Examples of the Cantor-Bendixson process on the reals

Harold Simmons

Mathematical Foundations Group, The University, Manchester, England
hsimmons @ manchester.ac.uk

A few months ago I was introducing some research students to the mysteries and delights or ordinal numbers, particularly ordinal iteration. To do that I was using the Cantor-Bendixson rank of closed subsets of various simple spaces. I needed some examoples to show that in the reals $\mathbb{R}$ these ranks exhaust the countable ordinals. I realized that I didn't have any such examples to hand. (It is some time since I had the pleasure of teaching any of this stuff.) After rummaging around the literature for a while I couldn't find what I was looking for, so I decided to sit down and sort out some examples for myself. (I admit I didn't rummage for very long, hours rather than days. So may be exactly what I was looking for is out there somewhere. Anyway, doing the work myself probably did me more good than taking a job lot off the shelf.)

This note is the result of that exercise. There is nothing very novel here, but it might be useful to some of you out there. I you find these notes useful, do let me know. I might be persuaded to expand them if enough people send me money. Also, if there are other collections of simple examples that I should have cited, again let me know.

To set the scene let's recall some of the well known background without going overboard on the generality.

Let $\mathbb{R}$ be the real numbers viewed as a topological space (with the metric topology). A subset $U \subseteq \mathbb{R}$ is open if it is the union of intervals (l, r) with rational end points l, r. A subset $X \subseteq \mathbb{R}$ is closed if its complement $X' = \mathbb{R} - X$ is open. Equivalently, a subset X is closedif for each cauchy sequence taken from X, the limit of that sequence is also in X.

Some causch sequencies are easier to handle thatn others. Let us say a cauchy sequence is docile if it is eventually constant, otherwise it is frisky. The limit of a docile sequence is its eventual value. The limit of a frisky sequence may not be so obvious.

Clearly, each point x of a closed set X is the limit of a docile sequence taken from that set (for example, the obvious constant sequence). There may be some points $x \in X$ which can not be the limit of any frisky sequence taken from X. Let's 'delimit' these points in a rather general fashion.

0.1 DEFINITION. Let X be a close subset of the reals $\mathbb{R}$.

A point $x \in X$ is isolated in X if

$$X \cap U = \{x\}$$

for some open set U.

Let

$$X^\bullet = \boldsymbol{lim}(X)$$

be the set of non-isolated points of X. These are the limit points of X. ■

For instance the whole space $\mathbb{R}$ has no isolated points, and neither does the empty set $\emptyset$, but for different reasons. The integers $\mathbb{Z}$ are closed in $\mathbb{R}$ and each point is isolated. More generally, a point x is the limit of some frisky sequence taken from X precisely when $x \in \boldsymbol{lim}(X)$. Thus it is worth taking a closer look at the operation $\boldsymbol{lim}(\cdot)$.

It is an easy exercise to show that $\boldsymbol{lim}(X)$ is closed and

$$\boldsymbol{lim}(X) \subseteq X \qquad \boldsymbol{lim}(X \cup Y) = \boldsymbol{lim}(X) \cup \boldsymbol{lim}(Y)$$

hold for each closed sets X, Y. This operation $\boldsymbol{lim}$ on closed sets is called the Cantor-Bendixson derivative on $\mathbb{R}$, or the CB-derivative for short. (An analogous operation can be set up on any topological space by exactly the same method.)

A closed set X is perfect if $\boldsymbol{lim}(X) = X$, that if is X has no isolated points. For instance both $\mathbb{R}$ and $\emptyset$ are perfect (for different reasons). The step $X \longmapsto \boldsymbol{lim}(X)$ is a first attempt at converting X into a perfect set. Unfortunately, the subset $\boldsymbol{lim}(X)$ need not be perfect. For instance consider the set X of all fractions

$$\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{r}{r+1}, \dots$$

together with the limit 1 of this frisky sequence. This set X is closed with

$$\boldsymbol{lim}(X) = \{1\}$$

but this is not perfect. However

$$\boldsymbol{lim}^2(X) = \boldsymbol{lim}(\boldsymbol{lim}(X)) = \boldsymbol{lim}(\{1\}) = \emptyset$$

which is perfect.

For a more general example, consider any ascending chain of rationals

$$a(0) < a(1) < a(2) < \cdots < a(i) < \cdots \quad (i \in \mathbb{N})$$

with

$$a(i) \longrightarrow a \quad \text{as } i \longrightarrow \infty$$

where a is any previously selected rational. The set

$$X = \{a(i) \mid i \in \mathbb{N}\} \cup \{a\}$$

is closed with

$$\boldsymbol{lim}(X) = \{a\}$$

and hence $\boldsymbol{lim}^2(X) = \emptyset$.

For any such sequence $a(\cdot)$ we may interpolate at each step an ascending chain of rationals

$$a(i-1) < b(i,0) < b(i,1) < b(i,2) < \cdots < b(i,j) < \cdots < a(i) \quad (j \in \mathbb{N})$$

with

$$b(i,j) \longrightarrow a(i) \quad \text{as } j \longrightarrow \infty$$

for each $i \in \mathbb{N}$. (For the case $i = 0$ we let $a(-1)$ be any sufficiently small rational.) For each $i \in \mathbb{N}$ the set

$$Y(i) = \{b(i,j) \mid j \in \mathbb{N}\} \cup \{a(i)\}$$

is closed with $\boldsymbol{lim}(Y(i)) = \{a(i)\}$. We may then check that the set

$$Y = \{b(i,j) \mid i, j \in \mathbb{N}\} \cup X$$

is closed with

$$\boldsymbol{lim}(Y) = X$$

and hence $\boldsymbol{lim}^2(Y) = \{a\}$ with $\boldsymbol{lim}^3(Y) = \emptyset$.

By continuing in this manner we can produce a closed set Z with

$$\boldsymbol{lim}^r(Z) = \{a\}$$

where $r \in \mathbb{N}$ and $a \in \mathbb{Q}$ are given at the outset. The aim of this note is to extend this kind of example into the transfinite, that is with r replaced by a countable ordinal.

0.2 DEFINITION. For each closed set $X \subseteq \mathbb{R}$ we let

$$X^{(0)} = X \quad X^{(\alpha+1)} = X^{(\alpha)\bullet} = \boldsymbol{lim}(X^{(\alpha)} \quad X^{(\lambda)} = \bigcap\{X^{(\alpha)} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. In other words we set

$$X^{(\alpha)} = \boldsymbol{lim}^\alpha(X)$$

using the ordinal iterates of $\boldsymbol{lim}$. ■

This attaches to each closed set a descending chain

$$X = X^{(0)} \supseteq X^{(1)} \supseteq \cdots \supseteq X^{(\alpha)} \supseteq \cdots \quad (\alpha \in \mathbb{O}\mathrm{rd})$$

of closed sets.

On cardinality grounds there is at least one ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$. In fact, once such an ordinal is achieved all larger ordinal have this property. The smallest such ordinal $\rho = \rho(X)$ is the CB-rank of X, and then $X^{(\rho)}$ is the perfect part of X. A closed set X is scattered if $X^{(\rho)} = \emptyset$. Above we saw examples of scattered sets of ranks $0, 1, 2, 3$, and we indicated how examples of each finite rank can be achieved. We investigate what can happen in the transfinite.

How big can the rank of a closed set be? More specifically, how big can the rank of a scattered set be?

At least one point is removed at each step

$$X^{(\alpha)} \mapsto X^{(\alpha+1)}$$

of the associated chain of a closed set X. Thus the rank of X must be strictly smaller than the next cardinal after $2^{\aleph_0}$, the cardinality of $\mathbb{R}$. However, we can do a lot better than that.

At each step some interval (l, r) with rational end points is used to remove a point. Furthermore, each such interval can be used no more than once in the whole process. Thus the rank of X must be countable. (Technically, we have just remembered that $\mathbb{R}$ is second countable.)

Can we lower the bound even further? No! To show that we prove the following.

0.3 THEOREM. *For each $b \in \mathbb{Q}$ and countable ordinal α, there is a closed set X with $X \subseteq \mathbb{Q}$ and $X^{(\alpha)} = \{b\}$.*

In particular, for each countable ordinal α there is a scattered set of rank $\alpha + 1$.

The proof of Theorem 0.3 will take a little while. In fact, we prove a more specific result of which Theorem 0.3 is an immediate consequence. To state that result we need some notation.

0.4 DEFINITION. Let $a < b$ be rationals.

We write

$$X \sqsubset (a, b]$$

if X is a closed subset of $(a, b]$ with $b \in X$.

For each ordinal α we write

$$X \sqsubset^{(\alpha)} (a, b]$$

if $X \sqsubset (a, b]$ with $X^{(\alpha)} = \{b\}$. ■

By definition, if $X \sqsubset^{(\alpha)} (a, b]$ then

$$X^{(\alpha)} = \{b\} \qquad X^{(\alpha+1)} = \emptyset$$

so that X is scattered with rank $\alpha + 1$. Theorem 0.3 is a consequence of the following.

0.5 THEOREM. *For each pair $a < b$ of rationals and for each countable ordinal α we have*

$$X \sqsubset^{(\alpha)} (a, b]$$

for some set X.

We prove this by describing an algorithm which, when supplied with the input data a, b, α will return an appropriate example X. The algorithm proceeds by recursion on α. Thus to generate a set X for some larger ordinal α the algorithm calls itself for certain smaller ordinals $\beta < \alpha$. There is a slight subtlety in that the parameters a and b are allowed to vary at the recursion calls.

Of course, as well as describing the algorithm we must also justify it. That is, we must prove that it does what we claim it does. That justification proceeds by induction on α.

The algorithm repeatedly uses one construction over and over again. Here is that construction.

0.6 CONSTRUCTION. Let $a < b$ be any pair of rationals and let

$$a < a(0) < b(0) < a(1) < \cdots < a(i) < b(i) < a(i+1) < \cdots < b \quad (i \in \mathbb{N}) \tag{1}$$

be any pair of interlacing rational sequences with

$$b(i) \longrightarrow b \quad \text{as } i \longrightarrow \infty$$

(and hence $a(i) \longrightarrow b$ as $i \longrightarrow \infty$). Suppose we have

$$X(i) \sqsubset (a(i), b(i)]$$

for each index $i \in \mathbb{N}$.

For this data let

$$X = \bigcup \{X(i) \mid i \in \mathbb{N}\} \cup \{b\} \tag{2}$$

to obtain $X \subseteq (a, b]$ with $b \in X$ and with

$$X \cap (a(i), b(i)] = X(i)$$

for each index i. We refer to X as the canonical set constructed from the data. ■

In this construction each component $X(i)$ is a closed set, and different components do not 'interfere' with each other. After a closer look we obtain the following.

0.7 LEMMA. *Construction 0.6 produces a closed set X.*

Proof. We need some information about the possible positions of the members of X.

Apart for b each member of X belongs to a component $X(i) \sqsubset (a(i), b(i)]$ for some unique index $i \in \mathbb{N}$. Let

$$l(i) = \inf X(i)$$

so that $l(i) \in X(i)$ (since $X(i)$ is closed) and $X(i) \subseteq [l(i), b(i)]$. Note that both

$$X \cap (a(i), l(i)) = \emptyset = X \cap (b(i), a(i+1)) \tag{3}$$

hold. Also we have

$$X \cap [a, a(0)] = \emptyset$$

at the lower end.

Consider any $y \in X^-$. We require $y \in X$. We have

$$a(0) \leq y \leq b$$

since $X \subseteq (a(0), b]$. Also if $y = b$ then we are done, so we may assume $y < b$.

Since $a(i) \longrightarrow b$ as $i \longrightarrow \infty$ we have

$$a(i) \leq y < a(i+1)$$

for some index i. We show that $y \in X(i)$.

We first get a better approximation of the position of y. Since $y \in X^-$, each open neighbourhood of y must meet X. Thus

$$y \notin (a(i), l(i)) \qquad y \notin (b(i), a(i+1))$$

by (3). Similarly, by considerring a small interval around $a(i)$ we see that $y \neq a(i)$. Thus we have

$$l(i) \leq y \leq b(i)$$

for this index.

If $y = l(i)$ or $y = b(i)$ then we are done (since both of these are in $X(i)$). Thus we may suppose

$$l(i) < y < b(i)$$

and use this to show $y \in X(i)^- = X(i)$.

Consider any open set U with $y \in U$. We show that $X(i)$ meets U.

We have

$$y \in (l(i), b(i)) \cap U$$

and this open set meet X (since $y \in X^-$). But

$$X \cap (l(i), b(i)) \cap U \subseteq X(i) \cap U$$

which gives the required result. ■

Let's look at a template for the proof of Theorem 0.5. For this first pass we give a full description of the algorithm, but there are a couple of gaps in the justification. These are indicated by [GAP], and will be filled in later.

Proof of Theorem 0.5.

For the base case, $\alpha = 0$, we set

$$X = \{b\}$$

and there is nothing left to do.

For the recursion step, $\alpha \mapsto \alpha+1$, we first select an interlacing chain (1) with $b(i) \longrightarrow b$ as $i \longrightarrow \infty$. By recursion we may apply the algorithm to each interval $(a(i), b(i)]$ to produce a set

$$X(i) \sqsubset^{(\alpha)} (a(i), b(i)]$$

for each index $i \in \mathbb{N}$. Using this we take the canonical set X as in (2).

[GAP] We may check that

$$X^{(\alpha)} = \bigcup\{X(i)^{(\alpha)} \mid i \in \mathbb{N}\} \cup \{b\}$$

holds.

For each $i \in \mathbb{N}$ we have

$$X(i)^{(\alpha)} = \{b(i)\}$$

so that

$$X^{(\alpha)} = \bigcup\{b(i) \mid i \in \mathbb{N}\} \cup \{b\}$$

and hence

$$X^{(\alpha+1)} = \{b\}$$

as required.

For the recursion leap to a countable limit ordinal λ we first select an ascending chain

$$\alpha(0) < \alpha(1) < \cdots < \alpha(i) < \cdots \quad (i \in \mathbb{N})$$

of ordinals with limit λ. We also select an interlacing chain (1) with $b(i) \longrightarrow b$ as $i \longrightarrow \infty$. By recursion we may apply the algorithm to each interval $(a(i), b(i)]$ to produce a set

$$X(i) \sqsubset^{(\alpha(i))} (a(i), b(i)]$$

for each index $i \in \mathbb{N}$. Using this we take the canonical set X as in (2).

[GAP] We may check that

$$X^{(\lambda)} = \bigcup\{X(i)^{(\lambda)} \mid i \in \mathbb{N}\} \cup \{b\}$$

holds.

For each $i \in \mathbb{N}$ we have

$$X(i)^{(\alpha(i))} = \{b(i)\}$$

so that

$$X(i)^{(\lambda)} \subseteq X(i)^{(\alpha(i)+1)} = \emptyset$$

and hence

$$X^{(\lambda)} = \{b\}$$

as required. ■

This proof is not yet complete for, as indicated, it contains a couple of gaps. These gaps can be filled in separately, and it is instructive to try this. The first gap, in the step case, is fairly straight forward although it does require a little biot of organization to deal with the various possibilities that arise. However, when we try to fill the gap in the leap case we find that we need to prove something more general. This result also deals with the step case.

0.8 LEMMA. *Suppose*

$$\alpha(0) \leq \alpha(1) \leq \cdots \leq \alpha(i) \leq \cdots \quad (i \in \mathbb{N})$$

is an ascending chain of ordinals with supremum α.

In Construction 0.6 suppose

$$X(i) \sqsubset^{(\alpha(i))} (a(i), b(i)]$$

for each $i \in \mathbb{N}$.

Under these conditions we have

$$X^{(\beta)} = \bigcup\{X(i)^{(\beta)} \mid i \in \mathbb{N}\} \cup \{b\}$$

for each ordinal $\beta \leq \alpha$.

Proof. We proceed by induction on β.

The base case, $\beta = 0$, is immediate (by the construction of X).

The induction step, $\beta \mapsto \beta + 1$, is the heart of the proof. Let

$$Y = \bigcup\{X(i)^{(\beta)} \mid i \in \mathbb{N}\} \cup \{b\} \qquad Z = \bigcup\{X(i)^{(\beta+1)} \mid i \in \mathbb{N}\} \cup \{b\}$$

so the induction hypothesis is

$$X^{(\beta)} = Y \cup \{b\}$$

and we must show that $X^{(\beta+1)} = Z \cup \{b\}$. Of course, we assume $\beta + 1 \leq \alpha$.

Since $\beta < \alpha$ we have $\beta \leq \alpha(i)$ for almost all $i \in \mathbb{N}$. Thus

$$b(i) \in X(i)^{(\alpha(i))} \subseteq X(i)^{(\beta)} \subseteq X^{(\beta)}$$

for almost all i. Since $b(i) \longrightarrow b$ this gives $b \in X^{(\beta)\bullet} = X^{(\beta+1)}$. In particular, we have

$$Z \cup \{b\} \subseteq X^{(\beta+1)}$$

and it remains to prove the converse inclusion.

Consider any $y \in X^{(\beta+1)}$. If $y = b$ then we are done. Thus we may suppose

$$y \in X^{(\beta+1)} - \{b\} \subseteq X^{(\beta)} - \{b\} \subseteq Y$$

and we require $y \in Z$. Since $y \in Y$ we have

$$y \in X(k)^{(\beta)}$$

for some $k \in \mathbb{N}$. We show that $y \in X(k)^{(\beta+1)}$.

From the position of $X(k)$ we have either

$$a(k) < y < b(k) \quad \text{or} \quad y = b(k)$$

and we deal with the alternatives separately.

For the first altternative suppose

$$y \in U(k) = (a(k), b(k))$$

and, by way of contradiction, suppose $y \notin X(k)^{(\beta+1)}$. Since $y \in X(k)^{(\beta)}$ this means that y is isolated in $X(k)^{(\beta)}$, and hence

$$X(k)^{(\beta)} \cap V = \{y\}$$

for some open set V. But now

$$y \in X^{(\beta)} \cap U(k) \cap V \subseteq X(k)^{(\beta)} \cap V = \{y\}$$

to show that y is isolated in $X^{(\beta)}$. This is not so since $y \in X^{(\beta+1)}$.

For the first altternative suppose $y = b(k)$. We consider the relationship between $\alpha(k)$ and β.

If $\alpha(k) < \beta$ then

$$y \in X(k)^{(\beta)} \subseteq X(k)^{(\alpha(k)+1)} = \emptyset$$

which can not be.

Suppose $\beta = \alpha(k)$. Let J be the set of all indexes $j \in \mathbb{N}$ with $\alpha(j) = \beta$. Thus $k \in J \subseteq \mathbb{N}$. For each $j \in J$ we have

$$X(j)^{(\beta)} = \{b(j)\}$$

and hence

$$Y = \{b(j) \,|\, j \in J\} \cup \{X(i)^{(\beta)} \,|\, i \in \mathbb{N} - J\}$$

where some of the right hand components may be empty). This decomposition shows that each $b(j)$ is an isolated point of $X^{(\beta)}$. In particular, $y = b(k) \notin X^{(\beta+1)}$, which is not so.

This shows that $\beta < \alpha(k)$, so that

$$y = b(k) \in X(k)^{(\alpha(k)} \subseteq X(k)^{(\beta+1)}$$

as required.

This completes the induction step.

Finally we make the induction leap to a limit ordinal $\lambda \leq \alpha$. We have

$$X^{(\lambda)} = \bigcap \{X^{(\beta)} \,|\, \beta < \lambda\}$$

and, by the induction hypothesis, we have

$$X^{(\beta)} = \bigcup \{X(i)^{(\beta)} \mid i \in \mathbb{N}\} \cup \{b\}$$

for each $\beta < \lambda$. In particular

$$X(i)^{(\lambda)} \subseteq X(i)^{(\beta)} \subseteq X^{(\beta)} \qquad b \in X^{(\beta)}$$

for each $i \in \mathbb{N}$ and $\beta < \lambda$. This gives

$$X(i)^{(\lambda)} = \bigcap \{X(i)^{(\beta)} \mid \beta < \lambda\} \subseteq X^{(\lambda)}$$

and so an inclusion

$$X^{(\lambda)} \subseteq \bigcup \{X(i)^{(\lambda)} \mid i \in \mathbb{N}\} \cup \{b\}$$

is required.

Consider any $y \in X^{(\lambda)}$. If $y = b$ then we are done. Thus we may suppose $y \neq b$, and hence $y \in X^{(\beta)}$ for each $\beta < \lambda$. This with the induction hypothesis shows that

$$(\forall \beta < \lambda)(\exists i \in \mathbb{N})[y \in X(i)^{(\beta)}]$$

holds. But the components $X(i)$ are pairwise disjoint so, in fact,

$$(\exists i \in \mathbb{N})(\forall \beta < \lambda)[y \in X(i)^{(\beta)}]$$

holds. With this index i we have

$$y \in \bigcap \{X(i)^{(\beta)} \mid \beta < \lambda\} = X(i)^{(\lambda)}$$

as required.

This completes the whole proof. ■

In fact, this completes the proof of Theorem 0.5, and hence of Theorem 0.3.

You may be wondering why at certain places I have insisted that some real numbers are actually rational. For instance, in Construction 0.6 there seems to be no good reason that the interlacing sequences should be rational.

The import of Theorem 0.5 is that there are some rather complicated closed sets of real numbers. That's to be expected, you maight say, for there are some rather complicated real numbers. However, Theorem 0.3 shows that exactly the same kind of complicated behaviour arises even when we stay within the rationals. That makes the arrationality of $\sqrt{2}$ look a bit dull, doesn't it?