The extended Cantor-Bendixson analysis of trees

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Abstract

The Hausdorff analysis of chains is an instance of the Cantor-Bendixson analysis of topological spaces. Using the methods of point-free topology I obtain a considerable extension of Hausdorff's methods applicable to all trees.

Contents

1 Introduction

Depending on your interests, there are different reasons to look at this paper.

Suppose you have combinatorial interests. You will know that Hausdorff instigated a structure theory of chains (linearly ordered sets), and this has been developed quite extensively since then. The book [4] is a fairly comprehensive account of these methods (as they then stood). Some of these methods can be applied to posets and lattices. The book [3] describes some of these methods with an eye on applications to lattices of submodules. A brief survey of the standard Hausdorff analysis as required here is given at the end of this section. For the time being we remember that a chain is scattered if it does not include a copy of the rationals $\mathbb{Q}$ (as a chain).

In this paper I extend the Hausdorff method to give an analysis of (non-well founded) trees, that is posets in which the set of predecessors of any element is linearly ordered. Furthermore, I show how these results are related to point-free topology.

The combinatorial analysis is developed in Sections 4 and 5. With a bit of give and take it is possible to read these without knowing much about the background given in Sections 2 and 3, but then the wider picture will be missed.

What is this wider picture?

Let ***Top*** be the category of topological spaces and continuous maps. This is connected to another, more algebraic, category ***Frm***. A a frame is a complete lattice of a certain kind. These are the objects of ***Frm***, and the arrows are the appropriate morphisms. (A frame is nothing more than a complete heyting algebra, but this description suggests the wrong morphisms for ***Frm***.) For each topological space S the topology $\mathcal{O}S$ is a frame. For each continuous map ϕ the inverse image function $\phi^{\leftarrow}$ on open sets is a frame morphism. This sets up a contravariant functor ***Top*** $\longrightarrow$ ***Frm***. (This functor is one half of a contravariant adjunction between the two categories. The other functor ***Frm*** $\longrightarrow$ ***Top***, the point-space construction, is not needed here.)

In some respects the category ***Frm*** is nicer that ***Top***. It contains many objects that are not space-like, it can be analysed by algebraic methods, and it provides tools that are not so obvious from a ***Top*** perspective. In this paper I use one such tool, an extended version of the Cantor-Bendixson process.

The category ***CBA*** of complete boolean algebras and complete morphisms sits inside ***Frm***. We may ask when a frame H is reflective, that is reflected into ***CBA***. Trivially, each complete boolean algebra is reflective. Almost trivially, each finite frame is reflective. However, since the free complete boolean algebra on infinitely many generators doesn't exist, there are frames that are not reflective. So what it is that makes a frame reflective?

Each frame H has an embedding into an associated frame NH

$$H \longrightarrow NA$$

its assembly. The members of NH are essentially the congruences on H, but viewed in a different way. They are certain functions on H called nuclei.

The construction $N(\cdot)$ is functorial, and can be iterated along the ordinals to produce the assembly tower

$$H \longrightarrow NH \longrightarrow N^2H \longrightarrow \cdots \longrightarrow N^\alpha H \longrightarrow \cdots$$

of H. This can continue into the transfinite and, in some cases, never stops. In particular, the frame H is reflective precisely when its assembly tower stabilizes, and then the eventual frame $N^\theta H$ is the boolean reflection. The height θ of this tower gives us some information about the discreteness properties of H. In this paper we don't need to look at the tower beyond level 2.

What has this got to do with spaces and trees, the CB-process, and the Hausdorff analysis of chains? To get an idea of the answer let's look at some of the known results.

Suppose S is a T_0 space. The topology $\mathcal{O}S$ is boolean precisely when S is discrete. The assembly $N\mathcal{O}S$ is boolean precisely when S is scattered, which is why the CB process is relevant. I will extend the CB process to investigate $N^2\mathcal{O}S$ for some amenable spaces.

Suppose S is a poset, and let $\mathcal{O}S$ be its Alexandroff topology of all upper section of S. The topology $\mathcal{O}S$ is boolean precisely when S is discrete (that is when the carried comparison is just equality). The assembly $N\mathcal{O}S$ is boolean precisely when S is scattered, that is when each non-empty lower section of S has a maximal member. What properties ensure that $N^2\mathcal{O}S$ is boolean? I don't know the answer to that for an arbitrary poset.

Suppose S is a chain. The topology $\mathcal{O}S$ (of final sections) is boolean precisely when S is a singleton or empty. The assembly $N\mathcal{O}S$ is boolean precisely when S is scattered. Remember that S is scattered as a space precisely when it does not include a copy of $\mathbb{Q}$, that is it is scattered as a chain.

(You may suspect that for a space S, if $\mathcal{O}S$ is reflective then the reflection is just the power set of S. This is not the case. In general $N\mathcal{O}S$ is not a topology. Even when it is a topology, it need not be the power set of S; it is the topology of the front-space of S. A frame having a boolean first assembly is an analogue of a module being semiartinian, and we find that the Cantor-Bendixson rank of a space corresponds to the Loewy length of a module. A frame having a boolean second assembly is a kind of analogue of a module having a Gabriel rank. However, the process of assigning such a rank to a space does not seem to be a common topological activity.)

In this paper I analyse the situation somewhere between the poset and the chain case.

Suppose S is a tree with its final section topology $\mathcal{O}S$. A consequence of the analysis can be stated as follows. (This result was first stated as Theorem 8.1 of [7], but a proof has never appeared until now.)

> *For a tree* S *the second assembly* $N^2\mathcal{O}S$ *is boolean precisely when* S *does not include a copy of the rational* $\mathbb{Q}$ *and does not include a copy of the Cantor tree* $\mathbb{C}$*. The frame* $\mathcal{O}S$ *is reflective precisely when* $N^2\mathcal{O}S$ *is boolean.*

In other words, for a tree S the assembly tower of $\mathcal{O}S$ either stops at $N^2\mathcal{O}S$ or continues for ever. Note also, that for a chain S, if $N^2\mathcal{O}S$ is boolean then, in fact, $N\mathcal{O}S$ is boolean.

The technique is to lift the standard CB process on $\mathcal{O}S$ up to $N\mathcal{O}S$. This can be seen as an extension of the Hausdorff analysis.

Section 2 contains the frame-theoretic background. This is not a comprehensive account, but enough is given to understand the more general picture.

Section 3 first gives a point-free version of the standard CB process. This is then lifted up a level to obtain the extended process. Definition 3.10 gives a succinct description of the topological content of the extended process. We could take this as the starting point of the combinatorial analysis, but then it would seem a little ad hoc.

Section 4 begin this analysis by restricting the attention to posets. This could be done without mentioning frames or topologies, but again it would then seem ad hoc.

Section 5 refines this analysis so that it is applicable to trees. The technique is to study a relation

$$\mathcal{G} \vdash (\lhd, \lambda)$$

where $\mathcal{G}$ is a certain kind of family of lower sections of the parent tree, $\lhd$ is a 'separator' on the tree, and λ is an ordinal (usually a limit ordinal). I set up a list of five rules for manipulating the relation, and these provide a calculus through which we can measure certain combinatorial properties of the parent tree.

In the remainder of this section I will, as promised, set down the background information about the structure theory of chains.

By definition, a chain A is a linear ordered set. As usual we write $<$ and $\leq$ for the strict and the unstrict comparison. To analyse A we look for copies of various test chains that are included in A. Such a copy need not be a convex part of A, so we may need to iteratively collapse various intervals to a point before the copy becomes visible.

1.1 DEFINITION. A separator on a chain A is a relation $\lhd$ such that

$$a \lhd b \Longrightarrow a < b \qquad x \leq a \lhd b \leq y \Longrightarrow x \lhd y$$

hold for all $x, y, a, b \in A$. ■

The strict comparison is the largest separator on A, whereas the empty relation is the smallest separator. We look at how we may pass from one to the other.

The test chains we use are the rationals $\mathbb{Q}$ and the countable ordinals. Let $\Omega = \omega_1$ be the least uncountable ordinal. We search for, possibly non-convex, copies of ordinals $\theta < \Omega$ in a chain.

1.2 DEFINITION. Let A be a chain and let $\lhd$ be a separator on A.

(a) For an ordinal $\theta < \Omega$, we say an interval $[a, b]$ of A $\lhd$-includes θ if there is an indexed family $(x(i) \mid i < \theta)$ of elements such that

$$a \le x(i) \lhd x(j) \le b$$

for all $i < j < \theta$.

(b) We set

$$a \lhd' b \iff a < b \text{ and the interval } [a,b] \ \lhd\text{-includes each } \theta < \Omega$$

(for $a, b \in A$) to obtain a separator $\lhd'$ on A. ■

By considering $2 < \Omega$ we see that

$$a \lhd' b \implies a \lhd b$$

holds, and hence $\lhd'$ is smaller than $\lhd$. However, $\lhd$ can be stable, that is

$$a \lhd' b \iff a \lhd b$$

can hold for all $a, b \in A$. For instance, on $\mathbb{Q}$ we see that $<$ is stable. On each countable ordinal (viewed as a chain) the comparison $<$ is not stable since, in fact, $<'$ is empty. On $\Omega + 1$ we have

$$a <' b \iff a < b = \Omega$$

and then $<''$ is empty.

1.3 DEFINITION. For each chain A we generate the ordinal indexed family $<^\alpha$ of separators by

$$x <^0 y \iff x < y \qquad x <^{\alpha+1} y \iff x \, (<^\alpha)' \, y \qquad x <^\lambda y \iff (\forall \alpha < \lambda)[x <^\alpha y]$$

for each ordinal α, limit ordinal λ, and $x, y \in A$. (A simple calculation shows that $<^\lambda$ is a separator.) ■

On cardinality grounds there is some ordinal ∞ such that $<^\infty$ is stable, and then $<^\infty$ is the largest stable separator on A. Let $\sigma(A)$ be the smallest possible value of ∞. We may call $\sigma(A)$ the stable length of A.

We have seen that $\sigma(A)$ can be quite small, since $\sigma(\mathbb{Q}) = 0$. For each $1 < \theta < \Omega$ we have $\sigma(\theta) = 1$, and $\sigma(\Omega + 1) = 2$. By taking suitably larger ordinals we see that $\sigma(A)$ can take arbitrarily large ordinal values.

As $\mathbb{Q}$ shows, the largest stable separator of a chain need not be empty. As each ordinal shows, this largest stable separator can be empty. We say a chain A is scattered if it does not include a copy of $\mathbb{Q}$.

1.4 PROPOSITION. *A chain A is scattered if and only if $<^{\sigma(A)}$ is empty.*

We need a simple consequence of this.

Suppose the chain is scattered, and so does not include a copy of $\mathbb{Q}$. Then

$$a <^\alpha b \implies \alpha < \sigma(A)$$

holds for each ordinal α and $a, b \in A$. Eventually, in Section 5, we will extend this rank $\sigma(\cdot)$ to certain non-linear posets.

2 Frame-theoretic background

In this section we gather together the required background from point-free topology. There will be no proofs, for most of these can be found in [2].

2.1 DEFINITION. A frame $(H, \wedge, \top, \bigvee, \bot)$ is a complete lattice, with distinguished attributes as indicated, such that

$$a \wedge \bigvee X = \bigvee \{a \wedge x \,|\, x \in X\}$$

for all $a \in H$ and $X \subseteq H$. A frame morphism (between frames H and K)

$$H \xrightarrow{\quad f \quad} K$$

is a function f which respects the distinguished attributes. ■

Frames and frame morphisms form the category ***Frm*** of frames. It is a simple exercise to show that each frame morphism f has a right adjoint g

$$H \xrightarrow{\quad f \quad} K \qquad H \xleftarrow{\quad g \quad} K$$

given by

$$f(x) \leq y \iff x \leq g(y)$$

for $x \in H$ and $y \in K$. This adjoint g is not a frame morphism.

The category has some nice and some unusual features. As expected, monics are just the injective morphisms, and surjective morphisms are epic but not conversely. A bi-morphism, a mono-epic, need not be an isomorphism. In fact, there are some frames with arbitrarily large bi-morphic extensions. We will see some of these shortly.

The initial interest in ***Frm*** arose because of its connection with topological spaces.

Let ***Top*** be the category of topological spaces and continuous maps. For each space S the topology $\mathcal{O}S$ of open sets is a frame, and for each continuous map ϕ, as on the left,

$$T \xrightarrow{\quad \phi \quad} S \qquad \mathcal{O}S \xrightarrow{\quad \phi^{\leftarrow} \quad} \mathcal{O}T$$

the inverse image function $\phi^{\leftarrow}$, as on the right, is a frame morphism. This sets up a contravariant functor from ***Top*** to ***Frm***. There is an adjoint to this, the point space functor, but we don't need that here. We use topologies to provide concrete examples of frames (but these are by no means the only examples).

Each frame H carries a (binary) implication operation $\cdot \supset \cdot$ given by

$$x \leq (a \supset b) \iff x \wedge a \leq b$$

(for $x, a, b \in H$). This has all the usual properties of intuitionistic implication. Setting

$$\neg a = a \supset \bot$$

gives the negation on H. This is a pseudo-complement. In particular

$$\neg\neg a \neq a \qquad a \vee \neg a \neq \top$$

can hold. An element $a \in H$ is complemented if it there is some $b \in H$ such that

$$a \wedge b = \bot \qquad a \vee b = \top$$

holds. Thus a is complemented if and only if $\neg\neg a = a$, and then $\neg a$ is its complement. A frame is boolean when every element is complemented. In the usual way the set $H_{\neg\neg}$ of complemented elements can be structured as a complete lattice. This turns out to be a boolean frame with a frame morphism $H \longrightarrow H_{\neg\neg}$ given by $a \longmapsto \neg\neg a$.

For a subset A of a topological space S we write A', A^-, A° for the complement, the closure, and the interior of A, respectively. Notice that for $U, V \in \mathcal{O}S$

$$(U \supset V) = (U' \cup V)^\circ \qquad \neg U = U^{-\prime} \qquad \neg\neg U = U^{-\circ}$$

give the implication, negation, and double negation on $\mathcal{O}S$. In particular, $\mathcal{O}S_{\neg\neg}$ is just the boolean algebra of regular open sets.

2.2 DEFINITION. A **derivative** on a frame H is a function $f : H \longrightarrow H$ which is inflationary and monotone, that is

$$x \leq f(x) \leq f(y)$$

for all $x, y \in H$ with $x \leq y$.

A **closure operation** on H is a derivative f which is idempotent, that is if $f^2 = f$ (where f^2 is $f \circ f$).

A **pre-nucleus** on H is a derivative f which satisfies

$$f(x \wedge y) = f(x) \wedge f(y)$$

for all $x, y \in H$.

A **nucleus** on H is an idempotent pre-nucleus. ■

The nuclei on a frame H are the important gadgets. On the whole the derivatives, pre-nuclei, and closure operations are gadgets which help us to study nuclei. However, in this section we will construct a derivative which is interesting in its own right.

We write $f, g, h, \ldots$ for derivatives on H, but we write $j, k, l, \ldots$ for nuclei.

Derivatives (and hence closure operations and nuclei) on a frame H are partially ordered by the pointwise comparison. Thus

$$f \leq g \iff (\forall x \in H)[f(x) \leq g(x)]$$

for derivatives f, g. This gives us three complete lattices.

Each derivative f has an idempotent closure. We set

$$f^0 = id_H \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda(x) = \bigvee\{f^\alpha(x) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. It is easy to check that this is an ascending chain of derivatives. On cardinality grounds there is some sufficiently large ordinal ∞ such that f^∞ is idempotent, and hence a closure operation.

If f is a pre-nucleus then so is f^α for each ordinal α, and hence f^∞ is a nucleus.

The derivative f and its closure f^∞ are directly related via their fixed points. Thus

$$f(a) = a \iff f^\infty(a) = a$$

for each $a \in H$. In some calculations this is a convenient way of passing from f to f^∞.

2.3 **DEFINITION.** For each frame morphism with its adjoint

$$H \xrightarrow{\ f\ } K \qquad H \xleftarrow{\ g\ } K$$

the composite $j = g \circ f$ on H is called the **kernel nucleus** of f. ■

As the name suggest, each kernel nucleus is a nucleus. Furthermore, each nucleus arises in this way. Given a nucleus j on H the fixed set

$$H_j = \{x \in H \mid j(x) = x\} = j[H]$$

is a frame in its own right. The square brackets '$[\cdot]$' indicate the direct image across the function. It can be seen that the assignment $x \mapsto j(x)$ is a frame morphism $H \longrightarrow H_j$, and j is the kernel nucleus of this. (This is a simple generalization of the construction $H \longrightarrow H_{\neg\neg}$, but the target need not be boolean.) Infima and implications computed in H_j agree with those computed in H, but suprema need not agree.

Subsets of H which arise as fixed sets have a simple characterization.

2.4 **LEMMA.** *A subset K of a frame is the fixed set H_j of some nucleus j if and only if*

$$\top \in K \qquad \bigwedge X \in K \qquad (a \supset x) \in K$$

for all $X \subseteq K, x \in K$ and $a \in H$.

For a frame H we say a subset K of H is an **F-set** if it satisfies these conditions. Thus it is the fixed set of some nucleus. When H is a topology we speak of an **F-family** of open sets. In Section 3 we will introduce the dual notion of a **G-family** of closed sets.

2.5 **EXAMPLES.** (a) For each $a \in H$ for a frame H

$$u_a(x) = a \vee x \qquad v_a(x) = (a \supset x)$$

defines two nuclei u_a and v_a. These are the respective kernels of the morphism

$$x \mapsto a \vee x : H \longrightarrow [a, \top] \qquad x \mapsto a \wedge x : H \longrightarrow [\bot, a]$$

where each interval is viewed as a frame in its own right.

(b) For each pair $a \leq b$ of elements of a frame H let

$$l(x) = (b \vee x) \wedge (a \supset x)$$

(for $x \in H$). A simple calculation shows that l is a nucleus. In fact, $l = u_b \wedge v_a$.

(c) For each $a \in H$ set

$$w_a(x) = ((x \supset a) \supset a)$$

(for $x \in H$). Again a simple calculation shows that w_a is a nucleus. For the particular case $a = \bot$ this is just double negation. ■

There are a couple of useful tricks with nuclei.

2.6 LEMMA. *For each nucleus j on a frame H we have*

$$j(x) \supset j(a) = x \supset j(a) \qquad j \leq w_a \Longleftrightarrow j(a) = a$$

for all $x, a \in H$.

We will first construct a derivative ***der*** and a pair of nuclei δ, ϵ on an arbitrary frame H. We then try to determine what these are for a topology $\mathcal{O}S$. We find that ***der*** and δ are well known, but ϵ is rather more enigmatic. We try to throw some light on ϵ.

Although ***Frm*** is connected to ***Top***, it is far richer in structure. Here is one gadget that is missing from point-sensitive topology.

2.7 DEFINITION. The assembly NH of a frame H is the poset of all nuclei on H under the pointwise comparison. ■

It is not too hard to see that NH is a complete lattice. In fact, for each $J \subseteq NH$ the infimum $\bigwedge J$ is computed pointwise, that is

$$(\bigwedge J)(x) = \bigwedge\{j(x) \mid j \in J\}$$

(for $x \in H$). Computations of suprema (even binary suprema) can be rather more complicated. Because of this it comes as a bit of a surprise to find that it is also a frame.

2.8 PROPOSITION. *The assembly NH of all nuclei on a frame H is itself a frame. The construction $N(\cdot)$ is functorial and there is a natural embedding $H \longrightarrow NH$. This embedding is a bi-morphism which universally solves the complementation problem for H. Thus for any frame morphism $H \longrightarrow K$ where each image value is complemented in K, there is a unique factorization through $H \longrightarrow NH$.*

The embedding $H \longrightarrow NH$ is just the assignment $a \mapsto u_a$, and we find that u_a, v_a are a complementary pair in NH. Furthermore, NH is $\bigvee$-generated by the family of all nuclei $v_a \wedge u_b$ (for $a, b \in H$).

The structure of NH is rather more complicated than that of H. In general NH does not arise as a topology. When it does we know what it is. In such a case we find that the parent frame H is a topology $\mathcal{O}S$ and then NH is the topology O^fS of the front space of S. The front space is formed be declaring that each originally open set should be clopen. This is a very crude way to solve the complementation problem for $\mathcal{O}S$. In general for a space S the assembly $N\mathcal{O}S$ is not a topology.

2.9 EXAMPLES. (a) Let's look at three topologies on the real line.

$$\mathcal{O}_l\mathbb{R} \longrightarrow \mathcal{O}_m\mathbb{R} \longrightarrow \mathcal{O}_r\mathbb{R}$$

The smallest one $\mathcal{O}_l\mathbb{R}$ consists of the extremes $\emptyset, \mathbb{R}$ and all intervals $(-\infty, a)$ for $a \in \mathbb{R}$. In fact, $a \mapsto (-\infty, a)$ sets up an isomorphism between the completed line $\overline{\mathbb{R}}$ and $\mathcal{O}_l\mathbb{R}$. Thus we have a linear frame. The middle one $\mathcal{O}_m\mathbb{R}$ is just the metric topology. The largest one $\mathcal{O}_r\mathbb{R}$ is generated by all intervals $[a, b)$ for elements $a \leq b$ of $\mathbb{R}$.

Notice how the insertion $\mathcal{O}_l\mathbb{R} \longrightarrow \mathcal{O}_r\mathbb{R}$ solves the complementation problem for $\mathcal{O}_l\mathbb{R}$. In fact $\mathcal{O}_r\mathbb{R}$ is just the front space topology of $\mathcal{O}_l\mathbb{R}$ and we find that $N\mathcal{O}_l\mathbb{R}$ is isomorphic to $\mathcal{O}_r\mathbb{R}$. In particular, here we have a frame $H = \mathcal{O}_l\mathbb{R}$ which is the topology of a rather simple space, where its assembly NH is the topology of what looks like a fairly simple space, but which includes a rather complicated part, the metric topology of the reals.

(b) Let S be a spectral space, the spectrum of a distributive lattice L. Let ${}^*S$ be the patch space of S. This is the spectrum of the boolean closure of L. The two spaces S and ${}^*S$ have the same points, and the topology $\mathcal{O}{}^*S$ is be formed on S be simply declaring that each originally compact open set should be clopen. This is a more refined version of the front space construction. Notice how the embedding $\mathcal{O}S \longrightarrow \mathcal{O}{}^*S$ only partially solves the complementation problem for $\mathcal{O}S$. However, with a little effort we can construct a factorization of the assembly embedding

$$\mathcal{O}S \longrightarrow \mathcal{O}^*S \longrightarrow N\mathcal{O}S$$

where both components are bi-morphisms. ■

The subcategory of ***Frm*** of boolean frames and frame morphisms is the category ***CBA*** of complete boolean algebras and complete morphisms. At first sight this seems a nice subcategory, but we soon run into smallness problems. In particular, this subcategory ***CBA*** is not reflective in ***Frm*** (and we will see why shortly).

2.10 DEFINITION. A frame H is reflective if there is a morphism $H \longrightarrow A$ to a boolean frame which is universal for all such morphisms, that is if each morphism $H \longrightarrow B$ to a boolean frame factors uniquely through the selected one $H \longrightarrow A$. ■

In [1] Isbell observed there are non-reflective frames. We will look at his example shortly. Since there are non-reflective frames, we must ask several questions.

Which frames are reflective? In particular, which spaces have a reflective topology? When it exists, what does this reflection look like?

Let us say a frame morphisms is tight if it is monic and the pushout across any morphism produces a monic. It is easy to check that tight monics are closed under composition and transfer across pushouts. The canonical embedding $H \longrightarrow NH$ is tight. Futhermore, this embedding is an isomorphism precisely when H is boolean.

By iterating the assembly construction we produce an ordinal indexed chain

$$H \longrightarrow NH \longrightarrow N^2H \longrightarrow \cdots \longrightarrow N^{\alpha}H \longrightarrow \cdots$$

of frame embeddings. (The passage across limit ordinals is the colimit in ***Frm***.) Each embedding $H \longrightarrow N^{\alpha}H$ is tight which leads to a smallness condition for being reflective.

2.11 PROPOSITION. *For each frame H the following are equivalent.*

(a) H is reflective.

(b) There is a tight monic $H \longrightarrow A$ to some boolean frame A.

(c) H has just a set of epimorphic images.

(d) The assembly tower stabilizes, there is an ordinal α such that $N^{\alpha}H$ is boolean.

This result rephrases the existence of a reflection for a frame H to the more concrete question of when its assembly tower stabilizes, and this leads to more questions. When is NH boolean? When is N^2H boolean? When is $N^{\omega}H$ boolean?

In this paper we show that for a large class of spaces S, if $\mathcal{O}S$ is reflective then, in fact, $N^2\mathcal{O}S$ is boolean. We see how this is related to an extended version of the usual Cantor-Bendixson analysis of S.

We need the generic example of a non-reflective frame given by Isbell.

2.12 EXAMPLE. Consider Cantor space

$$\mathsf{C} = {}^{\omega}\{0,1\}$$

with the product topology lifted from the discrete 2-point space. Let $\mathbf{2}$ be Sierpinski space, the 2-point space with just three open sets. Consider the space $\mathsf{D} = {}^{\omega}\mathbf{2}$ with the product topology lifted from $\mathbf{2}$. It can be checked that $\mathcal{O}\mathsf{D}$ is the free frame on ω generators. Furthermore, D is spectral, and is the spectrum of the free distributive lattice on ω generators. Its patch space is just C. Now, if $\mathcal{O}\mathsf{D}$ is reflective, then its reflection is the free complete boolean algebra on ω generators, which doesn't exist. Thus $\mathcal{O}\mathsf{D}$ is not reflective, and neither is $\mathcal{O}\mathsf{C}$ (since $\mathcal{O}\mathsf{D} \longrightarrow \mathcal{O}\mathsf{C}$ is tight). ■

Cantor space C can be used to show several other spaces have non-reflective frames. By suitably embedding C into a space S we obtain a surjective morphism $\mathcal{O}S \longrightarrow \mathcal{O}\mathsf{C}$ which shows that $\mathcal{O}S$ is not reflective. In this way, by a suitable splitting argument, we can show that the metric topology on $\mathbb{R}$ is not reflective, hence neither is the completed line $\overline{\mathbb{R}}$. We will develop this method in Section 5.

3 The extended Cantor-Bendixson analysis

In this section we construct a certain nucleus ϵ on an arbitrary frame H. We then look at this for the case of a topology, and we find that this spatial version is rather curious. To construct ϵ we first produce a certain derivative $\boldsymbol{der}$ and an associated nucleus δ on H. In the spatial case these are quite familiar gadgets.

For elements $a \leq b$ of a frame H we say the interval $[a, b]$ is boolean if, as a frame, it is boolean. Thus $[a, b]$ is boolean if for each $a \leq x \leq b$ there is some $a \leq y \leq b$ such that

$$x \wedge y = a \qquad x \vee y = b$$

hold. We consider how we can collapse each boolean interval to a singleton.

3.1 DEFINITION. For each element a of a frame H we set

$$\boldsymbol{der}(a) = \bigvee\{x \in H \mid \text{the interval } [a, x] \text{ is boolean}\}$$

to produce a function $\boldsymbol{der}$ on H. ■

Since each singleton interval is boolean, the function $\boldsymbol{der}$ is inflationary. An easy calculation shows that $\boldsymbol{der}$ is monotone, and hence $\boldsymbol{der}$ is a derivative. For reasons that will become clear, we call $\boldsymbol{der}$ the CB-derivative on H. Another easy calculation shows that the interval $[a, \boldsymbol{der}(a)]$ is boolean, and hence $\boldsymbol{der}(a)$ is the largest element b such that $[a, b]$ is boolean.

We need another couple of descriptions of $\boldsymbol{der}$.

For the first of these we use the relation $\lessdot$ on H given by

$$a \lessdot y \iff (a \leq y) \text{ and } ((y \supset a) = a)$$

(for $a, y \in H$). Late in this section we see the topological significance of this relation.

3.2 LEMMA. *For each* $a \in H$ *we have* $\boldsymbol{der}(a) = \bigwedge\{y \in H \mid a \lessdot y\}$.

Proof. Fix a and for convenience let

$$b = \bigwedge\{y \in H \mid a \lessdot y\}$$

be the element of interest.

Consider elements x, y where $a \lessdot y$ and $[a, x]$ is boolean. Let $u = x \wedge y$. Since $a \leq u \leq x$, there is some $a \leq v \leq x$ such that

$$y \wedge v = u \wedge v = a \qquad u \vee v = x$$

hold. These give first $v \leq (y \supset a) = a \leq u$ and then $u = x$ to show $x \leq y$. Since x and y are arbitrary this gives $\boldsymbol{der}(a) \leq b$.

For the converse comparison it suffices to show that the interval $[a, b]$ is boolean. To this end consider any $a \leq x \leq b$. Let $z = (x \supset a)$, so that $a \leq z$ and $x \wedge z = a$. Then

$$(x \vee z) \supset a = (x \supset a) \wedge (z \supset a) = z \wedge (z \supset a) = z \wedge a = a$$

so that $a \lessdot x \vee z$ and hence $b \leq x \vee z$. With this we may check that $y = b \wedge z$ is the required complement of x in $[a, b]$. ■

For each $j \in NH$ the fixed set H_j is a frame in its own right, and so carries its own CB-derivative $\boldsymbol{der}_{H_j}$ (say). We may compose this with the canonical quotient $H \longrightarrow H_j$ to obtain another derivative $\boldsymbol{der}_j$ on H. Thus

$$\boldsymbol{der}_j = \boldsymbol{der}_{H_j} \circ j$$

and, as a function $\boldsymbol{der}_j : H \longrightarrow H$, we see that $\boldsymbol{der}_j$ is a derivative on H. Eventually we are interested in the idempotent closure

$$\delta_j = \boldsymbol{der}_j^{\infty}$$

of $\boldsymbol{der}_j$, but before that we need to look inside $\boldsymbol{der}_j$.

For each $a \in H$, the value $\boldsymbol{der}_j(a)$ is the largest element b of H with $b = j(b)$ and where the interval $[j(a), b]$ of H_j is boolean. Since infima and implication on H_j agree with those on H, for each $a \in H$ we have

$$\boldsymbol{der}_j(a) = \bigwedge\{y \in H_j \mid j(a) \lessdot y\}$$

by Lemma 3.2.

3.3 LEMMA. *For each $j \in NH$ the derivative $\boldsymbol{der}_j$ is a pre-nucleus.*

Proof. We produce a different description of $\boldsymbol{der}_j$ which ensures the required result. (This description was pointed out to me by Richard Squires.)

For each $x \in H$ let

$$e_x = j \circ u_x \circ v_x \circ j$$

to obtain a pre-nucleus (since it is a composite of nuclei). We have

$$e_x(a) = j(x \vee (x \supset j(a)))$$

for each $a \in H$. Let

$$e = \bigwedge \{e_x \,|\, x \in H\}$$

be the pointwise infimum of all these e_x. This e is a pre-nucleus, and we show $\boldsymbol{der}_j = e$.

Consider any $a \in H$ and $y \in H_j$ with $j(a) \lessdot y$. Thus

$$j(y) = y \qquad j(a) \leq y \qquad y \supset j(a) = j(a)$$

and hence

$$y \vee (y \supset j(a)) = y \vee j(a) = y$$

so that $e_y(a) = jy = y$. Taking the infimum over y gives $e(a) \leq \boldsymbol{der}_j(a)$.

For the converse consider any $a, x \in H$ and let

$$y = e_x(a) = jz \quad \text{where} \quad z = x \vee (x \supset j(a))$$

so that $y \in H_j$ and $ja \leq z \leq y$. A simple calculation shows that

$$(x \vee (x \supset b)) \supset b = b$$

for each $b \in H$, so that (using Lemma 2.6)

$$(y \supset j(a)) = (x \supset j(a)) = j(a)$$

to give $j(a) \lessdot y$. Thus $\boldsymbol{der}_j(a) \leq y = e_x(a)$ which leads to the required result. ■

This result has an immediate consequence.

3.4 COROLLARY. *For each $j \in NH$ the idempotent closure $\delta_j = \boldsymbol{der}_j^\infty$ is a nucleus.*

This construction sets up an operation $j \mapsto \delta_j$ on NH. By analysing this we get a nice surprise. We use another description of $\boldsymbol{der}_j$.

The assembly NH is itself a frame, and so carries an implication operation. Thus, for $j, k \in NH$ there is a nucleus $(k \supset j)$ on H. We use the following.

3.5 LEMMA. *For each $j \in NH$ and $a \in H$*

$$\boldsymbol{der}_j(a) = (w_{j(a)} \supset j)(a)$$

holds.

Proof. Fix a and for convenience let $b = der_j(a)$ and $k = (w_{j(a)} \supset j)$. Note that $a \leq j(a) \leq k(a)$, and hence $k(j(a)) = k(a)$.

For each $y \in H_j$ with $j(a) \lessdot y$ we have $j(y) = y$ and $w_{j(a)}(y) = \top$ so that

$$k(a) \leq k(y) = (w_{j(a)} \wedge k)(y) \leq j(y) = y$$

and hence $k(a) \leq b$. For the converse comparison let

$$l(x) = (b \vee x) \wedge (j(a) \supset x)$$

to obtain a nucleus l on H. We show that $l \leq k$ and then evaluate at $j(a)$ to get $b \leq k(a)$.

A simple calculation gives $c \lessdot z \vee (z \supset c)$ for each $c, z \in H$. In particular, we have

$$j(a) \lessdot (j(x) \vee (j(x) \supset j(a)) = (j(x) \vee (x \supset j(a))$$

and hence

$$b \vee x \leq j(x) \vee (x \supset j(a))$$

holds. This gives

$$(b \vee x) \wedge w_{j(a)}(x) \leq j(x) \vee j(a)$$

and hence

$$l(x) \wedge w_{j(a)}(x) \leq j(x)$$

which is $l \wedge w_{j(a)} \leq j$, so that $l \leq k$, as required. ■

The particular case $j = \bot_{NH}$ of this result gives $\boldsymbol{der}(a) = (\neg w_a)(a)$ which is the description I used in the original work described in [5].

As a frame NH carries its own CB-derivative $\boldsymbol{Der}$. By Lemma 3.2 this is given by

$$\boldsymbol{Der}(j) = \bigwedge\{k \in NH \mid j \lessdot k\}$$

where $\lessdot$ is the appropriate relation on NH.

This brings to the result which drives the machinery used in this paper.

3.6 THEOREM. *For each frame H we have*

$$\delta_j = \boldsymbol{der}_j^{\infty} = \boldsymbol{Der}(j)$$

for each $j \in NH$.

Proof. Fix $j \in NH$.

Consider first any $k \in NH$ with $j \lessdot k$, that is $j \leq k$ and $(k \supset j) = j$. For arbitrary $x \in H$ let $a = k(x)$. Then $j(a) \leq k(a) = a$ so that $j \leq k \leq w_a$ and hence

$$(w_a \supset j) \leq (k \supset j) = j$$

holds. By Lemma 3.5 this gives

$$\boldsymbol{der}_j(a) = (w_a \supset j)(a) \leq j(a) = a$$

so that $\delta_j(a) = a$, and hence

$$\delta_j(x) \leq \delta_j(a) = a = k(x)$$

to give $\delta_j \leq k$. The arbitrary choice of k now gives $\delta_j \leq \boldsymbol{Der}(j)$.

Conversely, for arbitrary $x \in H$ consider $a = \delta_j(x)$. We have $j(a) \leq \delta_j(a) = a$ so that $j(a) = a$ and $j \leq w_a$. Also

$$(w_a \supset j)(a) = \boldsymbol{der}_j(a) = a$$

so that $j \leq (w_a \supset j) \leq w_a$ and hence

$$(w_a \supset j) = (w_a \supset j) \wedge w_a = w_a \wedge j = j$$

to verify $j \lessdot w_a$. Thus $\boldsymbol{Der}(j) \le w_a$ and hence

$$\boldsymbol{Der}(j)(x) \le w_a(x) \le w_a(a) = a = \delta_j(x)$$

so that $\boldsymbol{Der}(j) \le \delta_j$, as required. ■

This result gives us a whole hierarchy of nuclei on H. We want to study one of these. Let

$$\mathsf{d}_0(H) = \boldsymbol{der}(\bot_H)$$

so that $[\bot_H, \mathsf{d}_0(H)]$ is the largest boolean lower interval of H. In particular

$$H \text{ is boolean} \iff \mathsf{d}_0(H) = \top_H$$

holds. This is true for every frame H, so we may apply it to the assembly NH. Notice that

$$\mathsf{d}_0(NH) = \boldsymbol{Der}(\bot_{NH}) = \delta$$

by Theorem 3.6. Let

$$\mathsf{d}_1(H) = \mathsf{d}_0(NH)(\bot_H) = \delta(\bot_H)$$

so that

$$NH \text{ is boolean} \iff \mathsf{d}_0(NH) = \top_{NH} \iff \mathsf{d}_1(H) = \top_H$$

using the previous equivalence on NH. Next we apply this equivalence to the assembly NH to obtain some information about the second assembly N^2H. Let $\Delta = \boldsymbol{Der}^\infty$ be the CB-nucleus of NH, and set

$$\epsilon = \Delta(\bot_{NH}) = \mathsf{d}_1(NH)$$

to obtain a particular nucleus ϵ on H. Let

$$\mathsf{d}_2(H) = \epsilon(\bot_H)$$

so that

$$N^2H \text{ is boolean} \iff \epsilon = \mathsf{d}_1(NH) = \top_{NH} \iff \mathsf{d}_2(H) = \top_H$$

is an instance of the second equivalence.

We could take this further to obtain nuclei which are related to properties of the higher level assemblies. We don't do that here, for the nucleus ϵ is the object of interest.

It is time we explained some of the terminology.

Consider the spatial case where $H = \mathcal{O}S$ is the topology of an T_0 space. What are the gadgets $\lessdot, \boldsymbol{der}, \delta, \epsilon$? It is more convenient to work with the family $\mathcal{C}S$ of closed sets of S. The first three gadgets are familiar, but ϵ is not. We work with the dual complements.

Thus, let $\Subset$ be the relation on $\mathcal{C}S$ given by

$$Y \Subset X \iff X' \lessdot Y'$$

for $X, Y \in \mathcal{C}S$. If we remember how the implication on $\mathcal{O}S$ is computed we see that

$$Y \Subset X \iff (Y \subseteq X) \text{ and } X = (X - Y)^-$$

in other words Y is an inessential part of X. We do the same trick with the operations.

3.7 DEFINITION. Let $\boldsymbol{lim}, \boldsymbol{per}, \boldsymbol{enig}$ be the operations on $\mathcal{C}S$ given by

$$\boldsymbol{lim}(X) = \boldsymbol{der}(X')' \qquad \boldsymbol{per}(X) = \delta(X')' \qquad \boldsymbol{enig}(X) = \epsilon(X')'$$

for each $X \in \mathcal{C}S$ (where S is a space). ■

The names of these operations should indicate what they are, especially the first two.

3.8 THEOREM. *Let S be a T_0 space. Then*

$$\boldsymbol{lim}(X) = \textit{ the set of limit points of } X \qquad \boldsymbol{per}(X) = \textit{ the perfect part of } X$$

for each $X \in \mathcal{C}S$.

Proof. Fix $X \in \mathcal{C}S$. By translating Lemma 3.2 we see that

$$\boldsymbol{lim}(X) = \big(\bigcup\{Y \in \mathcal{C}S \mid Y \Subset X\}\big)^-$$

holds.

Consider first any point $p \in \boldsymbol{lim}(X)$ and any open set U with $p \in U$. The above description of $\boldsymbol{lim}(X)$ provides some closed set $Y \Subset X$ and some point $q \in Y \cap U$. Also $p \in X = (X - Y)^-$ so there is some point $r \in (X - Y) \cap U$. Since $q \in Y$ and $r \notin Y$ we have $q \neq r$, and hence $X \cap U$ has at least two members. Thus p is not an isolated point of X, it is a limit point.

Conversely, consider any point $p \in X - \boldsymbol{lim}(X)$ and let

$$U(p) = (X - p^-)^{-\prime} = (X' \cup p^-)^\circ$$

to obtain an open set. Notice first that

$$X \cap U(p) \subseteq X \cap (X' \cup p^-) = X \cap p^- = p^-$$

(since $p \in X$). Also, if $X \cap U(p)$ is empty then

$$X \subseteq U(p)' = (X - p^-)^-$$

so that $p^- \Subset X$ and hence

$$p \in p^- \subseteq \boldsymbol{lim}(X)$$

which is a contradiction. Thus $\emptyset \neq X \cap U(p) \subseteq p^-$. Consider any $r \in X \cap U(p)$. Then $r \in U(p) \cap p^-$ so that $p \in U(p)$.

This argument shows that

$$p \in X \cap U(p) \subseteq p^-$$

for each $p \in X - \boldsymbol{lim}(X)$. We use this to show

$$X \cap \boldsymbol{lim}(X)' \cap U(p) = (X - \boldsymbol{lim}(X)) \cap U(p) = \{p\}$$

for each $p \in X - \boldsymbol{lim}(X)$, and hence such a p is isolated in X.

For $p \in X - \boldsymbol{lim}(X)$, consider any $q \in (X - \boldsymbol{lim}(X)) \cap U(p)$. Using the associated open set $U(q)$ we have $q \in U(q) \cap p^-$, and hence $p \in U(q)$. This gives

$$p \in X \cap U(q) \subseteq q^- \qquad q \in U(q) \cap p^- \subseteq p^-$$

where the left hand inclusion follows by a second use of the first observation. In particular $p \in q^-$ and $q \in p^-$, to give $p^- = q^-$ and hence $p = q$ since S is T_0.

This completes the description of $\boldsymbol{lim}(X)$. Also $\boldsymbol{per} = \boldsymbol{lim}^\infty$, and

$$\boldsymbol{lim}(X) = X \iff X \text{ is perfect}$$

so that $\boldsymbol{per}(X)$ is the largest perfect subset of X, as required. ■

This result explains why we call $\boldsymbol{der}$ the CB-derivative and δ the CB-nucleus of a frame. But what is ϵ?

3.9 DEFINITION. For a T_0 space S let

$$X \in \mathcal{D}S \iff \boldsymbol{per}(X) = X \qquad X \in \mathcal{E}S \iff \boldsymbol{enig}(X) = X$$

for $X \in \mathcal{C}S$ to obtain $\mathcal{E}S \subseteq \mathcal{D}S \subseteq \mathcal{C}S$. Also set

$$X_0 = \boldsymbol{lim}(S) \qquad X_1 = \boldsymbol{per}(S) \qquad X_2 = \boldsymbol{enig}(S)$$

to obtain closed sets $X_2 \subseteq X_1 \subseteq X_0$. ■

By Theorem 3.8 we see that X_0 is the set of limit points of S and X_1 is the perfect part of S. Furthermore, $\mathcal{D}S$ is the set of perfect subsets of S. In general I have no idea what the the set X_2 is, and I have even less idea what the operation $\boldsymbol{enig}$ is. In the remainder of this paper we will look at some examples of these gadgets for a family of rather feeble spaces. Notice that the general equivalences given above translate into

$$\begin{aligned} \mathcal{O}S \text{ is boolean} &\iff X_0 = \emptyset \iff S \text{ is discrete} \\ N\mathcal{O}S \text{ is boolean} &\iff X_1 = \emptyset \iff S \text{ is scattered} \\ N^2\mathcal{O}S \text{ is boolean} &\iff X_2 = \emptyset \iff S \text{ is ????} \end{aligned}$$

so a description of X_2 is not without interest.

Nuclei on $\mathcal{O}S$ correspond to the F-families of open sets. By taking the dual complement we obtain the equivalent notion of a G-family of closed sets. Each such G-family $\mathcal{G}$ corresponds to a nucleus j on $\mathcal{O}S$ which inflates to a larger nucleus $\boldsymbol{Der}(j)$ and this corresponds to a smaller G-family $\mathcal{G}^+$. Here are the appropriate definitions.

3.10 DEFINITION. Let S be a topological space.

(a) A G-family for S is a a collection $\mathcal{G} \subseteq \mathcal{C}S$ of closed subsets such that $\emptyset \in \mathcal{G}$ and

$$\left(\bigcup \mathcal{X}\right)^- \in \mathcal{G} \qquad (X - Y)^- \in \mathcal{G}$$

for each $\mathcal{X} \subseteq G$ and $X \in \mathcal{G}, Y \in \mathcal{C}S$.

(b) For each G-family $\mathcal{G}$ let $\mathcal{G}^+$ be the family of closed sets $X \in \mathcal{G}$ such that

$$X = \left(\bigcup\{Y \in \mathcal{G} \mid Y \subset X\}\right)^-$$

holds. ■

The step $\mathcal{G} \mapsto \mathcal{G}^+$ corresponds to hitting a nucleus with $\boldsymbol{Der}$. Accordingly we say a G-family $\mathcal{G}$ is perfect if $\mathcal{G}^+ = \mathcal{G}$. Thus $\mathcal{E}S$ is the largest perfect G-family. By iterating the process $(\cdot)^+$ we obtain a descending chain of G-families. Thus we set

$$(\mathcal{C}S)^{(0)} = \mathcal{C}S \qquad (\mathcal{C}S)^{(\alpha+1)} = (\mathcal{C}S)^{(\alpha)+} \qquad (\mathcal{C}S)^{(\lambda)} = \bigcap\{(\mathcal{C}S)^{(\alpha)} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. For instance $\mathcal{D}S = (\mathcal{C}S)^{(1)}$ and $\mathcal{E}S = (\mathcal{C}S)^{(\infty)}$ for some ordinal ∞. The size of this ordinal ∞ is the extended CB-rank of S.

What is this family $\mathcal{E}S$ and how big can ∞ be? In the remainder of this paper we produce some interesting examples of $\mathcal{E}S$, and show that ∞ can get arbitrarily large.

4 Alexandroff topologies

In our search for an interesting example of the enigmatic family $\mathcal{E}S$ of a space S we will need to calculate with the operation $(\cdot)^+$ on G-families. A glance at Definition 3.10(b) shows that this involves handling the closure of the union of closed sets. This is not always easy. To make life simpler we will restrict our attention to those spaces for which arbitrary unions of closed sets are already closed. These are the Alexandroff spaces.

Let S be an arbitrary poset. The set $\mathcal{O}S$ of upper sections of S is the Alexandroff topology on S. The lower sections form the corresponding family $\mathcal{C}S$ of closed sets. We will obtain some information about the families $\mathcal{D}S$ and $\mathcal{E}S$.

4.1 LEMMA. *For $X, Y \in \mathcal{C}S$ (where S is a poset) with $Y \subseteq X$, the equivalence*

$$Y \Subset X \iff (\forall y \in Y)(\exists z \in X - Y)[y < z]$$

holds.

Proof. Suppose first that $y \in Y \Subset X$ and let $U = \uparrow y \in \mathcal{O}S$ be the principal upper section above y. Then $y \in U \cap (X - Y)^-$ so that U meets $X - Y$, which produces the required $z \in X - Y$.

Suppose the escape property holds, consider any $U \in \mathcal{O}S$ and any $x \in X \cap U$. If $x \notin Y$ then $x \in (X - Y) \cap U$. If $x \in Y$ then there is some $x < z \in X - Y$ and hence $z \in (X - Y) \cap U$. In both cases $X - Y$ meets U, so that $x \in (X - Y)^-$, as required. ■

Since $\mathcal{D}S = (\mathcal{C}S)^+$ this gives

$$X \in \mathcal{D}S \iff (\forall x \in X)(\exists y \in X)[x < y]$$

(for $X \in \mathcal{C}S$). This is the well known description of the perfect sets of S as those lower sections which have no maximal members. We work towards a similar description of $\mathcal{E}S$.

For $x \in X \in \mathcal{C}S$ let

$$X^x = \{y \in X \mid x < y\}$$

to produce a subset of X. Thus

$$X \in \mathcal{D}S \iff (\forall x \in X)[X^x \neq \emptyset]$$

and our eventual description of $\mathcal{E}S$ will have the form

$$X \in \mathcal{E}S \iff (\forall x \in X)[X^x \text{ is large}]$$

where the meaning of 'large' must be suitably interpreted.

Alexandroff spaces can be used to generate more complicated spaces. As as space a poset S may not be sober. By passing to the sober reflection we produce its missing points, and then we may restrict the topology to just these points.

4.2 CONSTRUCTION. For a poset S let S be the family of non-empty, directed, perfect subsets of S. Thus the members p of S are those non-empty lower sections which are directed and have no maximal members. Using a hull-kernel construction we convert S into a space and set up a surjective frame morphism

$$\mathcal{O}S \xrightarrow{\ d\ } \mathcal{O}\mathsf{S}$$

(which does not arise as the inverse image function of a continuous map $\mathsf{S} \longrightarrow S$).

For each $U \in \mathcal{O}S$ consider

$$p \in d(U) \iff U \text{ meets } p$$

(for $p \in \mathsf{S}$) to produce a subset of S. It is easy to check that

$$d(U) \cap d(V) = d(U \cap V) \qquad d(\bigcup \mathcal{U}) = \bigcup d[\mathcal{U}]$$

for each $U, V \in \mathcal{O}S$ and $\mathcal{U} \subseteq \mathcal{O}S$. Thus $\mathcal{O}\mathsf{S} = d[S]$ is a topology on S and d is a surjective frame morphism, as indicated. For each $a \in S$ let $I_a = d(\uparrow a)$ so that

$$p \in I_a \iff a \in p$$

(for $p \in \mathsf{S}$) and $d(U) = \bigcup\{I_a \mid a \in I\}$, to show that $\{I_a \mid a \in S\}$ is a base for $\mathcal{O}\mathsf{S}$.

For each $X \in \mathcal{C}S$ let

$$\boldsymbol{ker}(X) = \bigcup\{p \in \mathsf{S} \mid p \subseteq X\}$$

to produce a closed subset of X. A bit of symbol shuffling shows that for $X, Y \in \mathcal{C}S$

$$d(Y') \subseteq d(X') \iff \boldsymbol{ker}(X) \subseteq \boldsymbol{ker}(Y)$$

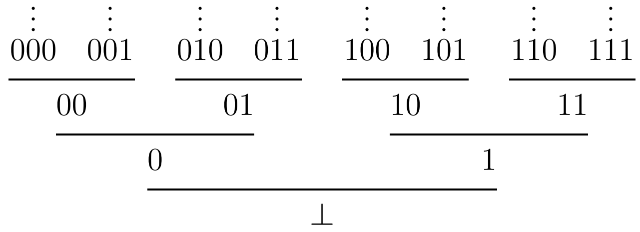
and hence $\boldsymbol{ker}(\cdot)$ is the dual complement of the kernel nucleus of d. But now

$$X = \boldsymbol{ker}(X) \iff (\forall a \in X)(\exists p \in \mathsf{S})[a \in p \subseteq X] \iff X \in \mathcal{D}S$$

so that we have a concrete realization of the CB-nucleus of $\mathcal{O}S$. The kernel nucleus of the morphism d is just the CB-nucleus δ of $\mathcal{O}S$, and hence $(\mathcal{O}S)_\delta \cong \mathcal{O}\mathsf{S}$. ■

We use two instances of this construction.

4.3 EXAMPLE. Let $\mathbb{C}$ be the full binary splitting tree. We think of this as



finite sequences from $\{0,1\}$ ordered by extension. The set C of ideal points of $\mathbb{C}$ is just the set of infinite branches of this tree, the functions $\mathbb{N} \longrightarrow \{0,1\}$. We may check that this C with the induced topology is nothing more than Cantor space. This shows that $(\mathcal{O}\mathbb{C})_\delta = \mathcal{O}\mathsf{C}$ and hence $\mathcal{O}\mathbb{C}$ is not reflective. ■

A similar construction can be carried out on any full splitting tree of height ω.

4.4 EXAMPLE. Consider the rationals $\mathbb{Q}$ as a poset (in fact, a linear order). Let $\mathcal{O}\mathbb{Q}$ be the upper section topology (*not* the metric topology). Since each lower section of $\mathbb{Q}$ is directed we see that the set Q of ideal points is just the set of metric-open lower sections of $\mathbb{Q}$. Thus for Q we may take the reals $\mathbb{R}$ (together with a point at infinity). Each $p \in \mathbb{R}$ is identified with the lower section $(-\infty, p) \cap \mathbb{Q}$ of $\mathbb{Q}$, and for each $a \in \mathbb{Q}$ we have $I_a = (a, \infty)$. In this case the family of all these I_a is the whole of the topology $\mathcal{O}\mathbb{R}$ induced on $\mathbb{R}$. This, of course, is not the metric topology $\mathcal{O}_m\mathbb{R}$. It can be shown that the canonical embedding $\mathcal{O}\mathbb{R} \longrightarrow N\mathcal{O}\mathbb{R}$ factorizes through the the insertion $\mathcal{O}\mathbb{R} \longrightarrow \mathcal{O}_m\mathbb{R}$. From this we see that $\mathcal{O}\mathbb{R}$ is not reflective and, in fact, is a perfect frame. Thus $\mathcal{O}\mathbb{Q}$ is not reflective since $(\mathcal{O}\mathbb{Q})_\delta$ is perfect and isomorphic to the completed real line. ■

We will use the two frames $\mathcal{O}\mathbb{C}$ and $\mathcal{O}\mathbb{Q}$ to show that many other posets S produce non-reflective frames. To do this we construct a suitable embedding of $\mathbb{C}$ or $\mathbb{Q}$ in S.

4.5 DEFINITION. For posets S and T, a **T-obstruction in S** is a function $\phi : T \longrightarrow S$ such that

$$a \leq b \Longleftrightarrow \phi(a) \leq \phi(b)$$

for each $a, b \in T$. In other words a T-obstruction in S is simply a copy of T in S. ■

The idea is that we think of T as a test poset such as $\mathbb{C}$ or $\mathbb{Q}$. This will have an associated space T of ideal points with a frame morphism

$$\mathcal{O}T \xrightarrow{\quad d \quad} \mathcal{O}\mathsf{T}$$

as constructed above (for S). When ϕ is a T-obstruction in S, the implication '$\Longrightarrow$' ensures that ϕ is a continuous map, and so there is an induced frame morphism $\phi^{\leftarrow}$, as on the left

$$\mathcal{O}S \xrightarrow{\quad \phi^{\leftarrow} \quad} \mathcal{O}T \qquad\qquad \mathcal{O}S \xrightarrow{\quad f = d \circ \phi^{\leftarrow} \quad} \mathcal{O}\mathsf{T}$$

which may be composed with d to produce a frame morphism f, as on the right, with which we test S. It is easy to check that

$$p \in f(U) \Longleftrightarrow (\exists b \in T)[b \in p \text{ and } \phi(b) \in U]$$

for $p \in \mathsf{T}$ and $U \in \mathcal{O}S$.

4.6 LEMMA. *For each obstruction $\phi : T \longrightarrow S$ in a poset S, the induced frame morphism f is surjective.*

Proof. For $a \in T$ it suffices to produce $U_a \in \mathcal{O}S$ with $f(U_a) = I_a \in \mathcal{O}T$. We use

$$x \in U_a \Longleftrightarrow (\forall b \in T)[x \leq \phi(b) \Longrightarrow a \leq b]$$

(for $x \in S$). For $p \in \mathsf{T}$ the unravelling of $p \in f(U_a)$ from above gives some

$$b \in p \qquad x = \phi(b) \in U_a \qquad x \leq \phi(a)$$

so that $a \leq b \in p$, to give $a \in p$. Conversely, we have $\phi(a) \in U_a$ since ϕ is an obstruction. Thus if $a \in p$ then $b = a$ witnesses that $p \in f(U_a)$. ■

From now on we consider only $\mathbb{C}$-obstructions or $\mathbb{Q}$-obstructions in a poset. We often refer to these as 'obstructions', as in the next result and definition.

4.7 COROLLARY. *If a poset S has an obstruction, then $\mathcal{O}S$ is not reflective.*

Each obstruction in a poset S gives us members of $\mathcal{E}S$.

4.8 DEFINITION. Given an obstruction ϕ in a poset S we set

$$X(\phi) = \{x \in S \mid (\exists a \in T)[x < \phi(a)]\}$$

to produce some $X(\phi) \in \mathcal{C}S$.

For $x \in X \in \mathcal{C}S$ we say x **supports an obstruction in** X if there is some obstruction ϕ with $x \in X(\phi) \subseteq X$.

Let $\mathcal{B}S$ be the set of all those $X \in \mathcal{C}S$ which are everywhere obstructed, that is each $x \in X$ supports an obstruction in X. ■

We have $\emptyset \in \mathcal{B}S$ and this may be the only member of $\mathcal{B}S$ (for consider $\mathbb{N}$ as a poset). However, if $\mathcal{B}S$ is non-empty then (by Corollary 4.7) $\mathcal{O}S$ is non-reflective.

4.9 LEMMA. *For each poset S, the family $\mathcal{B}S$ is a perfect G-family.*

Proof. Almost trivially $\mathcal{B}S$ is closed under arbitrary unions. Thus it suffices to show

$$X \in \mathcal{B}S, Y \in \mathcal{C}S \Longrightarrow (X - Y)^{-} \in \mathcal{B}S$$

holds, and that $\mathcal{B}S$ is perfect.

To this end consider any $X \in \mathcal{B}S, Y \in \mathcal{C}S$ and $x \in (X - Y)^{-}$. There is some $x \leq y \in X - Y$, and hence some obstruction $\phi : T \longrightarrow S$ with some $a \in T$ where $x \leq y < \phi(a)$ and $X(\phi) \subseteq X$. Let U be any upper section of T such that $a < b$ for all $b \in U$ and (as a poset) U is isomorphic to T. Such a U exists since, in our case, T is either $\mathbb{C}$ or $\mathbb{Q}$. The restriction $\psi = \phi|_U$ is also an obstruction in S, and $y \in X(\psi) \subseteq X$. Note also that for $b \in U$ we have $y < \phi(a) \leq \phi(b) = \psi(b)$, so that $\psi(b) \notin Y$ (since $y \notin Y$). Thus $\psi[U] \subseteq X - Y$, and hence $x \in X(\psi) \subseteq (X - Y)^{-}$, which gives $(X - Y)^{-} \in \mathcal{B}S$.

To show that $\mathcal{B}S$ is perfect consider any obstruction ϕ in S and any $x \in X(\phi)$. We construct another obstruction ψ with $x \in X(\psi) \mathrel{\subset\!\!\!\cdot} X(\phi)$, from which we obtain the required result. There are two cases.

For the first case suppose that $\phi : \mathbb{C} \longrightarrow S$ is an obstruction with $x < \phi(\xi)$ where $\xi \in \mathbb{C}$. Consider the function $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ generated recursively by

$$\theta(\bot) = \xi \qquad \theta(\nu i) = \theta(\nu)0i \quad (\text{ for } i = 0, 1)$$

for each $\nu \in \mathbb{C}$. Let $\Theta = \theta[\mathbb{C}]$. This Θ is an isomorphic copy of $\mathbb{C}$ but is not a subtree of $\mathbb{C}$. As a subset Θ sits above ξ, has a left-handed bias, and skips every other level.

$$\begin{array}{ccc} \theta(\nu 0) & \theta(\nu 1) & \\ \cline{1-2} & \lambda & \rho \\ \cline{2-3} & \theta(\nu) & \end{array}$$

Notice that for ν, λ, ρ as shown, we have $\rho \not\leq \theta(\mu)$ for each $\mu \in \mathbb{C}$.

Let ψ be the composite $\phi \circ \theta$. This is a $\mathbb{C}$-obstruction in S with $x \in X(\psi) \subseteq X(\phi)$. Consider any $y \in X(\psi)$. We have $y < \psi(\nu) = \phi(\theta(\nu))$ for some $\nu \in \mathbb{C}$. Referring to the diagram above let $z = \phi(\rho)$. Then $y < z \in X(\phi)$. Also $z \notin X(\psi)$, for otherwise there is some $\mu \in \mathbb{C}$ with $\phi(\rho) = z < \phi(\theta(\mu))$ and hence $\rho < \theta(\mu)$, which is a contradiction.

Thus, for this case, we have $X(\psi) \Subset X(\phi)$, as required,

For the second case suppose the $\phi : \mathbb{Q} \longrightarrow S$ is an obstruction and that $x < \phi(r)$ where $r \in \mathbb{Q}$. The interval $I = (-\infty, r) \cap \mathbb{Q}$ is isomorphic to $\mathbb{Q}$, so that restriction $\psi = \phi|_I$ is also an obstruction in S, and clearly $x \in X(\psi) \subseteq X(\phi)$. Furthermore, $\phi(r)$ witnesses that fact that $X(\psi) \Subset X(\phi)$ to complete this case. ■

This result gives us a fairly decent part of $\mathcal{E}S$.

4.10 COROLLARY. *For each poset S the inclusion $\mathcal{B}S \subseteq \mathcal{E}S$ holds.*

We would like to improve this inclusion to an equality. I don't know if this is true in general but it is the case for a large family of posets. This is the topic of the next section. Before that let's isolate the problem with arbitrary posets.

4.11 LEMMA. *For each poset S and $x \in X \in \mathcal{C}S$, one of the following holds.*

- *(i)* $X^x = \emptyset$
- *(ii) x supports a $\mathbb{C}$-obstruction in X*
- *(iii) there is some $y \in X^x$ such that X^y is directed*

Proof. Consider $x \in X \in \mathcal{C}S$ and suppose neither (i) nor (iii) hold. We produce a $\mathbb{C}$-obstruction $\phi : \mathbb{C} \longrightarrow S$.

Since (iii) does not hold, we know that

$$(\forall y \in X^x)(\exists y_0, y_1 \in X^y)(\forall z \in X)[y_0 \not\leq z \text{ or } z \not\leq z]$$

holds. Thus, since X^x is non-empty, there is lots of splitting in X^x.

Let $\phi(\perp)$ be any member of X^x. We now generate a function $\phi : \mathbb{C} \longrightarrow S$ such that $\phi(\nu i) = \phi(\nu)_i$ for each $\nu \in \mathbb{C}$ and $i = 0, 1$. Here the two elements $\phi(\nu)_i$ are chosen using the splitting property. Clearly $\phi(\nu) \leq \phi(\nu i)$ (since $y_i \in X^y$) to show that ϕ is monotone. To show that ϕ is an obstruction we first show the following.

$$(*) \quad (\forall \nu, \mu \in \mathbb{C})\big[(\exists z \in X)[\phi(\nu), \phi(\mu) \leq z] \Longrightarrow \nu \leq \mu \text{ or } \mu \leq \nu\big]$$

Consider incomparable $\nu, \mu \in \mathbb{C}$. There is some $\lambda \in \mathbb{C}$ with $\lambda 0 \leq \nu$ and $\lambda 1 \leq \mu$ (say). Let $y = \phi(\lambda)$ so that $y_0 = \phi(\lambda)_0 = \phi(\lambda 0) \leq \phi(\nu)$. Similarly $y_1 \leq \phi(\mu)$. But now there can be no $z \in X$ with $\phi(\nu), \phi(\mu) \leq z$, for otherwise $y_0, y_1 \leq z$, which is not so.

To verify that ϕ is an obstruction consider any $\nu, \mu \in \mathbb{C}$ with $\phi(\nu) \leq \phi(\mu)$. By $(*)$ we have either $\nu \leq \mu$ (as required) or $\mu < \nu$. If $\mu < \nu$ and then $\mu < \mu 0 \leq \nu$ (say). We also have $\phi(\nu) \leq \phi(\mu) \leq \phi(\mu 1)$, so that (by a second use of $(*)$) either $\nu \leq \mu 1$ or $\mu 1 \leq \nu$. But now either $\mu 0 \leq \nu \leq \mu 1$ or both $\mu 0, \mu 1 \leq \nu$, and each of these is contradictory. ■

This highlights the problem with an arbitrary poset S. We would like to use clause (iii) (of Lemma 4.11) to produce a $\mathbb{Q}$-obstruction in S. This is much easier if we know that the set X^x is a chain (not just directed). Hence we force this to be the case.

5 Trees

A **tree** is a poset S such that

$$x, y \leq z \Longrightarrow x \leq y \text{ or } y \leq x$$

for all $x, y, z \in S$. (We don't need a tree to be well founded.) This restriction ensures that each directed set is a chain, and hence Lemma 4.11 becomes the following.

5.1 LEMMA. *For each tree S and $x \in X \in \mathcal{C}S$, one of the following holds.*

(i) $X^x = \emptyset$

(ii) *x supports a $\mathbb{C}$-obstruction in X*

(iii) *there is some $y \in X^x$ such that X^y is a chain*

The plan now is to look for a copy of $\mathbb{Q}$ in the chain produced by clause (iii). This copy need not be a convex subset, so we need a method of collapsing the irrelevant gaps.

5.2 DEFINITION. Let S be a tree.

(a) A **separator** on S is a relation $\lhd$ such that

$$a \lhd b \Longrightarrow a < b \qquad x \leq a \lhd b \leq y \Longrightarrow x \lhd y$$

holds for all $x, y, a, b \in S$.

(b) Given a separator $\lhd$ on S, an ordinal λ, and $x \in X \in \mathcal{C}S$, we say x **$\lhd$-supports** λ **in** X if there is a chain $(x(i) \mid i < \lambda)$ of elements of X such that

$$x \leq x(i) \lhd x(j)$$

for all $i < j < \lambda$.

(c) Let

$$\mathfrak{S}(x, X, \lhd, \lambda)$$

abbreviate 'Either x supports $\mathbb{C}$ in X or x $\lhd$-supports λ in X'.

(d) For a G-family $\mathcal{G}$ of S let

$$\mathcal{G} \vdash (\lhd, \lambda)$$

mean that $\mathfrak{S}(x, X, \lhd, \lambda)$ for all $x \in X \in \mathcal{G}$. ■

For instance

$$\mathcal{G} \vdash (\lhd, 1)$$

for any G-family $\mathcal{G}$ and separator $\lhd$. Also we find that

$$\mathcal{D}S \vdash (<, 2)$$

holds.

We will derive various rules for operating the relation $\vdash$. These are listed in Table 1. Once we have these we can quite quickly obtain a description of $\mathcal{E}S$ for a tree S. Let's go through these rules in turn to see what they mean. We will verify the rules later once we have extracted the information we want.

(A) For each non-zero ordinal λ, if $\mathcal{G} \vdash (\lhd, \lambda + 1)$ then $\mathcal{G} \vdash (\lhd, \lambda\omega)$.

(B) For each limit ordinal λ, if $\mathcal{G} \vdash (\lhd, \lambda)$ then $\mathcal{G}^{+} \vdash (\lhd, \lambda\omega)$.

(C) If $\mathcal{G}^{+} \vdash (\lhd, 2)$ then $\mathcal{G}^{(\alpha)} \vdash (\lhd, \omega^{\alpha})$ for each $\alpha < \Omega$.

(D) If $\mathcal{G}^{+} \vdash (\lhd, 2)$ then $\mathcal{G}^{(\Omega)+} \vdash (\lhd', 2)$.

(E) $(\mathcal{C}S)^{(\Omega\alpha+1)} \vdash (<^{\alpha}, 2)$

Table 1: The $\vdash$ rules

Rule (A) shows that in general we may assume that λ is either 2 or a limit ordinal.

Rule (B) provides the crucial information about the operation $(\cdot)^{+}$ on G-sets.

Even with just these two rules we can obtain some non-trivial information. For instance, since $\mathcal{D}S \vdash (<, 2)$ and $(\mathcal{C}S)^{(1)} = \mathcal{D}S$, a use of (A) gives

$$(\mathcal{C}S)^{(1)} \vdash (<, \omega)$$

and then repeated use of (B) gives

$$(CS)^{(r)} \vdash (<, \omega^{r})$$

for each $r < \omega$. For an application of this suppose S is a tree of height $< \omega^{\omega}$. By definition (of height) there is some $r < \omega$ such that no $x \in S$ can $<$-support ω^{r}. Thus

$$x \in X \in (\mathcal{C}S)^{(r)} \Longrightarrow x \text{ supports } \mathbb{C} \text{ in } X$$

and hence $\mathcal{E}S = \mathcal{B}S = (\mathcal{C}S)^{(r)}$.

In particular, recall that for Cantor space C we have $\mathcal{C}\mathsf{C} = (\mathcal{C}\mathbb{C})^{+}$. Thus, since $\mathbb{C}$ has height $< \omega^{2}$, we have $(\mathcal{C}\mathsf{C})^{+} = (\mathcal{C}\mathsf{C})^{++} = \mathcal{E}\mathbb{C}$, so that

$$(\mathcal{C}\mathsf{C})^{++} = (\mathcal{E}\mathbb{C})^{+} = \mathcal{E}\mathbb{C} = (\mathcal{C}\mathsf{C})^{+} \qquad \mathcal{E}\mathsf{C} = \mathcal{D}\mathsf{C} = (\mathcal{C}\mathsf{C})^{+} \qquad \boldsymbol{enig} = \boldsymbol{per}$$

which perhaps indicates why ***enig*** has not been recognized before.

Rule (C) extends the possibilities for such calculations. This rule shows that

$$(\mathcal{C}S)^{(\alpha)} \vdash (<, \omega^{\alpha})$$

for all $\alpha < \Omega$ (where $\Omega = \omega_1$).

Since $\omega^{\Omega} = \Omega$ we might expect that

$$(\mathcal{C}S)^{(\Omega)} \vdash (<, \Omega)$$

but this is false. Suppose S is a countable chain. Thus neither $\mathbb{C}$ nor Ω are supported in S. The above 'result' would show that $\mathcal{E}S = (\mathcal{C}S)^{(\Omega)} = \{\emptyset\}$ and hence $N^{2}\mathcal{O}S$ is boolean. The example $S = \mathbb{Q}$ contradicts this.

To continue we need to generalize an idea used in Section 1.

5.3 DEFINITION. For a separator $\lhd$ on a tree S let

$$x \lhd' y \iff (\forall \theta < \Omega)[x \lhd\text{-supports } \theta \text{ in } {\downarrow} y]$$

to produce a new relation $\lhd'$ on S. ■

Clearly $\lhd'$ is also a separator and

$$x \lhd' y \implies x \lhd y$$

holds. Using this, rule (D) is the way to make an Ω-leap.

As with the analysis of chains, the trick is to iterate the construction $(\cdot)'$. Notice the similarity between this and Definition 1.3.

5.4 DEFINITION. For a tree S let $(<^{\alpha} \mid \alpha \text{ an ordinal})$ be the long sequence of separators on S given by

$$x <^0 y \iff x < y \qquad x <^{\alpha+1} y \iff x \,(<^{\alpha})'\, y \qquad x <^{\lambda} y \iff (\forall \alpha < \lambda)[x <^{\alpha} y]$$

for each ordinal α, limit ordinal λ, and $x, y \in S$. ■

With this (E) completes the list of rules.

To use these rules we need a way of bounding the exponent α on the relation $<^{\alpha}$. Recall that at the end of Section 1 we attached an ordinal measure $\sigma(A)$ to each scattered chain. Here we attach two ordinal measures to a tree.

5.5 DEFINITION. Let S be a tree. We set

$$\sigma(S) = \sup\{\sigma(A) \mid A \text{ is a scattered chain from } S\} \qquad \tau(S) = \Omega\sigma(S) + 1$$

to assign two ordinals to S. Here Ω is ω_1, the first uncountable ordinal. ■

Finally we can prove the main result of this paper.

5.6 THEOREM. *Let S be a tree with $\sigma = \sigma(S)$. Then*

$$X \in \mathcal{B}S \iff (\forall x \in X)\mathfrak{S}(x, X, <^{\sigma}, 2)$$

for each $X \in \mathcal{C}S$, and

$$\mathcal{E}S = \mathcal{B}S = (\mathcal{C}S)^{(\tau)}$$

where $\tau = \tau(S)$.

Proof. Suppose first that $x \in X \in \mathcal{B}S$. Then either x supports $\mathbb{C}$ in X or x supports $\mathbb{Q}$ in X. In the second case there is some $y \in X^x$ such that $x \lhd^{\alpha} y$ for all ordinal α (since this is a property of $\mathbb{Q}$). Thus, in both cases $\mathfrak{S}(x, X, <^{\sigma}, 2)$.

Conversely suppose that $(\forall x \in X)\mathfrak{S}(x, X, <^{\sigma}, 2)$ and consider any $x \in X$. We may assume that x does not support $\mathbb{C}$ in X (for otherwise we are done). Thus there is some $y \in X^x$ such that $A = X^y$ is a chain. We have $\mathfrak{S}(y, X, <^{\sigma}, 2)$, so there is some $z \in X$ with $y <^{\sigma} z$. This shows that S is not scattered, for otherwise, by the remark at the end of Section 1 gives $\sigma < \sigma(A) \leq \sigma(S) = \sigma$. Thus A includes a copy of $\mathbb{Q}$, and hence x supports $\mathbb{Q}$ in X.

In general we have $BS \subseteq ES \subseteq (\mathcal{C}S)^{(\tau)}$. The first part with rule (E) gives $(\mathcal{C}S)^{(\tau)} \subseteq \mathcal{B}S$ which is enough to complete the proof. ■

Recalling the equivalences of Section 3 we see that this result gives us some information about the reflective properties of trees.

5.7 COROLLARY. *For a tree S the frame $\mathcal{O}S$ is reflective if an only if $N^2\mathcal{O}S$ is boolean.*

To conclude it remains to verify the five rules.

Proof of (A). Suppose $\mathcal{G} \vdash (\lhd, \lambda+1)$ and consider any $x \in X \in \mathcal{G}$. We may assume that x does not support a $\mathbb{C}$ obstruction (for otherwise we are done). By hypothesis we have $\mathfrak{S}(x, X, \lhd, \lambda)$, so there is a $\lhd$-chain

$$x \leq x(0) \lhd x(1) \lhd \cdots \lhd x(i) \lhd \cdots \lhd x(\lambda) \qquad (i < \lambda)$$

of elements of X. In particular $x(\lambda) \in X$, so we may iterate to produce a chain

$$x = y(0) < y(1) < \cdots < y(r) < \cdots \qquad (r < \omega)$$

of elements of X where for each $r < \omega$ the interval $[y(r), y(r+1)]$ includes an ascending $\lhd$-chain of length $\lambda + 1$ with $y(r+1)$ as the last element. This whole family of elements is an ascending $\lhd$-chain of length $\lambda\omega$, as required. ■

Proof of (B). Suppose $\mathcal{G} \vdash (\lhd, \lambda)$ (where λ is a limit ordinal) and consider any $x \in X \in \mathcal{G}^+$. By (A) it suffices to show $\mathfrak{S}(x, X, \lhd, \lambda+1)$. We may assume that x does not support a $\mathbb{C}$ obstruction in X, so from Lemma 5.1 there is some $y \in X^x$ such that X^y is a chain. Since $y \in X \in \mathcal{G}^+$, there is some $Y \in \mathcal{G}$ with $y \in Y \subset X$, and hence there is some $z \in X - Y$ with $y < z$. Now $y \in Y \in \mathcal{G}$ so that, by hypothesis, $\mathfrak{S}(y, Y, \lhd, \lambda)$ and hence there is a chain

$$y \leq y(0) \lhd y(1) \lhd \cdots \lhd y(i) \lhd \cdots \qquad (i < \lambda)$$

of elements of Y. Each $y(i)$ is a member of X^y, as is z. Also $z \not\leq y(i)$ (for otherwise $z \in Y$) so that $y(i) < z$. But now, since λ is a limit ordinal, we may set $y(\lambda) = z$ to obtain the required $\lhd$-chain in X of length $\lambda + 1$. ■

Proof of (C). We proceed by induction on α. By (A) and (B) it suffices to consider the leap to a limit ordinal $\lambda < \Omega$.

Suppose $X \in \mathcal{G}^{(\lambda)}$. By the induction hypothesis we have $\mathfrak{S}(z, X, \lhd, \omega^\alpha)$ for each $z \in X$ and each $\alpha < \lambda$. Consider any $x \in X$. As usual we may assume that x does not support a $\mathbb{C}$-obstruction in X, so there is some $y \in X^x$ such that X^y is a chain. Note that for each $z \in X^y$ and each $\alpha < \lambda$ we have that z $\lhd$-supports $\omega^{\alpha+1}$ in X, and hence z $\lhd$-supports $\omega^\alpha + 1$ in X.

Since λ is countable it is the supremum of some ascending chain $(\alpha(r) \mid r < \omega)$ of ordinals. Thus, by the remarks above we can obtain an ascending chain

$$x < y(0) < y(1) < \cdots < y(r) < \cdots \qquad (r < \omega)$$

of elements of X such that for each $r < \omega$ the interval $[y(r), y(r+1)]$ includes an ascending $\lhd$-chain of length $\omega^{\alpha(r)} + 1$ with $y(r+1)$ as the last element. Thus we obtain an ascending $\lhd$-chain of length

$$\omega^{\alpha(0)} + \omega^{\alpha(1)} + \omega^{\alpha(2)} + \cdots = \sup\{\omega^{\alpha(r)} \mid r < \omega\} = \omega^\lambda$$

as required. ■

Proof of (D). Consider any $x \in X \in \mathcal{G}^{(\Omega)+}$. As usual we may assume that x does not support a $\mathbb{C}$-obstruction in X, so there is some $y \in X^x$ such that X^y is a chain. Since $y \in X \in \mathcal{G}^{(\Omega)+}$ we have some $Y \in \mathcal{G}^{(\Omega)}$ with $y \in Y \subsetdot X$, and hence there is some $z \in X - Y$ with $y < z$. We show that $y \lhd' z$.

Consider any $\alpha < \Omega$. Since $y \in Y \in \mathcal{G}^{(\alpha)}$, rule (C) gives us a $\lhd$-chain

$$y < y(0) \lhd y(1) \lhd \cdots \lhd y(i) \lhd \cdots$$

of elements of Y of length ω^α. Each $y(i)$ is a member of the chain X^y, as us z. Also $z \nleq y(i)$ (since $z \notin Y$) so that $y(i) < z$. Thus y $\lhd$-supports ω^α (and hence α) in $\downarrow z$, as required. ■

Proof of (E). We proceed by induction on α. The initial case, $\alpha = 0$, is just the standard description of $\mathcal{D}S$. The induction step, $\alpha \mapsto \alpha + 1$, follows by rule (D) with $\mathcal{G} = (\mathcal{C}S)^{(\Omega\alpha)}$. It remains to make the induction leap to a limit ordinal λ.

Consider any $x \in X \in (\mathcal{C}S)^{(\Omega\lambda)+}$. By the usual reduction we may suppose that X^x is a chain. Now there is some $Y \in (\mathcal{C}S)^{(\Omega\lambda)}$ with $x \in Y \subsetdot X$, and hence some $y \in X - Y$ with x, y. We show that $x <^\lambda y$.

For each $\alpha < \lambda$ we have $x \in Y \in (\mathcal{C}S)^{(\Omega\lambda)} \subseteq (\mathcal{C}S)^{(\Omega\alpha+1)}$ so the induction hypothesis gives some $z \in Y$ with $z <^\alpha z$. But y and z are members of the chain X^x and $y \nleq z$ (since $y \notin Y$) so that $x <^\alpha z < y$, and hence $x <^\alpha y$, as required. ■

This completes the analysis.

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