Modifier Logics Based on Graded Modalities

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Modifier logics are considered as generalizations of "classical" modal logics. Thus modifier logics are so-called multimodal logics. Multimodality means here that the basic logics are modal logics with graded modalities. The interpretation of modal operators is more general, too. Leibniz’s motivating semantical ideas (see [8], p. 20-21) give justification to these generalizations. Semantics of canonical frames forms the formal semantic base for modifier logics. Several modifier systems are given. A special modifier calculus is combined from some "pure" modifier logics. Creating a topological semantics to this special modifier logic may give a basis to some kind of fuzzy topology. Modifier logics of S4-type modifiers will give a graded topological interior operator systems, and thus we have a link to fuzzy topology.

Keywords: graded modality, modifier, modifier logic, modifier calculus

1. Introduction

We suppose that the reader is familiar with usual notation and terminology of logic.

First, consider some fundamental things from history and some motivating things. Lemmon [8, p. 20 - 21] describes Leibniz’s basic ideas for motivating the idea of "classical modal logics". He says: "Leibniz’s suggestion now becomes: a sentence is necessarily true (in this world) if that sentence is true in all worlds alternative to this world." See Lemmon [8, p. 20] to check the detailed analysis about what the concept "alternative to this world" means. Lemmor [8, p.21] continues: "Actually, in many connections it is intuitively simpler to think of world t as accessible from world u rather than alternative to u. This at least has the merit of avoiding the temptation to suppose that alternatives is a symmetric relation between worlds - that if t is alternative to u, then u must be alternative to t. Indeed, we shall not assume that each world is accessible from itself, or even that to each world there is at least one accessible world; there may be accessibility-isolated worlds. We shall find that to many such assumptions about the accessibility relation between worlds there correspond distinctive modal sentences which come ou: valid precisely because we have made those assumptions. If necessity means truth in all accessible worlds, then possibility will mean truth in some accessible world. Thus our remarks about the vagueness of the notion of necessity, and the various more precise accounts of it, may be repeated mutatis mutandis for the notion of possibility.

As we see, intuitive ideas for modal logics start from the concepts necessary and possible. Also these concepts are not truth-functional, because considering truth in a world needs also other worlds accessible from that world. Already Aristotle considered the question of presuppositions for a sentence be necessarily true, or possibly true. He somehow gave probabilistic meanings to these concepts. If we interpret the concept "necessary" to be "certain", the probabilistic meaning would be more clear. But this interpretation is just a special case. The reason for this may be the fact that these modal concepts has closely been related to many-valuedness already in Aristotle’s time - and also cue to him. Also Lukiskiewicz continued this interpretation. He used Aristotle’s way to motivate the idea of his three-valued logic with probabilistic examples where the concept “true” (may be accidentally) had the fundamental role in such a way that tomorrow we can see, whether the sentence "Tomorrow we will have a naval battle" were true or not yesterday. Before yesterday it is uncertain. Anyway, the time is actually not present in the formalism of these logics. Also early considering of modal logics did not include the deontic aspect. As we have learned later, a deontic logic needs some further operators in addition to the usual ones. As we see above. Lemmon says that vagueness is associated with the notion of necessity. As we know, vagueness does not mean the same as probability. It contains also features of fuzziness. When we consider mathematical results from above mentioned intuitive ideas, i.e. formal semantics of modal logics, especially canonical frames, we see that these results are more general than just the ideas about mathematical models of necessity and possibility. The author has strongly come into the thought: that the most general linguistic interpretations for operators resulting from the formal semantics equipped with the considerations above are substantiating operator and weakening operator. We can also call them substantiating modifier and weakening modifier, respectively. The concepts necessity and possibility are corresponding instances of these modifiers. Thus we have for example modifiers of T-like, S4-style, S5-style etc. corresponding modal systems T, S4, S5 etc., respectively. These labels or names do not have any special interpretations, like probabilistic one, as their burdens. The mathematical analysis of the formal semantics does not take any such interpretations for granted.

The term modifier appears in many different connections and in many different meaning. Here we use it in the meaning of either a substantiating or weakening op-
erator having some fuzzy features. For example, some hedges can be modifiers in our purposes.

**Example.** Consider a situation that illustrates modifiers, accessibility relation, and possible worlds. We describe what kind of spices we can associate with the sentence

(a) "Mr. M can speak Japanese." For example, we can operate with hedges "well" and 'more or less well" to this sentence. Thus we have

(b) "Mr. M can speak Japanese well."

This sentence presupposes something more from Mr. M's ability to speak Japanese. The other hedge gives

(c) "Mr. M can speak Japanese more or less well."

This sentence does not presuppose so much from Mr. M as the sentence (a). The hedges "well" and "more or less well" modify the original sentence (a). The first hedge "well" is apparently substantiating and the second one "more or less well" is weakening. If this separation with hedges is too rough, we can make it more dense by adding some further modifiers, say, "rather well", "very well", "extremely not well", and "extremely well". So, we have the set of substantiating modifiers

$$\text{Mod} = \{\text{well, very well, extremely well}\}$$

And the corresponding set of weakening modifiers

$$\text{Mod}^* = \{\text{more or less well, rather well, extremely not well}\}$$

Now we have more sentences associated with (a):

(d) "M: M can speak Japanese rather well."

(e) "Mr. M can speak Japanese very well."

(f) 'Mr. M can speak Japanese extremely well."

(g) "Mr. M can speak Japanese extremely not well."

We may consider the hedges in Mod* to be the duals of those in Mod, i.e. the dual of "well" is (well)* = not well, that of "more or less well" is (more or less well)* = "very well", that of "extremely well" is (extremely well)" = 'extremely not well'! We can translate this into propositional modifier language (see the definitions below) as follows. The set of non-weakening modifiers is

$$\text{Mod} = \{F_0, F_1, F_2, F_3\}$$

i.e. $F_0$ stands for "well", $F_1$ for "very well", and $F_3$ for "extremely well", $F_0$ is the identifying operator. We need it mainly for technical reasons. The set of non-substantiating modifiers is

$$\text{Mod}^* = \{F_0^*, F_1^*, F_3^*, F_3^*\}$$

i.e. $F_1^*$ stands for "more or less well", $F_3^*$ for "rather well", and $F_3^*$ for "extremely not well". $F_0^*$ is again the identifying operator. Identifying operators in modifier logic are self-dual, i.e. $F_0$ is the dual of itself. Thus $F_0 = F_3^*$. See formal bases below. Next. Consider some possible worlds related to this case. These worlds can be sets of different situations, where natural language is needed, and especially, in this case the language is Japanese. Suppose we have a set of possible worlds, say,

$$W = \{w_1, w_2, \ldots, w_n\}.$$