Obstacles on the Way to Qualitative Spatial Reasoning with Description Logics: Some Undecidability Results

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Abstract

We summarize the results we obtained on the extensions of $\mathcal{ALC}$ with composition-based role inclusion axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$. A set of these role axioms is called a role box. The original motivation for this work was to develop a description logic suitable for qualitative spatial reasoning problems. We quickly define and discuss the DLs $\mathcal{ALC}_{\mathcal{RA}}$, $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, $\mathcal{ALC}_{\mathcal{RASG}}$ and $\mathcal{ALCN}_{\mathcal{RASG}}$. All but $\mathcal{ALC}_{\mathcal{RASG}}$ are shown to be undecidable, and $\mathcal{ALC}_{\mathcal{RASG}}$ is of limited utility, even though it is still as expressive as $\mathcal{ALC}_{\mathcal{R}^+}$. $\mathcal{ALC}_{\mathcal{RASG}}$ has shown to be EXPTIME-complete, due to *associativity of role boxes*, which is an important requirement.

1 Introduction and Motivation

At DL 2000, we presented the DL $\mathcal{ALC}_{\mathcal{RA}}$, extending $\mathcal{ALC}$ with a set of composition based role inclusion axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$([10]). Such a set was called a "role box", and additionally, global role disjointness was required. We then examined the satisfiability problem of $\mathcal{ALC}$ concept terms w.r.t. a set of the proposed role axioms and conjectured (not proved) to have found a decision procedure for $\mathcal{ALC}_{\mathcal{RA}}$ in forms of a tableau calculus. However, the presented calculus was incomplete since it lacked an appropriate blocking condition, but we were optimistic enough to claim that such a blocking condition could be found. In the meantime we have found that $\mathcal{ALC}_{\mathcal{RA}}$ is undecidable. This paper summarizes the results we found so far.

The motivation for our work can be sketched as follows: In the field of qualitative spatial reasoning, the so-called RCC-family of spatial reasoning calculi is well-known ([6]). RCC descriptions of spatial scenes (arrangements of spatial objects in the world) focus on topological properties and relationships

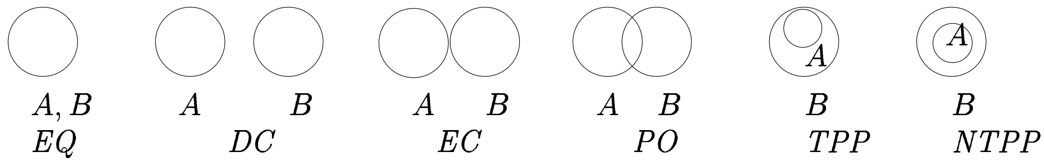

Figure 1: RCC8 Relationships, f.l.t.r.: Equal, Disconnected, Externally Connected, Partial Overlap, Tangential Proper Part, Non-Tangential Proper Part. To be read as $EQ(A, B)$, etc. TPP and $NTPP$ have inverses: $TPPI$ and $NTPPI$.

between spatial objects. For example, an object a can be *externally connected* to another object b ("$EC(a, b)$"), or be a *non-tangential proper part* of another object ("$NTPP(a, b)$"), with the intended meaning that object a is inside of object b, but does not touch its border from the inside. In the case of RCC8, we are equipped with 8 *base relations* that describe purely topological aspects of the scene (see Fig. 1). To describe more general, vague, incomplete or even missing information, disjunctions of base relations are used, called *non-base relations*. The set of base relations does usually (e.g. in the case of RCC8 or RCC5) have the so-called *JEPD-property*, which means that these relations are *j*ointly *e*xhaustive and *p*airwise *d*isjoint: between *every* two spatial objects *exactly one* of the RCC8 relations holds (a missing relation therefore corresponds to the disjunction of all base relations).

Given that we do not have a spatial depiction of a scene from which we can "read off" the relationships, but only a purely symbolic relational description of that scene (e.g. like $\{NTPP(a, b), NTPP(b, c)\}$), we want at least to be able to recognize inconsistent descriptions and, in the case of missing or incomplete information, want to be able to deduce implied relationships. For example, $\{NTPP(a, b), NTPP(b, c), EC(a, c)\}$ should be inconsistent, and $\{NTPP(a, b), NTPP(b, c)\} \models \{NTPP(a, c)\}$ should be deduced. In the QSR community, a central line of research is concerned with constraint propagation techniques to achieve this. The heart of constraint propagation is the so-called *composition table* which is used to solve the following basic inference problem: given three objects a, b and c and the relations $R(a, b), S(b, c)$ between them, what can be deduced about the possible relationships between a and c? For example, in the case of RCC8, the composition table contains the relation $\{DC, EC, PO, TPP, TPPI\}$ as the entry for $EC \circ EC$ (a disjunction or non-base relation), and $\{NTPP\}$ as the entry for $NTPP \circ NTPP$.

Considering FOPL as a representation language, it is obvious that composition table-based reasoning can be captured with FOPL formulas of the form $\forall x, y, z : (S(x, y) \wedge T(y, z) \Rightarrow R_1(x, z) \vee \cdots \vee R_n(x, z))$ ("composition table entry") and $\forall x, y : R_1(x, y) \oplus R_2(x, y) \oplus \ldots \oplus R_n(x, y)$ ("JEPD" property, if $R_1 \ldots R_N$ is the set of all base relations), $\forall x, y : R(x, y) \Leftrightarrow R^{-1}(y, x)$ (inverse relation), etc. It is well known that 2 variables are sufficient if one translates $\mathcal{ALC}$ concept descriptions into FOPL. Decidability of $\mathcal{ALC}$ follows immediately then,

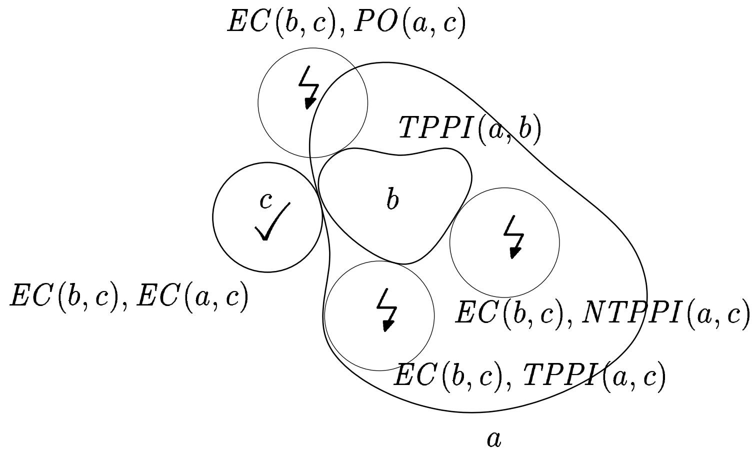

Figure 2: Illustration of an "intended spatial model" of *special_figure*

since the two-variable fragment of FOPL is decidable. However, as the above translation shows, it is likely that we are leaving the two-variable fragment of FOPL if composition table based reasoning capabilities are added.

Considering this in a DL framework, we are using $\mathcal{ALC}$ as the starting point of our investigation and consider the *concept satisfiability problem* w.r.t. a set of role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, enforcing $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$ on the models $\mathcal{I}$. The relationships of, for example, RCC8, correspond to roles of $\mathcal{ALC}$ now. Considering the TBox

$$
\begin{array}{lcl}
circle & \dot{\sqsubseteq} & figure \\
figure\_touching\_a\_figure & \doteq & figure \sqcap \exists EC.figure \\
special\_figure & \doteq & figure \sqcap \\
 & & \forall PO.\neg figure \sqcap \\
 & & \forall NTPPI.\neg figure \sqcap \\
 & & \forall TPPI.\neg circle \sqcap \\
 & & \exists TPPI.(figure \sqcap \exists EC.circle)
\end{array}
$$

we want to deduce that $figure\_touching\_a\_figure$ subsumes $special\_figure$ (see Fig. 2), or equivalently, that $figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \neg(figure \sqcap \exists EC.figure)$ is unsatisfiable w.r.t. a role box $\mathfrak{R}$ corresponding to the RCC8 composition table, and this is indeed the case, since the role box would contain the axiom $TPPI \circ EC \sqsubseteq EC \sqcup PO \sqcup TPPI \sqcup NTPPI \in \mathfrak{R}$.

In the following we will define the considered DLs and summarize the results and the main ideas behind the proofs. At least one of the undecidability results is not new (undecidability of $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$), since the undecidability is, in principle, already known in the modal logics community and has been "rediscovered" by the author (undecidability of context-free inclusion modal logics, see also [1], [3]). To the best of our knowledge, the other results can be called "new".

2 Summary of Obtained Results

We are considering $\mathcal{ALC}$ concept satisfiability w.r.t. role boxes; e.g. if C is an $\mathcal{ALC}$ concept and $\mathfrak{R}$ is a role box, we ask whether there is a model $\mathcal{I}$ of $(C, \mathfrak{R})$. Let $\mathcal{N_C}$ be the set of concept names, and $\mathcal{N_R}$ be the set of role names:

Definition 1 (Role Axioms, Role Box, Admissible Role Box) If $S, T, R_1, \ldots, R_n \in \mathcal{N_R}$, then the expression $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, $n \geq 1$, is called a *role axiom*. If $ra = S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, then $\mathsf{pre}(ra) =_{def} (S, T)$ and $\mathsf{con}(ra) =_{def} \{R_1, \ldots, R_n\}$. If $n = 1$, then ra is called a *deterministic* role axiom. In this case we also write $T = \mathsf{con}(ra)$ instead of $T \in \mathsf{con}(ra)$. A finite set $\mathfrak{R}$ of role axioms is called a *role box*. Let $\mathsf{roles}(ra) =_{def} \{S, T, R_1, \ldots, R_n\}$, and $\mathsf{roles}(\mathfrak{R}) =_{def} \bigcup_{ra \in \mathfrak{R}} \mathsf{roles}(ra)$. A role box $\mathfrak{R}$ is called *deterministic*, iff it contains only deterministic role axioms. A role box $\mathfrak{R}$ is called *functional*, iff $\forall ra_1, ra_2 \in \mathfrak{R} : \mathsf{pre}(ra_1) = \mathsf{pre}(ra_2) \Rightarrow ra_1 = ra_2$. We can then use the function $\mathsf{ra}(S, T) = ra$ to refer to the unique role axiom ra with $\mathsf{pre}(ra) = (S, T)$ and define $\mathsf{con}(S, T) =_{def} \mathsf{con}(\mathsf{ra}(S, T))$. A role box $\mathfrak{R}$ is called *complete*, iff $\forall R, S \in \mathsf{roles}(\mathfrak{R}) : \exists ra \in \mathfrak{R} : \mathsf{pre}(ra) = (R, S)$. A role box $\mathfrak{R}$ is called *admissible* iff it is deterministic, functional, complete, and *associative*: $\forall R, S, T : \mathsf{con}(\mathsf{con}(R, S), T) = \mathsf{con}(R, \mathsf{con}(S, T))$. The role box $\mathfrak{R}$ is called *admissible for the concept* C iff $\mathfrak{R}$ is admissible and additionally, $\mathsf{roles}(C) \subseteq \mathsf{roles}(\mathfrak{R})$ ($\mathsf{roles}(C)$ returns the set of roles used within the concept term C).

According to the classes of allowed role boxes, we define $\mathcal{ALC_{RA}}$, $\mathcal{ALC_{RA^{\ominus}}}$ and $\mathcal{ALC_{RASG}}$ as follows:

- In $\mathcal{ALC_{RA^{\ominus}}}$ we allow all role boxes.
- In $\mathcal{ALC_{RA}}$ we also allow all role boxes, but we require that all roles must be interpreted as disjoint, see below. Of course, certain classes of role boxes can be singled out beforehand (we did this in previous work, but not here for the sake of brevity).
- In $\mathcal{ALC_{RASG}}$ we allow only role boxes that are *admissible* (see above) w.r.t. the considered concept C. Like in $\mathcal{ALC_{RA}}$, we require that all roles must be interpreted as disjoint. An admissible role box can be seen as defining the operation table of a $\mathcal{S}$emi-$\mathcal{G}$roup (therefore the suffix $\mathcal{SG}$). For example, if we consider the operation table of "+" modulo 4 on the natural numbers

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

and interpret "+" as "$\circ$", and assign for each number i a unique role name R_i, we get an admissible role box. For example, $(\exists R_1.\exists R_2.C) \sqcap \forall R_3.\neg C$ is unsatisfiable w.r.t. this role box.

Of course, there are $(C, \mathfrak{R})$ which are satisfiable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, but unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}}$. $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ and $\mathcal{ALC}_{\mathcal{RA}}$ are different languages, since role disjointness is not enforceable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ (disjointness of roles is not "modally definable"). All three DLs are powerful enough to allow for the *internalization of GCIs* and can deal with general TBoxes. Considering $\mathcal{ALC}_{\mathcal{RASG}}$, it becomes clear that there is always a model in which all roles are interpreted as disjoint, and therefore, it doesn't really matter that the roles should be interpreted as disjoint (the same holds for plain $\mathcal{ALC}$, since $\mathcal{ALC}$ has the *Tree Model Property*). However, the models are different. The *semantics* is specified in the usual way:

Definition 2 (Interpretation) An *interpretation* $\mathcal{I} =_{def} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, called the domain of $\mathcal{I}$, and an interpretation function $\cdot^{\mathcal{I}}$ that maps every concept name to a subset of $\Delta^{\mathcal{I}}$, and every role name to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function $\cdot^{\mathcal{I}}$ is extended inductively for complex concepts by using the usual $\mathcal{ALC}$–equations (e.g. $(C \sqcap D)^{\mathcal{I}} =_{def} C^{\mathcal{I}} \cap D^{\mathcal{I}}$, etc.).

In case of $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RASG}}$, we additionally require that for all roles $R, S \in \mathcal{N_R}$, $R \neq S$: $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$. All roles are interpreted as disjoint then.

As usually, we say that an interpretation $\mathcal{I}$ is a model of a concept C, written $\mathcal{I} \models C$, iff $C^{\mathcal{I}} \neq \emptyset$. An interpretation $\mathcal{I}$ is a model of a role axiom $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, written $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, iff $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a model of a role box $\mathfrak{R}$, written $\mathcal{I} \models \mathfrak{R}$, iff for all role axioms $ra \in \mathfrak{R}$: $\mathcal{I} \models ra$.

Theorem 1 $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, $\mathcal{ALC}_{\mathcal{RA}}$, and $\mathcal{ALCN}_{\mathcal{RASG}}$ are undecidable ($\mathcal{ALCN}_{\mathcal{RASG}}$ is $\mathcal{ALC}_{\mathcal{RASG}}$ plus *unqualified number restrictions*).

Of course, we cannot give the full proofs here, but we can try to convince the reader; please refer to [9], [7], [8]. None of the considered logics has the finite model property.

The $\mathcal{ALC}_{\mathcal{RA}^\ominus}$-proof ([9]) is given by a reduction from the (undecidable) non-empty intersection problem of a special class of context-free grammars which are similar to context-free grammars in Chomsky Normal Form. The basic idea is to mimic the "top down"-derivation of words as done by the grammars with the role axioms in a "bottom up"-style. Let $\mathcal{G}_1 = (\mathcal{V}_1, \Sigma_1, \mathcal{P}_1, S_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Sigma_2, \mathcal{P}_2, S_2)$ be two context-free grammars such that $\mathcal{P}_i \subseteq \mathcal{V}_i \times ((\mathcal{V}_i \cup \Sigma_i) \times (\mathcal{V}_i \cup \Sigma_i))$ (as usual, $\mathcal{V}$ are the non-terminal symbols, Σ is the set of terminal symbols, $\mathcal{P}$ are the production rules, and S is the starting symbol). W.l.o.g. we assume $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. It is undecidable whether $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$ ($\mathcal{L}(\mathcal{G}_i)$ is the language generated by $\mathcal{G}_i$). For $i \in \{1, 2\}$, we define the role boxes $\mathfrak{R}_i =_{def} \{ B \circ C \sqsubseteq A \mid A \to B\ C \in \mathcal{P}_i \}$.
Let $\Sigma =_{def} \Sigma_1 \cup \Sigma_2$ and $\mathfrak{R} =_{def} \mathfrak{R}_1 \cup \mathfrak{R}_2$. Let $R_? \notin \mathsf{roles}(\mathfrak{R})$, and let $\mathfrak{R}' =_{def} \mathfrak{R} \cup \{ R \circ S \sqsubseteq R_? \mid R, S \in (\{R_?\} \cup \mathsf{roles}(\mathfrak{R})), \neg\exists ra \in \mathfrak{R} : \mathsf{pre}(ra) = (R, S) \}$ be the *completion* of $\mathfrak{R}$. Then, $(E, \mathfrak{R}')$ is satisfiable iff $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$, where

$$E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap \forall S_1.C \sqcap \forall S_2.D, \text{ with}$$
$$X =_{def} \sqcap_{a \in \Sigma} \exists a.\top \text{ and}$$
$$Y =_{def} \sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.(X \sqcap \neg(C \sqcap D)).$$

Please note that the role box performs a "bottom up parsing" of the words $w \in \Sigma^+$, whose presence is enforced by the interplay of X and Y, and that S_1 and S_2 are the starting symbols of the two grammars. Inspecting $\mathfrak{R}'$ and E it becomes clear that already $\mathcal{ALU}_{\mathcal{RA}^\ominus}$ with *deterministic* role boxes is undecidable.

The $\mathcal{ALC}_{\mathcal{RA}}$ proof ([8]) is more involved since the disjointness requirement has to be fulfilled; the "trick" to propagate $C \sqcap D$ with $\forall S_1.C \sqcap \forall S_2.D$ to yield the unsatisfiability does not work, since already $S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$ must hold. The exploited grammars therefore have to ensure that each word w of role names $w \in \Sigma^+$ can be derived by *at most one non-terminal of one of the grammars.* The proof works by transforming a Post's Correspondence Problem into two special grammars with the required properties which are again transformed into a role box that does not violate the disjointness requirement:

Definition 3 (PCP) A *Post's Correspondence Problem (PCP)* K over an alphabet $\mathcal{A}$ is given by a finite set of pairs $K = \{(x_1, y_1), (x_2, y_2), \ldots (x_k, y_k)\}$, where x_i, y_i are (non-empty!) words over a given alphabet $\mathcal{A}$: $x_i, y_i \in \mathcal{A}^+$. A *solution* to a PCP is sequence of indices $(i_1, i_2, \ldots, i_n) \in \{1 \ldots k\}$ with $n \geq 1$ such that $x_{i_1} x_{i_2} \ldots x_{i_n} = y_{i_1} y_{i_2} \ldots y_{i_n}$.

For example, the PCP $K = \{(1, 101), (10, 00), (011, 11)\}$ has the solution (1,3,2,3), since $x_1 x_3 x_2 x_3 = \boxed{1}\boxed{011}\boxed{10}\boxed{011} = 101110011 = \boxed{101}\boxed{11}\boxed{00}\boxed{11} = y_1 y_3 y_2 y_3$. In the following it suffices to consider (sufficiently large) PCPs with $\mathcal{A} = \{0, 1\}$.

Definition 4 (Auxiliary Definitions) Let $x \in \mathcal{A}^+$, $x = a_1 \ldots a_n$. We define $|x| =_{def} n$, $\mathsf{first}(x) =_{def} a_1$, and $\mathsf{rest}(x) =_{def} a_2 \ldots a_n$. Let $\mathsf{postfixes}(x) =_{def} \{\, w \mid \exists v \in \mathcal{A}^\star : x = vw, w \neq \epsilon \,\}$ (e.g. $\mathsf{postfixes}(1011) = \{1011, 011, 11, 1\}$).

Given a PCP K we construct the grammars $\mathcal{G}_{1,K}$ and $\mathcal{G}_{1,K}$ which have the following important properties: firstly, we have $\mathcal{L}(\mathcal{G}_{1,K}) \cap \mathcal{L}(\mathcal{G}_{2,K}) = \emptyset$, since words in $\mathcal{G}_{1,K}$ have the form $i_{i_n}\# \ldots \# i_{i_2} \# i_{i_1} \# x_{i_1} \# x_{i_2} \# \ldots \# x_{i_n} \#$, and words in $\mathcal{G}_{2,K}$ have the form $\# i_{i_n} \# \ldots \# i_{i_2} \# i_{i_1} \# y_{i_1} \# y_{i_2} \# \ldots \# y_{i_n}$. Additionally, the PCP K has the solution $(i_1, \ldots, i_n)$ iff $\# i_{i_n} \# \ldots \# i_{i_2} \# i_{i_1} \# x_{i_1} \# x_{i_2} \# \ldots \# x_{i_n} \# \in (\{\#\}\mathcal{L}(\mathcal{G}_{1,K})) \cap (\mathcal{L}(\mathcal{G}_{2,K})\{\#\})$. Consequently, K has no solution iff $(\{\#\}\mathcal{L}(\mathcal{G}_{1,K})) \cap (\mathcal{L}(\mathcal{G}_{2,K})\{\#\}) = \emptyset$. Emptiness for this language is therefore undecidable. Thirdly, whenever a word w is derivable by *some* non-terminal $A \in \mathcal{V}_i$ such that $A \xrightarrow{+} w$, then there is no other non-terminal $B \in \mathcal{V}_i$ with $A \neq B$ such that also $B \xrightarrow{+} w$, and this does even hold if we put the two set of production rules together in one "union" grammar (w.l.o.g. we can assume $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$). Let K be the PCP of size k, and let $\mathcal{A}' =_{def} \mathcal{A} \cup \{i_1, \ldots, i_k\}$. Now, $\mathcal{G}_{1,K}$ and $\mathcal{G}_{2,K}$ are defined as follows:

- $\mathcal{G}_{1,K} = (\mathcal{V}_1, \mathcal{A}' \cup \{\#\}, \mathcal{P}_1, S_1)$

$$
\begin{aligned}
\mathcal{V}_1 = \ & \{S_1\} \cup \{\, \boxed{a\#} \mid a \in \mathcal{A}' \,\} \cup \\
& \{\, \boxed{w\#} \mid x \in \{x_1, \ldots, x_k\} \text{ (note: } x_1, \ldots, x_k \text{ of the PCP } K\text{)} \\
& \qquad w \in \mathsf{postfixes}(x) \,\} \cup \\
& \{\, \boxed{S_1 x\#} \mid x \in \{x_1, \ldots, x_k\} \,\} \\
\mathcal{P}_1 = \ & \{\, \boxed{a\#} \to a\# \mid a \in \mathcal{A} \,\} \cup \\
& \{S_1 \to \boxed{i_1\#}\ \boxed{x_1\#} \mid \cdots \mid \boxed{i_k\#}\ \boxed{x_k\#}\} \cup \\
& \{S_1 \to \boxed{i_1\#}\ \boxed{S_1 x_1\#} \mid \cdots \mid \boxed{i_k\#}\ \boxed{S_1 x_k\#}\} \cup \\
& \{\boxed{S_1 x_1\#} \to S_1\ \boxed{x_1\#}, \ldots, \boxed{S_1 x_k\#} \to S_1\ \boxed{x_k\#}\,\} \cup \\
& \{\, \boxed{x\#} \to \boxed{\mathsf{first}(x)\#}\ \boxed{\mathsf{rest}(x)\#} \mid n \in 1 \ldots k, \\
& \qquad\qquad x \in \mathsf{postfixes}(x_n), |x| \geq 2 \,\}
\end{aligned}
$$

- $\mathcal{G}_{2,K} = (\mathcal{V}_2, \mathcal{A}' \cup \{\#\}, \mathcal{P}_2, S_2)$

$$
\begin{aligned}
\mathcal{V}_2 = \ & \{S_2\} \cup \{\, \boxed{\#a} \mid a \in \mathcal{A}' \,\} \cup \\
& \{\, \boxed{\#w} \mid y \in \{y_1, \ldots, y_k\}, \text{ (note: } y_1, \ldots, y_k \text{ of the PCP } K\text{)} \\
& \qquad w \in \mathsf{postfixes}(y) \,\} \cup \\
& \{\, \boxed{S_2\#y} \mid y \in \{y_1, \ldots, y_k\} \,\} \\
\mathcal{P}_2 = \ & \{\, \boxed{\#a} \to \#a \mid a \in \mathcal{A}' \,\} \cup \\
& \{S_2 \to \boxed{\#i_1}\ \boxed{\#y_1} \mid \cdots \mid \boxed{\#i_k}\ \boxed{\#y_k}\} \cup \\
& \{S_2 \to \boxed{\#i_1}\ \boxed{S_2\#y_1} \mid \cdots \mid \boxed{\#i_k}\ \boxed{S_2\#y_k}\} \cup \\
& \{\boxed{S_2\#y_1} \to S_2\ \boxed{\#y_1}, \ldots, \boxed{S_2\#y_k} \to S_2\ \boxed{\#y_k}\,\} \cup \\
& \{\, \boxed{\#y} \to \boxed{\#\mathsf{first}(y)}\ \boxed{\#\mathsf{rest}(y)} \mid n \in 1 \ldots k, \\
& \qquad\qquad y \in \mathsf{postfixes}(y_n), |y| \geq 2 \,\}
\end{aligned}
$$

Applied to the example PCP K we get

- $\mathcal{G}_{1,K} = (\mathcal{V}_1, \{\#, 0, 1, i_1, i_2, i_3\}, \mathcal{P}_1, S_1)$, with

$$
\begin{aligned}
\mathcal{V}_1 = \{\ & S_1, \boxed{0\#}, \boxed{1\#}, \boxed{i_1\#}, \boxed{i_2\#}, \boxed{i_3\#}, \\
& \boxed{10\#}, \boxed{011\#}, \boxed{11\#}, \\
& \boxed{S_1 1\#}, \boxed{S_1 10\#}, \boxed{S_1 011\#}\ \} \\
\mathcal{P}_1 = \ & \{\boxed{0\#} \to 0\#, \boxed{1\#} \to 1\#, \boxed{i_1\#} \to i_1\#, \boxed{i_2\#} \to i_2\#, \boxed{i_3\#} \to i_3\#\} \cup \\
& \{S_1 \to \boxed{i_1\#}\ \boxed{1\#}, S_1 \to \boxed{i_2\#}\ \boxed{10\#}, S_1 \to \boxed{i_3\#}\ \boxed{011\#}\} \cup \\
& \{S_1 \to \boxed{i_1\#}\ \boxed{S_1 1\#}, S_1 \to \boxed{i_2\#}\ \boxed{S_1 10\#}, S_1 \to \boxed{i_3\#}\ \boxed{S_1 011\#}\} \cup \\
& \{\boxed{S_1 1\#} \to S_1 \boxed{1\#}, \boxed{S_1 10\#} \to S_1 \boxed{10\#}, \boxed{S_1 011\#} \to S_1 \boxed{011\#}\} \cup \\
& \{\boxed{10\#} \to \boxed{1\#}\ \boxed{0\#}, \boxed{011\#} \to \boxed{0\#}\ \boxed{11\#}, \boxed{11\#} \to \boxed{1\#}\ \boxed{1\#}\}, \text{ and}
\end{aligned}
$$

- $\mathcal{G}_{2,K} = (\mathcal{V}_2, \{\#, 0, 1, i_1, i_2, i_3\}, \mathcal{P}_2, S_2)$ with

$$
\begin{aligned}
\mathcal{V}_2 = \{\ & S_2, \boxed{\#0}, \boxed{\#1}, \boxed{\#i_1}, \boxed{\#i_2}, \boxed{\#i_3}, \\
& \boxed{\#10}, \boxed{\#011}, \boxed{\#11}, \\
& \boxed{S_2\#101}, \boxed{S_2\#00}, \boxed{S_2\#11}\ \} \\
\mathcal{P}_2 = \ & \{\boxed{\#0} \to \#0, \boxed{\#1} \to \#1, \boxed{\#i_1} \to \#i_1, \boxed{\#i_2} \to \#i_2, \boxed{\#i_3} \to \#i_3\} \cup \\
& \{S_2 \to \boxed{\#i_1}\ \boxed{\#101}, S_2 \to \boxed{\#i_2}\ \boxed{\#00}, S_2 \to \boxed{\#i_3}\ \boxed{\#11}\} \cup \\
& \{S_2 \to \boxed{\#i_1}\ \boxed{S_2\#101}, S_2 \to \boxed{\#i_2}\ \boxed{S_2\#00}, S_2 \to \boxed{\#i_3}\ \boxed{S_2\#11}\} \cup \\
& \{\boxed{S_2\#101} \to S_2 \boxed{\#101}, \boxed{S_2\#00} \to S_2 \boxed{\#00}, \boxed{S_2\#11} \to S_2 \boxed{\#11}\} \cup \\
& \{\boxed{\#101} \to \boxed{\#1}\ \boxed{\#01}, \boxed{\#01} \to \boxed{\#0}\ \boxed{\#1}, \\
& \ \boxed{\#00} \to \boxed{\#0}\ \boxed{\#0}, \boxed{\#11} \to \boxed{\#1}\ \boxed{\#1}\}.
\end{aligned}
$$

One can easily verify that $i_3\#i_2\#i_3\#i_1\#1\#0\#1\#1\#1\#0\#0\#1\#1\# \in \mathcal{L}(\mathcal{G}_{1,K})$ and $\#i_3\#i_2\#i_3\#i_1\#1\#0\#1\#1\#1\#0\#0\#1\#1 \in \mathcal{L}(\mathcal{G}_{2,K})$, as required, since $(1,3,2,3)$ is a solution of K.

Continuing the proof sketch, define the role box $\mathfrak{R}_{\mathfrak{K}}$ as

$$\mathfrak{R}_{\mathfrak{K}} =_{def} \{ B \circ C \sqsubseteq A \mid A \rightarrow B\ C \in \mathcal{P}_1 \cup \mathcal{P}_2 \},$$

and define its *completion* $\mathfrak{R}'_{\mathfrak{K}}$ using a new role $R_? \notin \mathsf{roles}(\mathfrak{R}_{\mathfrak{K}})$ as

$$\mathfrak{R}'_{\mathfrak{K}} =_{def} \mathfrak{R}_{\mathfrak{K}} \cup \{ R \circ S \sqsubseteq \sqcup_{T \in (\mathsf{roles}(\mathfrak{R}_{\mathfrak{K}}) \cup R_?)} T \mid R, S \in (\{R_?\} \cup \mathsf{roles}(\mathfrak{R}_{\mathfrak{K}})), \neg \exists ra \in \mathfrak{R}_{\mathfrak{K}} : \mathsf{pre}(ra) = (R, S) \}.$$

We then have the following:

$(E, \mathfrak{R}'_{\mathfrak{K}})$ is satisfiable (in $\mathcal{ALC}_{\mathcal{RA}}$) iff $(\{\#\}\mathcal{L}(\mathcal{G}_{1,K})) \cap (\mathcal{L}(\mathcal{G}_{2,K})\{\#\}) = \emptyset$,

and the concept E is defined as

$$E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap (\forall \#.\forall S_1.C) \sqcap (\forall S_2.\forall \#.D), \text{ with}$$
$$X =_{def} \sqcap_{a \in \Sigma} \exists a.\top \text{ and}$$
$$Y =_{def} \sqcap_{R \in \mathsf{roles}(\mathfrak{R}'_{\mathfrak{K}})} \forall R.(X \sqcap \neg(C \sqcap D)).$$

The $\mathcal{ALCN}_{\mathcal{RASG}}$-proof ([7]) is by reduction from the well-known undecidable *Domino Problem:* A *domino system* $\mathcal{DOM}$ is a triple $(\mathcal{D}, \mathcal{H}, \mathcal{V})$, where $\mathcal{D} = \{d_1, \ldots, d_n\}$ is a non-empty set of so-called *domino types*, $\mathcal{V} \subseteq \mathcal{D} \times \mathcal{D}$ is the vertical matching relation, and $\mathcal{H} \subseteq \mathcal{D} \times \mathcal{D}$ is the horizontal matching relation. A *solution* of a domino system is a function $\mathsf{f} : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{D}$ (in the following we assume that $0 \in \mathbb{N}$) such that the matching relationships of the domino types are respected, i.e. for all $(i,j) \in \mathbb{N} \times \mathbb{N}$: $(\mathsf{f}(i,j), \mathsf{f}(i+1,j)) \in \mathcal{H}$ and $(\mathsf{f}(i,j), \mathsf{f}(i,j+1)) \in \mathcal{V}$. Let $\mathfrak{R}$ be the role box corresponding to the following table:

$\circ$	R_X	R_Y	R_Z	R_U
R_X	R_U	R_Z	R_U	R_U
R_Y	R_Z	R_U	R_U	R_U
R_Z	R_U	R_U	R_U	R_U
R_U	R_U	R_U	R_U	R_U

That is, $\mathfrak{R} =_{def} \{R_X \circ R_X \sqsubseteq R_U, R_X \circ R_Y \sqsubseteq R_Z, \ldots\}$. Let $\mathcal{DOM} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ and further assume, that $\mathcal{D} \subseteq \mathcal{N}_\mathcal{C}$. Now, define the concept C as follows:

$$\begin{aligned}
C &=_{def} X \sqcap (\forall R_X.X) \sqcap (\forall R_Y.X) \sqcap (\forall R_Z.X) \sqcap (\forall R_U.X), \text{ where} \\
X &=_{def} M \sqcap (\geq R_X\ 1) \sqcap (\geq R_Y\ 1) \sqcap \\
&\qquad (\leq R_X\ 1) \sqcap (\leq R_Y\ 1) \sqcap (\leq R_Z\ 1), \text{ and} \\
M &=_{def} \sqcup_{D_i \in \mathcal{D}} (D_i \sqcap (\sqcap_{D_j \in \mathcal{D}, D_i \neq D_j} \neg D_j)) \sqcap \\
&\qquad \sqcap_{D_i \in \mathcal{D}} (D_i \Rightarrow (\ \forall R_X.(\sqcup_{(D_i,D_j) \in \mathcal{H}} D_j) \sqcap \\
&\qquad\qquad \forall R_Y.(\sqcup_{(D_i,D_j) \in \mathcal{V}} D_j)))
\end{aligned}$$

Then, $(C, \mathfrak{R})$ is satisfiable iff the domino system $\mathcal{DOM} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ has a solution. Please note that $\mathfrak{R}$ is an admissible role box for C.

Theorem 2 $\mathcal{ALC}_{\mathcal{RASG}}$ is decidable and EXPTIME complete.

Again, the details of the proof can be found in [7]. Surprisingly, satisfiability of $\mathcal{ALC}_{\mathcal{RASG}}$-concepts w.r.t. admissible role boxes can be reduced to concept satisfiability w.r.t. general TBoxes in $\mathcal{ALC}$. Does it make sense to define a restricted logic like $\mathcal{ALC}_{\mathcal{RASG}}$ and even define a tableau calculus ([7]) for it? First of all, we did not define $\mathcal{ALC}_{\mathcal{RASG}}$ to get a DL of utmost utility, but solely to investigate where the borderline between decidability and undecidability of the considered DLs lies. Secondly, it should be noted that $\mathcal{ALC}_{\mathcal{RASG}}$ is still at least as expressive as $\mathcal{ALC}_{\mathcal{R}^+}$. The associativity of role boxes is a strong, but in our opinion important requirement. For example, the composition tables of RCC8 and RCC5 are also associative, but in a more general sense than required here, since $\mathcal{ALC}_{\mathcal{RASG}}$ does not admit role axioms with disjunctions. However, if one introduces for each (disjunctive) non-base relation a new role name (e.g. $EC - DC$ for the disjunction of EC and DC) and rewrites the composition tables in this way (this yields an exponential blow-up in the size of the table), one gets indeed tables that are *admissible for* $\mathcal{ALC}_{\mathcal{RASG}}$. However, the resulting logic is of course incomplete now, since the connection between the base relations and their disjunctive non-base relations are missing now (e.g. it is by no way granted that $(EC - DC)^{\mathcal{I}} = EC^{\mathcal{I}} \cup DC^{\mathcal{I}}$, etc.), and the disjointness requirement becomes meaningless then, too.

3 Discussion & Future Work

It should be noted that the original idea to quantify over "defined roles" corresponding to spatial relationships is already present in the work of Cohn in [2], were he suggests to use a pair of modal operators for each RCC8 relationship, e.g. $\Box_{EC}$, $\Diamond_{EC}$, but this idea has not been carried further. $\mathcal{ALC}_{\mathcal{RA}}$ (with inverse roles etc.) has also been considered in [5] as an "alternative" to $\mathcal{ALCRP}(\mathcal{D})$ ([4]) for qualitative spatial reasoning problems, but no proofs were given. One of the motivations to consider $\mathcal{ALC}_{\mathcal{RA}}$ was to define a logic that would overcome the deficiencies of the logic $\mathcal{ALCRP}(\mathcal{S}_2)$ (a special $\mathcal{ALCRP}(\mathcal{D})$ instantiation for qualitative spatial reasoning problems, see [4]) which somehow suffers from very strong syntax-restrictions that are necessary to ensure its decidability.

What we have shown so far is that undecidability in case of $\mathcal{ALC}_{\mathcal{RA}}$ arises in the *general* case. It is still open whether special instantiations of $\mathcal{ALC}_{\mathcal{RA}}$ with the composition tables of RCC8 or RCC5 might be decidable (let's call these DLs $\mathcal{ALC}_{\mathcal{RCC}8}$ and $\mathcal{ALC}_{\mathcal{RCC}5}$). There are indeed some good reasons to believe this, since the composition tables obey strong laws. For example, the *expanded* RCC composition tables are *admissible.* We admit that a DL, in order to be really useful for QSR applications, must also have *inverse roles.* Since TPP is the inverse of $TPPI$, we should require that $TPP^{\mathcal{I}} = (TPPI^{\mathcal{I}})^{-1}$ in all models $\mathcal{I}$, but this is (again) future work. It doesn't make sense to work on $\mathcal{ALCI}_{\mathcal{RCC}8}$

unless we know if $\mathcal{ALC}_{\mathcal{RCC}8}$ is decidable. Note that *disjoint roles* are used to enforce the pairwise exclusiveness of base relations. Another open point is how truly-spatial models can be constructed (e.g. where $\Delta^{\mathcal{I}}$ is a set of internally connected two-dimensional regions).

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