On Axiomatic Products of PDL and S5: Substitution, Tests and Knowledge*

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1 Introduction

Propositional dynamic logic (*PDL* for short) [5] is an expressive, powerful and convenient logical tool to reason about programs or actions. It has found applications in various fields of computer science, which range from program verification to multi-agent systems. In computer science applications, the modal logic *S5* is generally accepted as an adequate representation of the notion of knowledge. For example, in agent based applications the *S5* formula $\Box\varphi$ can be read to mean 'the agent knows φ'. Thus, combinations of *PDL* and *S5* are meaningful when we want to reason about dynamic and epistemic information. In modal logic different forms of combinations of logics have been investigated. The simplest form of combination of two (or more) logics is their fusion, or independent join. It is well-known [9, 6] that fusions of logics inherit many of the good properties of the individual logics, including soundness, completeness, the finite model property and decidability. Another form of combination of two logics is their product. With products the situation is more varied and complicated than with fusions. First of all, products can be defined in two ways: axiomatically and semantically [3]. Whereas in fusions there is no interdependence between the operators of the different modal dimensions, in products the modal operators are commuting. This complicates matters, so that for products there is no preservation theorem of the generality as for fusions. In fact, the particular type of interaction between the

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modal opertors in products makes it much more difficult to obtain positive results regarding completeness, the finite model property and decidability for products. However advanced technique to deal with products have been developed [3], see also [8]. These are techniques for investigating the properties of products of *modal logics*. The product of *PDL* and *S5* is a combination of a *dynamic logic* and a modal logic. Therefore, the techniques developed for standard modal logics need to be elaborated and can be applied to products involving *PDL* only with a lot of care.

An issue we look at in this paper concerns the uncertainty as to how substitutivity should be defined in the product of *PDL* and *S5*. Because *PDL* has a two-sorted language over actions and propositions, a key question is the following. In axiom schemata, do we allow substitution of all action terms into action variables, or do be allow only substitution of atomic action terms into action variables? If the answer is 'yes' we speak of *full substitutivity*, whereas if the answer is 'no' we speak of *weak substitutivity*. For *PDL* we can prove that

(1) weak substitutivity implies full substitutivity.

We regard this as a good property, because it allows us to reason about all actions in a uniform way.

In this paper we focus on axiomatically defined products of *PDL* and *S5*. In particular, we explore all possible definitions of axiomatic products of *PDL* and *S5* with respect to full and weak substitutivity. For each definition we consider the problems of completeness, the small model property, decidability and the admissibility of the full substitution rule (Sections 3 and 4). In tackling these problems we use a filtration method which is essentially a combination of the filtration method for products of *S5* in [3] and the Fisher-Ladner filtration technique developed for *PDL* [2] but with slight modifications. We prove that under full substitutivity the *S5* operator in the product of *PDL* and *S5* is vacuous, which implies the product of *PDL* and *S5* fails to satisfy the property (1). As a remedy we propose and discuss a new definition for the *PDL* test operator which allows us to satisfy all the properties desirable in an axiomatic product of *PDL* and *S5*, namely completeness, the small model property, decidability and the admissibility of full substitutivity (Section 6). We also analyse the product of *test-free PDL* and *S5* (Section 5).

2 Main definitions

In the product of *PDL* and *S5* one dimension is represented by dynamic logic operators and the other dimension by the epistemic modality of *S5*. Thus, the language $\mathcal{L}$ we consider is an extension of the language of *PDL* [5] with a new modal operator $\Box$ representing knowledge. Formally, the language $\mathcal{L}$ is defined over the following primitive types: a countable set $\mathrm{Var} = \{p, q, r, \ldots\}$ of propositional variables and a countable set $\mathrm{AtAc} = \{a, b, c, \ldots\}$ of atomic action variables. The connectives in $\mathcal{L}$ are the Boolean connectives, $\to$ and $\bot$, the dynamic logic connectives, $\cup$ (non-deterministic choice), ; (sequential execution), $*$ (repetition), ? (test) and the modal operators $[\_]$ and $\Box$. The set For of formulae and the set Ac of action terms in $\mathcal{L}$ are the smallest sets that satisfy the following conditions.

- $\mathrm{AtAc} \subseteq \mathrm{Ac}$, $\mathrm{Var} \cup \{\bot\} \subseteq \mathrm{For}$.
- If ϕ and ψ are formulae in For and α and β are action terms in Ac then $\phi?$, α^*, $\alpha \cup \beta$, $\alpha;\beta$ are action terms in Ac, and $\phi \to \psi$, $\Box\phi$, $[\alpha]\phi$, are formulae in For.

As usual we use the following abbreviations: $\neg\phi$ for $\phi \to \bot$, $\phi \vee \psi$ for $\neg\phi \to \psi$, $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, $\langle\alpha\rangle\phi$ for $\neg[\alpha]\neg\phi$, and $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$.

By definition, an *atomic action* is an action variable, and a *semi-atomic action* is an atomic action or a test action $\phi?$.

By a *theory* in $\mathcal{L}$ we understand any subset of For which is closed under the following standard rules:

$$\phi,\ \phi \to \psi \vdash \psi \qquad\qquad \phi \vdash [\alpha]\phi \qquad\qquad \phi \vdash \Box\phi$$

Generally axioms and theorems of a logic are assumed true for all instantiations for the atomic symbols. However, in this paper we distinguish between two variants of the substitution rule. The *weak substitution rule* allows the substitution of arbitrary formulae for the atomic propositional symbols but does not allow substitution for atomic action symbols. By contrast, the *full substitution rule* allows both kinds of substitutions, i.e. both propositional substitutions and action substitutions.

A *logic* in $\mathcal{L}$ is a theory which is closed under the full substitution rule and a *weak logic* in $\mathcal{L}$ is a theory closed under the weak substitution rule. Weak logics are notationally discerned by a subscript $_w$.

Let Γ and Δ be any subsets of For. By $\Gamma \oplus \Delta$ (resp. $(\Gamma \oplus \Delta)_w$) we denote the least logic (resp. the least weak logic) which contains both Γ and Δ. According to this notation, the *fusion* of *PDL* and *S5* is denoted by *PDL* $\oplus$ *S5*. For an axiomatisation of *PDL* we refer to [5], and axiomatisations of *S5* can be found in [1, 9].

Proposition 1 $PDL \oplus S5 = (PDL \oplus S5)_w$, i.e. the full substitution rule is admissible in $(PDL \oplus S5)_w$.

PDL $\oplus$ *S5* models are combinations of the familiar Kripke models for *PDL* and *S5*, that is, a *PDL* $\oplus$ *S5 model* is a tuple $\langle S, Q, R, \models\rangle$ such that $\langle S, Q, \models\rangle$ is a *PDL* model [5] and $\langle S, R, \models\rangle$ is an *S5* model [1]. Here, S is a non-empty set of states, R is an equivalence relation on S, $\models$ is the usual truth relation on the Kripke model and Q is a mapping from the set of actions to the set of binary relations on S which satisfies the following conditions.

$$Q(\alpha \cup \beta) = Q(\alpha) \cup Q(\beta) \qquad Q(\alpha;\beta) = Q(\alpha) \circ Q(\beta)$$
$$Q(\alpha^*) = Q(\alpha)^* \qquad Q(\phi?) = \{(s, s) \in S^2 \mid s \models \phi\}$$

($\circ$ denotes relational composition, and $*$ is the reflexive and transitive closure operator on relations.)

The products of *PDL* and *S5* we consider in this paper are extensions of *(test-free)* *PDL* $\oplus$ *S5* with the following axioms.

(*NL*)	$[a]\Box p \to \Box[a]p$
(*PR*)	$\Box[a]p \to [a]\Box p$

(*NL* is short for no learning, and *PR* for perfect recall.) These axioms are Sahlqvist formulae and their first-order equivalents are:

(com^r)	Right commutativity:	$R \circ Q(a) \subseteq Q(a) \circ R$
(com^l)	Left commutativity:	$Q(a) \circ R \subseteq R \circ Q(a)$

Note, that in the case of logics with full substitution a ranges over all the actions, whereas for weak logics a ranges over the set of atomic actions only and in this case we refer to these properties as com^r_w and com^l_w.

We consider the logics:

$$[PDL, S5] =^{\mathrm{def}} PDL \oplus S5 \oplus \{NL, PR\}$$
$$[PDL, S5]_w =^{\mathrm{def}} (PDL \oplus S5 \oplus \{NL, PR\})_w$$
$$[\textit{test-free } PDL, S5] =^{\mathrm{def}} \textit{test-free } PDL \oplus S5 \oplus \{NL, PR\}$$
$$[\textit{test-free } PDL, S5]_w =^{\mathrm{def}} (\textit{test-free } PDL \oplus S5 \oplus \{NL, PR\})_w$$

This class contains all possible definitions of axiomatic products of *(test-free)* *PDL* and *S5*.

By definition, a $[PDL, S5]$ *model* (resp. $[\textit{test-free } PDL, S5]$ *model*) is a $PDL \oplus S5$ model (resp. test-free $PDL \oplus S5$ model) which satisfies properties com^r and com^l. Similarly, a $[PDL, S5]_w$ *model* (resp. $[\textit{test-free } PDL, S5]_w$ *model*) is a $PDL \oplus S5$ model (resp. test-free $PDL \oplus S5$ model) which satisfies properties com^r_w and com^l_w.

Note 1 Completeness for the logics under consideration with respect to the corresponding classes of models is not a straightforward consequence of the Sahlqvist theorem because the canonical models for these logics do not satisfy the property $Q(\alpha^*) = Q(\alpha)^*$. To get this property we need to modify the canonical models by the method of Fischer and Ladner [2, 5], and, moreover, take care that left and right commutativity are preserved.

3 $[PDL, S5]$

We say that a modal logic L *admits the elimination of* $\Box$, if the formula $p \leftrightarrow \Box p$ belongs to L.

Lemma 2 Let L be a normal modal logic with reflexive modality $\Box$ satisfying one of the axiom schemata $\Box(p \to q) \to (p \to \Box q)$ or $(p \to \Box q) \to \Box(p \to q)$. Then, L admits the elimination of $\Box$.

Proof. Assume the first axiom schema hold for L. As L is closed under the substitution rule, the formula $\Box(p \to p) \to (p \to \Box p)$ belongs to L. The left side of the implication is always true. Thus, this formula is classically equivalent to $p \to \Box p$. Because $\Box$ is reflexive, the logic contains the formula $\Box p \to p$. Consequently, $\Box p \leftrightarrow p$ is true. If the second axiom schema holds in L then substitute $\neg q$ for p. So, $(\neg q \to \Box q) \to \Box(\neg q \to q)$ is in L.

This formula is equivalent to $(\neg q \to \Box q) \to \Box q$. The following equivalences complete the proof of the lemma. $((\neg q \to \Box q) \to \Box q) \leftrightarrow (\neg(q \vee \Box q) \vee \Box q) \leftrightarrow ((\neg q \wedge \neg\Box q) \vee \Box q) \leftrightarrow ((\neg q \vee \Box q) \wedge (\neg\Box q \vee \Box q)) \leftrightarrow (q \to \Box q)$. $\Box$

Lemma 3 If an extension L of $PDL \oplus S5$ contains any of the axioms NL or PR and the full substitution rule is admissible in L, then L admits the elimination of the $\Box$ operator.

Proof. Consider, for instance, the axiom NL, $[a]\Box p \to \Box[a]p$, and substitute a test expression $q?$ for a. This gives $[q?]\Box p \to \Box[q?]p$. Since $[q?]p \leftrightarrow (q \to p)$ belongs to PDL, $(q \to \Box p) \to \Box(q \to p)$ is in L. In the case of PR, $\Box(q \to p) \to (q \to \Box p)$ is in L. Finally, apply Lemma 2. $\Box$

The previous lemma implies that $[PDL, S5]$ admits the elimination of $\Box$. Hence:

Theorem 4 $[PDL, S5]$ is deductively equivalent to PDL.

Theorem 5 (Complexity) $[PDL, S5]$ is EXPTIME-complete.

Moreover, it also follows that the class of all $[PDL, S5]$ models is just a class of all $PDL \oplus S5$ models in which the relation R corresponding to the $\Box$-operator is a congruence relation on the set of states. Then the following theorems hold.

Theorem 6 (Completeness) $[PDL, S5]$ is complete with respect to the class of all $[PDL, S5]$ models.

Theorem 7 (Small Model Theorem) Let ϕ be a formula of $\mathcal{L}$ and n be the number of symbols in ϕ. If ϕ is satisfiable in some $[PDL, S5]$ model then it is satisfiable in $[PDL, S5]$ model with no more than 2^n states.

4 $[\boldsymbol{PDL}, \boldsymbol{S5}]_w$

Theorem 8 $[PDL, S5]_w \neq [PDL, S5]$.

Proof. Let M_0 be a model $\langle S, Q, R, \models \rangle$ with $S = \{0, 1\}$, $Q(a) = \emptyset$ for any atomic action a, R is the universal relation on S, $0 \models p$ and $1 \models \neg p$. It is easy to see that NL and PR are true in M_0 for all atomic actions. But the both $[\neg p?]\Box p \to \Box[\neg p?]p$ and $\Box[\neg p?]p \to [\neg p?]\Box p$ are false in the state 0. $\Box$

By the way we note that the example in the proof is also suitable for proving that the full substitution rule is not admissible in the semantical product of *PDL* and *S5*, if is defined as in [8].

Combining the filtration method described in [3] for products of K and *S5* with the method of Fisher-Ladner for *PDL* (see [2, 5]) we obtain the following theorems.

Theorem 9 (Completeness) $[PDL, S5]_w$ is complete with respect to the class of all $[PDL, S5]_w$ models.

Theorem 10 (Small Model Theorem) Let ϕ be a formula of $\mathcal{L}$ and n be the number of symbols in ϕ. If ϕ is satisfiable in some $[PDL, S5]_w$ model then it is satisfiable in some $[PDL, S5]_w$ model with no more than $2^n \cdot 2^{2^n}$ states.

Theorem 11 (Decidability) $[PDL, S5]_w$ is decidable.

5 Test-free products

Lemma 12 Let M be a $PDL \oplus S5$ model which satisfies com^r_w. Then, com^r is true in M for all test-free actions.

Proof. By induction on the length of an action term α. The base case holds by the assumptions of the lemma. We skip the easy case where $\alpha = \beta \cup \gamma$. For the case $\alpha = \beta;\gamma$, by taking into account the induction hypothesis, we obtain $Q(\beta;\gamma) \circ R = Q(\beta) \circ Q(\gamma) \circ R \subseteq Q(\beta) \circ R \circ Q(\gamma) \subseteq R \circ Q(\beta) \circ Q(\gamma) = R \circ Q(\beta;\gamma)$. For $\alpha = \beta^*$ the argument is similar. □

The following lemma can be proved in a similar way.

Lemma 13 Let M be an $PDL \oplus S5$ model which satisfies com^l_w. Then, com^l is true in M for all test-free actions.

As a consequence:

Theorem 14 $[\textit{test-free } PDL, S5]_w = [\textit{test-free } PDL, S5]$.

This result together with those from the previous section give us:

Theorem 15 (Completeness) $[\textit{test-free } PDL, S5]$ is complete with respect to the class of all $[\textit{test-free } PDL, S5]$ models.

Theorem 16 (Small Model Theorem) Let ϕ be a test-free formula of $\mathcal{L}$ and n be the number of symbols in ϕ. If ϕ is satisfiable in some [*test-free PDL*, *S5*] model then it is satisfiable in some [*test-free PDL*, *S5*] model with no more than $2^n \cdot 2^{2^n}$ states.

Theorem 17 (Decidability) [*test-free PDL*, *S5*] is decidable.

6 A new definition and semantics for test

From a logical perspective the *trivial* elimination of the *S5* operator in the product [*PDL*, *S5*] is unsatisfactory. The reason for the elimination of the *S5* operator is the implicit connection between the test operator and $\Box$ in the commutativity axioms under full substitutivity. This suggests the problem may be twofold. (i) Assuming full substitutivity is inappropriate for the commutativity axioms. But, the full substitution rule gives us the possibility to reason uniformly about all actions, in the same way as we reason about all propositions in any logic. *PDL* is closed under the full substitution rule and, thus, fits this paradigm. So, perhaps, the problem is rather that, (ii) the semantics of the test operator is not defined well enough when test interacts with $\Box$. A solution we propose here is to define an alternative semantics for test such that in the resulting logic, $\Box$ and test interact in a way so that weak substitutivity implies full substitutivity.

Therefore, we introduce a new operator, denoted by ⁇, as replacement for the standard test operator. This is intended to remedy the problem with standard test in the presence of an epistemic modal operator. The new operator can be interpreted as an *epistemic test* operator. The intuition of p⁇ is an action which can be succesfully accomplished only if p is *known* in the current state. The result of this action is an arbitary state within the same knowledge cluster. Thus, p⁇ is the action of confirming the agents's own knowledge. In contrast, with the usual test operator the agent has the capability to confirm truths rather than knowledge. Philosophically, this is a strong property of an agent; we believe, too strong. In agent based applications the new interpretation of test is more suitable than the usual test operator.

The logical apparatus is the same as previously with the obvious changes. The symbol ⁇ is used in the superscript to indicate a replacement of the operator ? by ⁇. Let $[PDL, S5]^{⁇}$ and $[PDL, S5]^{⁇}_{w}$ be the logics in $\mathcal{L}^{⁇}$ obtained from $[PDL, S5]$ and $[PDL, S5]_w$, respectively, by replacing the usual test axiom with:

$$[p⁇]q \leftrightarrow \Box(\Box p \to q).$$

In accordance with this axiom, the formula $[p⁇]q$ can be read as 'q is known with respect to p being known'. Thus, we think of the modal operator $[\_⁇]$ as the operator of *relative knowledge*.

Using the elimination of second-order quantifiers [4, 7] it is easy to find the corresponding semantic definition for the new operator. Thus, a $(PDL \oplus S5)^{⁇}$ model is a tuple $\langle S, Q, R, \models\rangle$ satisfying all the properties of $PDL \oplus S5$ model, except the meaning of ⁇ is specified by:

$$Q(\phi⁇) =^{\text{def}} \{(s,t) \in R \mid t \models \Box\phi\}.$$

This induces the notions of $[PDL, S5]_w^{⁇}$ and $[PDL, S5]^{⁇}$ models as expected.

The definition of ⁇ still allows the elimination of $\Box$ but this time the elimination is not trivial.

Proposition 18 $\Box p \leftrightarrow [\top⁇]p \in (PDL \oplus S5)^{⁇}$.

Proposition 19 The formulae $[p⁇]\Box q \to \Box[p⁇]q$ and $\Box[p⁇]q \to [p⁇]\Box q$ belong to $(PDL \oplus S5)^{⁇}$, and, consequently, $[PDL, S5]_w^{⁇}$ and $[PDL, S5]^{⁇}$.

Applying the filtration technique and using Results 12, 13 and 19 we can prove the next theorem, and obtain the usual consequences of filtration.

Theorem 20 $[PDL, S5]_w^{⁇} = [PDL, S5]^{⁇}$.

Theorem 21 (Completeness) $[PDL, S5]^{⁇}$ is complete with respect to the class of all $[PDL, S5]^{⁇}$ models.

Theorem 22 (Small Model Theorem) Let ϕ be a formula of $\mathcal{L}^{⁇}$ and n be the number of symbols in ϕ. If ϕ is satisfiable in some $[PDL, S5]^{⁇}$ model then it is satisfiable in $[PDL, S5]^{⁇}$ model with no more than $2^n \cdot 2^{2^n}$ states.

Theorem 23 (Decidability) $[PDL, S5]^{⁇}$ is decidable.

How does the standard test operator of *PDL* relate to the new one? It turns out that there is a simulation of *PDL* in $(PDL \oplus S5)^{⁇}$. Define the translation σ from formulae of *PDL* to For$^{⁇}$ by the following:

$$\begin{array}{ccc} \sigma p = \Box p & \sigma\bot = \bot & \sigma a = a \\ \sigma(\psi?) = (\sigma\psi)⁇ & \sigma(\alpha \cup \beta) = \sigma\alpha \cup \sigma\beta & \sigma(\alpha;\beta) = \sigma\alpha;\sigma\beta \\ \sigma(\alpha^*) = (\sigma\alpha)^* & \sigma(\phi \to \psi) = \Box(\sigma\phi \to \sigma\psi) & \sigma([\alpha]\psi) = \Box[\sigma\alpha]\sigma\psi \end{array}$$

Theorem 24 For any $\Box$-free formula ϕ in $\mathcal{L}$, $\phi \in PDL$ iff $\sigma\phi \in (PDL \oplus S5)^{⁇}$.

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