Automated Synthesis of Tableau Calculi

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Abstract. This paper presents a method for synthesising sound and complete tableau calculi. Given a specification of the formal semantics of a logic, the method generates a set of tableau inference rules which can then be used to reason within the logic. The method guarantees that the generated rules form a calculus which is sound and constructively complete. If the logic can be shown to admit finite filtration with respect to a well-defined first-order semantics then adding a general blocking mechanism produces a terminating tableau calculus. The process of generating tableau rules can be completely automated and produces, together with the blocking mechanism, an automated procedure for generating tableau decision procedures for logics. For illustration we show the workability of the approach on propositional intuitionistic logic and the description logic $\mathcal{ALCO}$.

1 Introduction

In this paper we are interested in the problem of automatically generating a tableau calculus for a logic. We assume that the logic is defined by a high-level specification of the formal semantics. Our aim is to turn this into a set of inference rules that provide a sound and complete tableau calculus for the logic. For a decidable logic we want to generate a terminating calculus. In previous work we have described a framework for turning sound and complete tableau calculi into decision procedures [7]. The prerequisites for this to work are that the logic admits an effective finite model property shown by a filtration argument, and that (i) the tableau calculus is sound and constructively complete, and (ii) a weak form of subexpression property holds for tableau derivations. Constructive completeness is a slightly stronger notion than completeness and means that for every open branch in a tableau there is a model which reflects all the expressions (formulae) occurring on the branch. The subexpression property says that every expression in a derivation is a subexpression of the input expression with respect to a finite subexpression closure operator.

In order to be able to exploit the 'termination through blocking' results in [7], in this paper, our goal is to produce tableau calculi that satisfy the prerequisites (i) and (ii). It turns out that provided that the semantics of the logic is well-defined in a certain sense, the subexpression property can be imposed on the generated calculus. Crucial is the separation of the syntax of the logic from the 'extras' in the meta-language needed for the semantic specification of the logic. The process can be completely automated and gives, together with the

unrestricted blocking mechanism and the results in [6, 7], an automated procedure for generating tableau decision procedures for logics, whenever they have an effective finite model property with respect to a well-defined first-order semantics.

That the generated calculi are constructively complete has the added advantage that models can be effectively generated from open, finished branches in tableau derivations. This means that the synthesised tableau calculi can be used for model building purposes.

The method works as follows. The user defines the formal semantics of the given logic in a many-sorted first-order language so that certain well-definedness conditions hold. The method automatically reduces the semantic specification of the logic to Skolemised implicational forms which are then rewritten as tableau inference rules. Combined with some default closure and equality rules, this provides a sound and constructively complete calculus for the logic. Under certain conditions it is then possible to refine the rules. If the logic can be shown to admit finite filtration, then the generated calculus can be automatically turned into a terminating calculus by adding the unrestricted blocking mechanism from [6].

The method is intended to be as general as possible, and cover as many logics as possible. Our main applications are non-classical logics and description logics. As case studies we consider the application of the method to propositional intuitionistic logic (e.g. [4]) and the description logic $\mathcal{ALCO}$ [1]. Intuitionistic logic provides a nearly perfect example because the semantics of the logical connectives is not Boolean and the semantics is restricted by a background theory. In addition, the logic is simple. $\mathcal{ALCO}$ is the extension of the description logic $\mathcal{ALC}$ with individual concepts (or nominals).

The paper is structured as follows. Section 2 defines the apparatus for specifying the semantics of the logic of interest. Section 3 is about synthesising tableau rules. In Section 4 we prove that the generated rules form a sound and constructively complete calculus for the logic. Section 5 discusses ways of refining the rules in order to reduce branching and redundancy in the syntax of the calculus. In Section 6 the approach is applied to intuitionistic logic and $\mathcal{ALCO}$. We conclude with a discussion of the method.

2 Specifying the Semantics of the Logic

For the sake of generality we assume the logic for which we want to develop a tableau calculus is a many-sorted logic.

Let $\mathsf{Sorts} \stackrel{\text{def}}{=} \{0, 1, \ldots, N\}$ be an index set of sorts and Conn a countable set of the logical connectives of the logic. Every connective σ in Conn is associated with a tuple $(i_1, i_2, \ldots, i_{m+1}) \in \mathsf{Sorts}^{(m+1)}$, where $m \geq 0$. The last argument i_{m+1} is the sort of the expression obtained by applying σ to expressions of sorts $i_1, i_2, \ldots, i_m$, respectively. We say that σ is an m-ary connective of sort $(i_1, i_2, \ldots, i_{m+1})$.

By $\mathcal{L}$ we denote an *abstract sorted language* defined over an alphabet given by a set of sorts Sorts, a set of connectives Conn, a countable set of variable

symbols $\{p^i_j \mid i \in \mathsf{Sorts}, j \in \omega\}$, and a countable set of constant symbols $\{q^i_j \mid i \in \mathsf{Sorts}, j \in \omega\}$. $\mathcal{L}$ is defined as a set of *expressions* over the alphabet closed under the connectives in Conn. More formally, let $\mathcal{L} \stackrel{\text{def}}{=} \bigcup_{i \in \mathsf{Sorts}} \mathcal{L}^i$, where each $\mathcal{L}^i$ denotes a *set of expressions of sort* i defined as the smallest set of expressions satisfying the following conditions:

- All variables p^i_j and all constants q^i_j in the alphabet belong to $\mathcal{L}^i$.
- For every connective $\sigma \in \mathsf{Conn}$ of sort $(i_1, i_2 \ldots, i_{m+1})$, $\sigma(E_1, \ldots, E_m)$ belongs to $\mathcal{L}^{i_{m+1}}$, whenever $E_1, \ldots, E_m$ are expressions of sorts $i_1, \ldots, i_m$ and belong to $\mathcal{L}^{i_1}, \ldots, \mathcal{L}^{i_m}$, respectively.

Symbols, expressions and connectives in $\mathcal{L}$ are also referred to as $\mathcal{L}$-symbols, $\mathcal{L}$-expressions and $\mathcal{L}$-connectives. We usually refer to expressions in $\mathcal{L}^0$ as *individuals*, expressions in $\mathcal{L}^1$ as *concepts*, and expressions in $\mathcal{L}^2$ as *roles*.

For an $\mathcal{L}$-expression E, the notation $E(p_1, \ldots, p_m)$ indicates that $p_1, \ldots, p_m$ are variables in the expression E. $E(E_1, \ldots, E_m)$ denotes the expression obtained by uniformly substituting E_i into p_i, for all $i = 1, \ldots, m$. Similarly, if X is a set of $\mathcal{L}$-expressions depending on variables $p_1, \ldots, p_m$, we indicate this as $X(p_1, \ldots, p_m)$ and denote by $X(E_1, \ldots, E_m)$ the set of expressions which are instances of expressions from X under uniform substitution of expressions $E_1, \ldots, E_m$ into $p_1, \ldots, p_m$, respectively.

Let $\prec$ be any transitive ordering $\prec$ on $\mathcal{L}$-expressions, E an $\mathcal{L}$-expression, and X a set of $\mathcal{L}$-expressions. We define

$$\mathsf{sub}_\prec(E) \stackrel{\text{def}}{=} \{E' \mid E' \prec E\} \quad \text{and} \quad \mathsf{sub}_\prec(X) \stackrel{\text{def}}{=} \bigcup_{E \in X} \mathsf{sub}_\prec(E).$$

We usually write $\mathsf{sub}_\prec(E_1, \ldots, E_m)$ rather than $\mathsf{sub}_\prec(\{E_1, \ldots, E_m\})$.

The language in which the semantics of the given logic is specified, is a sorted first-order language with equality, denoted by $\mathsf{FO}(\mathcal{L})$. Formally, $\mathsf{FO}(\mathcal{L})$ is an extension of the language $\mathcal{L}$ with: one additional sort, additional symbols, the usual connectives and quantifiers of first-order logic, and the equality predicate. The sorts of $\mathsf{FO}(\mathcal{L})$ are $\mathsf{Sorts} \cup \{N+1\} = \{0, \ldots, N, N+1\}$. We call the additional sort $N+1$ the *designated sort*, and symbols that operate on this sort, *designated symbols*. The additional symbols comprise of a countable set of variable symbols $\{x, y, z, x_0, y_0, z_0, \ldots\}$ of the designated sort, a countable set of constants $\{a, b, c, a_0, b_0, c_0, \ldots\}$ of the designated sort, function symbols $\{f, g, h, f_0, g_0, h_0, \ldots\}$ mapping argument terms to terms of sort $N+1$, and a countable set of constant predicate symbols $\{P, Q, R, P_0, Q_0, R_0, \ldots\}$ of the designated sort (i.e., argument terms are required to be terms of sort $N+1$). In addition, $\mathsf{FO}(\mathcal{L})$ contains intersort symbols denoted by $\nu_0, \ldots, \nu_N$, i.e., one for each sort in Sorts. The intersort symbols are used to define the semantics of the connectives of the logic. In particular, ν_0 is a unary function symbol of sort $(0, N+1)$, and each of the remaining ν_i is a predicate symbol of sort $(i, N+1, \ldots, N+1)$, with arity $i+1$. Furthermore, for every sort we assume the presence of a binary predicate symbol functioning as equality predicate for

that sort. For reasons of simplicity, we use one symbol, namely $\approx$, for each of the equality predicates.

We fix some more common notation. $\overline{w}$ denotes a sequence of first-order variables: $\overline{w} \stackrel{\text{def}}{=} w_1, \ldots, w_n$. Similarly, $\forall\overline{w}$ denotes the universal quantifier prefix on all variables $w_1, \ldots, w_n$: $\forall\overline{w} \stackrel{\text{def}}{=} \forall w_1 \cdots \forall w_n$. For any set S of formulae, $\forall S$ denotes the universal closure of S, i.e., the set $\forall S \stackrel{\text{def}}{=} \{\forall\overline{w}\ \phi(\overline{w}) \mid \phi(\overline{w}) \in S\}$. For every first-order formula ψ we let

$$\sim\psi \stackrel{\text{def}}{=} \begin{cases} \psi', & \psi = \neg\psi', \\ \neg\psi, & \text{otherwise.} \end{cases}$$

Formulae of $\mathsf{FO}(\mathcal{L})$ in which all occurrences of the $\mathcal{L}$-variables p^i_j (of sorts $0, \ldots, N$) are free are called $\mathcal{L}$-*open* formulae. Similarly, any $\mathcal{L}$-open formula is an $\mathcal{L}$-*open sentence* if it does *not* have free occurrences of variables of the designated sort $N+1$.

For any set S of $\mathcal{L}$-open formulae in $\mathsf{FO}(\mathcal{L})$ and a set X of $\mathcal{L}$-expressions, let $S{\restriction}X$ be the set of substitution instances of formulae in S under substitutions into the variables of $\mathcal{L}$ which do not contain expressions outside X. Formally,

$$S{\restriction}X \stackrel{\text{def}}{=} \{\phi(E_1, \ldots, E_m) \mid \phi(p_1, \ldots, p_m) \in S \text{ and all } \mathcal{L}\text{-expressions occurring in } \phi(E_1, \ldots, E_m) \text{ belong to } X\}.$$

The semantics of $\mathcal{L}$ is specified in $\mathsf{FO}(\mathcal{L})$ as follows. Each expression in $\mathcal{L}$ is interpreted as a term in $\mathsf{FO}(\mathcal{L})$. In particular, each variable symbol p^i_j in $\mathcal{L}^i$ is interpreted as a variable of sort i in $\mathsf{FO}(\mathcal{L})$, each constant symbol q^i_j in $\mathcal{L}^i$ is interpreted as a constant of sort i in $\mathsf{FO}(\mathcal{L})$, and every connective σ is interpreted as a function of the same sort as σ.

An $\mathcal{L}$-*structure* is a tuple $\mathcal{I} \stackrel{\text{def}}{=} (\mathcal{L}^0, \ldots, \mathcal{L}^N, \Delta^{\mathcal{I}}, \nu^{\mathcal{I}}_0, \ldots, \nu^{\mathcal{I}}_N, a^{\mathcal{I}}, \ldots, P^{\mathcal{I}}, \ldots)$ where $\Delta^{\mathcal{I}}$ is a non-empty set, $\nu_0(\ell)^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for every individual $\ell \in \mathcal{L}^0$, $\nu^{\mathcal{I}}_n \subseteq \mathcal{L}^n \times (\Delta^{\mathcal{I}})^n$, for $0 < n \leq N$. $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ and $P^{\mathcal{I}} \subseteq (\Delta^{\mathcal{I}})^m$, where m is the arity of P. Observe that an $\mathcal{L}$-structure $\mathcal{I}$ is a first-order model (interpretation) of the language $\mathsf{FO}(\mathcal{L})$. For simplicity we omit the sets $\mathcal{L}^0, \ldots, \mathcal{L}^N$ and simply write $\mathcal{I} = (\Delta^{\mathcal{I}}, \nu^{\mathcal{I}}_0, \ldots, \nu^{\mathcal{I}}_N, a^{\mathcal{I}}, \ldots, P^{\mathcal{I}}, \ldots)$.

A *valuation* in $\mathcal{I}$ is a mapping ι from the set of variables and constants of $\mathsf{FO}(\mathcal{L})$ to $\mathcal{L} \cup \Delta^{\mathcal{I}}$ such that $\iota(p^i_j), \iota(q^i_j) \in \mathcal{L}^i$, and $\iota(x_j), \iota(a_j) \in \Delta^{\mathcal{I}}$. We say that a valuation ι in an $\mathcal{L}$-structure is *canonical* if every variable and constant of any sort $i = 0, \ldots, N$ is interpreted by itself, that is, $\iota(p^i_j) = p^i_j$ and $\iota(q^i_j) = q^i_j$ for every variable p^i_j and constant q^i_j of the language $\mathcal{L}$. This means that a canonical valuation of every term of any sort $i = 0, \ldots, N$ is the term itself. It is not difficult to see that any $\mathcal{L}$-open formula ϕ is satisfiable in an $\mathcal{L}$-structure iff it is satisfiable in an $\mathcal{L}$-structure under a canonical valuation. We write $S \models_c S'$ for sets of formulae S and S', if, for every $\mathcal{L}$-structure $\mathcal{I}$ and a canonical valuation ι in $\mathcal{I}$, $\mathcal{I}, \iota \models S$ implies $\mathcal{I}, \iota \models S'$. Similarly, we write $\mathcal{I} \models_c S$ iff there is a canonical valuation ι such that $\mathcal{I}, \iota \models S$.

$$\forall x\ (x \approx x) \qquad \forall x \forall y\ (x \approx y \to y \approx x) \qquad \forall x \forall y \forall z\ (x \approx y \wedge y \approx z \to x \approx z)$$
$$\forall x_1 \cdots \forall x_n \forall y_i\ (P(x_1, \ldots, x_n) \wedge x_i \approx y_i \to P(x_1, \ldots x_{i-1}, y_i, x_{i+1}, x_n))$$
$$\forall p \forall x_1 \cdots \forall x_n \forall y_i\ (\nu_n(p, x_1, \ldots, x_n) \wedge x_i \approx y_i \to \nu_n(p, x_1, \ldots x_{i-1}, y_i, x_{i+1}, x_n))$$
$$\forall p_1 \cdots \forall p_m \forall x_1 \cdots x_n \forall y_i\ (x_i \approx y_i \to$$
$$f(p_1, \ldots, p_m, x_1, \ldots, x_n) \approx f(p_1, \ldots, p_m, x_1, \ldots x_{i-1}, y_i, x_{i+1}, \ldots, x_n))$$

Fig. 1. Equality axioms in $\mathsf{FO}(\mathcal{L})$.

We say that a concept C is *satisfiable* in $\mathcal{I}$ if there is an $a \in \Delta^{\mathcal{I}}$ such that $(C, a) \in \nu_1^{\mathcal{I}}$, or equivalently $\mathcal{I} \models_c \exists x\ \nu_1(C, x)$. A concept C is *valid* in $\mathcal{I}$ if $\mathcal{I} \models_c \forall x\ \nu_1(C, x)$.

Let S be a set of $\mathcal{L}$-open sentences in $\mathsf{FO}(\mathcal{L})$. A formula ϕ^σ in the language of S *defines the connective* σ with respect to S if it does not contain σ and the following holds:

$$(*) \qquad \forall S \models \forall p_1 \ldots \forall p_m\ \forall \overline{x}\ (\nu_n(\sigma(p_1, \ldots, p_m), \overline{x}) \equiv \phi^\sigma(p_1, \ldots, p_m, \overline{x})).$$

Here $p_1, \ldots, p_m$ are variables of appropriate sorts which match the signature of σ, and n is the result sort of σ (for $\overline{x} = (x_1, \ldots, x_n)$). We also say S *defines* σ. The $\mathcal{L}$-open sentence $\forall \overline{x}\ (\nu_n(\sigma(p_1, \ldots, p_m), \overline{x}) \equiv \phi^\sigma(p_1, \ldots, p_m, \overline{x})$ is said to be a *σ-definition (in S)*.

By definition, a *(first-order) semantic specification* of $\mathcal{L}$ is a set of $\mathcal{L}$-open sentences in $\mathsf{FO}(\mathcal{L})$ that defines the connectives of $\mathcal{L}$. Given a semantic specification S, we use the notation S^0 for the subset of $\mathcal{L}$-open sentences defining the connectives of $\mathcal{L}$.

For the sake of generality, we always include the usual equality axioms, listed in Figure 1, in a semantic specification. This ensures that $\approx$ is a congruence on every sort in any first-order interpretation of $\mathsf{FO}(\mathcal{L})$.

We consider a semantic specification S to be *normalised* if it consists of three disjoint parts. More specifically, $S = S^+ \cup S^- \cup S^b$, where S^+, S^-, and S^b are disjoint sets of sentences satisfying the following:

(n1) S^+ is a set of $\mathcal{L}$-open sentences ξ_+^E of the form:

$$\xi_+^E \stackrel{\text{def}}{=} \forall \overline{x}\ (\nu_n(E(p_1, \ldots, p_m), \overline{x}) \to \phi_+^E(p_1, \ldots, p_m, \overline{x})),$$

(n2) S^- is a set of $\mathcal{L}$-open sentences ξ_-^E of the form:

$$\xi_-^E \stackrel{\text{def}}{=} \forall \overline{x}\ (\phi_-^E(p_1, \ldots, p_m, \overline{x}) \to \nu_n(E(p_1, \ldots, p_m), \overline{x})),$$

(n3) None of the $\mathcal{L}$-open sentences in the set S^b contain complex $\mathcal{L}$-expressions.

In this definition, we suppose that multiple sentences of the form (n1) (resp. (n2)) for the same expression E in S^+ and S^- are all equivalently reduced to a single

sentence ξ^E_+ (resp. ξ^E_-). The intuition is that S^+ and S^- define the semantics of the connectives. S^+ defines it for positive occurrences of expressions E (with free variables $p_1, \ldots, p_m$), while S^- defines it for negative occurrences of expressions E. We refer to S^b as the *background theory* of the semantics S (for $\mathcal{L}$).

It can be seen that the set $S^0 \cup S^b$ is a semantic specification which can be turned into normalised form by decomposing each connective definition in S^0 into two implications. In fact, S^0 and $S^+ \cup S^-$ play the same role in axiomatising $\mathcal{L}$-connectives in $\mathsf{FO}(\mathcal{L})$ modulo the background theory S^b.

For every $\mathcal{L}$-expression E, let

$$\Phi^E_+ \stackrel{\text{def}}{=} \{\phi^F_+(E_1, \ldots, E_m, \overline{x}) \mid E = F(E_1, \ldots, E_m) \text{ for some } \xi^{F(p_1,\ldots,p_m)}_+ \text{ from } S\},$$
$$\Phi^E_- \stackrel{\text{def}}{=} \{\phi^F_-(E_1, \ldots, E_m, \overline{x}) \mid E = F(E_1, \ldots, E_m) \text{ for some } \xi^{F(p_1,\ldots,p_m)}_- \text{ from } S\}.$$

Thus, Φ^E_+ (resp. Φ^E_-) is the set of instantiations of succedents (resp. antecedents) of positive (resp. negative) specifications in S, where the antecedents (resp. succedents) are unifiable with the given expression E.

The expression specifications in any normalised semantics S induce an ordering $\prec$ on expressions as follows. Let $\prec$ be the smallest transitive ordering satisfying: $E' \prec E$ whenever there is a sentence $\xi^{F(p_1,\ldots,p_m)}_+$ or a sentence $\xi^{F(p_1,\ldots,p_m)}_-$ such that $E = F(E_1, \ldots, E_m)$, for some $\mathcal{L}$-expressions $E_1, \ldots, E_m$, and E' occurs in $\phi^F_+(E_1, \ldots, E_m, \overline{x})$ or $\phi^F_-(E_1, \ldots, E_m, \overline{x})$, respectively. The reflexive closure of $\prec$ is denoted by $\preceq$. Because we can assume that S^0 is also a normalised semantic specification, it similarly induces an ordering $\prec_0$ which is assumed to be well-founded.

Usually the semantics is defined by induction in terms of definitions of the semantics of the connectives and the primitives in the logic which is lifted to arbitrary $\mathcal{L}$-expressions. This is equivalent to assuming a well-founded ordering on expressions of $\mathcal{L}$. For any reasonable definition such a well-founded ordering exists. Thus, although it is not difficult to imagine formulae ϕ^σ such that $\prec_0$ is not well-founded, we assume that the ϕ^σ are chosen in such a way that it is possible to lift the semantics of $\mathcal{L}$-primitives to all $\mathcal{L}$-expressions, i.e., $\prec_0$ is well-founded.

Recall that S^0 denotes the set of $\mathcal{L}$-open sentences that define the $\mathcal{L}$-connectives. A semantic specification S is *well-defined* iff

(wd1) $\forall S^0, \forall S^b \models \forall S$,

(wd2) the ordering $\prec$ is well-founded, and

(wd3) for every expression $E = \sigma(E_1, \ldots, E_m)$,
$$\forall S^0, S^b \restriction \mathsf{sub}_\prec(E) \models_c \forall \overline{x}\Big(\Big(\bigwedge \Phi^E_+ \rightarrow \phi^\sigma(E_1, \ldots, E_m, \overline{x})\Big) \wedge \Big(\phi^\sigma(E_1, \ldots, E_m, \overline{x}) \rightarrow \bigvee \Phi^E_-\Big)\Big).$$

It is not difficult to see that condition (wd3) follows from the following first-order condition:

$$\text{(wd3')} \qquad \forall S^0, S^b \restriction \mathsf{sub}_{\prec}(\sigma(\overline{p})) \models_c \forall \overline{x}\Big(\Big(\bigwedge \Phi_+^{\sigma(\overline{p})} \rightarrow \phi^{\sigma}(\overline{p},\overline{x})\Big) \wedge \Big(\phi^{\sigma}(\overline{p},\overline{x}) \rightarrow \bigvee \Phi_-^{\sigma(\overline{p})}\Big)\Big).$$

There are different semantic specifications which describe the same class of $\mathcal{L}$-structures. As we have just noted, some semantic specifications already allow the lifting of the semantics from atomic expressions to all $\mathcal{L}$-expressions. We assume that $S^0 \cup S^b$ is such a specification and implicitly accommodates $\mathcal{L}$-connectives. According to this definition, a well-defined semantics S is equivalent to $S^0 \cup S^b$ modulo the background theory S^b. This is ensured by condition (wd1) and the assumption that S defines all $\mathcal{L}$-connectives in S^0. Further, through condition (wd2), S imposes its own inductive structure on $\mathcal{L}$-expressions. Finally, condition (wd3) specifies a correlation between S and S^0 on instances of $\mathcal{L}$-expressions. It can be seen that $S^0 \cup S^b$ is a well-defined semantic specification itself.

A *(propositional) logic* L over the language $\mathcal{L}$ is a subset of concepts in $\mathcal{L}$ which is closed under arbitrary substitutions of primitive symbols with complex expressions of the same sorts. A logic L is *first-order definable* iff there is a semantic specification S_L such that L coincides with the set of all concepts that are valid in all $\mathcal{L}$-structures satisfying $\forall S_L$, i.e., $L = \{C \in \mathcal{L}^1 \mid \forall S_L \models_c \forall x\, \nu_1(C, x)\}$.

For a fixed semantic specification S_L of logic L, if $\mathcal{I}$ is an $\mathcal{L}$-structure satisfying S_L then by definition $\mathcal{I}$ is a *model of* L or simply a L-*model* (with respect to S_L).

The following are examples first-order definable logics and all have a normalised semantic specification according to the above definitions:

- most description logics, including $\mathcal{ALCO}$, $\mathcal{ALBO}$ [6], $\mathcal{SHOIQ}$ [2];
- most propositional modal logics, including K, $\mathsf{S4}$, $\mathsf{KD45}$;
- propositional intuitionistic logic [4];
- the logic of metric and topology [3].

3 Synthesising a Tableau Calculus

A *tableau calculus* is a set of tableau inference rules. In general, a *tableau inference rule* is a rule of the form

$$\frac{X}{X_1 \mid \cdots \mid X_m},$$

where both the numerator X and all denominators X_i ($m \geq 0$) are finite sets of atomic formulae or negated atomic formulae in the language $\mathsf{FO}(\mathcal{L})$. The formulae in the numerator are called *premises*, while the formulae in the denominators are called *conclusions*. The numerator and all the denominators are non-empty,

but m may be zero, in which case the rule is a *closure rule*. Closure rules are also written $X/\bot$. If $m > 1$, the rule is a branching rule.

Inference steps are performed as usual. A rule is applied to a set of (ground) literals in a branch of a tableau derivation, if the literals are instances of the premises of the rule. Then, in the case of a non-branching rule, the corresponding (ground) instances of the conclusions of the rule are added to the branch. In the case of a branching rule the branch is split into several branches and the corresponding (ground) instances of the conclusions are added to each branch.

Let T denote a tableau calculus and C a concept. We take an arbitrary constant a of the designated sort which does not occur in the rules of T.[1] We denote by $T(C)$ a finished tableau derivation built by starting with the formula $\nu_1(C, a)$ as input and applying the rules of T. That is, all branches in the tableau derivation are fully expanded and all applicable rules of T have been applied in $T(C)$. As usual we assume that all the rules of the calculus are applied *non-deterministically in a tableau derivation*. A branch of a tableau derivation is *closed* if a closure rule has been applied, otherwise the branch is called *open*. The tableau derivation $T(C)$ is *closed* if all its branches are closed and $T(C)$ is *open* otherwise. The calculus T is *sound* iff for any concept C, C is unsatisfiable whenever $T(C)$ is closed. T is *complete* iff for any concept C, C is satisfiable (has a model) whenever $T(C)$ is open. T is said to be *terminating* if every finished open tableau derivation in T has a finite open branch.

Let L be a first-order definable propositional logic over $\mathcal{L}$ and S_L a well-defined semantic specification of L. We now describe how tableau rules can be synthesised from S_L. If S_L is not already normalised we first normalise it. Thus assume $S_L = S_L^+ \cup S_L^- \cup S_L^b$. Now take a positive specification ξ_+^E in S_L^+. Eliminate quantifiers using Skolemisation and equivalently rewrite ξ_+^E into the following implicational form

$$\nu_n(E(p_1, \ldots, p_m), x_1, \ldots, x_n) \rightarrow \bigvee_{j=1}^{J} \bigwedge_{k=1}^{K_j} \psi_{jk},$$

where each ψ_{jk} denotes a literal. This is always possible. The implication is now turned into the rule:

$$\rho_+(\xi_+^E) \stackrel{\text{def}}{=} \frac{\nu_n(E(p_1, \ldots, p_m), x_1, \ldots, x_n), \quad y_1 \approx y_1, \quad \ldots, \quad y_s \approx y_s}{\psi_{11}, \quad \ldots, \quad \psi_{1K_1} \mid \cdots \mid \psi_{J1}, \quad \ldots, \quad \psi_{JK_J}},$$

where $y_1, \ldots, y_s$ denote the free variables in ψ_{jk} which are not among the variables $x_1, \ldots, x_n$. Essentially, the antecedent of the implication has become the main premise in the nominator and the succedent was appropriately turned into the denominators of the rule. We say the rule *corresponds* to ξ_+^E. Analogously a tableau rule is be generated for each negative specification ξ_-^E in S_L^-. The corresponding rules have the form

$$\rho_-(\xi_-^E) \stackrel{\text{def}}{=} \frac{\neg\nu_n(E(p_1, \ldots, p_m), x_1, \ldots, x_n), \quad y_1 \approx y_1, \quad \ldots, \quad y_s \approx y_s}{\psi_{11}, \quad \ldots, \quad \psi_{1K_1} \mid \cdots \mid \psi_{J1}, \quad \ldots, \quad \psi_{JK_J}}.$$

[1] a is a Skolem constant for the $\mathsf{FO}(\mathcal{L})$-formula $\exists x\ \nu_1(C, x)$

This is obtained by Skolemising the contrapositive of ξ_-^E and then equivalently rewriting it to an implication of the form

$$\neg\nu_n(E(p_1,\ldots,p_m),x_1,\ldots,x_n) \rightarrow \bigvee_{j=1}^{J}\bigwedge_{k=1}^{K_j}\psi_{jk},$$

where each ψ_{jk} denotes a literal. We refer to the rules $\rho_-(\xi_-^E)$ and $\rho_-(\xi_-^E)$ as *decomposition rules*.

For example, the generated decomposition rules for the existential restriction operator in the description logic $\mathcal{ALC}$ are:

$$\frac{\nu_1(\exists r.p,x)}{\nu_2(r,x,f(p,x)),\quad \nu_1(p,f(p,x))}, \qquad \frac{\neg\nu_1(\exists r.p,x)}{\neg\nu_2(r,x,y)\ \mid\ \neg\nu_1(p,y)}.$$

These are not the familiar rules used in standard description logic tableau systems, but in Section 5 we see how to get those.

The sentences in the background theory of S_L are turned into rules by first equivalently transforming them into Skolemised disjunctive normal form. More specifically, let ξ be an arbitrary sentence in S_L^b. It is first equivalently rewritten to

$$(**) \qquad \bigvee_{j=1}^{J}\bigwedge_{k=1}^{K_j}\psi_{jk},$$

where each ψ_{jk} denotes a literal, and is then turned into the corresponding rule, namely

$$\rho(\xi) \stackrel{\text{def}}{=} \frac{p_1\approx p_1,\ \ldots,\ p_m\approx p_m,\ \ x_1\approx x_1,\ \ldots,\ x_n\approx x_n}{\psi_{11},\ \ldots,\ \psi_{1K_1}\ \mid\ \cdots\ \mid\ \psi_{J1},\ \ldots,\ \psi_{JK_J}}.$$

The $p_1,\ldots,p_m,x_1,\ldots,x_n$ are the variables that are free in $(**)$. Rules corresponding to sentences in S_L^b are called *theory rules*.

The equality axioms in the background theory are treated slightly unusually. In our formalisation the equality predicate(s) are also used as domain predicate(s) to keep track of the ground terms that occur in the tableau branches. The *equality rules* are the rules listed in Figure 2. The rules on the left ensure that expression(s) $t\approx t$ are treated as domain predicate(s) and appear in every branch of a tableau for every term t in the branch. They also state reflexivity of the equality predicate(s). The rules in the right column ensure that $\approx$ is a congruence relation for predicates on terms occurring in a branch. The last rule is a congruence rule for function symbols f occurring in a branch including Skolem function symbols.

We use T_L to denote the generated calculus. In summary, it consists of these rules.

(t1) The decomposition rules $\rho_+^\sigma(\xi)$ and $\rho_-^\sigma(\eta)$ corresponding to all positive specifications ξ in S_L^+ and all negative specifications η in S_L^-.

$$\frac{P(x_1,\ldots,x_n)}{x_1 \approx x_1,\ \ \ldots,\ \ x_n \approx x_n} \qquad \frac{x \approx y}{y \approx x}$$

$$\frac{\neg P(x_1,\ldots,x_n)}{x_1 \approx x_1,\ \ \ldots,\ \ x_n \approx x_n} \qquad \frac{x \approx y,\ \ y \approx z}{x \approx z}$$

$$\frac{\nu_n(p,x_1,\ldots,x_n)}{p \approx p,\ \ x_1 \approx x_1,\ \ \ldots,\ \ x_n \approx x_n} \qquad \frac{P(x_1,\ldots,x_n),\ \ x_i \approx y_i}{P(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)}$$

$$\frac{\neg\nu_n(p,x_1,\ldots,x_n)}{p \approx p,\ \ x_1 \approx x_1,\ \ \ldots,\ \ x_n \approx x_n} \qquad \frac{\nu_n(p,x_1,\ldots,x_n),\ \ x_i \approx y_i}{\nu_n(p,x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)}$$

$$\frac{p_1 \approx p_1,\ \ \ldots,\ \ p_m \approx p_m,\ \ x_1 \approx x_1,\ \ \ldots,\ \ x_i \approx y_i,\ \ \ldots,\ \ x_n \approx y_n}{f(p_1,\ldots,p_m,x_1,\ldots,x_n) \approx f(p_1,\ldots,p_m,x_1,\ldots x_{i-1},y_i,x_{i+1},x_n)}$$

Fig. 2. Equality congruence rules for predicates and functions occurred in S_L.

(t2) The theory rules $\rho(\zeta)$ corresponding to all sentences ζ in the background theory S_L^b.
(t3) The equality rules of Figure 2.
(t4) The *closure rules* (for every $n = 1,\ldots,N$, and every constant predicate symbol P in S_L):

$$\frac{\nu_n(p,\overline{x}),\ \ \neg\nu_n(p,\overline{x})}{\bot}, \qquad \frac{P(\overline{x}),\ \ \neg P(\overline{x})}{\bot}.$$

Note for each connective there are exactly two decomposition rules in the calculus T_L, one for unnegated occurrences and one for negated occurrences.

4 Ensuring Soundness, Subexpression Property, and Constructive Completeness

It is not difficult to see that the generated tableau T_L has the following *subexpression property* which is a necessary condition for termination of the calculus enhanced by the unrestricted blocking rule mechanism [6, 7]. Let $\preceq$ be a reflexive and transitive ordering of $\mathcal{L}$-expressions. Following [7], we say that a tableau calculus T is *compatible with* $\mathsf{sub}_\preceq$ (or has *subexpression property* with respect to $\preceq$) iff for every concept C, all $\mathcal{L}$-expressions occurring in the tableau derivation $T(C)$ belong to $\mathsf{sub}_\preceq(C)$. Since every rule of T_L is monotone with respect to $\prec$, i.e., every expression in any conclusion of a rule is not greater with respect to $\prec$ than $\mathcal{L}$-expressions in the premises of the rule, we can conclude that T_L has subexpression property with respect to the reflexive closure of the ordering $\prec$. Thus, the subexpression property holds automatically for the calculus T_L generated from a well-defined semantics S_L.

It is possible to prove that every rule of the generated calculus T_L preserves satisfiability of $\mathsf{FO}(\mathcal{L})$-formulae. That is, if all premises of a rule are true in an L-model $\mathcal{I}$ (under a canonical valuation) then the conclusions of some branch are also true. This is not difficult to see because the definitions of the rules mimic the specified semantics. Hence, the following soundness theorem holds.

Theorem 1 (Soundness). *T_L is sound for L, i.e., every concept satisfiable in an L-model has an open T_L tableau.*

Now, we prove constructive completeness of T_L. Let $\mathcal{B}$ denote an arbitrary branch in a T_L-tableau derivation. We define the following relation $\sim_{\mathcal{B}}$ with respect to $\mathcal{B}$: $t \sim_{\mathcal{B}} t' \stackrel{\text{def}}{\iff} t \approx t' \in \mathcal{B}$, for any ground terms t and t' of the designated sort $N+1$ in $\mathcal{B}$. Let $\|t\| \stackrel{\text{def}}{=} \{t' \mid t \sim_{\mathcal{B}} t'\}$ be the equivalence class of an element t. The presence of the rules of Figure 2 ensure that $\sim_{\mathcal{B}}$ is a congruence relation on all designated ground terms in $\mathcal{B}$.

We say a model $\mathcal{I}$, under a (canonical) valuation ι, *reflects* an expression E occurring in a branch $\mathcal{B}$ iff for every ground terms $t_1, \ldots, t_n$

- $(E, \iota(t_1), \ldots, \iota(t_n)) \in \nu_n^{\mathcal{I}}$ whenever $\nu_n(E, t_1, \ldots, t_n) \in \mathcal{B}$, and
- $(E, \iota(t_1), \ldots, \iota(t_n)) \notin \nu_n^{\mathcal{I}}$ whenever $\neg\nu_n(E, t_1, \ldots, t_n) \in \mathcal{B}$.

Similarly, $\mathcal{I}$ *reflects* predicate constant P from $\mathcal{B}$ under a (canonical) valuation ι in $\mathcal{I}$ iff for every ground terms $t_1, \ldots, t_n$

- $(\iota(t_1), \ldots, \iota(t_n)) \in P^{\mathcal{I}}$ whenever $P(t_1, \ldots, t_n) \in \mathcal{B}$, and
- $(\iota(t_1), \ldots, \iota(t_n)) \notin P^{\mathcal{I}}$ whenever $\neg P(t_1, \ldots, t_n) \in \mathcal{B}$.

A model $\mathcal{I}$ *reflects* branch $\mathcal{B}$ under a valuation ι if $\mathcal{I}$ reflects all predicate constants and expressions occurring in $\mathcal{B}$ under ι.

A tableau calculus T_L is said to be *constructively complete* (for L) iff for any given concept C that is satisfiable, if $\mathcal{B}$ is an open branch in the tableau derivation $T_L(C)$ then there is an L-model $\mathcal{I}$ such that:

(m1) The domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$ is the set of the equivalence classes $\|t\|$ for each ground term t occuring in $\mathcal{B}$.

(m2) $\mathcal{I}$ reflects $\mathcal{B}$ under the *canonical projection valuation* π defined by $\pi(t) \stackrel{\text{def}}{=} \|t\|$, for every ground term t occuring in $\mathcal{B}$.

It is clear that if T_L is constructively complete then T_L is complete for L.

Suppose now that S_L is a semantic specification and $\prec_0$ is a well-founded ordering on $\mathcal{L}$-expressions induced by the set S_L^0 of the definitions of the connectives of the form $(*)$ with respect to S_L.

Let $\mathcal{B}$ be an open branch in a finished tableau derivation in T_L. Define interpretations of predicate symbols in $\mathcal{I}(\mathcal{B})$ by induction on $\prec_0$ as follows:

- For every n-ary constant predicate symbol P in S_L

$$P^{\mathcal{I}(\mathcal{B})} \stackrel{\text{def}}{=} \{(\|t_1\|, \ldots, \|t_n\|) \mid P(t_1, \ldots, t_n) \in \mathcal{B}\}.$$

- For every $n = 1, \ldots, N$ the interpretation $\nu_n^{\mathcal{I}(\mathcal{B})}$ of the ν_n symbols is defined as the smallest subset of $\mathcal{L}^n \times (\Delta^{\mathcal{I}(\mathcal{B})})^n$ satisfying both the following, for every variable or constant p of the sort n, every connective σ, and any expressions $E_1, \ldots, E_m$:

$$(p, \|t_1\|, \ldots, \|t_n\|) \in \nu_n^{\mathcal{I}(\mathcal{B})} \iff \nu_n(p, t_1, \ldots, t_n) \in \mathcal{B},$$

$$(\sigma(E_1, \ldots, E_m), \|t_1\|, \ldots, \|t_n\|) \in \nu_n^{\mathcal{I}(\mathcal{B})} \iff \mathcal{I}(\mathcal{B}) \models_c \phi^{\sigma}(E_1, \ldots, E_m, \|t_1\|, \ldots, \|t_n\|).$$

In what follows, we say that$\mathcal{I}(\mathcal{B})$ reflects an expression E (a predicate P, or a branch $\mathcal{B}$) if $\mathcal{I}(\mathcal{B})$ reflects E (P, or $\mathcal{B}$, respectively) under the canonical projection valuation π, and usually omit any explicit reference to π.

A consequence of the definition of $\mathcal{I}(\mathcal{B})$ is that the definitions of the connectives are valid in $\mathcal{I}(\mathcal{B})$:

Lemma 1. *$\mathcal{I}(\mathcal{B}) \models \forall S_L^0$.*

Lemma 2. *Let X be any set of expressions which occur in $\mathcal{B}$. Suppose $\mathcal{I}(\mathcal{B})$ reflects all the expressions from X. Then $\mathcal{I}(\mathcal{B}) \models S_L^b \restriction X$.*

Proof. Consider any $\xi \in S_L^b$ and suppose

$$\xi(p_1, \ldots, p_m) \equiv \forall x_1 \cdots \forall x_n \bigvee_{j=1}^{J} \bigwedge_{k=1}^{K_j} \psi_{jk}(p_1, \ldots, p_m, x_1, \ldots, x_n).$$

Let $E_1, \ldots, E_m$ be any expressions from X and $t_1, \ldots, t_n$ any ground terms of sort $N+1$ occurring in $\mathcal{B}$. By rule $\rho(\xi)$, there is a $j = 1, \ldots, J$ such that all literals $\psi_{jk}(E_1, \ldots, E_m, t_1, \ldots, t_n)$, for $k = 1, \ldots, K_j$, are in the branch $\mathcal{B}$. Since S_L^b does not contain complex expressions of the language $\mathcal{L}$ we have that $\mathcal{I}(\mathcal{B}) \models_c \psi_{jk}(E_1, \ldots, E_m, \|t_1\|, \ldots, \|t_n\|)$ by the assumptions of the lemma for every $k = 1, \ldots, K_j$. This implies that $\mathcal{I}(\mathcal{B}) \models_c \xi(E_1, \ldots, E_m, \|t_1\|, \ldots, \|t_n\|)$.

Corollary 1. *$\mathcal{I}(\mathcal{B}) \models_c S_L^b$.*

Proof. Immediately from the definition of $\mathcal{I}(\mathcal{B})$ and the closure rules we get that $P(t_1, \ldots, t_n) \in \mathcal{B}$ implies $(\|t_1\|, \ldots, \|t_n\|) \in P^{\mathcal{I}(\mathcal{B})}$, $\neg P(t_1, \ldots, t_n) \in \mathcal{B}$ implies $(\|t_1\|, \ldots, \|t_n\|) \notin P^{\mathcal{I}(\mathcal{B})}$, $\nu_n(p, t_1, \ldots, t_n) \in \mathcal{B}$ implies $(p, \|t_1\|, \ldots, \|t_n\|) \in \nu_n^{\mathcal{I}(\mathcal{B})}$, and $\neg\nu_n(p, t_1, \ldots, t_n) \in \mathcal{B}$ implies $(p, \|t_1\|, \ldots, \|t_n\|) \notin \nu_n^{\mathcal{I}(\mathcal{B})}$ for every constant predicate symbol P, $n = 0, \ldots, N$, and primitive p of sort n. Thus, by Lemma 2 $\mathcal{I}(\mathcal{B}) \models_c S_L^b$.

Lemma 3. *$\mathcal{I}(\mathcal{B})$ reflects the branch $\mathcal{B}$.*

Proof. By simultaneous induction on the well-founded ordering $\prec$ we show that for all $n = 1, \ldots, N$, for every E, and all designated ground terms $t_1, \ldots, t_n$ (of sort $N+1$) in $\mathcal{B}$, we have that

- $(E, \|t_1\|, \ldots, \|t_n\|) \in \nu_n^{\mathcal{I}(\mathcal{B})}$ whenever $\nu_n(E, t_1, \ldots, t_n) \in \mathcal{B}$, and
- $(E, \|t_1\|, \ldots, \|t_n\|) \notin \nu_n^{\mathcal{I}(\mathcal{B})}$ whenever $\neg\nu_n(E, t_1, \ldots, t_n) \in \mathcal{B}$.

$E = p$. This case follows from the definition of $\mathcal{I}(\mathcal{B})$.

$E = \sigma(E_1, \ldots, E_m)$. Suppose $\nu_n(E, t_1, \ldots, t_n) \in \mathcal{B}$. Let ξ_+^F be such that $E = F(F_1, \ldots, F_m)$ for some $F_1, \ldots, F_m$ and the Skolemised form of corresponding ϕ_+^F is as follows

$$\phi_+^F(p_1, \ldots, p_m) \equiv \bigvee_{j=1}^{J} \bigwedge_{k=1}^{K_j} \psi_{jk}(p_1, \ldots, p_m).$$

Then by rule $\rho_+(\xi_+^F)$ there is a $j = 1, \ldots, J$ such that all the literals $\psi_{jk}(F_1, \ldots, F_m, t_1, \ldots, t_n)$ for $k = 1, \ldots, K_j$ are in $\mathcal{B}$. Further, for every expression $E'(F_1, \ldots, F_m)$ which occurs in $\psi_{jk}(F_1, \ldots, F_m, t_1, \ldots, t_n)$, $k = 1, \ldots, K_j$, we have $E'(F_1, \ldots, F_m) \prec F(F_1, \ldots, F_m) = E$. Thus, by the induction hypothesis $\mathcal{I}(\mathcal{B}) \models_c \psi_{jk}(F_1, \ldots, F_m, \|t_1\|, \ldots, \|t_n\|)$ for every $k = 1, \ldots, K_j$. Consequently, $\mathcal{I}(\mathcal{B}) \models_c \phi_+^F(F_1, \ldots, F_m, \|t_1\|, \ldots, \|t_n\|)$ and, hence, $\mathcal{I}(\mathcal{B}) \models_c \Phi_+^E(\|t_1\|, \ldots, \|t_n\|)$. By Lemma 2, $\mathcal{I}(\mathcal{B}) \models_c S_L^b\restriction\mathsf{sub}_\prec(E)$. Since $\mathcal{I}(\mathcal{B}) \models \forall S_L^0$, we obtain $\mathcal{I}(\mathcal{B}) \models_c \phi^\sigma(E_1, \ldots, E_m, \|t_1\|, \ldots, \|t_n\|)$ and, therefore, by the definition of $\mathcal{I}(\mathcal{B})$, we have $(E, \|t_1\|, \ldots, \|t_n\|) \in \nu_n^{\mathcal{I}(\mathcal{B})}$.
The second implication for negative literals is proved similarly.

As a consequence we obtain the following theorem.

Theorem 2 (Constructive completeness). *T_L is constructively complete.*

Proof. We only need to prove that $\mathcal{I}(\mathcal{B}) \models \forall S_L$. However, this follows from $\forall S_L^0, \forall S_L^b \models \forall S_L$ since $\mathcal{I}(\mathcal{B}) \models \forall S_L^b$ by Lemma 3 and Lemma 2.

5 Refining the Synthesised Calculus

In order to get inference rules that have better properties, in this section we introduce two refinements.

The first refinement reduces the number of branches of a rule by constraining the rule with additional premises rather than deriving new conclusions. Suppose r is a tableau rule of a sound and constructively complete tableau calculus T_L:

$$r \stackrel{\text{def}}{=} \frac{X}{X_1 \mid \cdots \mid X_m}.$$

For some $i \in \{1, \ldots, m\}$ suppose $X_i = \{\psi_1, \ldots, \psi_k\}$. Without loss of generality we can assume that $i = 1$. Consider the rules r_j with $j = 1, \ldots, k$ defined by

$$r_j \stackrel{\text{def}}{=} \frac{X \cup \{\sim\psi_j\}}{X_2 \mid \cdots \mid X_m}.$$

Let T_L' be the tableau calculus obtained from T_L by replacing rule r by the rules $r_1, \ldots, r_k$. It is clear that T_L' is sound and has the subexpression property. In general, T_L' is not constructively complete. However analysis of the proofs of Lemma 3 and Lemma 2 shows that the following theorem is true.

Theorem 3. *Let $\mathcal{B}$ be an open branch in a T_L'-tableau. Assume that for every set Y of $\mathcal{L}$-expressions the following holds.*

If all expressions from Y are reflected in $\mathcal{I}(\mathcal{B})$ then for every $E_1, \ldots, E_l \in Y$,

(†) $X(E_1, \ldots, E_l, t_1, \ldots, t_n) \subseteq \mathcal{B}$ *implies*
$\mathcal{I}(\mathcal{B}) \models X_i(E_1, \ldots, E_l, \|t_1\|, \ldots, \|t_n\|)$ *for some* $i = 1, \ldots, m$.

Then, $\mathcal{B}$ is reflected in $\mathcal{I}(\mathcal{B})$.

Corollary 2. *If the condition of Theorem 3 holds for every open branch $\mathcal{B}$ of any T'_L-tableau then T'_L is constructively complete.*

The condition (†) follows by the same induction argument from the following condition:

$$(\ddagger)\quad \begin{array}{l}\text{if } X(E_1,\ldots,E_l,t_1,\ldots,t_n)\subseteq\mathcal{B} \text{ and } \mathcal{I}(\mathcal{B})\not\models X_1(E_1,\ldots,E_l,\|t_1\|,\ldots,\|t_n\|)\\ \text{then } X_i(E_1,\ldots,E_l,t_1,\ldots,t_n)\subseteq\mathcal{B}, \text{ for some } i=2,\ldots,m.\end{array}$$

Generalising this refinement to moving up more than one conclusion is not difficult. The formulation of Theorem 3 does not change in this case.

Consider the generated rule for negative occurrences of the existential restriction operator given on p. 9. In most description logics it can be transformed to the more often seen rule:

$$\frac{\neg\nu_1(\exists r.p,x),\quad \nu_2(r,x,y)}{\neg\nu_1(p,y)}.$$

In order to preserve constructive completeness, condition (‡) is usually proved by induction on $\prec$ which in its turn inductively implies condition (†).

The rules for equality used in Section 3 (given in Figure 2) are already in refined form. The rules that would be produced from the semantic specification of equality in Figure 1 have a different form. For example, the congruence rule

$$\frac{\nu_n(p,\overline{x}),\quad x_i\approx y_i}{\nu_n(p,x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)}$$

is refined from the following rule (without sacrificing constructive completeness):

$$\frac{x_1\approx x_1,\quad \ldots,\quad x_n\approx x_n,\quad y_i\approx y_i}{\neg\nu_n(p,\overline{x})\ \mid\ x_i\not\approx y_i\ \mid\ \nu_n(p,x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)}.$$

Another example of a generated rule and a refinement are (for a transitive role constant R):

$$\frac{x\approx x,\quad y\approx y,\quad z\approx z}{\neg R(x,y)\ \mid\ \neg R(y,z)\ \mid\ R(x,z)},\qquad\qquad \frac{R(x,y),\quad R(y,z)}{R(x,z)}.$$

Condition (†) holds in this case since, by definition of $\mathcal{I}(\mathcal{B})$, $\mathcal{I}(\mathcal{B})$ reflects all atomic predicate constants in the branch $\mathcal{B}$.

The second refinement we describe exploits the expressivity of the logic. Suppose that the language $\mathcal{L}$ of the logic L is expressive enough to represent its own semantics. That is, assume that for every $n=0,\ldots,N$ and every n-ary predicate constant P occuring in S_L, there are concepts $C^+_n(p,\ell_1,\ldots,\ell_n)$, $C^-_n(p,\ell_1,\ldots,\ell_n)$, $D^+_P(\ell_1,\ldots,\ell_n)$, and $D^-_P(\ell_1,\ldots,\ell_n)$, depending on variable p

of sort n and variables $\ell_1, \ldots, \ell_n$ of sort 0, such that

$$\begin{aligned}
&\forall S_L \models \forall x \left(\nu_1(C_n^+(p, \ell_1, \ldots, \ell_n), x) \to \nu_n(p, \nu_0(\ell_1), \ldots, \nu_0(\ell_n))\right), \\
&\forall S_L \models \forall x \left(\nu_1(C_n^-(p, \ell_1, \ldots, \ell_n), x) \to \neg\nu_n(p, \nu_0(\ell_1), \ldots, \nu_0(\ell_n))\right), \\
&\forall S_L \models \forall x \left(\nu_1(D_P^+(\ell_1, \ldots, \ell_n), x) \to P(\nu_0(\ell_1), \ldots, \nu_0(\ell_n))\right), \\
&\forall S_L \models \forall x \left(\nu_1(D_P^-(\ell_1, \ldots, \ell_n), x) \to \neg P(\nu_0(\ell_1), \ldots, \nu_0(\ell_n))\right).
\end{aligned}$$

It is worth noting that because the equality theory is included in the semantics, S_L, we have the following:

$$\begin{aligned}
&\forall S_L \models \forall x \left(\nu_1(D_{\approx}^+(\ell_1, \ell_2), x) \to \nu_0(\ell_1) \approx \nu_0(\ell_2)\right), \\
&\forall S_L \models \forall x \left(\nu_1(D_{\approx}^-(\ell_1, \ell_2), x) \to \nu_0(\ell_1) \not\approx \nu_0(\ell_2)\right).
\end{aligned}$$

In this case we are able to express all tableau rules for L in the language $\mathcal{L}$ itself. We only need to replace every positive occurrence of $\nu_n(E, x_1, \ldots, x_n)$ in T_L with $C_n^+(E, \ell_1, \ldots, \ell_n)$, every (negative) occurrence of $\neg\nu_n(E, x_1, \ldots, x_n)$ in T_L with $C_n^-(E, \ell_1, \ldots, \ell_n)$, and, similarly, all the predicate constants P need to be replaced with occurrences of D_P^+ or D_P^- depending on the polarity of P. In fact, the sort $N+1$ of $\mathsf{FO}(\mathcal{L})$ can be reflected in the sort 0.

A slight difficulty is caused by Skolem functions in $\mathsf{FO}(\mathcal{L})$ occurring in the tableau calculus, since for them there could be no corresponding function symbols in $\mathcal{L}$. It can be solved by introducing new connective f_g into the language $\mathcal{L}$ for every (Skolem) function and constant g of $\mathsf{FO}(\mathcal{L})$ in a way that for every $(p_1, \ldots, p_m, \ell_1, \ldots, \ell_n)$, $f_g(p_1, \ldots, p_m, \ell_1, \ldots, \ell_n)$ is a term of the sort 0 and its semantics is defined by

$$\nu_0(f_g(p_1, \ldots, p_m, \ell_1, \ldots, \ell_n)) \stackrel{\text{def}}{=} g(p_1, \ldots, p_m, \nu_0(\ell_1), \ldots, \nu_0(\ell_n)).$$

An alternative is to introduce unique, new individual names (for every $p_1, \ldots, p_m$, $\ell_1, \ldots, \ell_n$) instead of new connectives.

For example, in the description logic $\mathcal{ALCO}$ with the full support of individuals, the expressions $C_2^+(r, \ell_1, \ell_2)$, $C_2^-(r, \ell_1, \ell_2)$ for any atomic role r, and $D_{\approx}^+(\ell_1, \ell_2)$, $D_{\approx}^-(\ell_1, \ell_2)$ for individual equality are:

$$\begin{aligned}
C_2^+(r, \ell_1, \ell_2) &\stackrel{\text{def}}{=} \ell_1 : \exists r.\{\ell_2\}, & D_{\approx}^+(\ell_1, \ell_2) &\stackrel{\text{def}}{=} \ell_1 : \{\ell_2\}, \\
C_2^-(r, \ell_1, \ell_2) &\stackrel{\text{def}}{=} \ell_1 : \neg\exists r.\{\ell_2\}, & D_{\approx}^-(\ell_1, \ell_2) &\stackrel{\text{def}}{=} \ell_1 : \neg\{\ell_2\}.
\end{aligned}$$

Thus, the language of the tableau calculus can be significantly simplified. For instance, the (refined) rules for the existential restriction operator become:

$$\frac{\ell : \exists r.p}{\ell : \exists r.\{f(r,p,\ell)\}, \quad f(r,p,\ell) : p}, \qquad \frac{\ell : \neg\exists r.p, \quad \ell : \exists r.\{\ell'\}}{\ell' : \neg p}.$$

6 Case studies

As illustration of the synthesis method this section presents two case studies.

Connective definitions
$\forall x\ \big(\nu_1(\bot, x) \equiv \bot\big)$
$\forall x\ \big(\nu_1(p_1 \wedge p_2, x) \equiv \nu_1(p_1, x) \wedge \nu_1(p_2, x)\big)$
$\forall x\ \big(\nu_1(p_1 \vee p_2, x) \equiv \nu_1(p_1, x) \vee \nu_1(p_2, x)\big)$
$\forall x\ \big(\nu_1(p_1 \rightarrow p_2, x) \equiv \forall y\ \big(R(x, y) \rightarrow (\nu_1(p_1, y) \rightarrow \nu_1(p_2, y))\big)\big)$
Background theory of a partial ordering R and ν_1
$\forall x\ R(x, x)$
$\forall x \forall y\ (R(x, y) \wedge R(y, x) \rightarrow x \approx y)$
$\forall x \forall y \forall z\ (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
$\forall x \forall y\ \big(\nu_1(p, x) \wedge R(x, y) \rightarrow \nu_1(p, y)\big)$

Fig. 3. Specification of semantics of intuitionistic logic.

6.1 Synthesising Tableaux for Intuitionistic Logic

Intuitionistic logic is an example of a logic where ν_n cannot be expressed in the language of the logic. It also provides an example of a logics for which a background theory is an essential part of the definition of the semantics.

The language of intuitionistic logic is a one-sorted language defined over a countable set of propositional symbols $p_1^1, p_2^1, \ldots$, and the standard connectives are $\rightarrow, \vee, \wedge, \bot$. The semantic specification in $\mathsf{FO}(\mathcal{L})$ is given in Figure 3 (cf. [4]). R is the designated predicate symbol representing the partial ordering of the background theory. For intuitionistic logic the orderings $\prec_0$ and $\prec$ coincide. The ordering $\prec$ on subexpressions induced by the semantic definition of the connectives is the smallest ordering satisfying: $E_1 \prec E_1 \sigma E_2$ and $E_2 \prec E_1 \sigma E_2$, for each $\sigma \in \{\rightarrow, \vee, \wedge\}$ and all intuitionistic formulae E_1 and E_2.

The tableau rules generated by our approach are those listed in Figure 4. The rules are compatible with the ordering $\prec$. Together with the equality rules, they form a calculus which is sound and constructively complete for propositional intuitionistic logic. This is an immediate consequence of Theorems 1 and 2. Refining the generated rules yields the rules listed in Figure 5. Using Theorem 3 we conclude that these rules together with the equality rules provide a sound and constructively complete tableau calculus for intuitionistic logic.

A tableau decision procedure is obtained if the calculus is enhanced with the blocking mechanism of [6, 7]. This follows from the results in [7], the soundness and constructive completeness of the calculus, the subexpression property and the fact that intuitionistic logic admits finite filtrations.

6.2 Synthesising Tableaux for $\mathcal{ALCO}$

The syntax of $\mathcal{ALCO}$ is defined over the sorts $\mathcal{L}^0$, $\mathcal{L}^1$, and $\mathcal{L}^2$, i.e., over the sorts of individuals, concepts, and roles, respectively. A (minimal) set of logical connectives for $\mathcal{ALCO}$ consists of the following symbols: $\{\cdot\}$ (*singleton*) of sort $(0, 1)$, $\neg$ (*negation*) of sort $(1, 1)$, $\sqcup$ (*union*) of sort $(1, 1, 1)$, and $\exists$ (*existential concept restriction*) of sort $(2, 1, 1)$. We follow traditional notation for the *concept*

Decomposition rules:

$$\frac{\nu_1(\bot, x)}{\bot} \qquad \frac{\nu_1(p_1 \wedge p_2, x)}{\nu_1(p_1, x),\ \ \nu_1(p_2, x)} \qquad \frac{\neg\nu_1(p_1 \wedge p_2, x)}{\neg\nu_1(p_1, x)\ \mid\ \neg\nu_1(p_2, x)}$$

$$\frac{\neg\nu_1(\bot, x)}{\neg\bot} \qquad \frac{\nu_1(p_1 \vee p_2, x)}{\nu_1(p_1, x)\ \mid\ \nu_1(p_2, x)} \qquad \frac{\neg\nu_1(p_1 \vee p_2, x)}{\neg\nu_1(p_1, x),\ \ \neg\nu_1(p_2, x)}$$

$$\frac{\nu_1(p_1 \to p_2, x)}{\neg R(x, y)\ \mid\ \neg\nu_1(p_1, y)\ \mid\ \nu_1(p_2, y)}$$

$$\frac{\neg\nu_1(p_1 \to p_2, x)}{R(x, f(p_1, p_2, x)),\ \ \nu_1(p_1, f(p_1, p_2, x)),\ \ \neg\nu_1(p_2, f(p_1, p_2, x))}$$

Theory rules:

$$\frac{x \approx x}{R(x, x)} \qquad \frac{x \approx x,\ \ y \approx y}{\neg R(x, y)\ \mid\ \neg R(y, x)\ \mid\ x \approx y} \qquad \frac{x \approx x,\ \ y \approx y,\ \ z \approx z}{\neg R(x, y)\ \mid\ \neg R(y, z)\ \mid\ R(x, z)}$$

$$\frac{p \approx p,\ \ x \approx x,\ \ y \approx y}{\neg\nu_1(p, x)\ \mid\ \neg R(x, y)\ \mid\ \nu_1(p, y)}$$

Closure rules:

$$\frac{\nu_1(p, x),\ \ \neg\nu_1(p, x)}{\bot} \qquad \frac{R(x, y),\ \ \neg R(x, y)}{\bot}$$

Fig. 4. Generated tableau rules for intuitionistic logic.

expressions, i.e., every concept expression C has one of the following forms:

$$C \stackrel{\text{def}}{=} p_i \mid \{\ell_i\} \mid \neg C \mid C \sqcup D \mid \exists r_i.C.$$

p_i is a propositional variable in the set $\mathcal{L}^1$, ℓ_i is a (individual) variable in $\mathcal{L}^0$, and r_i is a (role) variable in $\mathcal{L}^2$. The semantics of $\mathcal{ALCO}$ is defined in Figure 6. As can be seen, the background theory of the $\mathcal{ALCO}$-semantics is empty. As in the case of intuitionistic logic, $\prec = \prec_0$ and the ordering is the usual subexpression ordering defined as the smallest transitive ordering satisfying: $\ell \prec \{\ell\}$, $C \prec \neg C$, $C \prec (C \sqcup D)$, $D \prec (C \sqcup D)$, and $C \prec \exists r.C$ for all expressions ℓ, C, and D of appropriate sorts.

The tableau rules generated by our approach are those listed in Figure 7. The generated calculus also includes the equality congruence rules and by Theorem 2, it is sound and constructive complete.

Further, the rule

$$\frac{\neg\nu_1(\exists r.p, x)}{\neg\nu_2(r, x, y)\ \mid\ \neg\nu_1(p, y)}$$

can be refined to

$$\frac{\neg\nu_1(\exists r.p, x),\ \ \nu_2(r, x, y)}{\neg\nu_1(p, y)}$$

without any damage to constructive completeness, by using Theorem 3. This is because the condition (‡) can be proved by induction on the ordering $\prec$ for the refined calculus.

Decomposition rules:

$$\frac{\nu_1(\bot, x)}{\bot} \qquad \frac{\nu_1(p_1 \wedge p_2, x)}{\nu_1(p_1, x), \quad \nu_1(p_2, x)} \qquad \frac{\neg\nu_1(p_1 \wedge p_2, x)}{\neg\nu_1(p_1, x) \mid \neg\nu_1(p_2, x)}$$

$$\frac{\nu_1(p_1 \vee p_2, x)}{\nu_1(p_1, x) \mid \nu_1(p_2, x)} \qquad \frac{\neg\nu_1(p_1 \vee p_2, x)}{\neg\nu_1(p_1, x), \quad \neg\nu_1(p_2, x)} \qquad \frac{\nu_1(p_1 \to p_2, x), \quad R(x, y), \quad \nu_1(p_1, y)}{\nu_1(p_2, y)}$$

$$\frac{\neg\nu_1(p_1 \to p_2, x)}{R(x, f(p_1, p_2, x)), \quad \nu_1(p_1, f(p_1, p_2, x)), \quad \neg\nu_1(p_2, f(p_1, p_2, x))}$$

Theory rules:

$$\frac{x \approx x}{R(x, x)} \qquad \frac{R(x, y), \quad R(y, x)}{x \approx y} \qquad \frac{R(x, y), \quad R(y, z)}{R(x, z)} \qquad \frac{\nu_1(p, x), \quad R(x, y)}{\nu_1(p, y)}$$

Closure rules:

$$\frac{\nu_1(p, x), \quad \neg\nu_1(p, x)}{\bot}$$

Fig. 5. Refined tableau rules for intuitionistic logic.

Connective definitions
$\forall x \ \big(\nu_1(\{\ell\}, x) \equiv \nu_0(\ell) \approx x\big)$
$\forall x \ \big(\nu_1(\neg p, x) \equiv \neg\nu_1(p, x)\big)$
$\forall x \ \big(\nu_1(p_1 \sqcup p_2, x) \equiv \nu_1(p_1, x) \vee \nu_1(p_2, x)\big)$
$\forall x \ \big(\nu_1(\exists r.p_1, x) \equiv \exists y \ \big(\nu_2(r_1, x, y) \wedge \nu_1(p_1, y)\big)\big)$

Fig. 6. Specification of semantics of $\mathcal{ALCO}$.

Decomposition rules:

$$\frac{\nu_1(\{\ell\}, x)}{\nu_0(\ell) \approx x} \qquad \frac{\neg\nu_1(\{\ell\}, x)}{\nu_0(\ell) \not\approx x} \qquad \frac{\nu_1(\neg p, x)}{\neg\nu_1(p, x)} \qquad \frac{\neg\nu_1(\neg p, x)}{\nu_1(p, x)}$$

$$\frac{\nu_1(p_1 \vee p_2, x)}{\nu_1(p_1, x) \mid \nu_1(p_2, x)} \qquad \frac{\neg\nu_1(p_1 \vee p_2, x)}{\neg\nu_1(p_1, x), \quad \neg\nu_1(p_2, x)}$$

$$\frac{\nu_1(\exists r.p, x)}{\nu_2(r, x, f(r, p, x)), \quad \nu_1(p, f(r, p, x))} \qquad \frac{\neg\nu_1(\exists r.p, x)}{\neg\nu_2(r, x, y) \mid \neg\nu_1(p, y)}$$

Closure rule:

$$\frac{\nu_1(p, x), \quad \neg\nu_1(p, x)}{\bot}$$

Fig. 7. Generated tableau rules for $\mathcal{ALCO}$.

Finally, we enrich the language of the logic by the colon : connective of the sort $(0, 1, 1)$ with the definition:

$$\forall x \ \big(\nu_1(\ell : p, x) \equiv \nu_1(p, \nu_0(\ell))\big),$$

Decomposition rules:

$$\frac{\ell : \neg\neg p}{\ell : p} \qquad \frac{\ell : (p_1 \sqcup p_2)}{\ell : p_1 \mid \ell : p_2} \qquad \frac{\ell : \neg(p_1 \sqcup p_2)}{\ell : \neg p_1, \quad \ell : \neg p_2}$$

$$\frac{\ell : \exists r.p}{\ell : \exists r.\{f(r,p,\ell)\}, \quad f(r,p,\ell) : p} \qquad \frac{\ell : \neg\exists r.p, \quad \ell : \exists r.\{\ell'\}}{\ell' : \neg p}$$

Closure rule:

$$\frac{\ell : p, \quad \ell : \neg p}{\bot}$$

Fig. 8. Refined tableau rules for $\mathcal{ALCO}$.

and also introduce connectives which correspond to Skolem function symbols into the language of the logic. This allows us to find the expressions for defining predicates $\approx$, ν_1 and ν_2 in the language of the logic:

$$C_1^+(p,\ell) \stackrel{\text{def}}{=} \ell : p, \qquad C_2^+(r,\ell_1,\ell_2) \stackrel{\text{def}}{=} \ell_1 : \exists r.\{\ell_2\}, \qquad D_{\approx}^+(\ell_1,\ell_2) \stackrel{\text{def}}{=} \ell_1 : \{\ell_2\},$$
$$C_1^-(p,\ell) \stackrel{\text{def}}{=} \ell : \neg p, \quad C_2^-(r,\ell_1,\ell_2) \stackrel{\text{def}}{=} \ell_1 : \neg\exists r.\{\ell_2\}, \quad D_{\approx}^-(\ell_1,\ell_2) \stackrel{\text{def}}{=} \ell_1 : \neg\{\ell_2\}.$$

Thus, we can simplify the notation of the tableau calculus and obtain the more usual looking labelled tableau calculus given in Figure 8.

Assuming that the given input concept is C, the refined tableau starts with the concept $\ell : C$ for ℓ a fresh individual constant which corresponds to the Skolem constant introduced for the existential quantifier in $\exists x\, \nu_1(C,x)$.

Similarly to intuitionistic logic, because $\mathcal{ALCO}$ admits finite filtrations, a tableau decision procedure for $\mathcal{ALCO}$ is obtained if the calculus is enhanced with the blocking mechanism of [6, 7].

7 Discussion and Conclusions

The method introduced in this paper automatically produces a sound and constructively complete tableau calculus from a semantic first-order specification of a many-sorted logic. The method is directly applicable to many non-classical logics and covers many types of ground tableau calculi commonly found in the literature. These include two types of tableau calculi for relations satisfying extra theory conditions which can be accommodated either by structural rules or propagation rules.

The results of the paper can be regarded as a mathematical formalisation and generalisation of tableau development methodologies. Our formalisation is based on and provides the basis for the implementation of tableau decision procedures for various modal and description logics in the MetTeL system [8]. The formalisation separates the creative part of tableau calculus development which needs to be done by a human developer and the automatic part of the development process which can be left to an automated (currently first-order) prover and

an automated tableau synthesiser. The creative part is the specification of the logic so that the conditions of well-foundness of the orderings $\prec_0$ and $\prec$ hold. The automatic part deals with verification of the first-order conditions (wd1) and (wd3'), and the generation of tableau rules from the well-defined semantics provided by a developer. Then, the developer can transform the generated rules to optimised form by applying Theorem 3.

For known modal and description logics conditions (wd1) and (wd3) are simple to check, even trivial in many cases. In fact, a developer usually implicitly formalises the logic's semantics S in such way that $S = S^0 \cup S^b$. This is the case for almost all of known logics. If the specification of the semantics satisfies $S = S^0 \cup S^b$ then conditions (wd1) and (wd3) hold trivially and the orderings $\prec_0$ and $\prec$ coincide. This means the ordering used for the specification of the semantics of the logical connectives (which is usually well-founded), is enough for tableau synthesis.

This paper also presents a general method for proving (constructive) completeness of tableau calculi. In addition, the generated rules can be transformed to an optimal form provided that the special condition (†) has been proven by induction on the ordering $\prec$ for the refined calculus.

With enough expressivity for representing the basics of the semantics within the logic it is possible to simplify the language of the tableau. In this case, the obtained calculus is similar to tableau calculi for description logics with full support of individuals, hybrid modal logic, and labelled tableau calculi. Otherwise, the calculus has the same flavour as the standard tableau calculus for intuitionistic logic, where every node of a tableau is characterised by two complementary sets of true and false formulae (concepts).

As case studies we considered tableau synthesis for propositional intuitionistic logic and the description logic $\mathcal{ALCO}$ with the full support for individuals and the colon : operator. We believe the approach is also applicable to most known first-order definable modal and description logics.

The tableau calculi generated are Smullyan-type tableau calculi, i.e., ground semantic tableau calculi. We believe that other types of tableau calculi can be generated using the same techniques. Exploiting the known relationships to other deduction methods and the underlying ideas of [5] we expect synthesis of non-tableau approaches is possible as well, but this is future work. In [5] we have shown that it is possible to synthesise tableau calculi for modal logics by translation to first-order logic and first-order resolution. In this framework the semantic specification of a logic is transformed into clausal form and then a set of inference rules. Soundness and completeness of the generated calculus follows from the soundness and completeness of the simulating resolution refinement used. This approach has several advantages, but in this paper we have taken a different, more direct approach. Rather than proceeding via simulation by resolution we have shown that tableau rules can be generated directly from the specification of the logic.

Our future goal is to further reduce human involvement in the development of calculi by finding appropriate automatically verifiable conditions for optimal calculi to be generated.

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