

On the Undecidability of Logics with Converse, Nominals, Recursion and Counting

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Abstract

The evolution of Description Logics (DLs) and Propositional Dynamic Logics produced a hierarchy of decidable logics with multiple maximal elements. It would be desirable to combine different maximal logics into one super-logic, but then inference may turn out to be undecidable. Then it is important to characterize the decidability threshold for these logics. In this perspective, an interesting open question pointed out by Sattler and Vardi [20] is whether inference in a hybrid μ -calculus with restricted forms of graded modalities is decidable, and which complexity class it belongs to. In this paper we improve a previous result [5] and prove that this calculus and the corresponding DL $\mu\mathcal{ALCCIO}_{fa}$ are undecidable. We show also that nested fixpoints are not necessary for undecidability.

Key words: Description logics, hybrid μ -calculus, regular roles, graded modalities, number restrictions.

1 Introduction

Description logics are popular knowledge representation languages, with important applications to the semantic web, software engineering and heterogeneous databases [1]. Description logics (DLs) are strictly related to propositional dynamic logics (PDLs) [19,9], that play an important role in software and protocol verification based on automated reasoning techniques. The analogies between the two frameworks are so tight that DLs and PDLs can be regarded as syntactic variants of the same family of logics.

The simplest DLs can be easily embedded into a fragment of L^2 , that is, first-order logic with two variables. Application requirements led researchers

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to extend these basic logics with more expressive constructs, such as generalized forms of quantification—called *number restrictions* in DLs and *graded modalities* in PDL [11,7]—, fixpoints and *nominals* [13,17,18,12,10,3]. In DLs, nominals provide a means to denote individuals, while in PDLs nominals are interpreted as unique labels for possible worlds. Modal logics with nominals are called *hybrid*.

The search for a tradeoff between expressiveness and complexity produced—and keeps on extending—a hierarchy of decidable logics with multiple maximal elements. Currently, two of the maximal decidable DLs are $\mu\mathcal{ALC}\mathcal{IO}$ (featuring fixpoints and nominals [20,4]) and $\mu\mathcal{ALC}\mathcal{Q}$ (featuring fixpoints and number restrictions). The corresponding PDLs are the *hybrid μ -calculus* and the *μ -calculus with graded modalities*, respectively [20,16].

Of course, it would be desirable to combine the features of different maximal logics into one super-logic. For example, a combination of $\mu\mathcal{ALC}\mathcal{IO}$ and $\mu\mathcal{ALC}\mathcal{Q}$ would help in describing the functional behavior of e-Services (cf. [4] and related comments on $\mathcal{SDL}(\mathcal{X})$ in Section 4). However, in the super-logic, inference may turn out to be too complex, and in particular undecidable. Therefore, it is important to investigate the decidability threshold for this family of logics.

A related, interesting question pointed out by Sattler and Vardi [20] is whether inference in the union of the hybrid μ -calculus and the μ -calculus with graded modalities is decidable, and which complexity class it belongs to. More precisely, Sattler and Vardi mention a slightly simpler logic: a hybrid μ -calculus with *deterministic programs*. Deterministic programs are a special case of graded modality, whose counterpart in DLs are *features*, i.e., functional roles.

A partial negative answer to the above question was given in [5]. There, it was proved that the description logic $\mu\mathcal{ALC}\mathcal{IO}_f$, supporting nominals, fixpoints and injective functional roles is undecidable. Injective roles were obtained by declaring inverse roles to be functional. In this paper we strengthen this result in several ways:

- We prove that even if functionality assertions are restricted to atomic roles only, roles can be forced to be injective, at least over an infinite subdomain.
- With this result, the description logic $\mu\mathcal{ALC}\mathcal{IO}_{fa}$, featuring fixpoints, nominals and functionality assertions over *atomic* roles is proved to be undecidable. This result is then rephrased for the corresponding dynamic logic, namely, a hybrid μ -calculus with converse programs, where only atomic programs can be declared to be deterministic.
- We prove that nested fixpoints are not necessary for undecidability. The undecidability proof in [5], on the contrary, made use of nested fixpoints.

In the next section we recall the basic notions about DLs and the μ -calculi.

Section 3 is devoted to the undecidability proof for $\mu\mathcal{ALCTO}_{fa}$ and the hybrid μ -calculus with deterministic atomic programs. Finally, Section 4 concludes the paper with a discussion of these results and some directions for further research.

2 Preliminaries

The vocabulary of the description logics we deal with in this paper consists of the following pairwise disjoint countable sets of symbols: a set of *atomic concepts* \mathbf{At} , a set of *nominals* \mathbf{Nom} , a set of *concept variables* \mathbf{Var} , and a set of *atomic roles* \mathbf{AR} .

The set of *roles* is the smallest superset of \mathbf{AR} such that if R, R' are roles then R^- , $R \sqcup R'$, and R^+ are roles.

Let R be a role, $X \in \mathbf{Var}$ and $n \in \mathbb{N}$. The set of *concepts* is the smallest superset of $\mathbf{At} \cup \mathbf{Nom} \cup \mathbf{Var}$ such that if C, C', D are concepts, then $\neg C$, $C \sqcap D$, $\exists.C$, $\exists^{\leq n} R.C$, and $\mu X.C'$ are concepts, provided that all the free occurrences¹ of X in C' lie within the scope of an even number of operators \neg and $\exists^{\leq n}$.² A concept is *closed* iff it has no free occurrences of any variable.

Semantics is based on interpretations of the form $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where $\Delta^{\mathcal{I}}$ is a set of *individuals* and $\cdot^{\mathcal{I}}$ is an interpretation function mapping each $A \in \mathbf{At} \cup \mathbf{Nom}$ on some $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and each $R \in \mathbf{AR}$ on some $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Furthermore, nominals must be mapped on singletons. A *valuation on \mathcal{I}* is a function $\rho : \mathbf{Var} \rightarrow \wp(\Delta^{\mathcal{I}})$. As usual, $\rho[X/S]$ denotes the valuation such that $\rho[X/S](X) = S$ and for all $Y \neq X$, $\rho[X/S](Y) = \rho(Y)$. The meaning of inverse roles is

$$(R^-)^{\mathcal{I}} = \{ \langle y, x \rangle \mid \langle x, y \rangle \in R^{\mathcal{I}} \},$$

while $(R \sqcup R')^{\mathcal{I}} = R^{\mathcal{I}} \cup R'^{\mathcal{I}}$, and $(R^+)^{\mathcal{I}}$ denotes the transitive closure of $R^{\mathcal{I}}$.

The meaning of compound concepts is determined by pairs (\mathcal{I}, ρ) . By $\#S$ we

¹ A variable occurrence is free if it is not in the scope of operator μ .

² Usually, nesting of $\exists^{\leq n}$ and role operators is restricted, too, in order to avoid compound expressions that easily lead to undecidability. We do not restrict syntax; however, in our results we apply $\exists^{\leq n}$ only to atomic roles.

denote the cardinality of a set S .

$$\begin{aligned}
A_\rho^\mathcal{I} &= A^\mathcal{I} & (A \in \text{At} \cup \text{Nom}) & & (\neg C)_\rho^\mathcal{I} &= \Delta^\mathcal{I} \setminus C_\rho^\mathcal{I} \\
X_\rho^\mathcal{I} &= \rho(X) & (X \in \text{Var}) & & (C \sqcap D)_\rho^\mathcal{I} &= C_\rho^\mathcal{I} \cap D_\rho^\mathcal{I} \\
(\exists R.C)_\rho^\mathcal{I} &= \{x \mid \exists y. \langle x, y \rangle \in R^\mathcal{I} \wedge y \in C_\rho^\mathcal{I}\} \\
(\exists^{\leq n} R.C)_\rho^\mathcal{I} &= \{x \mid \#\{y \mid \langle x, y \rangle \in R^\mathcal{I} \wedge y \in C_\rho^\mathcal{I}\} \leq n\} \\
(\mu X.C)_\rho^\mathcal{I} &= \bigcap \{S \subseteq \Delta^\mathcal{I} \mid C_{\rho[X/S]}^\mathcal{I} \subseteq S\}.
\end{aligned}$$

Sometimes, subscript ρ will be omitted when it applies to a closed concept (i.e., such that all variables are bound by μ).

In order to improve readability, given a concept $C(X_1, \dots, X_n)$ with free variables X_1, \dots, X_n , and given sets of individuals of $\Delta^\mathcal{I}$, S_1, \dots, S_n , we shall abbreviate $C(X_1, \dots, X_n)_{\rho[X_1/S_1] \dots [X_n/S_n]}^\mathcal{I}$ with $C_\rho^\mathcal{I}(S_1, \dots, S_n)$.

A concept C is *satisfiable* iff there exists an interpretation \mathcal{I} such that $C^\mathcal{I} \neq \emptyset$.

Other standard constructs can be derived from the above concepts. We use the symbol \triangleq to define abbreviations.

$$\begin{aligned}
\top &\triangleq A \sqcup \neg A \quad (\text{for some } A \in \text{At}) & \forall R.C &\triangleq \neg \exists R. \neg C \\
\perp &\triangleq \neg \top & \exists^{\geq n+1} R.C &\triangleq \neg \exists^{\leq n} R.C \\
C \sqcup D &\triangleq \neg(\neg C \sqcap \neg D) & \nu X.C &\triangleq \neg \mu X. \neg C[X/\neg X].
\end{aligned}$$

Here $C[X/\neg X]$ is the concept obtained from C by replacing all free occurrences of X with $\neg X$.

The syntactic restriction on concept variables makes every concept $C(X_1, \dots, X_n)$ with free variables X_1, \dots, X_n monotonic with respect to X_1, \dots, X_n , that is, for all \mathcal{I} and ρ , if $S_i \subseteq S'_i$ for $1 \leq i \leq n$, then $C_\rho^\mathcal{I}(S_1, \dots, S_n) \subseteq C_\rho^\mathcal{I}(S'_1, \dots, S'_n)$. Under this restriction, $\mu X.C(X)$ and $\nu X.C(X)$ denote exactly the least and the greatest fixpoints of $C(X)$, that can be characterized with the standard iterative constructions. In particular, $(\mu X.C(X))_\rho^\mathcal{I} = \bigcup_{\alpha < \beta} X_\alpha$, where β is a suitable ordinal and $X_0, \dots, X_\alpha, \dots \subseteq \Delta^\mathcal{I}$ is the monotonically nondecreasing, transfinite sequence defined below:

$$\begin{aligned}
X_0 &= \emptyset \\
X_{\alpha+1} &= C_\rho^\mathcal{I}(X_\alpha) \\
X_\lambda &= \bigcup_{\alpha < \lambda} X_\alpha \quad \text{where } \lambda \text{ is a limit ordinal.}
\end{aligned}$$

If C is *continuous in X* , then $\beta = \omega$ (where ω denotes the least transfinite ordinal). Recall that C is continuous in X iff for all monotonic sequences $X_0, \dots, X_i, \dots \subseteq \Delta^{\mathcal{I}}$ with $i < \omega$,

$$C_{\rho}^{\mathcal{I}}\left(\bigcup_{i \geq 0} X_i\right) = \bigcup_{i \geq 0} C_{\rho}^{\mathcal{I}}(X_i). \quad (1)$$

The following lemma will be needed later on. Recall that a concept is in *negation normal form* (NNF) iff negation is applied only to atomic concepts.

Lemma 1 *If C is in NNF and contains only the operators $\neg, \sqcap, \sqcup, \exists$, then C is continuous in every variable X .*³

Proof. By structural induction on C . The base case ($C \in \text{At} \cup \text{Nom} \cup \text{Var}$) is trivial. For the induction step, let $X_0, \dots, X_i, \dots \subseteq \Delta^{\mathcal{I}}$ be any increasing chain, and consider all the possible shapes of C .

If $C = \neg A$, then A is an atom (by hypothesis), and syntactic restrictions impose that $A \notin \text{Var}$, that is, $C \in \text{At} \cup \text{Nom}$. Then $C_{\rho[X/S]}^{\mathcal{I}} = A^{\mathcal{I}}$ is constant w.r.t. S and hence equation (1) is trivially satisfied.

If $C = D \sqcap E$, then

$$\begin{aligned} C_{\rho[X/\bigcup_{i \geq 0} X_i]}^{\mathcal{I}} &= D_{\rho[X/\bigcup_{i \geq 0} X_i]}^{\mathcal{I}} \cap E_{\rho[X/\bigcup_{i \geq 0} X_i]}^{\mathcal{I}} = \\ &= \bigcup_{i \geq 0} D_{\rho[X/X_i]}^{\mathcal{I}} \cap \bigcup_{i \geq 0} E_{\rho[X/X_i]}^{\mathcal{I}} \quad (\text{by induction hyp.}) \\ &= \bigcup_{i \geq 0} \left(D_{\rho[X/X_i]}^{\mathcal{I}} \cap E_{\rho[X/X_i]}^{\mathcal{I}} \right) \quad (\text{by monotonicity of } D, E, \{X_i\}_{i \geq 0}) \\ &= \bigcup_{i \geq 0} \left(C_{\rho[X/X_i]}^{\mathcal{I}} \right) \end{aligned}$$

so equation (1) holds. The proof for $C = D \sqcup E$ is similar.

Finally, let $C = \exists R.D$. Then,

$$\begin{aligned} C_{\rho[X/\bigcup_{i \geq 0} X_i]}^{\mathcal{I}} &= \left\{ x \mid \text{there exists } \langle x, y \rangle \in R^{\mathcal{I}} \text{ s.t. } y \in D_{\rho[X/\bigcup_{i \geq 0} X_i]}^{\mathcal{I}} \right\} = \\ &= \left\{ x \mid \text{there exists } \langle x, y \rangle \in R^{\mathcal{I}} \text{ s.t. } y \in \bigcup_{i \geq 0} D_{\rho[X/X_i]}^{\mathcal{I}} \right\} \quad (\text{by ind. hyp.}) \\ &= \bigcup_{i \geq 0} \left\{ x \mid \text{there exists } \langle x, y \rangle \in R^{\mathcal{I}} \text{ s.t. } y \in D_{\rho[X/X_i]}^{\mathcal{I}} \right\} \end{aligned}$$

³ Recall that C must always satisfy the syntactic restriction on concept variables.

$$= \bigcup_{i \geq 0} C_{\rho[X/X_i]}^{\mathcal{I}}.$$

This completes the proof. \square

Remark 2 *The above lemma can be extended to concepts with least fixpoints. We do not include the extended version here because we are deliberately avoiding any use of nested fixpoints.*

An *assertion* has the form $C \sqsubseteq D$, where C and D are closed concepts. Assertion $C \sqsubseteq D$ is satisfied by \mathcal{I} (equivalently, \mathcal{I} is a model of the assertion) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. A *TBox* is a finite set of assertions. Symmetric pairs of assertions such as $C \sqsubseteq D$ and $D \sqsubseteq C$ will be abbreviated by $C \equiv D$. A TBox is *satisfiable* iff it has a model, that is, an interpretation \mathcal{I} that satisfies all the assertions in the TBox. A TBox \mathcal{T} *entails* $C \sqsubseteq D$ if every model of \mathcal{T} satisfies $C \sqsubseteq D$.

The description logic \mathcal{ALC} is a fragment of the logic described so far, freely generated by atomic concepts, atomic roles, \neg , \sqcap and $\exists R.C$ (plus all the constructs definable from these). In \mathcal{ALC} , $\text{Nom} = \emptyset$.

By convention, the name of a description logic contains \mathcal{ALC} if the logic extends \mathcal{ALC} . Moreover, the name contains an \mathcal{I} if inverse roles (R^-) are supported, an \mathcal{O} if $\text{Nom} \neq \emptyset$, a \mathcal{Q} if number restrictions ($\exists^{\leq n} R.C$) are supported, and a μ if fixpoints are supported. For example, $\mathcal{ALC}\mathcal{I}\mathcal{O}$ denotes the extension of \mathcal{ALC} with inverse roles and nominals. Subscript f , as in $\mu\mathcal{ALC}\mathcal{I}\mathcal{O}_f$, indicates that all roles are functions. Note that functional roles are a special case of number restriction, as they can be expressed with axioms of the form $\top \sqsubseteq \exists^{\leq 1} R.\top$. We use subscript fa to specify that these assertions (equivalently, functionality restrictions) are applied to all the atomic roles and no compound role. If the logic supports role operators besides inversion, we list those operators as superscripts. For example $\mathcal{ALC}\mathcal{I}^{\cup,+}$ denotes the extension of $\mathcal{ALC}\mathcal{I}$ with union and transitive closure over roles.

Description logics can be regarded as variants of the propositional μ -calculi. Individuals correspond to possible worlds and roles correspond to accessibility relations. Atomic concepts play the role of propositional symbols. In particular, $\mu\mathcal{ALC}\mathcal{I}\mathcal{O}_{fa}$ can be embedded into the hybrid μ -calculus with deterministic atomic programs and graded modalities ($\langle n, P \rangle F$ and $[n, P]F$) via the following satisfiability-preserving translation [19,9]. For all propositions p , and for all $n > 0$,

$$\begin{aligned} \varepsilon(p) &= p & \varepsilon(F \sqcap G) &= \varepsilon(F) \wedge \varepsilon(G) \\ \varepsilon(\neg F) &= \neg \varepsilon(F) & \varepsilon(\exists^{\geq n} P.F) &= \langle n-1, P \rangle \varepsilon(F) \\ \varepsilon(\mu X.F) &= \mu X.\varepsilon(F). \end{aligned}$$

Moreover, functional roles are mapped on deterministic programs (whose accessibility relation is a function), and nominals are mapped on their equivalents (called nominals, too), that in PDL terms are propositional symbols that are true in exactly one world. Program o denotes the universal program whose accessibility relation consists of all pairs of possible worlds. The reader is referred to [20,16,8] for further details.

3 Undecidability of $\mu\mathcal{ALCCIO}_{fa}$ and of the corresponding hybrid μ -calculus

This section is devoted to the proof of the following theorem.

Theorem 3 *In $\mu\mathcal{ALCCIO}_{fa}$, concept satisfiability, TBox satisfiability and TBox entailment are all undecidable.*

We find it convenient to prove this theorem by first reducing domino problems to TBox satisfiability, and then extending this result to the other decision problems.

Recall that domino problems consist in placing tiles on an infinite grid, satisfying a given set of constraints on adjacent tiles. Formally, a domino problem is a structure $\mathcal{D} = \langle T, H, V \rangle$, where T is a finite set of *tile types* and $H, V \subseteq T^2$ specify which tiles can be adjacent horizontally and vertically, respectively. A *solution to \mathcal{D}* is a *tiling*, that is, a function $\tau : \mathbb{N}^2 \rightarrow T$, such that

- (1) if $\tau(x, y) = t$ and $\tau(x + 1, y) = t'$ then $(t, t') \in H$, and
- (2) if $\tau(x, y) = t$ and $\tau(x, y + 1) = t'$ then $(t, t') \in V$.

The existence of a solution for a given domino problem is known to be undecidable (cf. [2]).

Domino problems are reduced to reasoning problems by characterizing (i) the grid and (ii) correct tilings. Formally, the grid is a structure $\mathcal{G} = \langle \mathbb{N}^2, h^{\mathcal{G}}, v^{\mathcal{G}} \rangle$, where $h^{\mathcal{G}}(x, y) = (x + 1, y)$, and $v^{\mathcal{G}}(x, y) = (x, y + 1)$, for all $x, y \in \mathbb{N}$.

In description logics, $h^{\mathcal{G}}$ and $v^{\mathcal{G}}$ can be denoted by two roles. For technical reasons, we use roles l^- and v , respectively (l stands for “left”, v for “vertical”). If the two roles characterize the grid correctly (see Figure 1), then characterizing the solutions of a domino problem is easy, even within simple (and decidable)

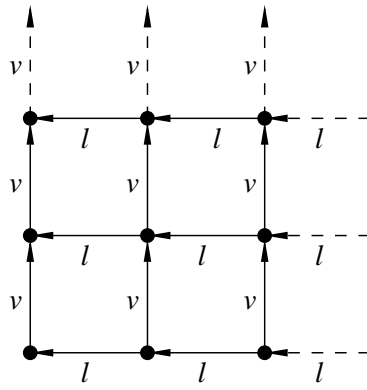


Fig. 1. Modeling the grid \mathcal{G} with roles v and l

description logics such as \mathcal{ALC} , by means of the following assertion:

$$\begin{aligned}
\top \sqsubseteq & \left(\bigsqcup_{t \in T} C_t \right) & (2) \\
& \sqcap \left(\bigsqcap_{t \in T} \bigsqcap_{t' \in T \setminus \{t\}} \neg(C_t \sqcap C_{t'}) \right) \sqcap \\
& \bigsqcap_{t \in T} \left[\neg C_t \sqcup \left(\exists l^-. \bigsqcup_{(t,t') \in H} C_{t'} \sqcap \exists v. \bigsqcup_{(t,t') \in V} C_{t'} \right) \right].
\end{aligned}$$

Here for each tile type t , a distinct concept name C_t is introduced. Assertion (2) basically states that each individual is a tile (first line), that each tile has one type (second line), and that the tiling preserves the constraints specified by H and V (third line).

The real problem is characterizing the grid, because there is no direct way to force l^- and v to commute. Here we shall provide a projective characterization of the grid variant $\mathcal{G}' = \langle \mathbb{N}, (h^-)^{\mathcal{G}}, v^{\mathcal{G}} \rangle$ illustrated in Figure 1, that is, we shall capture a class of expanded interpretations (i.e., interpretations defined over a set of roles larger than $\{h, v\}$) whose projection over l and v is isomorphic to \mathcal{G}' . Intuitively, this means simply that we are going to use auxiliary roles to model the grid.

Informally speaking, we are going to check whether the horizontal and vertical roles commute by means of a fixpoint whose constructive characterization corresponds to a visit of the grid along diagonals directed north-west (Figure 4).

The next subsection is devoted to the proof of some auxiliary technical lemmas. Then Section 3.2 contains the main results (grid characterization and undecidability).

3.1 Technical lemmas

First some terminology and notation.

The *restriction* of a binary relation R to a set D , denoted by $R \downarrow D$, is the relation

$$R \downarrow D = \{\langle x, y \rangle \mid \langle x, y \rangle \in R \text{ and } x \in D\}.$$

Given an interpretation \mathcal{I} and a role R , an R -path from x_0 to x_n is a sequence x_0, \dots, x_n such that $\langle x_i, x_{i+1} \rangle \in R^{\mathcal{I}}$, for $0 \leq i < n$.

Given an interpretation \mathcal{I} , the set of individuals R -reachable from a concept C , denoted by $\text{reach}(C, R)$, is the set of all x such that there exists an R -path from y to x with $y \in C^{\mathcal{I}}$. Sometimes, in the following, we will slightly abuse notation and denote with a nominal the unique member of its extension.

Proposition 4 *For all concepts C and roles R , $\text{reach}(C, R)$ is definable in $\mu\mathcal{ALCCTO}_{fa}$.*

Proof. It is immediate to see that $\text{reach}(C, R)$ is defined by

$$\mu X.[C \sqcup \exists R^- .X].$$

□

Henceforth, we shall sometimes abuse notation and abbreviate the fixpoint $\mu X.[C \sqcup \exists R^- .X]$ to $\text{reach}(C, R)$. The context shall clarify whether a specific occurrence of $\text{reach}(C, R)$ denotes the syntactic or the semantic notion.

We say that a set $S \subseteq \Delta^{\mathcal{I}}$ is an *isolated R -chain from x_0* if there exists an enumeration x_0, \dots, x_i, \dots ($i \geq 0$) of S satisfying the following conditions:

- I1. $\langle x_i, x_{i+1} \rangle \in R^{\mathcal{I}}$, for all consecutive elements x_i, x_{i+1} in the sequence,
- I2. there is no element $x \in \Delta^{\mathcal{I}}$ such that $\langle x, x_0 \rangle \in R^{\mathcal{I}}$,
- I3. $R^{\mathcal{I}} \downarrow S$ is a function,
- I4. $(R^-)^{\mathcal{I}} \downarrow S$ is a function.

Lemma 5 *Given a nominal N and a functional atomic role R , the class of all interpretations where $\text{reach}(N, R)$ is an infinite isolated R -chain from N can be characterized in $\mu\mathcal{ALCCTO}_{fa}$.*

Proof. The desired class of interpretations can be characterized with the following assertions:

$$N \sqsubseteq \forall R^- . \perp \quad (3)$$

$$\text{reach}(N, R) \sqsubseteq \exists R . \top \quad (4)$$

$$\text{reach}(N, R) \sqsubseteq \mu Y . [N \sqcup (\exists R^- . Y \sqcap \forall R^- . Y)]. \quad (5)$$

We have to prove that an arbitrary interpretation \mathcal{I} is a model of the above assertions iff $\text{reach}(N, R)$ in \mathcal{I} is an infinite isolated R -chain from N .

First let \mathcal{I} be an arbitrary model of the assertions. To simplify notation, in the following we identify each concept C with its interpretation $C^{\mathcal{I}}$. Recall that $\text{reach}(N, R)$ can be characterized by the concept $\mu X . [N \sqcup \exists R^- . X]$. It is not hard to see that the following sequence equals the standard iterative characterization of this fixpoint.

$$\begin{aligned} X_0 &= \emptyset \\ X_1 &= N \\ X_{\alpha+1} &= X_\alpha \sqcup \exists R^- . X_\alpha \quad (\alpha > 0) \\ X_\lambda &= \bigcup_{\alpha < \lambda} X_\alpha \quad (\lambda \text{ a limit ordinal}). \end{aligned} \quad (6)$$

By Lemma 1, $X_{\omega+1} = X_\omega$, that is, X_ω is the least fixpoint of the above sequence. By a straightforward induction it can be verified that for all ordinals $i < \omega$,

$$X_i = \{x_0, \dots, x_{i-1}\} \quad (7)$$

where $x_0 = N$ and each x_{j+1} ($0 \leq j < i$) is the unique element (by the functionality of R) such that $\langle x_j, x_{j+1} \rangle \in R$.

From the above properties of the sequence we immediately derive I1. Condition I2 is enforced by assertion (3). Condition I3 is an immediate consequence of the functionality of R . So we are left to prove that (i) the set $\{x_0, \dots, x_i, \dots\}$ is infinite (i.e., the sequence x_0, \dots, x_i, \dots is acyclic), and (ii) I4 holds. For this purpose, consider the following sequence, that equals the standard iterative construction of the fixpoint occurring in the right-hand side of (5).

$$\begin{aligned} Y_0 &= \emptyset \\ Y_1 &= N \\ Y_{\alpha+1} &= Y_\alpha \sqcup (\exists R^- . Y_\alpha \sqcap \forall R^- . Y_\alpha) \quad (\alpha > 0) \\ Y_\lambda &= \bigcup_{\alpha < \lambda} Y_\alpha \quad (\lambda \text{ a limit ordinal}). \end{aligned} \quad (8)$$

We claim that for all ordinals α , $Y_\alpha = X_\alpha$. The proof is by natural induction

on the above sequence. The claim is obvious for $\alpha \in \{0, 1\}$, and an immediate consequence of the induction hypothesis for all limit ordinals. Now consider the induction step for a successor ordinal $\alpha + 1$ with $\alpha > 0$. We have $Y_\alpha = X_\alpha$ by induction hypothesis. Then a simple inspection of (8) and (6) shows that

$$Y_{\alpha+1} \setminus Y_\alpha \subseteq (\exists R^-.X_\alpha \cap \forall R^-.X_\alpha) \subseteq (\exists R^-.X_\alpha) \subseteq X_{\alpha+1} \setminus X_\alpha. \quad (9)$$

Moreover, $X_{\alpha+1} \setminus X_\alpha$ contains at most one element, because for $\alpha + 1 < \omega$ we have $X_{\alpha+1} \setminus X_\alpha \subseteq \{x_\alpha\}$ by (7), and for $\alpha + 1 > \omega$, $X_{\alpha+1} \setminus X_\alpha = \emptyset$, since X_ω is the least fixpoint of the sequence $\{X_i\}_{i \geq 0}$. Then there are only two possibilities: either $Y_{\alpha+1} \setminus Y_\alpha = X_{\alpha+1} \setminus X_\alpha$, or $Y_{\alpha+1} \setminus Y_\alpha = \emptyset$. In the former case, the claim follows easily from the induction hypothesis. In the latter case, $Y_{\alpha+1}$ is the least fixpoint of the sequence $\{Y_\alpha\}_{\alpha \geq 0}$ and hence, by assertion (5),

$$X_{\alpha+1} \subseteq \mu X.[N \sqcup \exists R^-.X] \subseteq \mu Y.[N \sqcup (\exists R^-.Y \cap \forall R^-.Y)] = Y_{\alpha+1}.$$

The opposite inclusion ($Y_{\alpha+1} \subseteq X_{\alpha+1}$) follows easily from (9). This completes the proof of the claim.

Now we can prove that the sequence x_0, \dots, x_i, \dots is acyclic (and hence infinite). Suppose not, and let m be the least index such that $x_m = x_{m+k}$, for some $k > 0$. The cycle includes the edge $\langle x_{m+k-1}, x_m \rangle = \langle x_{m+k-1}, x_{m+k} \rangle \in R$, and hence,

$$\langle x_m, x_{m+k-1} \rangle \in R^-.$$

A first consequence is that $m > 0$, otherwise assertion (3) would be false (a contradiction). Now, by (7) and the claim, $x_m \in X_{m+1} = Y_{m+1}$. Note that by the minimality of m , x_0, \dots, x_m are pairwise distinct, and this implies $Y_{m+1} \setminus Y_m = \{x_m\}$. By (9)—that can be applied because $m > 0$ —it follows that x_m belongs to $\forall R^-.X_m$. But then, x_{m+k-1} belongs to $X_m = \{x_0, \dots, x_{m-1}\}$, and the minimality of m is contradicted. This proves that x_0, \dots, x_i, \dots is acyclic and hence the corresponding set of elements is infinite.

Using these properties, we can finally prove I4. Suppose that it does not hold, and let x_n be an element of $\text{reach}(N, R)$ that violates it, that is, for some pair of distinct elements y and z in $\Delta^{\mathcal{I}}$, both $\langle x_n, y \rangle \in R^-$ and $\langle x_n, z \rangle \in R^-$. By analogy with the acyclicity proof, it can be shown that assertion (3) ensures that $n > 0$ and then (9) implies $x_n \in (\forall R^-.X_n)$, therefore $y = x_j$ and $z = x_k$ for some $j \neq k$ smaller than n . Since R is functional, it follows that $x_{j+1} = x_n = x_{k+1}$, with $j + 1 \neq k + 1$. But then the sequence would be cyclic, a contradiction. This completes the proof that each model \mathcal{I} of the assertions belongs to the desired class of interpretations.

We are left to show that every interpretation where $\text{reach}(N, R)$ is an infinite isolated R -chain from N satisfies assertions (3)–(5). Let \mathcal{I} be such an interpretation.

By I2, there is no element x such that $\langle x, x_0 \rangle \in R$ and, therefore there is no element x such that $\langle N, x \rangle \in R^-$, thus assertion (3) is satisfied.

Since $\text{reach}(N, R)$ is infinite, I1 implies that for each element $x_i \in \text{reach}(N, R)$, there exists y such that $\langle x_i, y \rangle \in R$, thus assertion (4) is satisfied.

Next recall that $\text{reach}(N, R)$ is a shorthand for $\mu X.[N \sqcup \exists R^-.X]$. Then (5) is $\mu X.[N \sqcup \exists R^-.X] \sqsubseteq \mu X.[N \sqcup (\exists R^-.X \sqcap \forall R^-.X)]$. By I4, we have that for any $x_i \in \text{reach}(N, R)$, there exists one and only one $y \in \Delta^{\mathcal{I}}$ such that $\langle x_i, y \rangle \in R^-$. Hence, for any subset $X \sqsubseteq \text{reach}(N, R)$, $\exists R^-.X$ equals $\exists R^-.X \sqcap \forall R^-.X$. It follows easily that assertion (5) is satisfied. \square

Lemma 6 *Given a nominal N and a functional atomic role R , the class of interpretations where $\text{reach}(N, R^-)$ is an infinite isolated R^- -chain from N , can be projectively characterized in $\mu\mathcal{ALCCIO}_{fa}$.*

Proof. The projective characterization is based on an auxiliary functional role P such that the set $\text{reach}(N, P)$ is an infinite isolated P -chain from N , and such that $P \downarrow \text{reach}(N, R^-)$ equals $R^- \downarrow \text{reach}(N, R^-)$.

By Lemma 5, we can characterize all the interpretations where $\text{reach}(N, P)$ is an infinite isolated chain from N , by means of a set Γ of $\mu\mathcal{ALCCIO}_{fa}$ assertions. The additional assertions that force P and R^- to coincide in $\text{reach}(N, R^-)$ are the following:

$$\begin{aligned} \text{reach}(N, P) &\sqsubseteq \mu X.[N \sqcup ((\exists P^-.X) \sqcap (\exists R^-.X) \sqcap (\exists P^-. \forall R^-. \exists P^-.X))] & (10) \\ \text{reach}(N, R^-) &\sqsubseteq \text{reach}(N, P) & (11) \\ N &\sqsubseteq \forall R^-. \perp. & (12) \end{aligned}$$

First we show that in any model \mathcal{I} of Γ and (10)–(12), the set $\text{reach}(N, R^-)$ is an infinite isolated R^- -chain from N . As in the previous Lemma, we identify each concept C with $C^{\mathcal{I}}$ to simplify notation.

If \mathcal{I} satisfies Γ , then there exists an enumeration x_0, \dots, x_i, \dots of $\text{reach}(N, P)$ satisfying the instance of properties I1–I4 obtained by replacing R with P and S with $\text{reach}(N, P)$. We shall refer to these properties by I1 ^{P} –I4 ^{P} . We are going to show that the same enumeration satisfies also the instance of properties I1–I4 where R is replaced with R^- and $S = \text{reach}(N, R^-)$. These properties shall be denoted by I1 ^{$-$} –I4 ^{$-$} .

Consider the fixpoint in the right-hand side of assertion (10). It is not hard to see that the first ω steps of its standard iterative construction coincide with

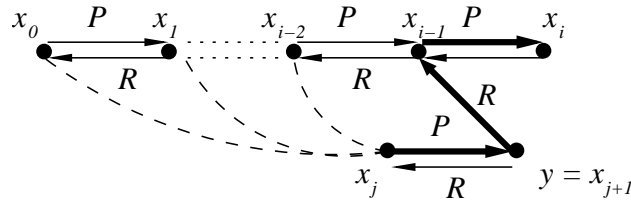


Fig. 2. Checking that R is injective with $x_i \in (\exists P^- . \forall R^- . \exists P^- . X_i)$

the ω -chain $X_0 \subseteq X_1 \subseteq \dots X_i \subseteq \dots$, where $X_0 = \emptyset$, $X_1 = N$ and for all $i > 0$,

$$X_{i+1} = X_i \sqcup ((\exists P^- . X_i) \cap (\exists R . X_i) \cap (\exists P^- . \forall R^- . \exists P^- . X_i)). \quad (13)$$

We prove by induction that, for each ordinal i such that $0 \leq i < \omega$, the set X_i satisfies the following properties:

- P1. $X_i = \{x_0, \dots, x_{i-1}\}$, (where $\langle x_j, x_{j+1} \rangle \in P$, for all $0 \leq j < i - 1$)
- P2. $\langle x_j, y \rangle \in R^-$ if and only if $y = x_{j+1}$, for all $0 \leq j < i - 1$.

Base case. The properties are obvious for $i = 0, 1$.

Induction step. Assume that $i > 1$, and that properties P1 and P2 hold for X_i . By assertion (10), we have that $X_{i+1} \setminus X_i \neq \emptyset$ since $\text{reach}(N, P)$ is an infinite set and X_i is finite by induction hypothesis. Let y be any member of $X_{i+1} \setminus X_i$. By (13), y must be a member of $\exists P^- . X_i$, and hence there must be $j < i$ such that $\langle x_j, y \rangle \in P$. Moreover, by property P1 of the induction hypothesis, there exists only one individual $x \in X_i$ such that $\langle x, y \rangle \in P$ and $y \notin X_i$, and such an element is x_{i-1} . Therefore $x_j = x_{i-1}$ and (by the functionality of P) $y = x_i$. Since y is an arbitrary element of $X_{i+1} \setminus X_i$, it follows that $X_{i+1} = X_i \cup \{x_i\}$, which proves property P1.

Next we show that X_{i+1} satisfies property P2. We deal only with the unique case not directly covered by the corresponding induction hypothesis, that is, we prove that $\langle x_{i-1}, y \rangle \in R^-$ if and only if $y = x_i$. Recall that x_i is the unique member of $X_{i+1} \setminus X_i$, by P1. By conjunct $\exists R . X_i$ of Eq. (13), x_i must be connected by R to some element of X_i . This element must be x_{i-1} , because induction hypothesis P2 implies that any other member of X_i is connected by R^- -edges only to another element of X_i , and $x_i \notin X_i$. Therefore, $\langle x_i, x_{i-1} \rangle \in R$, and $\langle x_{i-1}, x_i \rangle \in R^-$. So, to complete the proof of P2, it suffices to show that there exists no $y \neq x_i$ such that $\langle x_{i-1}, y \rangle \in R^-$.

Assume that such a y exists. By Eq. 13, x_i belongs to $\exists P^- . \forall R^- . \exists P^- . X_i$, and this implies that there exists $x_j \in X_i$ ($j < i$) such that $\langle x_j, y \rangle \in P$ (see Figure 2). Then $y = x_{j+1}$, because P is functional.

Note that $j < i - 1$ (otherwise $y = x_i$, a contradiction), and hence (i) $x_j \neq x_{i-1}$ and (ii) induction hypothesis P2 tells us that $\langle x_j, y \rangle \in R^-$. Moreover, recall

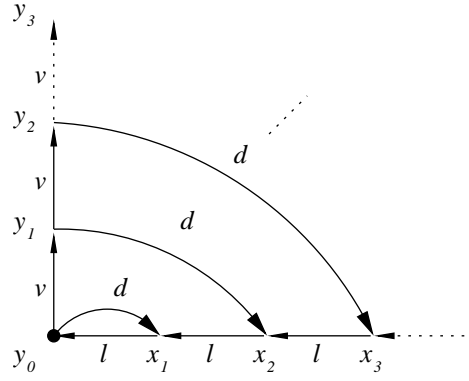


Fig. 3. Structure captured in Lemma 7

that $\langle x_{i-1}, y \rangle \in R^-$. It follows that R is not functional. This contradiction completes the proof of P2 and the whole induction.

Now we can use P2 and assertion (11) to prove that the sequence x_0, \dots, x_i, \dots is an enumeration of $\text{reach}(N, R^-)$ satisfying I1⁻–I4⁻ (and hence, $\text{reach}(N, R^-)$ is an infinite R^- chain from N).

Assertion (11) ensures that $\text{reach}(N, R^-) \subseteq \{x_0, \dots, x_i, \dots\}$. The opposite inclusion follows easily from P2. Then x_0, \dots, x_i, \dots is an enumeration of $\text{reach}(N, R^-)$. Now property I1⁻ is nothing but the if-part of P2. Property I2⁻ is enforced by assertion (12). Property I3⁻ is the only-if part of P2. Finally, property I4⁻ is an immediate consequence of the functionality of R .

To complete the projective characterization, we are only left to show that every interpretation \mathcal{I} of N, R such that $\text{reach}(N, R^-)$ is an infinite isolated R^- -chain from N , can be expanded to a model \mathcal{I}' of Γ and (10)–(12). Define \mathcal{I}' by setting $P^{\mathcal{I}'} = (R^-)^{\mathcal{I}}$. Then $\text{reach}(N, P)$ is an infinite P -chain from N , and hence Γ is satisfied. Moreover, since P equals R^- , it can be easily verified that the iterative construction $\{X_i\}_{i \geq 0}$ of the fixpoint in assertion (10) satisfies

$$X_i = \{(R^-)^j(N) \mid j < i\}, \quad (i > 0)$$

therefore the two sides of assertion (10) are equal and the assertion is satisfied. The truth of assertion (11) follows from the identity of P and R^- . The same equality and property I2^P imply that (12) is satisfied. \square

Next we show how to characterize the interpretations that contain the substructure illustrated in Figure 3.

Lemma 7 *Given a nominal N and three functional atomic roles l, v , and d it is possible to projectively characterize in μALCCIO_{fa} the class of interpretations where atomic concepts B_v and B_h equal $\text{reach}(N, v)$ and $\text{reach}(N, l^-)$,*

respectively, and:

- (1) $\text{reach}(N, v)$ and $\text{reach}(N, l^-)$ are infinite isolated v -chain and l^- -chain, respectively, from N ,
- (2) $\text{reach}(N, v) \cap \text{reach}(N, l^-) = N$, $l^- \downarrow \text{reach}(N, v) = \emptyset$ and $v^- \downarrow \text{reach}(N, l^-) = \emptyset$,
- (3) d is a function from $\text{reach}(N, v)$ to $\text{reach}(N, l^-) \setminus N$ such that for any individual x of $\text{reach}(N, v)$ and y of $\text{reach}(N, l^-)$, $\langle x, y \rangle \in d$ if and only if $\langle N, x \rangle \in v^i$ and $\langle N, y \rangle \in (l^-)^{i+1}$, for some $i \geq 0$.

Proof. By Lemma 5 and Lemma 6, with a set of $\mu\text{ALC}\text{IO}_{fa}$ assertions Γ we can projectively characterize the class of interpretations where $\text{reach}(N, v)$ and $\text{reach}(N, l^-)$ are infinite isolated chains. We denote by y_i (resp. x_i) the i -th individual of the v -chain (resp. l^- -chain) from N . The additional assertions needed are:

$$B_v \equiv \text{reach}(N, v) \tag{14}$$

$$B_h \equiv \text{reach}(N, l^-) \tag{15}$$

$$B_h \sqcap B_v \equiv N \tag{16}$$

$$B_v \sqsubseteq \forall l. \perp \tag{17}$$

$$B_h \sqsubseteq \forall v^-. \perp \tag{18}$$

$$B_v \equiv \exists d. \top \tag{19}$$

$$B_h \sqcap \neg N \equiv \exists d^-. \top \tag{20}$$

$$B_h \sqcup B_v \sqsubseteq \mu X. [N \sqcup \tag{21}$$

$$(B_h \sqcap (\exists l. [X \sqcap B_h]) \sqcap (\exists d^-. X)) \sqcup$$

$$(B_v \sqcap (\exists v^-. [X \sqcap B_v]) \sqcap (\exists v^-. \exists d. X))].$$

We have to prove that an arbitrary interpretation \mathcal{I} is a model of Γ and the above assertions if and only if \mathcal{I} satisfies the three conditions listed in the statement of the Lemma (that describe the structure illustrated in Figure 3). As before, we identify each concept C with $C^{\mathcal{I}}$ to improve readability.

First assume that \mathcal{I} is a model of the assertions. Condition 1 is enforced by Γ . Condition 2 is enforced by assertions (14)–(18). In the following we prove condition 3.

Let us consider the following ω -chain, that equals the standard iterative construction of the least fixpoint in the right-hand side of assertion (21). The chain consists of $X_0 \subseteq X_1 \subseteq \dots X_i \subseteq \dots$, where $X_0 = \emptyset$, $X_1 = \{y_0\} = N$ and

$$X_{i+1} = X_i \sqcup \tag{22}$$

$$(B_h \sqcap (\exists l. [X_i \sqcap B_h]) \sqcap (\exists d^-. X_i)) \sqcup$$

$$(B_v \cap (\exists v^-. [X_i \cap B_v])) \cap (\exists v^-. \exists d. X_i) \quad (23)$$

Let y_0, y_1, y_2, \dots and y_0, x_1, x_2, \dots be the enumerations of chains B_v and B_h , respectively; note that y_0 is the unique member of N . We prove by induction that, for each $i \geq 0$, X_i satisfies the following properties:

- P1. $X_i = \{y_0, x_1, y_1 \dots, y_{j-1}, x_j\}$, if $i = 2j$ for some $j > 0$, and
 $X_i = \{y_0, x_1, y_1 \dots, y_{j-1}, x_j, y_j\}$, if $i = 2j + 1$, for some $j \geq 0$;
P2. for any $x_k \in X_i$, with $k > 0$, it holds that $\langle y_{k-1}, x_k \rangle \in d$, and if $\langle x_k, z \rangle \in d^-$, with $z \in X_i$, then $z = y_{k-1}$.

Base case: The properties are obvious for $i = 0, 1$. Induction step. Assume that properties P1 and P2 hold for X_i , with $i > 0$. We start by observing that $X_{i+1} \setminus X_i \neq \emptyset$ as a consequence of assertion 21, because X_i is finite by induction hypothesis P1, and $B_v \sqcup B_h$ is infinite by (14) and (15).

To prove P1, we first treat the case in which i is even. If $i = 2j$, for some j , then X_i has the form $X_i = \{y_0, x_1, y_1 \dots, y_{j-1}, x_j\}$.

First we prove that $X_{i+1} \setminus X_i \subseteq B_v$, by showing that disjunct (22) does not produce any elements in $X_{i+1} \setminus X_i$. Suppose this is false, and assume $z \in X_{i+1} \setminus X_i$ and $z \notin B_v$. Clearly, z is not a member of (23); then z must be in (22). From the conjunct $\exists l. [X_i \cap B_h]$ of (22), it follows that $z = x_{j+1}$. Moreover, by the conjunct $\exists d^-. X_i$ in (22), there should be $y_k \in X_i$, with $k < j$, such that $\langle y_k, x_{j+1} \rangle \in d$. By induction hypothesis P2, and by the functionality of d , we have $x_{j+1} = x_{k+1}$. This is a contradiction because $k < j$ and we know that the chain $B_h = \text{reach}(N, l^-)$ contains no loops.

Therefore, it must be the case that $z \in B_v$ and z belongs to (23). Now, we can prove that $X_{i+1} \setminus X_i = \{y_j\}$. By conjunct $\exists v^-. [X_i \cap B_v]$ of (23), we have $\langle y_{j-1}, z \rangle \in v$, because all members of $X_i \cap B_v$ but y_{j-1} are connected by v to other members of X_i , and $z \notin X_i$. Clearly, $\langle y_{j-1}, z \rangle \in v$ implies $z = y_j$. Since z is an arbitrary member of $X_{i+1} \setminus X_i$, this proves that $X_{i+1} \setminus X_i = \{y_j\}$. This completes the proof of P1 when i is even.

Next, let $i = 2j + 1$, for some $j \geq 0$. By induction hypothesis,

$$X_i = \{y_0, x_1, y_1 \dots, y_{j-1}, x_j, y_j\}.$$

By analogy with the previous case, we are going to prove that disjunct (23) does not yield any elements in $X_{i+1} \setminus X_i$, and hence $X_{i+1} \setminus X_i \subseteq B_h$. Suppose not, and let $z \in X_{i+1} \setminus X_i$ and $z \notin B_h$. Clearly, z must belong to (23). From the conjunct $\exists v^-. [X_i \cap B_v]$ of (23), we have $\langle y_j, z \rangle \in v$ (because all members of X_i but y_j are connected by v to other members of X_i and $z \notin X_i$) which implies $z = y_{j+1}$. Therefore y_{j+1} belongs to $\exists v^-. \exists d. X_i$ in (23), and hence y_j belongs to $\exists d. X_i$ and there should be $x_k \in X_i$, with $k \leq j$, such that $\langle y_j, x_k \rangle \in d$.

By induction hypothesis P2, it follows that $y_j = y_{k-1}$ ($j \neq k-1$). This is a contradiction, because the sequence $\text{reach}(N, v) = y_0, \dots, y_i \dots$ should be acyclic.

This proves that $X_{i+1} \setminus X_i \subseteq B_h$. As a consequence of conjunct B_v of (23) and (16), we know also that $X_{i+1} \setminus X_i$ is contained in (22). Now we can prove that $X_{i+1} \setminus X_i = \{x_{j+1}\}$. Let z be any element in $X_{i+1} \setminus X_i$. By the above discussion, z belongs to B_h and (22). By analogy with the previous case, it can be proved that conjunct $\exists l.[X_i \cap B_h]$ of (22) connects z to some element of $X_i \cap B_h$ via an l -edge, and the latter element can only be x_i , so $z = x_{j+1}$. This completes the proof of property P1.

Next we show that X_{i+1} satisfies property P2. If $i+1$ is even ($i = 2j$), then P2 follows from the corresponding induction hypothesis, because X_i and X_{i+1} contain the same elements x_1, \dots, x_j . If i is odd and $i = 2j+1$, then the induction hypothesis covers all the elements but x_{j+1} , so we only have to show that (i) $\langle y_j, x_{j+1} \rangle \in d$, and (ii) if $\langle x_{j+1}, z \rangle \in d^-$, then $z = y_j$.

Recall that x_{j+1} belongs to B_h , so x_{j+1} must be a member of (22). The conjunct $\exists d^-.X_i$ of (22), assertion (19) and P1 imply that for some $k \leq j$, $\langle y_k, x_{j+1} \rangle \in d$. By property P2 of the induction hypothesis, we have that for all $k < j$, y_k is connected by d only to x_{k+1} , and $x_{k+1} \neq x_{j+1}$ because $k+1 < j+1$. Therefore, it must be $y_k = y_j$. This proves (i).

To prove (ii), assume it does not hold, that is, there exists $z \neq y_j$ such that $\langle x_{j+1}, z \rangle \in d^-$. By assertion (19), $z \in B_v$, so $z = y_k$ for some $k < j$, and $\langle y_k, x_{j+1} \rangle \in d$. However, by induction hypothesis P2, $\langle y_k, x_{k+1} \rangle \in d$, and $x_{k+1} \neq x_{j+1}$ because $k < j$. Then d is not functional, a contradiction.

This completes the proof of P2. Now the third condition in the Lemma's statement follows easily from assertions (19)–(20) and P2. More precisely, (19)–(20) force d to be a function from $\text{reach}(N, v)$ to $\text{reach}(N, l^-) \setminus N$, while P2 states that for any individuals $x \in \text{reach}(N, v)$ and $y \in \text{reach}(N, v)$, $\langle x, y \rangle \in d$ if and only if $\langle N, x \rangle \in v^i$ and $\langle N, y \rangle \in (l^-)^{i+1}$, for some $i \geq 0$.

To complete the characterization proof, we are only left to show that every interpretation \mathcal{I} belonging to the class defined in the statement of the Lemma, satisfies assertions (14)–(21). Then we know that \mathcal{I} can be expanded to a model of Γ by lemmas 5 and 6. Assertions (14)–(15) are satisfied by assumption. Assertions (16)–(18) are true as an immediate consequence of the constraints in condition 2. \mathcal{I} satisfies assertions (19)–(20) by the definition of d 's domain and range in condition 3 of the Lemma statement. Finally, consider (21), and the fixpoint in its right-hand side. By a straightforward induction it can be verified that the iterative construction $\{X_i\}_{i \geq 0}$ of the fixpoint is such that each X_i , with i even, contains the first $i/2$ elements of $\text{reach}(N, v)$ and $\text{reach}(N, l^-)$ (as in P1). It follows that the limit of the sequence covers all the nodes of B_v

and B_h , and hence assertion (21) is satisfied. \square

3.2 Main results

We are now ready to prove that the grid can be characterized.

Lemma 8 *The structure \mathcal{G}' illustrated in Figure 1 can be projectively characterized in $\mu\mathcal{ALCCTO}_{fa}$.*

Proof. By Lemma 7, with a set Γ of $\mu\mathcal{ALCCTO}_{fa}$ assertions we can projectively characterize the class of structures where

- (1) $B_v \equiv \text{reach}(N, v)$ and $B_h \equiv \text{reach}(N, l^-)$,
- (2) $\text{reach}(N, v)$ and $\text{reach}(N, l^-)$ are infinite isolated chains, whose members will be denoted by $e_{0,0}, e_{0,1}, \dots, e_{0,i}, \dots$ and $e_{0,0}, e_{1,0}, \dots, e_{i,0}, \dots$, respectively, (note that $N = \{e_{0,0}\}$),
- (3) $\text{reach}(N, v) \cap \text{reach}(N, l^-) = N$, $l \downarrow \text{reach}(N, v) = \emptyset$ and $v^- \downarrow \text{reach}(N, l^-) = \emptyset$,
- (4) $\langle x, y \rangle \in d$ iff $x = e_{0,i}$ and $y = e_{i+1,0}$, for some $i \geq 0$.

These properties characterize the horizontal and vertical borders of the grid, B_h and B_v , as well as the auxiliary role d that shall be of help to visit the grid. The grid structure of the internal nodes is imposed by the following additional $\mu\mathcal{ALCCTO}_{fa}$ assertion:

$$\begin{aligned} \top \sqsubseteq \mu X. [& N \sqcup & (24) \\ & ((\exists l.X) \cap (\exists d^-.X)) \sqcup \\ & ((\exists l^-. \exists v^-.X) \cap (\exists v^-. \exists l^-.X))]. \end{aligned}$$

We have to prove that the projection of any model \mathcal{I} of Γ and (24) on v and l^- is isomorphic to \mathcal{G}' . As usual, for all concepts C , we identify C with $C^{\mathcal{I}}$ to enhance readability.

First, some notation. For all functional roles R and S , let $R \circ S$ denote the composition of the two roles (where R is applied after S). In the following, we shall denote by $e_{i,j}$, with $i, j \geq 0$, the individual y such that $\langle N, y \rangle \in v^j \circ (l^-)^i$ (cf. Figure 4; this notation is clearly compatible with the one adopted in condition 2). Note that if $\langle N, y_1 \rangle, \langle N, y_2 \rangle \in v^j \circ (l^-)^i$, then $y_1 = y_2$, because l^- restricted to B_h is a function (by the properties of isolated chains) and v is a functional role.

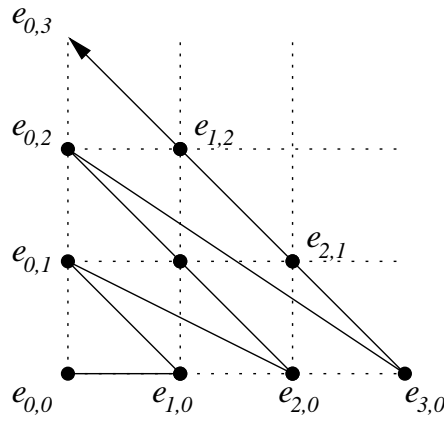


Fig. 4. Visiting the grid

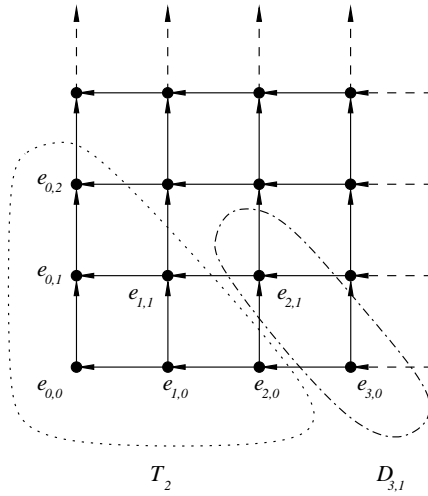


Fig. 5. Examples of T_k and $D_{k,n}$

For all $k \geq 0$, we denote by T_k the set of individuals $\{e_{i,j} : i + j \leq k\}$ (i.e. the triangle with vertices $e_{0,0}$, $e_{0,k}$, and $e_{k,0}$), and by $D_{k,n}$ the set of individuals $\{e_{i,j} : i + j = k, j \leq n\}$ (i.e. the first n elements of the right-to-left diagonal starting from individual $e_{k,0}$). Figure 5 illustrates T_k and $D_{k,n}$ with an example.

As usual, the iterative construction $X_0 \subseteq X_1 \subseteq \dots X_i \subseteq \dots$ of the fixpoint in (24) can be equivalently defined as follows: $X_0 = \emptyset$, $X_1 = N$ and for all $0 < i < \omega$,

$$X_{i+1} = X_i \sqcup ((\exists l.X_i) \cap (\exists d^-.X_i)) \sqcup \quad (25)$$

$$((\exists l^-. \exists v^-.X_i) \cap (\exists v^-. \exists l^-.X_i)) \quad (26)$$

The elements beyond ω equal X_ω by Lemma 1.

Informally speaking, we are going to prove that the sequence $X_0 \subseteq X_1 \subseteq \dots X_i \subseteq \dots$ “visits” the grid as illustrated in Figure 4, in the sense that at each step, $X_{i+1} \setminus X_i$ contains exactly the i -th visited element. If the underlying interpretation \mathcal{I} were not (an expansion of) a grid, then the visit would stop after a finite number of steps (that is, as soon as an element which is not properly connected to its neighbours is reached). But then infinitely many elements of the borders would not be included in the fixpoint, so assertion (24) would be violated.

To prove this formally, we are going to show that for all $i > 0$, the set X_i satisfies the following properties:

- Q1. either $X_i = T_k$ or $X_i = T_k \cup D_{k+1,n}$ for some $k \geq 0$ and $n < k + 1$;
- Q2. $\langle e_{i,j-1}, e_{i,j} \rangle \in v$, for all $e_{i,j} \in X_i \setminus B_h$ and $\langle e_{i,j}, e_{i-1,j} \rangle \in l$, for all $e_{i,j} \in X_i \setminus B_v$;
- Q3. $\langle e_{i,j}, e_{i-1,j+1} \rangle \in v \circ l$, for all $e_{i,j} \in X_i \setminus (\{last(X_i)\} \cup B_v)$;
- Q4. $\langle e_{i,j}, e_{i-1,j+1} \rangle \in l \circ v$, for all $e_{i,j} \in X_i \setminus (\{last(X_i)\} \cup B_v)$,

where $last(X_i)$ is the element $e_{k,j} \in X_i$ such that $k+j = \max\{s+t : e_{s,t} \in X_i\}$ and $j = \max\{t : e_{s,t} \in X_i, s+t = k+j\}$. Intuitively, $last(X_i)$ is the last visited element. If $X_i = T_k$, then $last(X_i) = e_{0,k}$. If $X_i = T_k \cup D_{k+1,n}$, then $last(X_i) = e_{k+1-n,n}$.

The base case holds trivially for $i = 0$, and for $i = 1$ since $X_1 = T_0$. For the induction step, let $i > 0$, and assume that properties Q1-Q4 hold for X_i .

Note that $X_{i+1} \setminus X_i \neq \emptyset$ as a consequence of assertion (24), since X_i is finite by induction hypothesis Q1, whereas \top is infinite because it contains the two infinite borders.

Now, X_i has either the form T_k or the form $T_k \cup D_{k+1,n}$, for some $k \geq 0$ and $n < k + 1$. Consider the former case first. Note that $last(X_i) = e_{0,k}$. Let z be any member of $X_{i+1} \setminus X_i$.

We claim that z is not an element of disjunct (26). Otherwise, if z belonged to (26), then it would be a member of $\exists v^- . \exists l^- . X_i$. This would imply the existence of $e_{i,j} \in X_i = T_k$ such that

$$\langle e_{i,j}, z \rangle \in v \circ l \tag{27}$$

where $z \notin X_i$. Then, $e_{i,j}$ should belong to $\{last(X_i)\} \cup B_v$, because all the other members of X_i , by induction hypothesis Q3, are connected by $v \circ l$ to a member of $T_k = X_i$. Since $last(X_i) = last(T_k) = e_{0,k} \in B_v$, $e_{i,j}$ belongs necessarily to B_v . But simultaneously, (27) implies that there is an edge $\langle e_{i,j}, y \rangle \in l$, thereby violating property $l \downarrow \text{reach}(N, v) = \emptyset$ in condition 3 (at the beginning of the proof). This proves the claim.

Therefore, z must belong to the other disjunct, (25), and hence, z belongs to the conjunct $\exists d^-.X_i$. Since the domain of d is B_v , for some $s \leq k$ we have $\langle e_{0,s}, z \rangle \in d$, from which we obtain $z = e_{s+1,0}$ (by condition 4). Moreover, $e_{0,s} = \text{last}(X_i) = e_{0,k}$, because for $s < k$, $e_{s+1,0}$ is already contained in $T_k = X_i$. Then $z = e_{k+1,0}$. Since this holds for an arbitrary $z \in X_{i+1} \setminus X_i$, we conclude $X_{i+1} \setminus X_i = \{e_{k+1,0}\}$, and property Q1 immediately follows, by noting that $X_{i+1} = X_i \cup \{e_{k+1,0}\} = T_k \cup D_{k+1,0}$.

Now we prove property Q2. The first part holds by definition of $e_{i,j}$. The second part ($\langle e_{i,j}, e_{i-1,j} \rangle \in l$) immediately follows from the structure of B_h .

To prove property Q3, it suffices to prove it for $\text{last}(X_i)$ since for all the other elements it holds by inductive hypothesis. The property holds vacuously because $\text{last}(X_i) = e_{0,k} \in B_v$. Property 4 is proved by a similar argument.

Next we prove Q1–Q4 for the case where $X_i = T_k \cup D_{k+1,n}$, with $n \leq k$. Under this hypothesis, we have that $\text{last}(X_i) = e_{k-n+1,n}$. Let $z \in X_{i+1} \setminus X_i$. We claim that z does not belong to disjunct (25). If this were not true, then z would be an element of conjunct $\exists d^-.X_i$ in (25). This implies that there exists $e_{0,j} \in X_i$ such that $\langle e_{0,j}, z \rangle \in d$, with $j \leq k$. Then $z = e_{j+1,0}$, with $j+1 \leq k+1$ (by condition 4). This fact leads to a contradiction, because $e_{j+1,0} \in X_i$ while $z \notin X_i$. The claim is proved.

It follows from the claim that z belongs to the other disjunct, (26). As a consequence, there are $x_1, x_2 \in X_i$ such that

$$\langle x_1, z \rangle \in l \circ v \text{ and } \langle x_2, z \rangle \in v \circ l. \quad (28)$$

We have $x_1, x_2 \in \{\text{last}(X_i)\} \cup B_v$, because all the other members of X_i would be connected to a member of X_i (while $z \notin X_i$), by induction hypotheses Q3 and Q4. Moreover, it cannot be the case that $x_1, x_2 \in B_v$, because there can be no outgoing l -edges from B_v (condition 3), while (28) implies that such edges exist if x_1 or x_2 belong to B_v . This proves that

$$x_1 = x_2 = \text{last}(X_i) = e_{k-n+1,n} \quad (29)$$

(i.e., v and l commute from $\text{last}(X_i)$).

Now, by induction hypothesis Q2, we have that $\langle e_{k-n+1,n}, e_{k-n,n} \rangle \in l$ and (by definition of $e_{k-n,n+1}$) $\langle e_{k-n,n}, e_{k-n,n+1} \rangle \in v$, which proves that $z = e_{k-n,n+1}$. Since this holds for an arbitrary z in $X_{i+1} \setminus X_i$, we conclude that $X_{i+1} = X_i \cup \{e_{k-n,n+1}\}$, and hence,

$$X_{i+1} = T_k \cup D_{k+1,n} \cup \{e_{k-n,n+1}\} = T_k \cup D_{k+1,n+1}.$$

This completes the proof of property Q1 for X_{i+1} .

Let us consider property Q2. The first part holds by definition of $e_{i,j}$. The second part, by induction hypothesis Q2, holds for all elements but the new element $e_{k-n,n+1}$. So, it suffices to prove that if $e_{k-n,n+1} \notin B_v$ (i.e. if $k-n \neq 0$), then $\langle e_{k-n,n+1}, e_{k-n-1,n+1} \rangle \in l$.

Consider $e_{k-n,n}$ (the element below the new element $e_{k-n,n+1}$). Since $k-n \neq 0$ and $e_{k-n,n} \neq e_{k-n+1,n} = \text{last}(X_i)$, we have $e_{k-n,n} \in X_i \setminus \{\text{last}(X_i)\} \cup B_v$. Then induction hypothesis Q4 and the functionality of l and v tell us that $\langle e_{k-n,n+1}, e_{k-n-1,n+1} \rangle \in l$, which proves Q2.

We are only left to prove properties Q3 and Q4. It suffices to prove the two properties for $\text{last}(X_i) = e_{k-n+1,n}$ (for the other elements the properties are implied by the corresponding inductive hypotheses), that is, we have to prove that $\langle e_{k-n+1,n}, e_{k-n,n+1} \rangle \in v \circ l$ and $\langle e_{k-n+1,n}, e_{k-n,n+1} \rangle \in l \circ v$.

Actually, we have already proved these facts while proving Q1, because they follow immediately from (28) and (29).

This completes the proof of Q1–Q4.

Now we use Q1–Q4 to prove that the projection of \mathcal{I} over l^- and v is a grid. By assertion (24), every domain element x belongs to some X_i , and hence, by property Q1, $x = e_{i,j}$, for some i and j . Moreover, Q2 ensures that x is properly connected to its neighbours on the left and below, namely, $e_{i-1,j}$ and $e_{i,j-1}$. Since v and l are functional, there can be no further edges. Then the projection of \mathcal{I} over l^- and v is isomorphic to \mathcal{G} .

To complete the characterization proof, we are only left to show that \mathcal{G}' can be expanded to a model of Γ and assertion (24). Let \mathcal{I} be the expansion of \mathcal{G}' where $d^{\mathcal{I}} = \{\langle (0, j), (j+1, 0) \rangle \mid j \in \mathbb{N}\}$. It is not hard to see that in the iterative construction $\{X_i\}_{i \geq 0}$ of the fixpoint, each X_i contains the first i elements of the grid according to the visit illustrated in Figure 4. It follows that the limit of the sequence covers all the nodes, and hence assertion (24) is satisfied. Moreover, \mathcal{I} satisfies Γ by Lemma 7. \square

Since μALCCIO_{fa} is powerful enough to projectively characterize the grid and the μALCCIO_{fa} assertion (2) is satisfied only by correct tilings, we derive the following theorem.

Theorem 9 *Satisfiability of μALCCIO_{fa} TBoxes is undecidable.*

We are left to extend this lemma to concept satisfiability and entailment. This is done through the following reductions.

Lemma 10 *In all extensions of $\mu\mathcal{ALCT}$,*

- (a) *TBox satisfiability can be reduced to concept satisfiability.*
- (b) *Concept unsatisfiability can be reduced to entailment.*

Proof. We use a standard technique based on greatest fixpoints (cf. [6]).

To prove (a), note that every TBox $\mathcal{T} = \{C_1 \sqsubseteq D_1, \dots, C_n \sqsubseteq D_n\}$ is equivalent to the TBox $\mathcal{T}' = \{\top \sqsubseteq C_{\mathcal{T}}\}$, where $C_{\mathcal{T}} = \prod_{i=1}^n (\neg C_i \sqcup D_i)$. Now let $D_{\mathcal{T}}$ be the concept $\nu X.[C_{\mathcal{T}} \sqcap \forall R_1.X \sqcap \forall R_1^-.X \sqcap \dots \sqcap \forall R_m.X \sqcap \forall R_m^-.X]$, where R_1, \dots, R_m are all the roles occurring in \mathcal{T} . Clearly, this fixpoint discards exactly the individuals e that violate the assertions of \mathcal{T} , as well as the individuals connected to such e by a composition of roles. If $D_{\mathcal{T}}^{\mathcal{I}} \neq \emptyset$, then the restriction of \mathcal{I} to $D_{\mathcal{T}}^{\mathcal{I}}$ must be a model of \mathcal{T} . Conversely, if \mathcal{I} is a model of \mathcal{T} , then $D_{\mathcal{T}}^{\mathcal{I}} = \Delta^{\mathcal{I}} \neq \emptyset$. It follows that \mathcal{T} is satisfiable iff $D_{\mathcal{T}}$ is. This completes the proof of (a).

To prove (b), note that C is unsatisfiable iff the empty TBox entails $C \sqsubseteq \neg \top$. \square

By Theorem 9 and Lemma 10, we conclude that the main result of this section, Theorem 3, holds.

Finally, with Theorem 3 and the standard embedding of description logics into propositional dynamic logics, we immediately obtain the following result.

Corollary 11 *Formula satisfiability in the hybrid μ -calculus with deterministic atomic programs is undecidable.*

4 Discussion and conclusions

Description logics evolved into a hierarchy of decidable logics with multiple maximal elements. Some support fixpoints, inverse roles, and either nominals or number restrictions (but not both, in the presence of fixpoints) [20,16]. Others support rich sets of role operators, including union and transitive closure.

The results of this paper show that the above features cannot be easily combined into one decidable logic. In particular, no decidable extension of \mathcal{ALCT} can simultaneously support fixpoints, nominals and number restrictions, even in the very special case where number restrictions are confined into functionality assertions for atomic roles, and fixpoint nesting is forbidden.

As a corollary, the hybrid μ -calculus with converse programs and deterministic atomic programs is proved to be undecidable.

These results have immediate implications on \mathcal{DLR}_μ [6], a rich DL with n -ary relations and fixpoints. It is known that $\mu\mathcal{ALCIO}$ can be embedded into \mathcal{DLR}_μ [6]. Then Theorem 3 implies that decidability is not preserved by extending \mathcal{DLR}_μ with nominals.⁴

It is interesting to compare the expressive power of fixpoints and transitive role closure (i.e., recursion over unary and binary relations). Consider the logic $\mathcal{ALCQ}^{\sqcup,+}$ (the extension of \mathcal{ALCQ} with unrestricted role union and transitive closure).

Theorem 12 $\mathcal{ALCQ}^{\sqcup,+}$ is undecidable.

This theorem can be proved by a simple adaptation of the construction in [15, Sec. 5]. That construction makes use of transitive roles and role inclusion; it is not hard to see that they can be replaced with $+$ and role union, respectively.

By substituting fixpoints for transitive closure in $\mathcal{ALCQ}^{\sqcup,+}$, we obtain a decidable logic, namely, $\mu\mathcal{ALCQ}^{\sqcup}$. Its decidability can be proved by showing that each expression in $\mu\mathcal{ALCQ}^{\sqcup}$ is equivalent to an expression in the decidable logic $\mu\mathcal{ALCQ}$, thanks to the equivalence

$$\exists^{\leq n}(R_1 \sqcup R_2).C \equiv \bigsqcup_{0 \leq k \leq n} (\exists^{\leq k} R_1.C \sqcap \exists^{\leq n-k} R_2.C),$$

by which role union can be eliminated.

Theorem 12 and the decidability of $\mu\mathcal{ALCQ}^{\sqcup}$ show that transitive closure is more powerful than fixpoints in the context of \mathcal{ALCQ}^{\sqcup} (the extension of \mathcal{ALCQ}^{\sqcup} with fixpoints is decidable, while the extension with $+$ is not).

An interesting question arising from our results concerns the family of *service description logics* $\mathcal{SDL}(\mathcal{X})$ [4]. These logics are analogous to \mathcal{DLR}_μ , in the sense that $\mathcal{SDL}(\mathcal{X})$ operates on mappings, and mappings can be regarded as n -ary relations. $\mathcal{SDL}(\mathcal{X})$ differs from \mathcal{DLR}_μ because the former supports set abstraction and composition, while \mathcal{DLR}_μ supports number restrictions. Service descriptions in $\mathcal{SDL}(\mathcal{X})$ are supposed to extend an underlying ontology written in a standard description logic \mathcal{X} (modelling concepts and roles only). The main reasoning tasks for $\mathcal{SDL}(\mathcal{X})$ are proved to be decidable by embedding $\mathcal{SDL}(\mathcal{X})$ into decidable extensions of both $\mu\mathcal{ALCIO}$ and \mathcal{X} . Unfortunately, by the undecidability of $\mu\mathcal{ALCIO}_{fa}$, this technique cannot be applied when \mathcal{X} supports number restrictions, or simply functional atomic roles. Then the (un)decidability of $\mathcal{SDL}(\mathcal{X})$, when \mathcal{X} supports number restrictions of some sort, remains an interesting open issue.

⁴ This application of our results constitutes an alternative proof of a known result (De Giacomo, personal communication).

We do not yet know whether inverse roles are essential to prove the undecidability of $\mu\mathcal{ALC}\mathcal{I}\mathcal{O}_{fa}$. In particular, the complexity of inference in $\mu\mathcal{ALC}\mathcal{O}\mathcal{Q}$ and its fragment $\mu\mathcal{ALC}\mathcal{O}_f = \mu\mathcal{ALC}\mathcal{O}_{fa}$ is currently unknown. Recall that the complexity of $\mu\mathcal{ALC}\mathcal{I}\mathcal{Q}$ is unknown, too. This leaves two gaps in our understanding of the decidability threshold below $\mu\mathcal{ALC}\mathcal{I}\mathcal{O}\mathcal{Q}$.

Acknowledgements

The author are grateful to the anonymous referees for their careful revision and their constructive suggestions.

References

- [1] F. Baader et al. *The Description Logic Handbook*. Cambridge university Press, 2003.
- [2] R. Berger. The Undecidability of the Domino Problem. *Mem. AMS*, 66:1-72, 1966.
- [3] P. Blackburn. Nominal tense logic and other sorted intensional frameworks. PhD Thesis, University of Edinburgh, 1990.
- [4] P.A. Bonatti. Towards Service Description Logics. In *Proc. of JELIA '02*, LNAI 2424, pages 74-85, Springer, 2002.
- [5] P.A. Bonatti. Undecidability results for description and dynamic logics with recursion and counting. In *Proc. of IJCAI'03*, pages 331-336, Morgan Kaufmann, 2003.
- [6] D. Calvanese, G. De Giacomo, M. Lenzerini. Reasoning in expressive description logics with fixpoints based on automata on infinite trees. In *Proc. of IJCAI'99*, pages 84-89, Morgan Kaufmann, 1999.
- [7] C. Cerato. General canonical models for graded normal logics. *Studia Logica* 49(2):242-252, 1990.
- [8] G. De Giacomo. *Decidability of class-based knowledge representation formalisms*. PhD Thesis, Università di Roma “La Sapienza”, 1995.
- [9] G. De Giacomo, M. Lenzerini. Boosting the correspondence between description logics and propositional dynamic logics. In *Proc. of the 12th National Conference on Artificial Intelligence (AAAI'94)*, pages 205-212, AAAI Press, 1994.
- [10] M. de Rijke. The modal logic of inequality. *The Journal of Symbolic Logic* 57:566-584, 1992.

- [11] M. Fattorossi Barnaba, F. De Caro. Graded modalities. *Studia Logica* 44(2):197-221, 1985.
- [12] G. Gargov, V. Goranko. Modal logic with names. *Journal of Philosophical Logic* 22(6):607-636, 1993.
- [13] G. Gargov, S. Passy. Determinism and looping in combinatory PDL. *Theoretical Computer Science* 61:259-277, 1988.
- [14] E. Grädel, M. Otto, E. Rosen. Undecidability results on two-variable logics. *Archive for Mathematical Logic*, 38:213-354, 1999.
- [15] I. Horrocks, U. Sattler, S. Tobies. Practical reasoning for very expressive description logics. *Logic Journal of the IGPL*, 8(3): 239-263, 2000.
- [16] O. Kupferman, U. Sattler, M.Y. Vardi. The Complexity of the Graded μ -Calculus. In *Proc. of 18th Conf. on Automated Deduction (CADE 2002)*, LNCS 2392, pages 423-437, Springer, 2002.
- [17] S. Passy, T. Tinchev. PDL with data constants. *Information Processing Letters* 20:35-41, 1985.
- [18] S. Passy, T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation* 93:263-332, 1991.
- [19] K. Schild. A correspondence theory for terminological logics: Preliminary report. In *Proc. of the 12th International Joint Conference on Artificial Intelligence (IJCAI'91)*, pages 466-471, Morgan Kaufmann, 1991.
- [20] U. Sattler, M.Y. Vardi. The hybrid μ -calculus. In *Proc. of the First Int.l Joint Conf. on Automated Reasoning (IJCAR'01)*, LNAI 2083, pages 76-91, Springer, 2001.