the specialization order of $\widehat{\mathbb{S}}$ is more or less the obvious one. In particular $\widehat{\mathbb{S}}$ is not T_1 , and is certainly not the desired space \mathbb{I} . For each $p \in \mathbb{I}$ let

$$\tilde{p} = \llbracket p, p \rrbracket$$

that is the filter on \mathbb{S} given by

$$(l, r) \in \tilde{p} \iff l < p < r$$

for $(l, r) \in S$. It doesn't take too long to check that the subspace

$$\{\tilde{p} \mid p \in (0, 1)\}$$

of $\widehat{\mathbb{S}}$ is homeomorphic to \mathbb{I} . Eventually this is how we will get at \mathbb{I} .

For a general space the CB-derivative **der** can be quite complicated and take a long time to close off to a nucleus. In particular, this is so for the space \mathbb{I} which is a subspace of the residue space of \mathbb{S} . So what do you make of the following?

8.17 LEMMA. *For the pathetic space \mathbb{S} the CB-derivative **der** is a nucleus.*

Proof. It is more convenient to work with the CB-process **cb**. Thus we show

$$\mathbf{cb}(X) \subseteq \mathbf{cb}^2(X)$$

for each $X \in \mathcal{CS}$. Remember that each such X is a \subseteq -upper section, and its isolated points are its \subseteq -minimal members.

Consider any $(a, b) \in \mathbf{cb}(X)$. Thus $(a, b) \in X$ and is not minimal in X . This gives us some interval $(l, r) \in X$ with

$$a \leq l < r \leq b$$

and either $a \neq l$ or $r \neq b$. By symmetry, let's suppose $a < l$, and consider any rational m with $a < m < l$.

We have $(l, r) \subseteq (m, b)$ and the two are not equal. Since $(l, r) \in X$ we have $(m, b) \in X$ and is not a minimal member of X . Thus $(m, b) \in \mathbf{cb}(X)$.

We have $(m, b) \subseteq (a, b)$ and both are in $\mathbf{cb}(X)$. Since $m \neq a$ this shows that (a, b) is not a minimal member of $\mathbf{cb}(X)$, and hence $(a, b) \in \mathbf{cb}^2(X)$, as required. ■

Theorem 8.16 show that to get at the interval \mathbb{I} the CB-nucleus is not half strong enough. We need to kill at lot more points, and there is one nice way of doing this.

8.18 DEFINITION. Given an interval $(a, b) \in \mathbb{S}$ we set

$$\text{ins}(a, b) = \{(m, n) \in \mathbb{S} \mid a < m < n < b\}$$

to produce the **inside** of (a, b) .

For each $U \in \mathcal{OS}$ let **out**(U) be given by

$$(a, b) \in \mathbf{out}(U) \iff \text{ins}(a, b) \subseteq U$$

(for $(a, b) \in \mathbb{S}$) to produce the **outside** of U . ■

Trivially **out** is an inflator on \mathcal{OS} . In fact, we have more.

8.19 LEMMA. *The operation **out** is a nucleus on \mathcal{OS} .*

Proof. Trivially, **out** is an inflator on \mathcal{OS} . To see that it is a pre-nucleus consider

$$(a, b) \in \mathbf{out}(U) \cap \mathbf{out}(V)$$

for $U, V \in \mathcal{OS}$. Then we have both

$$\text{ins}(a, b) \subseteq U \quad \text{ins}(a, b) \subseteq V$$

to give

$$\text{ins}(a, b) \subseteq U \cap V$$

and hence

$$(a, b) \in \mathbf{out}(U \cap V)$$

as required.

Finally, to show that **out** is idempotent consider

$$(a, b) \in \mathbf{out}^2(U)$$

for $(a, b) \in \mathbb{S}$ and $U \in \mathcal{OS}$. Thus

$$\text{ins}(a, b) \subseteq \mathbf{out}(U)$$

and we need to strengthen this to $\text{ins}(a, b) \subseteq U$ so that $(a, b) \in \mathbf{out}(U)$. Consider any $(l, r) \in \text{ins}(a, b)$. We have

$$a < m < l < r < n < b$$

for some rationals m, n , and then

$$(m, n) \in \text{ins}(a, b) \subseteq \mathbf{out}(U)$$

so that $(m, n) \in \mathbf{out}(U)$ to give

$$(l, r) \in \text{ins}(m, n) \subseteq U$$

as required. ■

We can produce \mathbf{out} from a poset. Consider the relation \Vdash_{out} where

$$(l, r) \Vdash_{\text{out}} \text{ins}(l, r)$$

are the only instances (for $(l, r) \in \mathbb{S}$). The inflator associated with the refining cover of this poset is nothing more than the nucleus \mathbf{out} . Thus, on its own, this poset hardly has any merit. However, later we will combine it with other posets.

I do not know a weaker poset which generates \mathbf{out} . Such a gadget might be interesting for there is something of a proximity construction going on here. The nucleus \mathbf{out} is a more rounded version of the CB-nucleus (and if you get the joke then you will know what I'm talking about).

8.20 LEMMA. *We have $\mathbf{der} \leq \mathbf{out}$.*

Proof. We verify

$$\mathbf{out}(X')' \subseteq \mathbf{cb}(X)$$

for $X \in \mathcal{CS}$. To this end consider

$$(a, b) \notin \mathbf{out}(X')$$

that is, such that

$$\text{ins}(a, b) \not\subseteq \mathbf{out}(X')$$

holds. This gives some interval $(l, r) \in X$ with $a < l < r < b$. In particular, (a, b) is a member of X , and it not a \subseteq -minimal member, to give $(a, b) \in \mathbf{cb}(X)$, as required. ■

We have quotients

$$\mathcal{OS} \longrightarrow (\mathcal{OS})_{\mathbf{der}} \longrightarrow (\mathcal{OS})_{\mathbf{out}}$$

and in Theorem 8.16 we have located the points of the central frame. The points of $(\mathcal{OS})_{\mathbf{out}}$ will be certain of these central points. Which ones?

8.21 THEOREM. *The points of $(\mathcal{OS})_{\mathbf{out}}$ are the filters $((m, n))$ for reals $0 < m \leq n < 1$.*

Proof. We show first that each filter $((m, n))$ is a point of $(\mathcal{OS})_{out}$. It is certainly a point of $(\mathcal{OS})_{der}$, so it suffices to show that its complement is in $(\mathcal{OS})_{out}$, that is

$$\mathbf{out}(((m, n))') \subseteq ((m, n))'$$

holds. By way of contradiction suppose there is some

$$(a, b) \in \mathbf{out}(((m, n))') \quad (a, b) \in ((m, n))$$

that is such that both

$$\text{ins}(a, b) \subseteq ((m, n))' \quad a < m \leq n < b$$

hold. The density of the rationals gives us some

$$a < l < m \leq n < r < b$$

and this interval (l, r) immediately leads to a contradiction.

For the second part consider any point of $(\mathcal{OS})_{out}$. This must be a point of $\widehat{\mathbb{S}}$ and so, by Theorem 8.16, it is one of

$$\llbracket m, n \rrbracket \quad ((m, n)) \quad ((m, n])$$

for some reals $0 \leq m \leq n \leq 1$. We show that if either of the two outer cases arise then

$$\begin{array}{ll} m \text{ is irrational} & n \text{ is irrational} \\ \llbracket m, n \rrbracket = ((m, n)) & ((m, n]) = ((m, n)) \end{array}$$

respectively. By symmetry we may deal with the left-hand case.

By way of contradiction, suppose $\pi = \llbracket m, n \rrbracket$ is a point of $(\mathcal{OS})_{out}$ with m rational. Since $n < 1$ there is a rational b with $n < b < 1$. Since m is rational we have $(m, b) \in \mathbb{S}$ and $\text{ins}(m, b) \subseteq \pi'$ (since

$$m < l < r < b \implies l \not\leq m$$

for each $(l, r) \in \mathbb{S}$). This gives

$$(m, b) \in \mathbf{out}(\pi') = \pi'$$

and hence either $m \not\leq m$ or $n \not\leq b$, which is the required contradiction. ■

It is now clear what we have to do. We have to find some way of squeezing the filters $((m, n))$ so that only those with $m = n$ are left. This is where the coverage technique comes into its own (for the required nucleus is not so easily written down).

The reals

We will use the rather more interesting, but still pathetic space \mathbb{S} to produce the real interval \mathbb{I} (with the metric topology) and its subspace of irrational numbers. In this section we concentrate on producing the reals. For each $p \in \mathbb{I}$ we have a point

$$\tilde{p} = ((p, p)) \in \widehat{\mathbb{S}}$$

and these give a subspace homeomorphic to \mathbb{I} . We rework this observation.

8.22 DEFINITION. For each $U \in \mathcal{OS}$ and $X \in \mathcal{CS}$ let

$$p \in \ddot{U} \iff \tilde{p} \text{ meets } U \quad p \in \ddot{X} \iff \tilde{p} \subseteq X$$

(for $p \in \mathbb{I}$) to produce subsets \ddot{U} and \ddot{X} of \mathbb{I} . ■

This is another instance of the hull-kernel construction.

Consider first $p \in \ddot{U} \subseteq \mathbb{I}$. There is some $(a, b) \in U$ with $(a, b) \in \tilde{p}$. But now, for each $q \in \mathbb{I}$ we have

$$a < q < b \implies (a, b) \in \tilde{q} \cap U \implies q \in \ddot{U}$$

so that \ddot{U} includes an interval around p . This shows that \ddot{U} is metric open in \mathbb{I} .

In the usual way we check that

$$\begin{array}{ccc} \mathcal{OS} & \longrightarrow & \mathcal{OI} \\ U & \longmapsto & \ddot{U} \end{array}$$

is a frame morphism. Furthermore, it is surjective. To see this consider any interval (a, b) of \mathbb{I} . With a confusion of notation we may view this as an element (a, b) of \mathbb{S} . Now consider the $U \in \mathcal{OS}$ generated by this $(a, b) \in \mathbb{S}$. Thus

$$(l, r) \in U \iff (l, r) \subseteq (a, b) \iff a \leq l < r \leq b$$

for $(l, r) \in \mathbb{S}$. A simple calculation shows that \ddot{U} is the interval we started with.

This exhibits \mathcal{OI} as a quotient $(\mathcal{OS})_\rho$ for the nucleus ρ on \mathcal{OS} characterized by

$$V \subseteq \rho(U) \iff \ddot{V} \subseteq \ddot{U}$$

for $U, V \in \mathcal{OS}$. In this subsection we describe a coverage which generates ρ .

8.23 LEMMA. For each $U \in \mathcal{OS}$ we have $\mathbf{out}(\ddot{U}) = \ddot{U}$.

Proof. It suffices to show that $\mathbf{out}(\ddot{U}) \subseteq \ddot{U}$ since this converse inclusion is immediate. To this end consider $p \in \mathbf{out}(\ddot{U})$. Since \tilde{p} meets $\mathbf{out}(U)$ we have

$$a < p < b \quad \text{ins}(a, b) \subseteq U$$

for some $(a, b) \in \mathbb{S}$. By the density of the rationals there are rational

$$a < l < p < r < b$$

so that

$$(l, r) \in \text{ins}(a, b) \subseteq U$$

to show that \tilde{p} meets U (at (l, r)), and hence $p \in \ddot{U}$, as required. ■

Since \mathbf{out} is a nucleus this gives us a lower bound.

8.24 COROLLARY. We have $\mathbf{out} \leq \rho$.

The tactic now is to combine \mathbf{out} with some other nucleus to produce ρ . But which one? This extra part is more complicated than \mathbf{out} , but can be got at from below.

8.25 DEFINITION. Let \Vdash_{lap} be the poset on \mathbb{S} which has just one kind of instance

$$(l, r) \Vdash_{\text{lap}} \{(l, n), (m, r)\}$$

where $l < r$ with $l \leq m < n \leq r$. This is the **overlapping poset**. ■

Notice that for any $(l, r) \in S$ we can take $l = m < n = r$ to show that

$$(l, r) \Vdash_{\text{lap}} \{(l, r)\}$$

holds. More generally this overlapping poset is bounded.

8.26 LEMMA. *The overlapping poset \Vdash_{lap} is an NG-poset.*

Proof. It suffices to show that \Vdash_{lap} is NG-stable. To this end consider

$$(a, b) \subseteq (l, r) \Vdash_{\text{lap}} \{(l, n), (m, r)\}$$

where

$$l \leq a < b \leq r \quad l \leq m < n \leq b$$

are rational. Consider the possible relationship between (a, b) and (m, n) . One of

$$m \leq a \quad a < m < n < b \quad b \leq n$$

must hold. The appropriate one of these gives

$$(a, b) \subseteq (m, r) \quad (a, n) \subseteq (l, n) \quad (m, b) \subseteq (m, r) \quad (a, b) \subseteq (l, n)$$

respectively, so that

$$(a, b) \Vdash_{\text{lap}} \{(a, b)\} \quad (a, b) \Vdash_{\text{lap}} \{(a, n), (m, b)\} \quad (a, b) \Vdash_{\text{lap}} \{(a, b)\}$$

witnesses the required result. ■

We turn this poset into an inflator in the usual way.

8.27 DEFINITION. For each $U \in \mathcal{OS}$ we use

$$(a, b) \in \mathbf{lap}(U) \iff (\exists m, n \in \mathbb{H}, a \leq m < n \leq b)[(a, n), (m, b) \in U]$$

(for $(a, b) \in S$) to produce a subset $\mathbf{lap}(U)$ of \mathbb{S} . ■

It shouldn't be a surprise to find that this is the associated inflator of \Vdash_{lap} , but let's look at some of the details of the verification of this.

It is not immediately clear that $\mathbf{lap}(U) \in \mathcal{OS}$ for $U \in \mathcal{OS}$. To see this suppose $(a, b) \in \mathbf{lap}(U)$ where this is witnessed by

$$(a, n), (m, b) \in U$$

for

$$a \leq m < n \leq b$$

and consider

$$a \leq l < r \leq b$$

so that $(l, r) \in \mathbf{lap}(U)$ is required. One of the cases

$$r \leq n \quad l < m < n < r \quad m \leq l$$

must hold. These give, respectively

$$(l, r) \in U \quad (l, n), (m, r) \in U \quad (l, r) \in U$$

so that

$$(l, r) \in U \subseteq \mathbf{lap}(U) \quad (l, r) \in \mathbf{lap}(U) \quad (l, r) \in U \subseteq \mathbf{lap}(U)$$

as required.

This shows that \mathbf{lap} is an operation on \mathcal{OS} , and it is immediate that it is inflationary and monotone. A couple more observations gives the following.

8.28 LEMMA. *The inflator \mathbf{lap} is the one associated with the refining cover of $\mathbb{H}_{\mathbf{lap}}$.*

Proof. Let d be the inflator associated with $\mathbb{H}_{\mathbf{lap}}$ and consider $U \in \mathcal{OS}$. We show $d(U) = \mathbf{lap}(U)$ via two inclusions.

Consider any $(a, b) \in d(U)$. Thus we have some

$$(a, b) \subseteq (l, r) \mathbb{H}_{\mathbf{lap}} Z \subseteq U$$

where

$$l \leq m < n \leq r \quad (l, n), (m, r) \in U \quad Z = \{(l, n), (m, r)\}$$

witness the instance of $\mathbb{H}_{\mathbf{lap}}$. Since $l \leq a < b \leq r$ we see that one of

$$m < a \quad a \leq m < n \leq b \quad b < n$$

must hold. These give, respectively,

$$(a, b) \subseteq (m, r) \in U \quad (a, b) \in \mathbf{lap}(U) \quad (a, b) \subseteq (l, n) \in U$$

each of which leads to $(a, b) \in \mathbf{lap}(U)$.

Conversely, suppose $(a, b) \in \mathbf{lap}(U)$. We have

$$(a, n), (m, b) \in U$$

for some $a \leq m < n \leq b$, so that

$$(a, b) \mathbb{H}_{\mathbf{lap}} Z \subseteq U$$

where

$$Z = \{(a, n), (m, b)\}$$

to give $(a, b) \in d(U)$. ■

Since $\mathbb{H}_{\mathbf{lap}}$ is a NG-poset the generated inflator \mathbf{lap} is stable. It is instructive to verify this directly. Thus consider

$$(a, b) \in \mathbf{lap}(U) \cap V$$

for $U, V \in \mathcal{OS}$. We have some $a \leq m < n \leq b$ with

$$(a, n), (m, b) \in U$$

so that, since V is a \subseteq -lower section,

$$(a, n), (m, b) \in U \cap V$$

to give

$$(a, b) \in \mathbf{lap}(U \cap V)$$

as required.

Because the construction of \mathbf{lap} is of finite character we have the following.

8.29 LEMMA. *The iterate \mathbf{lap}^ω is a nucleus.*

Proof. On general grounds we know that \mathbf{lap}^ω is a pre-nucleus. Thus the required result follows from an inclusion

$$\mathbf{lap}^{\omega+1}(U) \subseteq \mathbf{lap}^\omega(U)$$

for arbitrary $U \in \mathcal{OS}$. To this end consider any

$$(a, b) \in \mathbf{lap}^{\omega+1}(U)$$

so that

$$(a, n) \in \mathbf{lap}^\omega(U) \quad (m, b) \in \mathbf{lap}^\omega(U)$$

for some $a \leq m < n \leq b$, and hence the construction of \mathbf{lap}^ω gives

$$(a, n) \in \mathbf{lap}^r(U) \quad (m, b) \in \mathbf{lap}^r(U)$$

for some $r < \omega$. But now

$$(a, b) \in \mathbf{lap}^{r+1}(U) \subseteq \mathbf{lap}^\omega(U)$$

to give the required result. ■

We ought to show that no finite iterate of \mathbf{lap} is a nucleus.

8.30 EXAMPLE. For each strictly positive real number ℓ let $U(\ell)$ be the set of all intervals $(a, b) \in S$ with $(b - a) < \ell$. Trivially we have $U(\ell) \in \mathcal{OS}$ with $U(\ell) = S$ if $1 < \ell$. We show that

$$\mathbf{lap}(U(\ell)) = U(2\ell)$$

and hence no finite iterate \mathbf{lap}^r is a nucleus.

Consider first $(a, b) \in \mathbf{lap}(U(\ell))$. Then we have

$$a \leq m < n \leq b$$

for some rationals m, n where

$$(n - a) < \ell \quad (b - m) < \ell$$

hold. But now

$$(b - a) < (b - a) + (n - m) < 2\ell$$

and hence $(a, b) \in U(2\ell)$.

Conversely, consider $(a, b) \in U(2\ell)$. If $(a, b) \in U(\ell)$ then we are done. Thus we may suppose

$$\ell \leq (b - a) < 2\ell$$

and hence

$$(b - a) + 3\epsilon = 2\ell$$

for some strictly positive real ϵ . With

$$m = b - \ell + \epsilon \quad n = a + \ell - \epsilon$$

we check that

$$a \leq m < n \leq b$$

which since

$$(n - a) = \ell - \epsilon < \ell \quad (b - m) = \ell - \epsilon < \ell$$

gives the required result.

The three required comparisons translate to

$$a \leq b - \ell + \epsilon \quad b - \ell + \epsilon < a + \ell - \epsilon \quad a + \ell - \epsilon \leq b$$

which are

$$(\ell - \epsilon) \leq (b - a) \quad (b - a) + 2\epsilon < 2\ell \quad (\ell - \epsilon) \leq (b - a)$$

respectively. The choice of ϵ ensures these. ■

This inflator gives us another lower bound to the nucleus ρ .

8.31 LEMMA. *For each $U \in \mathcal{OS}$ we have $\mathbf{lap}^{\ddot{\ddot{}}}(U) = \ddot{\ddot{}}$.*

Proof. It suffices to show that $\mathbf{lap}^{\ddot{\ddot{}}}(U) \subseteq \ddot{\ddot{}}$ since this converse inclusion is immediate. To this end consider $p \in \mathbf{lap}^{\ddot{\ddot{}}}(U)$. Since \tilde{p} meets $\mathbf{lap}(U)$ we have some

$$a < p < b \quad (a, b) \in \mathbf{lap}(U)$$

and hence some

$$a \leq m < n \leq b \quad (a, n), (m, b) \in U$$

where a, m, n, b are rationals. We consider where the real p can lie.

If $p < n$ then $a < p < n$ and hence \tilde{p} meets U at (a, n) .

If $n \leq p$ then $m < p < b$ and hence \tilde{p} meets U at (m, b) .

In either case \tilde{p} meets U and hence $p \in \ddot{\ddot{}}$, as required. ■

A simple argument now shows that

$$\mathbf{lap}^{\ddot{\ddot{}}r}(U) = \ddot{\ddot{}}$$

for each $U \in \mathcal{OS}$ and $r < \omega$, to give the following.

8.32 COROLLARY. *We have $\mathbf{lap}^\omega \leq \rho$.*

This with Corollary 8.24 gives us two fairly complementary lower bounds to ρ . Our aim is to show that the join of these two nuclei is exactly ρ . Furthermore, we can generate this join using the pointwise union of the two inflators.

8.33 DEFINITION. For each $U \in \mathcal{O}\mathbb{S}$ let

$$\mathbf{real}(U) = \mathbf{out}(U) \cup \mathbf{lap}(U)$$

to produce an inflator \mathbf{real} on $\mathcal{O}\mathbb{S}$.

Since both the components \mathbf{out} and \mathbf{lap} are stable, then so is \mathbf{real} . Furthermore, the closure \mathbf{real}^∞ is the join $\mathbf{out} \vee \mathbf{lap}^\omega$ of the two related nuclei (as a nucleus). However, I do not know what the closure ordinal of \mathbf{real} is. This should be determined.

8.34 COROLLARY. We have $\mathbf{real}^\infty \leq \rho$.

Proof. Remember that the assignment

$$\begin{array}{ccc} \mathcal{O}\mathbb{S} & \longrightarrow & \mathcal{O}\mathbb{I} \\ U & \longmapsto & \ddot{U} \end{array}$$

is a frame morphism. In particular, for each $U \in \mathcal{O}\mathbb{S}$ we have

$$\mathbf{real}^{\ddot{\cdot}}(U) = \mathbf{out}^{\ddot{\cdot}}(U) \cup \mathbf{lap}^{\ddot{\cdot}}(U) = \ddot{U}$$

by Lemmas 8.23 and 8.31. A routine induction now gives

$$\mathbf{real}^{\ddot{\cdot}\alpha}(U) = \ddot{U}$$

for each ordinal α , and hence

$$\mathbf{real}^{\ddot{\cdot}\infty}(U) = \ddot{U}$$

which leads to the required result. ■

We must improve this comparison. We employ a splitting argument which seems to get at the essential content of the construction. For this we use the co-fixed sets of \mathbf{real} , the closed sets whose complements are fixed by \mathbf{real} .

8.35 DEFINITION. Let $\mathcal{R}\mathbb{S}$ be the family of those closed sets $Z \in \mathcal{C}\mathbb{S}$ for which both

$$(a, b) \in Z \implies (\forall m, n \in \mathbb{Q}, a \leq m < n \leq b)[(a, n) \in Z \text{ or } (m, b) \in Z]$$

$$(a, b) \in Z \implies (\exists l, r \in \mathbb{Q}, a < l < r < b)[(l, r) \in Z]$$

hold for each $(a, b) \in S$. ■

An exercise in taking the contrapositive a couple of times gives the following.

8.36 LEMMA. The family $\mathcal{R}\mathbb{S}$ is the collection of those $Z \in \mathcal{C}\mathbb{S}$ with $\mathbf{real}(Z') = Z'$.

We come now to the crucial splitting trick. For this we write

$$|(a, b)|$$

for the length $b - a$ of an interval $(a, b) \in S$.

8.37 LEMMA. (One step splitting) *For each situation*

$$(a, b) \in Z \in \mathcal{RS}$$

there are rationals

$$a < a' < b' < b$$

such that

$$(a', b') \in Z \quad |(a', b')| \leq 2/3|(a, b)|$$

hold.

Proof. Since $(a, b) \in Z \in \mathcal{RS}$ we first obtain rationals l, r with

$$a < l < r < b \quad (l, r) \in Z$$

by the second property of $Z \in \mathcal{RS}$. Now with the equal splitting

$$l < m < n < r$$

the first property of Z show that one of

$$(l, n) \in Z \quad (m, r) \in Z$$

holds. We set

$$a' = l \quad b' = n \quad a' = m \quad b' = r$$

respectively, to get $(a', b') \in Z$ with

$$|(a', b')| \leq 2/3|(l, r)| \leq 2/3|(a, b)|$$

as required. ■

With this we can produce a sandwiching argument for members of \mathcal{RS} .

8.38 LEMMA. (Witnessing) *For each situation*

$$(a, b) \in Z \in \mathcal{RS}$$

there is at least one $p \in \mathbb{I}$ such that

$$(a, b) \in \tilde{p} \subseteq Z$$

holds.

Proof. Starting from the given situation $(a, b) \in Z \in \mathcal{RS}$ we may iterate a use of Lemma 8.37 to produce a pair of strict ω -chains

$$a = a_0 < a_1 < \cdots < a_i < \cdots < b_i < \cdots < b_1 < b_0 = 0$$

where

$$(a_i, b_i) \in Z \quad |(a_i, b_i)| \leq (2/3)^i |(a, b)|$$

for each $i < \omega$.

By the completeness of \mathbb{I} there is a unique $p \in \mathbb{I}$ such that

$$\sup\{a_i \mid i < \omega\} = p = \inf\{b_i \mid i < \omega\}$$

holds. In particular, $(a_i, b_i) \in \tilde{p}$ for each i . Furthermore, for each $(l, r) \in \tilde{p}$, that is with $l < p < r$, there is some index i with

$$l < a_i < b_i < r$$

and hence $(l, r) \in Z$. ■

We're almost there.

8.39 THEOREM. *We have $\mathbf{real}^\infty = \rho$.*

Proof. By Corollary 8.34 it suffices to show

$$\rho(U) \subseteq \mathbf{real}^\infty(U)$$

for each $U \in \mathcal{OS}$. To this end let $V = \rho(U)$, so that

$$\ddot{V} \subseteq \ddot{U}$$

holds, and

$$V \subseteq \mathbf{real}^\infty(U)$$

is required. Let

$$X = U' \quad Y = V' \quad Z = \mathbf{real}^\infty(U)'$$

so that

$$\ddot{X} \subseteq \ddot{Y} \quad Z \subseteq X \quad Z \in \mathcal{RS}$$

hold and $Z \subseteq Y$ is required.

Consider any $(a, b) \in Z$. By Lemma 8.38 we have

$$(a, b) \in \tilde{p} \subseteq Z$$

for some $p \in \mathbb{I}$. But now

$$p \in \ddot{Z} \subseteq \ddot{X} \subseteq \ddot{Y}$$

to give

$$(a, b) \in \tilde{p} \subseteq Y$$

and hence $(a, b) \in Y$, as required. ■

We have generated the nucleus ρ from the pointwise join of two inflators each of which can be obtained from a poser. In the usual way the syntactic composite of these two posers generate the whole nucleus ρ .

Thus consider the poser \Vdash_{real} on \mathbb{S} whose instances are

$$(l, r) \Vdash_{\text{real}} Z$$

for $(l, r) \in S$ and where Z is one of

$$\{(l, n), (m, r)\} \quad \text{ins}(l, r)$$

with $l \leq m < n \leq r$ for the right hand case. Apart from a minor variation this is the poser used in [6] (which should not be a surprise since that's where I got it from).

The irrationals

We have created the real interval \mathbb{I} out of a fairly pathetic space \mathbb{S} . The interval \mathbb{I} has some interesting subspaces each of which produces a quotient of $\mathcal{O}\mathbb{I}$. Can any of these be obtained directly from \mathbb{S} ? You bet they can. Let \mathbb{J} be the set of irrational reals p with $0 < p < 1$. We follow fairly closely the construction of $\mathcal{O}\mathbb{I}$ to obtain $\mathcal{O}\mathbb{J}$.

8.40 DEFINITION. For each $U \in \mathcal{O}\mathbb{S}$ and $X \in \mathcal{C}\mathbb{S}$ let

$$p \in \ddot{U} \iff \tilde{p} \text{ meets } U \quad p \in \ddot{X} \iff \tilde{p} \subseteq X$$

(for $p \in \mathbb{J}$) to produce subsets \ddot{U} and \ddot{X} of \mathbb{J} . ■

In the usual way this sets up a surjective frame morphism

$$\begin{array}{ccc} \mathcal{O}\mathbb{S} & \longrightarrow & \mathcal{O}\mathbb{J} \\ U & \longmapsto & \ddot{U} \end{array}$$

to the canonical topology on \mathbb{J} . The kernel ζ of the quotient is characterized by

$$V \subseteq \zeta(U) \iff \ddot{V} \subseteq \ddot{U}$$

for $U, V \in \mathcal{O}\mathbb{J}$. The purpose of the subsection is to generate this nucleus using a poser.

We know that $\rho \leq \zeta$. We have a decomposition of ρ given by *out* and *lap*. Our tactic is to replace the component *lap* by something stronger.

8.41 DEFINITION. Let \Vdash_{spl} be the poser on \mathbb{S} which has two kind of instances

$$(l, r) \Vdash \{(l, r)\} \quad (l, r) \Vdash \{(l, m), (m, r)\}$$

where $l < r$ with $l < m < r$ for the second instance. This is the **splitting** poser. ■

Trivially, this poser is bounded. A few more observations gives the following.

8.42 LEMMA. *The splitting poser \Vdash_{spl} is an NG-poser.*

Proof. Since \Vdash_{spl} is bounded, it remains to check that it is NG-stable. There are two cases to check, but one of them is trivial. For the other we must show that if

$$(a, b) \subseteq (l, r) \Vdash_{\text{spl}} \{(l, m), (m, r)\}$$

then

$$(a, b) \Vdash_{\text{spl}} V$$

for some set V consisting of subintervals of (l, m) and (m, r) . We have

$$l \leq a < b \leq r \quad l < m < r$$

and we must worry about the position of m relative to a and b . There are three cases

$$l \leq a < b \leq m < r \quad l \leq a < m < b \leq r \quad l < m \leq a < b \leq r$$

which are dealt with by

$$(a, b) \Vdash_{\text{spl}} \{(a, b)\} \quad (a, b) \Vdash_{\text{spl}} \{(a, m), (m, b)\} \quad (a, b) \Vdash_{\text{spl}} \{a, b\}$$

respectively. ■

In the usual way this poser generates a stable inflator on $\mathcal{O}\mathbb{S}$.

8.43 DEFINITION. For each $U \in \mathcal{OS}$ we use

$$(a, b) \in \mathbf{spl}(U) \iff (a, b) \in U \text{ or } (\exists m \in \mathbb{H})[(a, m), (m, b) \in U]$$

(for $(a, b) \in S$) to produce a subset $\mathbf{spl}(U)$ of S . ■

A few moment's thought shows that we have an inflator \mathbf{spl} on \mathcal{OS} . It is mildly instructive to check that it is stable. Thus suppose

$$(a, b) \in \mathbf{spl}(U) \cap V$$

so that

$$(a, m), (m, b) \in U \quad (a, b) \in V$$

for some m . Since V is a \subseteq -lower section we have

$$(a, m), (m, b) \in U \cap V$$

and hence

$$(a, b) \in \mathbf{spl}(U \cap V)$$

as required. In fact, all this is a consequence of the following.

8.44 LEMMA. *The inflator \mathbf{spl} is the one associated with the refining cover of $\mathbb{I}_{\mathbf{spl}}$.*

Proof. After unravelling the various constructions we see that we must show that

$$(a, b) \in \mathbf{spl}(U)$$

precisely when one of

$$(a, b) \in U \quad (\exists l, r)[(a, b) \subseteq (l, r) \in U] \quad (\exists l, m, r)[(a, b) \subseteq (l, r) \text{ and } (l, m), (m, r) \in U]$$

holds. The verification of this is straight forward. ■

The inflator \mathbf{spl} is not a pre-nucleus. Let U be the set of intervals in S which do not contain $1/3$, let V be the set of intervals which do not contain $2/3$. Thus $U \cap V$ is the set of intervals which contain neither $1/3$ nor $2/3$. We have

$$(0, 1) \in \mathbf{spl}(U) \cap \mathbf{spl}(V) \quad (0, 1) \notin \mathbf{spl}(U \cap V)$$

by two splittings and since any splitting of $(0, 1)$ must pick up either $1/3$ or $2/3$.

The splitting inflator \mathbf{spl} has finite character, which keeps down its closure ordinal.

8.45 LEMMA. *The iterate \mathbf{spl}^ω is a nucleus.*

Proof. On general grounds we know that \mathbf{spl}^ω is a pre-nucleus. Thus it suffices to show it is idempotent. To this end consider any

$$(a, b) \in \mathbf{spl}^{\omega+1}(U)$$

so that

$$(a, m) \in \mathbf{spl}^\omega(U) \quad (m, b) \in \mathbf{spl}^\omega(U)$$

for some $a < m < b$, and hence the construction of \mathbf{spl}^ω gives

$$(a, m) \in \mathbf{spl}^r(U) \quad (m, b) \in \mathbf{spl}^r(U)$$

for some $r < \omega$. But now

$$(a, b) \in \mathbf{spl}^{r+1}(U) \subseteq \mathbf{spl}^\omega(U)$$

to give the required result. ■

The construction and analysis of \mathbf{spl} follows quite closely that of \mathbf{lap} . The next thing to do is to show that the closure ordinal of \mathbf{spl} is not finite. With \mathbf{lap} we did this by exhibiting appropriate examples. With \mathbf{spl} we can do a bit better than that.

8.46 LEMMA. *For each $l < \omega$, open set $U \in \mathcal{OS}$, and interval $(a, b) \in \mathbb{S}$ we have*

$$(a, b) \in \mathbf{spl}^l(U)$$

precisely when there is some $r \leq 2^l$ and an ascending chain

$$a = a_0 < a_1 < \cdots < a_r = b$$

with

$$(a_i, a_{i+1}) \in U$$

for each index $i < r$.

Proof. We proceed by induction on l with variation of the other parameters. The base case, $l = 0$, is trivial, so it remains to check the induction step, $l \mapsto l + 1$.

Suppose $(a, b) \in \mathbf{spl}^{l+1}(U)$. Then either $(a, b) \in \mathbf{spl}^l(U)$ or

$$(a, m), (m, b) \in \mathbf{spl}^l(U)$$

for some $a < m < b$. In the first case the induction hypothesis gives us a witnessing chain. In the second case the induction hypothesis gives us two witnessing chains

$$a = a_0 < a_1 < \cdots < a_r = m = b_0 < b_1 < \cdots < b_s = b$$

of $r + 1$ and $s + 1$ terms where $r, s \leq 2^l$. The composite chain has

$$(r + 1) + (s + 1) - 1 = r + s + 1$$

terms and $r + s \leq 2^{l+1}$, as required.

Conversely, suppose there is a chain

$$a = a_0 < a_1 < \cdots < b_{t-1} < b_t = b$$

where $t \leq 2^{l+1}$ and each component interval is in U . If $t \leq 2^l$ then $(a, b) \in \mathbf{spl}^l(U)$ by the induction hypothesis. If $2^l < t$ then we may set $r = 2^l$ and $m = a_r$ to split the given chain into two chains

$$a = a_0 < a_1 < \cdots < a_r = m = b_0 < b_1 < \cdots < b_s = b$$

where $r + s = t$ and hence $s \leq 2^l$. The induction hypothesis gives

$$(a, m), (m, b) \in \mathbf{spl}^l(U)$$

and hence $(a, b) \in \mathbf{spl}^{l+1}(U)$, as required. ■

As should be expected, there is a direct comparison between \mathbf{lap} and \mathbf{spl} .

8.47 LEMMA. We have $\mathbf{lap} \leq \mathbf{spl}$.

Proof. Consider any

$$(a, b) \in \mathbf{lap}(U)$$

for $U \in \mathcal{OS}$. Then

$$(a, n), (m, b) \in U$$

for some $a \leq m < n \leq b$. If $a = m$ then

$$(a, b) = (m, b) \in U$$

and we are done. Otherwise we have $a < m < b$ with

$$(a, m) \subseteq (a, n) \in U \quad (m, b) \in U$$

to give

$$(a, m), (m, b) \in U$$

and hence $(a, b) \in \mathbf{spl}(U)$, as required. \blacksquare

Towards the end of this subsection I will give some more information about the relationship between \mathbf{lap} and \mathbf{spl} . For now we continue directly to our goal.

8.48 LEMMA. For each $U \in \mathcal{OS}$ we have $\mathbf{spl}^{\dots}(U) = \ddot{U}$.

Proof. It suffices to show that $\mathbf{lap}^{\dots}(U) \subseteq \ddot{U}$ since this converse inclusion is immediate. To this end consider $p \in \mathbf{spl}^{\dots}(U)$. Since \tilde{p} meets $\mathbf{spl}(U)$ we have some

$$a < p < b \quad (a, b) \in \mathbf{lap}(U)$$

and hence either $(a, b) \in U$ (and we are done) or there is some rational m such that

$$a < m < b \quad (a, m), (m, b) \in U$$

hold. At this point we remember that p is irrational and a, m, b are rational, so that either $a < p < m$ or $m < p < b$, both of which lead to $p \in \ddot{U}$. \blacksquare

Observe how the irrationality of the points of \mathbb{J} is used in this proof. This is why we can get away with abutting covers whereas the analysis of \mathbb{I} needs overlapping covers.

In the usual way this result gives us a lower bound to ζ .

8.49 COROLLARY. We have $\mathbf{spl}^\omega \leq \zeta$.

This is going all too easy isn't it. In fact, this analysis is easier than that for \mathbb{I} .

8.50 DEFINITION. For each $U \in \mathcal{OS}$ let

$$\mathbf{irrt}(U) = \mathbf{out}(U) \cup \mathbf{spl}(U)$$

to produce an inflator \mathbf{irrt} on \mathcal{OS} .

As with *real*, this inflator *irrt* is stable and so generates a nucleus. Again as with *real*, I do not know what the closure ordinal of *irrt* is. This should be determined.

The following is a consequence of the comparisons $\mathit{out} \leq \rho \leq \zeta$ and Corollary 8.49.

8.51 LEMMA. *We have $\mathit{irrt}^\infty \leq \zeta$.*

To improve this comparison to an equality we look at the co-fixed sets of *irrt*.

8.52 DEFINITION. Let \mathcal{IS} be the family of those closed sets $Z \in \mathcal{CS}$ for which both

$$\begin{aligned} (a, b) \in Z &\implies (\forall m \in \mathbb{Q}, a < m < b)[(a, m) \in Z \text{ or } (m, b) \in Z] \\ (a, b) \in Z &\implies (\exists l, r \in \mathbb{Q}, a < l < r < b)[(l, r) \in Z] \end{aligned}$$

hold for each $(a, b) \in S$. ■

A routine use of taking the contrapositive gives the following.

8.53 LEMMA. *The family \mathcal{IS} is the collection of those $Z \in \mathcal{CS}$ with $\mathit{irrt}(Z') = Z'$.*

We come now to the crucial squeezing argument. Thus starting from

$$(a, b) \in Z \in \mathcal{IS}$$

we must obtain some *irrational* p such that

$$(a, b) \in \tilde{p} \subseteq Z$$

holds. Using more or less the same construction as in Lemma 8.38 we can produce a *real* witness p , but here we must find an *irrational* witness. This requires more effort.

As before we write $|(a, b)|$ for the length $b - a$ of $(a, b) \in S$.

8.54 LEMMA. (One step splitting) *For each situation*

$$(a, b) \in Z \in \mathcal{IS}$$

and rational $q \in \mathbb{I}$, there are rationals

$$a < a' < b' < b$$

such that

$$(a', b') \in Z \quad |(a', b')| \leq 1/2|(a, b)| \quad q \notin (a, b)$$

hold.

Proof. Given $(a, b) \in Z \in \mathcal{IS}$ with the mid point

$$m = \frac{a + b}{2}$$

we have one of

$$(a, m) \in Z \quad (m, b) \in Z$$

and then there are

$$a < l < r < m \quad m < l < r < b$$

such that

$$(l, r) \in Z \quad |(l, r)| < 1/2|(a, b)|$$

hold. If $q \notin (l, r)$ then we take $a' = l$ and $b' = r$. If $l < q < r$ then a second splitting gives one of

$$(l, q) \in Z \quad (q, r) \in Z$$

so that we may take

$$a' = l, b' = q \quad a' = q, b' = r$$

to omit q . ■

The trick now is to iterate this splitting where at each step we deal with a different $q \in \mathbb{H}$. We arrange the whole procedure so that eventually each $q \in \mathbb{H}$ is omitted.

Thus we assume given some enumeration

$$(q_i \mid i < \omega)$$

of \mathbb{H} . This enumeration has nothing to do with the standard comparison on \mathbb{H} .

8.55 LEMMA. (Witnessing) *For each situation*

$$(a, b) \in Z \in \mathcal{IS}$$

there is at least one $p \in \mathbb{J}$ such that

$$(a, b) \in \tilde{p} \subseteq Z \in \mathcal{IS}$$

holds.

Proof. Starting from the given situation $(a, b) \in Z \in \mathcal{IS}$ and using the assumed enumeration of \mathbb{H} , we may iterate a use of Lemma 8.54 to produce a pair of strict ω -chains

$$a = a_0 < a_1 < \cdots < a_i < \cdots < b_i < \cdots < b_1 < b_0 = 0$$

where

$$(a_i, b_i) \in Z \quad |(a_i, b_i)| \leq (1/2)^i |(a, b)| \quad q_i \notin (a_{i+1}, b_{i+1})$$

for each $i < \omega$. By the completeness of \mathbb{I} there is a unique $p \in \mathbb{I}$ such that

$$\sup\{a_i \mid i < \omega\} = p = \inf\{b_i \mid i < \omega\}$$

holds. In particular, $(a_i, b_i) \in \tilde{p}$ for each i . To see that p is irrational, suppose otherwise. Then $p = q_i$ for some index i , to give

$$p = q_i \notin (a_{i+1}, b_{i+1})$$

which is not so. Finally, a routine argument gives $\tilde{p} \subseteq Z$. ■

There is an interesting comparison between this proof and that of Lemma 8.38. Whereas that proof is a variant of a standard compactness argument, this one has the flavour of a Baire category argument. I do not know if this is just a superficial comparison or the greatest thing since sliced bread.

We can repeat the proof of Theorem 8.39 more or less word for word.

8.56 THEOREM. We have $\mathbf{irr}t^\infty = \zeta$.

Proof. By Lemma 8.51 it suffices to show

$$\zeta(U) \subseteq \mathbf{irr}t^\infty(U)$$

for each $U \in \mathcal{OS}$. To this end let $V = \rho(U)$, so that

$$\ddot{V} \subseteq \ddot{U}$$

holds, and

$$V \subseteq \mathbf{irr}t^\infty(U)$$

is required. Let

$$X = U' \quad Y = V' \quad Z = \mathbf{irr}t^\infty(U)'$$

so that

$$\ddot{X} \subseteq \ddot{Y} \quad Z \subseteq X \quad Z \in \mathcal{IS}$$

hold and $Z \subseteq Y$ is required.

Consider any $(a, b) \in Z$. By Lemma 8.55 we have

$$(a, b) \in \tilde{p} \subseteq Z$$

for some $p \in \mathbb{J}$. But now

$$p \in \ddot{Z} \subseteq \ddot{X} \subseteq \ddot{Y}$$

to give

$$(a, b) \in \tilde{p} \subseteq Y$$

and hence $(a, b) \in Y$, as required. ■

This result shows how we may use the coverage technique to produce the irrationals out of the rationals. We use the poser $\mathbb{I}_{\text{irrat}}$ on \mathbb{S} whose instances are

$$(l, r) \mathbb{I}_{\text{irrat}} Z$$

for $(l, r) \in S$ and where Z is one of

$$\{(l, r)\} \quad \{(l, m), (m, r)\} \quad \text{ins}(l, r)$$

with $l < m < r$ for the central case.

The difference between this construction of the irrationals and the construction of the reals given in the previous subsection is that here we use abutting covers whereas there we use overlapping covers. Can we isolate a more precise distinction between the two constructions? Perhaps what is needed is a careful analysis of the differences between the two inflators \mathbf{lap} and \mathbf{spl} , or between the two generating derivation systems. By Lemma 8.47 we have $\mathbf{lap} \leq \mathbf{spl}$. Here is another random fact that I know.

8.57 LEMMA. We have $\mathbf{spl}^2 \leq \mathbf{lap}^2 \circ \mathbf{spl}$.

Proof. Consider

$$(a, b) \in \mathbf{spl}^2(U)$$

for $U \in \mathcal{OS}$. Then either $(a, b) \in \mathbf{spl}(U)$ and we are done, or there is a splitting

$$(a, m), (m, b) \in \mathbf{spl}(U)$$

where $a < m < b$. There are now four cases to consider.

It could be that

$$(a, m), (m, b) \in U$$

but then $(a, b) \in \mathbf{spl}(U)$ and we are done.

It could be that

$$(a, l), (l, m) \in U \quad (m, b) \in U$$

for some $a < l < m < b$. But now

$$(a, l) \in U \subseteq \mathbf{spl}(U) \quad (l, b) \in \mathbf{spl}(U)$$

to give

$$(a, b) \in \mathbf{lap}(\mathbf{spl}(U))$$

and we are done.

There is a symmetric case where (m, b) is split. This is handled in the same way.

Finally it could be that

$$(a, l), (l, m) \in U \quad (m, r), (r, b) \in U$$

where

$$a < l < m < r < b$$

are the rationals involved. These give

$$(a, m), (l, b) \in \mathbf{spl}(U) \quad (l, r), (r, b) \in \mathbf{spl}(U)$$

so that

$$(a, r) \in \mathbf{lap}(\mathbf{spl}(U)) \quad (l, b) \in \mathbf{lap}(\mathbf{spl}(U))$$

and hence

$$(a, b) \in \mathbf{lap}^2(\mathbf{spl}(U))$$

as required. ■

This result can be used to compare the generated nuclei using the following.

8.58 COROLLARY. *For each $r < \omega$ we have $\mathbf{spl}^{r+1} \leq \mathbf{lap}^{2r} \circ \mathbf{spl}$.*

Proof. We proceed by induction over r . The base case $r = 0$ is trivial. For the induction step $r \mapsto r + 1$ a use of the induction hypothesis and Lemma 8.57 give

$$\mathbf{spl}^{r+2} \leq \mathbf{lap}^{2r} \circ \mathbf{spl}^2 \leq \mathbf{lap}^{2r} \circ \mathbf{lap}^2 \circ \mathbf{spl}$$

which is the required result. ■

I suspect that is more stuff lurking around here.

[Added 14-05-04: Since first writing these notes I have extended the results of the reals and the irrationals presented in this and the previous subsection. The closure ordinal of *real* and *irrt* are $\omega + 1$ and Ω , respectively. Details are given in [9].]

The spectrum of a ring

So far all the examples we have seen have used frames to construct spaces. In this subsection we use quantales to construct a space. To begin let's describe the space in a more standard way, and then we will construct it using coverages on poms.

Let R be a commutative and unital ring. Let

$$\mathbf{spec}R$$

be the set of prime ideals p of R . For each element $a \in R$ we use

$$p \in U(a) \iff a \notin p$$

(for $p \in \mathbf{spec}R$) to produce a subset $U(a)$ of R . We see that

$$U(a) \cap U(b) = U(ab)$$

for $a, b \in R$, and hence

$$\{U(a) \mid a \in R\}$$

is a base for a topology on $\mathbf{spec}R$. This is the **spectrum** of R .

It is essential here that the ring R is commutative. It does not make sense to talk about *the* spectrum of a non-commutative ring. There are several candidates and none of them occupies the central position that $\mathbf{spec}R$ does in the commutative case.

We need to index $\mathcal{O}\mathbf{spec}R$ in a slightly different way.

8.59 DEFINITION. Let $\mathcal{I}R$ be the family of all ideals of R . For each $I \in \mathcal{I}R$ we use

$$p \in \tilde{I} \iff I \not\subseteq p$$

(for $p \in \mathbf{spec}R$ to produce a subset of $\mathbf{spec}R$). ■

In other words

$$\tilde{I} = \bigcup \{U(a) \mid a \in I\}$$

to show that $\tilde{I} \in \mathcal{O}\mathbf{spec}R$. Conversely, for each $a \in R$ we have

$$U(a) = \langle \tilde{a} \rangle$$

where $\langle a \rangle$ is the principal ideal generated by a . This gives the following.

8.60 LEMMA. *The $\mathcal{I}R$ -indexed family*

$$\{\tilde{I} \mid I \in \mathcal{I}R\}$$

is precisely the topology $\mathcal{O}\mathbf{spec}R$.

We can view this assignment $(\tilde{\cdot})$ as a morphism of an appropriate kind.

The family $\mathcal{I}R$ of ideals is partially ordered by inclusion. For each collection \mathcal{J} of ideals the intersection $\bigcap \mathcal{J}$ is an ideal, and hence $\mathcal{I}R$ is complete. However, suprema are not just unions. In fact, for $\mathcal{J} \subseteq \mathcal{I}R$ we have

$$\bigvee \mathcal{J} = \sum \mathcal{J}$$

where $\sum \mathcal{J}$ is the ideal generated by $\bigcup \mathcal{J}$. This is just the closure of $\bigcup \mathcal{J}$ under summation. Given two ideals $I, J \in \mathcal{I}R$ we let IJ be the set of all sums of elements xy for $x \in I$ and $y \in J$, that is the ideal generated by all such pairs. This furnishes $\mathcal{I}R$ as a monoid, and the expected happens.

8.61 LEMMA. *The family $\mathcal{I}R$ is a quantale and the assignment*

$$\begin{array}{ccc} \mathcal{I}R & \longrightarrow & \mathcal{O}\text{spec}R \\ I & \longmapsto & \tilde{I} \end{array}$$

is a quantale morphism (to a frame).

Proof. For the first part, since $\mathcal{I}R$ is commutative, it suffices to show

$$I \left(\sum \mathcal{J} \right) = \sum \{IJ \mid J \in \mathcal{J}\}$$

for $I \in \mathcal{I}R$ and $\mathcal{J} \subseteq \mathcal{I}R$. The left hand side is the set of all

$$x(y_1 + \cdots + y_m)$$

for $x \in \mathcal{I}$ and where each y_i is a member of some $J \in \mathcal{J}$. The right hand side is the set of all sums

$$x_1y_1 + \cdots + x_my_m$$

where each $x_i \in I$ and where each y_i is a member of some $J \in \mathcal{J}$.

Notice that the improper ideal R is both the top and the unit of the quantale $\mathcal{I}R$.

For the morphism property it suffices to show

$$\widetilde{IJ} = \tilde{I} \cap \tilde{J} \quad \widetilde{\sum \mathcal{J}} = \bigcup \{\tilde{J} \mid J \in \mathcal{J}\}$$

for $I, J \in \mathcal{I}$ and $\mathcal{J} \subseteq \mathcal{I}R$. Only the inclusion

$$\widetilde{IJ} \subseteq \tilde{I} \cap \tilde{J}$$

offers much of a contest, and that follows by the prime property of the $p \in \text{spec}R$. ■

This exhibits $\mathcal{O}\text{spec}R$ as a quotient of a (commutative) quantale, and so is determined by a nucleus on that quantale. Which one? It is a standard construction which motivated a lot of the development of quantales, and is known to everyone even those who have never heard of quantales. As expected, since the quotient is concerned with the existence of points (of a frame) there is a choice principle involved.

8.62 LEMMA. (The separation principle) *Let M be a non-empty multiplicatively closed subset of R and let I be an ideal with $M \cap I = \emptyset$. Then*

$$M \cap p = \emptyset \quad I \subseteq p$$

for some $p \in \text{spec}R$.

Proof. The family

$$\{J \in \mathcal{I}R \mid M \cap J = \emptyset\}$$

contains the given ideal I and is closed under directed unions. By Zorn's Lemma the family contains a maximal member p with $I \subseteq p$. It suffices to show that p is prime.

By way of contradiction suppose p is not prime. Thus

$$xy \in p \quad x \notin p \quad y \notin p$$

for some $x, y \in R$. By the maximality of p both the ideals

$$p + \langle x \rangle \quad p + \langle y \rangle$$

meet M , and hence we have

$$a + xr = m \in M \quad b + ys = n \in M$$

for some $a, b \in P, r, s \in R$, and $m, n \in M$. Since M is multiplicatively closed this gives

$$ab + ays + bxr + xyrs = (a + xr)(b + ys) \in M$$

and

$$ab + ays + bxr + xyrs \in p$$

since a part of each component is in p . This is the contradiction. ■

Remember that the radical

$$\sqrt{I}$$

of an ideal I is the set of all $a \in R$ such that $a^m \in I$ for some $m \in \mathbb{N}$. Since R is commutative a use of the binomial expansion shows that \sqrt{I} is an ideal.

8.63 LEMMA. *We have*

$$\sqrt{I} = \bigcap \{p \in \text{spec}R \mid I \subseteq p\}$$

for each ideal $I \in \mathcal{I}R$.

Proof. The inclusion

$$\sqrt{I} \subseteq \bigcap \{p \in \text{spec}R \mid I \subseteq p\}$$

is a consequence of the prime property of the $p \in \text{spec}R$. For the converse consider

$$a \in R - \sqrt{I}$$

so that

$$\{a^m \mid m \in \mathbb{N}\} \cap I = \emptyset$$

where the left hand component of the intersection is multiplicatively closed. A use of Lemma 8.62 now gives some $p \in \text{spec}R$ with $I \subseteq p$ and $a \notin p$. ■

This result can be rephrased as follows.

8.64 THEOREM. *The kernel of the morphism*

$$\begin{array}{ccc} \mathcal{I}R & \longrightarrow & \mathcal{O}\text{spec}R \\ I & \longmapsto & \tilde{I} \end{array}$$

is the radical operation $\sqrt{(\cdot)}$.

Proof. It suffices to show that

$$\tilde{J} \subseteq \tilde{I} \iff J \subseteq \sqrt{I}$$

for all ideals $I, J \in \mathcal{I}R$. This follows by a couple of contrapositive rephrasings. ■

Although it may not be apparent why the following result is relevant, believe me it is when we try to take this example much further.

8.65 LEMMA. If j is any nucleus on $\mathcal{I}R$ for which $(\mathcal{I}R)_j$ is a frame, then $\sqrt{(\cdot)} \leq j$.

Proof. Since the quotient $(\mathcal{I}R)_j$ is a frame, we have

$$j(I) \cap j(J) \subseteq j(IJ)$$

for all $I, J \in \mathcal{I}$. In particular

$$j(I) = j(I^m)$$

for each $I \in \mathcal{I}R$ and $0 \neq m \in \mathbb{N}$. (One inclusion holds since $I^m \subseteq I$.) Thus we have

$$j(\langle a \rangle) = j(\langle a^m \rangle)$$

for each $a \in R$ and $0 \neq m \in \mathbb{N}$.

Now consider any $a \in \sqrt{I}$. Then $a^m \in I$ for some $m \in \mathbb{N}$. If $m = 0$ then $1 \in I$ so that $I = R$ and $a \in R = j(I)$. If $m \neq 0$ then

$$a \in j(\langle a \rangle) = j(\langle a^m \rangle) \subseteq j(I)$$

to give the required result. ■

We will use some pom R (obtained from the ring) and produce quotients

$$\mathcal{L}R \longrightarrow \mathcal{I}R \longrightarrow \mathcal{O}\text{spec}R$$

from the free quantale over R . We then generate both these using posers on the pom.

That's the idea but there is a bit a funny business to get round. In section 1 I said that the use of partially ordered monoids could be generalized to a use of pre-ordered monoids but it wasn't worth the effort of setting up all the general machinery. Well, here is one place where pre-orders are more convenient. I will set out the details in that form, but indicate how the same results can be obtained using a partial order.

We use the multiplicative monoid $(R, \cdot, 1)$ of the ring R .

8.66 DEFINITION. Let \preceq be the comparison on R given by

$$b \preceq a \iff (\exists r \in R)[b = ar]$$

for $a, b \in R$. ■

In other words

$$b \preceq a \iff b \in \langle a \rangle \iff \langle b \rangle \subseteq \langle a \rangle$$

where $\langle a \rangle$ is the principal ideal generated by a . A simple calculation shows that \preceq is a pre-order but, in general, is not a partial order. As usual we may turn the preset (R, \preceq) into a poset by factoring out the equivalence relation \approx given by

$$a \approx b \iff a \preceq b \preceq a \iff \langle a \rangle = \langle b \rangle$$

for $a, b \in R$. This gives the poset of principal ideals of R partially ordered by inclusion. We can lift the monoid structure up to this in an obvious fashion to obtain a pom. However, it is more convenient to let

$$(R, \preceq, \cdot, 1)$$

carry the posers.

A few moment's thought shows that a \preceq -lower section of R is just an R -closed subset. That is a subset $X \subseteq R$ such that $XR = X$ where

$$XR = \{xr \mid x \in X, r \in R\}$$

is the composite set. Let \mathcal{LR} be the family of all such sets (including the empty set). This is partially ordered by inclusion and closed under arbitrary unions, so that

$$(\mathcal{LR}, \subseteq, \bigcup, \emptyset)$$

is a \vee -semilattice. For each $X, Y \in \mathcal{LR}$ we set

$$XY = \{xy \mid x \in X, y \in Y\}$$

to obtain $XY \in \mathcal{LR}$. It is trivial to see that

$$(\mathcal{LR}, \cdot, R)$$

is a monoid. In fact, we have the following.

8.67 LEMMA. *The structure*

$$(\mathcal{LR}, \subseteq, \bigcup, \emptyset, \cdot, R)$$

is a commutative quantale.

It is an easy, but tedious exercise to show that this quantale is canonically isomorphic to the free quantale generated by the pom of principal ideals of R . Thus we use \mathcal{LR} , as constructed above, to carry the various inflators and nuclei that we need.

For each $X \in \mathcal{LR}$ let $\langle X \rangle$ be the ideal generated by X , that is the set of all sums

$$x_1 + \cdots + x_m$$

for $x_1, \dots, x_m \in X$. By convention, taking $m = 0$ gives $0 \in \langle X \rangle$, so $\langle \emptyset \rangle = \langle 0 \rangle = \{0\}$.

8.68 LEMMA. *The assignment $\langle \cdot \rangle$ is a nucleus on \mathcal{LR} .*

Proof. By routine calculations we see that $\langle \cdot \rangle$ is a closure operation on \mathcal{LR} . Furthermore, the distributive law ensures that

$$\langle X \rangle \langle Y \rangle = \langle XY \rangle$$

for $X, Y \in \mathcal{LR}$. ■

This nucleus give a quotient

$$(\mathcal{LR})_{\langle \cdot \rangle}$$

of \mathcal{LR} consisting of all those $X \in \mathcal{LR}$ with $\langle X \rangle = X$. In other words, it is carried by the family \mathcal{IR} of ideals of R . We find that the imposed structure on \mathcal{IR} is just the canonical one. This situation can be described as follows.

8.69 THEOREM. *The assignment*

$$\begin{array}{ccc} \mathcal{L}R & \longrightarrow & \mathcal{I}R \\ X & \longmapsto & \langle X \rangle \end{array}$$

is a quantale quotient with $\langle \cdot \rangle$ as its kernel.

This sets up the general context with two quotients

$$\begin{array}{ccc} \mathcal{L}R & \longrightarrow & \mathcal{I}R & \longrightarrow & \mathcal{O}\text{spec}R \\ X & \longmapsto & \langle X \rangle & & X & \longmapsto & \langle X \rangle \\ X & \longmapsto & \widetilde{\langle X \rangle} & & X & \longmapsto & \sqrt{\langle X \rangle} \end{array}$$

each with a kernel indicated to the right. We generate these using poser on R .

8.70 DEFINITION. Let \Vdash_+ be the poser on R which has two kinds of instances

$$0 \Vdash_+ \emptyset \quad x + y \Vdash_+ \{x, y\}$$

for $x, y \in R$.

For each $x \in \mathcal{L}R$ let

$$a \in \mathbf{add}(X) \iff a \in X \text{ or } a = 0 \text{ or } (\exists x, y \in X)[a = x + y]$$

(for $a \in R$) to produce a subset $\mathbf{add}(X)$ of R . ■

The following is no surprise and easy to prove.

8.71 LEMMA. *For each $X \in \mathcal{L}R$ we have $\mathbf{add}(X) \in \mathcal{L}R$, and \mathbf{add} is the inflator of $\mathcal{L}R$ generated from the poser \Vdash_+ . Furthermore \mathbf{add} is stable.*

Proof. This is a series of simple steps. Let's check that \mathbf{add} is stable. Thus suppose

$$a \in \mathbf{add}(X)Y$$

so that one of

$$a = xy \quad a = 0 \quad a = (x_1 + x_2)y$$

holds where $x, x_1, x_2 \in X$ and $y \in Y$. But then the corresponding one of

$$a = xy \quad a = 0 \quad a = x_1y + x_2y$$

holds, and hence $a \in \mathbf{add}(XY)$, as required. ■

The stable inflator \mathbf{add} need not be a pre-nucleus. Consider the polynomial ring

$$R = \mathbb{Z}[x_1, x_2, y_1, y_2]$$

and let

$$X = \langle x_1 \rangle \cup \langle x_2 \rangle \quad Y = \langle y_1 \rangle \cup \langle y_2 \rangle$$

to obtain $X, Y \in \mathcal{L}R$. Note that

$$XY = \langle x_1y_1 \rangle \cup \langle x_2y_1 \rangle \cup \langle x_1y_2 \rangle \cup \langle x_2y_2 \rangle$$

so that

$$(x_1 + x_2)(y_1 + y_2)$$

is in $\mathbf{add}(X)\mathbf{add}(Y)$ but not in $\mathbf{add}(XY)$.

Since \mathbf{add} has finite character we see that \mathbf{add}^ω is a nucleus. Furthermore, the fixed sets of \mathbf{add} are precisely the ideals. This more or less shows that following.

8.72 LEMMA. For each $X \in \mathcal{L}R$ we have $\mathbf{add}^\omega(X) = \langle X \rangle$.

This shows that the quotient

$$(\mathcal{L}R)_{\Vdash_+}$$

is nothing more than the quantale $\mathcal{I}R$ of ideals of R . To obtain the spectrum of R we need to strengthen the poset \Vdash_+ , but not by very much.

8.73 DEFINITION. Let \Vdash_{spec} be the poset on R which has three kinds of instances

$$0 \Vdash_{\text{spec}} \emptyset \quad x + y \Vdash_{\text{spec}} \{x, y\} \quad x \Vdash_{\text{spec}} \{x^2\}$$

for $x, y \in R$.

For each $X \in \mathcal{L}R$ let

$$a \in \mathbf{spec}(X) \iff a \in \mathbf{add}(X) \text{ or } a^2 \in X$$

(for $a \in R$) to produce a subset $\mathbf{spec}(X)$ of R . ■

This poset is a modest extension of \mathbf{add} . Similarly we can extend Lemma 8.71.

8.74 LEMMA. For each $X \in \mathcal{L}R$ we have $\mathbf{spec}(X) \in \mathcal{L}R$, and \mathbf{spec} is the inflator of $\mathcal{L}R$ generated from the poset \Vdash_{spec} . Furthermore \mathbf{spec} is stable.

Proof. This is a series of simple steps. Let's check that \mathbf{spec} is stable. Thus suppose

$$a \in \mathbf{spec}(X)Y$$

so that either $a \in \mathbf{add}(X)Y$ or $a = xy$ for some $x \in R$ with $x^2 \in X$ and some $y \in Y$. In the first case we have $a \in \mathbf{add}(XY) \subseteq \mathbf{spec}(XY)$ (since \mathbf{add} is stable). In the second case we have

$$a^2 = x^2y^2 \in XY$$

and hence $a \in \mathbf{spec}(XY)$, as required. ■

As with \mathbf{add} , this inflator \mathbf{spec} need not be a pre-nucleus, but in this case an appropriate example needs a little more thought.

Since \mathbf{spec} has finite character we see that \mathbf{spec}^ω is a nucleus. With a little more effort we obtain the following.

8.75 THEOREM. For each $X \in \mathcal{L}R$ we have $\mathbf{spec}^\omega(X) = \sqrt{\langle X \rangle}$.

Proof. We show first that

$$\mathbf{spec}(\sqrt{\langle X \rangle}) \subseteq \sqrt{\langle X \rangle}$$

for each $X \in \mathcal{L}R$. To this end consider any $a \in \mathbf{spec}(\sqrt{\langle X \rangle})$. Then one of

$$a \in \mathbf{add}(\sqrt{\langle X \rangle}) \subseteq \sqrt{\langle X \rangle} \quad a^2 \in \sqrt{\langle X \rangle}$$

holds, and hence $a \in \sqrt{\langle X \rangle}$ in both cases. From this a simple induction gives

$$\mathbf{spec}^r(\sqrt{\langle X \rangle}) \subseteq \sqrt{\langle X \rangle}$$

for each $r < \omega$ and hence

$$\mathbf{spec}^\omega(\sqrt{\langle X \rangle}) \subseteq \sqrt{\langle X \rangle}$$

holds.

For the converse consider $a \in \sqrt{\langle X \rangle}$. Then

$$a^m \in \langle X \rangle = \mathbf{add}^\omega(X)$$

for some $m < \omega$, and hence

$$a^m \in \langle X \rangle = \mathbf{add}^r(X) \subseteq \mathbf{spec}^r(X)$$

for some $r < \omega$. Consider the smallest m for which there is some r such that

$$a^m \in \mathbf{spec}^r(X)$$

holds. If $m \leq 1$ then we are done. (If $m = 0$ then $1 \in \mathbf{spec}^r(X)$ and so $\mathbf{spec}^r(X) = R$.)

If $1 < m$ then, since

$$a^{m+1} \in \mathbf{spec}^r(X)$$

there is some $n < m$ with

$$a^{2n} \in \mathbf{spec}^r(X)$$

and hence

$$a^n \in \mathbf{spec}^{r+1}(X)$$

which contradicts the minimality of m . Thus $m \leq 1$ as required. ■

This shows that the quotient

$$(\mathcal{L}R)_{\|\mathbf{spec}}$$

is nothing more than the topology $\mathcal{O}\mathbf{spec}R$ of the spectrum of R . Furthermore, the proof of this need nothing more than some fairly routine undergraduate mathematics.

9 Historical remarks

[to be written]

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