Relational Sheaves and Predicate Intuitionistic Modal Logic

Barnaby P. Hilken
Department of Computer Science
University of Manchester
Oxford Road, Manchester M13 9PL, UK
bhilken@cs.man.ac.uk

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Abstract

This paper generalises and adapts the theory of sheaves on a topological space to sheaves on a *relational space*: a topological space with a binary relation. The *relational bundles* on a relational space are defined as the continuous, relation-preserving functions into the space, and the *relational sections* of a relational bundle are defined as the relation-preserving partial sections. This defines a functor to the category of presheaves on the space, which has a left adjoint.

The presheaves which arise as the relational sections of a relational bundle are characterised by separation and patching conditions similar to those of a sheaf: we call them the *relational sheaves*. The relational bundles which arise from presheaves are characterised by local homeomorphism conditions: we call them the *local relational homeomorphisms*. The adjunction restricts to an equivalence between the categories of relational sheaves and local relational homeomorphisms.

The paper goes on to investigate the structure of these equivalent categories. They are shown to be quasi-toposes (thus modelling first-order logic), and to have enough structure to model a certain first-order modal logic described in a companion paper.

1 Introduction

The Kripke "many world" semantics of modal logic [11] models the modal connectives $\Box$ and $\Diamond$ in terms of a set S of "possible worlds" (or "states") and a binary relation $\twoheadrightarrow$ of "accessibility" between worlds (or "transition" between states). The truth of each proposition is determined separately at each world, and we write $t \Vdash \phi$ to mean that proposition ϕ holds at world t. In order to extend this structure to a model of first-order logic, we need a domain of "individuals" in which to interpret the elementary variables. Since the validity of propositions such as $t \Vdash x = y$ depends on the world t, it is natural also to allow the domain in which x and y are interpreted to depend on t. This leads us to consider, instead of a fixed set of individuals, a family of sets D_t, indexed by the set of worlds. Finally, as we move from one world t to a related world s, we must consider which individuals in D_t correspond to which individuals in D_s. In the most straightforward case there is a function $f_{ts} : D_t \to D_s$, but in the most general case (necessary when considering "reverse modalities") there is a relation $\twoheadrightarrow_{ts} \subseteq D_t \times D_s$ for each related pair $t \twoheadrightarrow s$.

The traditional approach to first-order modal logic is largely concerned with the syntactic problem of determining which world each variable should be interpreted in, and which set D_t each quantifier should range over. In order to avoid ambiguity and make the semantics coherent, most approaches (see, for example [3, 5]) restrict the relations $\twoheadrightarrow_{ts}$ to subset inclusions $D_t \subseteq D_s$. Our approach to this problem is to change the syntax (see [8] for details), thus allowing ourselves a more general class of models. The structure used to model first-order modal logic therefore consists of: a set S; a relation $\twoheadrightarrow \subseteq S \times S$; for each $t \in S$ a set D_t; for each related pair $t \twoheadrightarrow s$ a relation $\twoheadrightarrow_{ts} \subseteq D_t \times D_s$. The aim of this paper is to adapt this structure to the intuitionistic case.

In the author's recent paper [7], the Kripke semantics of propositional modal logic is extended to intuitionistic modal logic by considering a topology on the set of worlds, and interpreting propositions as open sets in the topology. This generalises the topological semantics of intuitionistic logic, and the main result of the paper is an extension of the duality between topological spaces and frames [10] to the modal case. The fibred structures used to model first-order modal logic suggest that we look for sheaf models of intuitionistic modal logic, generalising the sheaf semantics of first-order intuitionistic logic. This raises the question of how the sheaf structure (which describes how elements vary smoothly over the topology) and the relational structure (which describes how elements vary discretely over the relation) interact. This is the question addressed in this paper, and we answer it by developing a notion of "relational sheaf."

Various other approaches to predicate intuitionistic modal logic have been proposed in the literature (see, for example [12, 13, 14, 1, 4, 6]), often of a topological or category-theoretic nature. However, they are usually either limited to special cases (such as S4 modality or constant domains) or else have a very ad-hoc feel: the structures are defined specifically to model the logic, and have no natural examples or mathematical theory. The relational sheaves presented here generalise sheaves on a topological space in a natural way, by adding an arbitrary binary relation on the points. Just as sheaves describe mathematical structures which vary continuously over a space, relational sheaves describe structures which vary both continuously over a space and discretely over a relation. Since topological spaces with a binary relation abound in mathematics and computer science (the real numbers with their usual ordering, a Scott-domain with a transition relation), structures which vary over them are of intrinsic interest. That they have a rich mathematical theory (similar to that of sheaves) is one of the conclusions of this paper.

Our development of the theory follows that of traditional sheaf theory (see [15] or [2]). In Section 2 we describe bundles and presheaves over a relational space. We show that the "relational sections" of a bundle form a presheaf, that the fibres of a presheaf form a bundle, and that these two constructions are adjoint. In Section 3 we characterise those bundles which occur as the fibres of a presheaf as "local relational homeomorphisms," and those presheaves which occur as the relational sections of a bundle as "relational sheaves." We show that the adjunction restricts to an equivalence between these two, and conclude that these structures are a suitable generalisation of sheaves. In Section 4 we investigate the properties of the category of relational sheaves. We observe that the relational sheaves on a space form a quasitopos, and therefore have enough structure to model predicate logic. We investigate the properties of local relational homeomorphisms, and show that they have enough structure to model the modal logic presented in [8].

This paper shows that all the basic theory of sheaves on a topological space can be adapted to sheaves on a relational space, and that there are several interesting examples. In the author's opinion, this justifies the study of these structures, and suggests that they will form interesting models of first-order modal logic. The details of the semantics of this logic, and the problem of completeness will be presented in subsequent papers; the current paper develops the mathematical theory on which the others are based.

2 Presheaves and Bundles on a Relational Space

2.1 Relational Bundles

We start by recalling (from [7]) the definition of a relational space, and of several kinds of morphisms between such spaces.

Definition. A **relational space** $(S, \twoheadrightarrow)$ consists of a topological space S and a binary relation $\twoheadrightarrow \subseteq S \times S$.

A **continuous relational function** $f : (S, \twoheadrightarrow_S) \to (T, \twoheadrightarrow_T)$ is a continuous function $f : S \to T$ which satisfies

$$s \twoheadrightarrow_S s' \;\Rightarrow\; f(s) \twoheadrightarrow_T f(s') \tag{1}$$

A **continuous p-morphism** $f : (S, \twoheadrightarrow_S) \to (T, \twoheadrightarrow_T)$ is a continuous relational function which satisfies

$$f(s) \twoheadrightarrow_T t \;\&\; t \in U \in \mathfrak{O}(T) \;\Rightarrow\; \exists s' \in S.\ s \twoheadrightarrow_S s' \;\&\; f(s') \in U \tag{2}$$

A **relational homeomorphism** is a bijective continuous relational function, whose inverse is also a continuous relational function.

The category of relational spaces and continuous p-morphisms we denote RelSp.

The relational spaces form the base spaces for the various indexed structures which we are studying. The category RelSp and the semantics of propositional intuitionistic modal logic in relational spaces are studied in [7]; here we just repeat a couple of motivating examples.

Example 1. The real numbers $\mathbb{R}$ with their usual (metric) topology and the relation $\leqslant$.

This forms a natural model of time, familiar from Newtonian physics. As a model of intuitionistic temporal logic, the topology has an "observability" condition: anything true at time t must be true for some interval containing t.

Example 2. Let S be a domain-theoretic model of the state of a computer system, with the Scott topology. Let $s \twoheadrightarrow t$ if the system, when started in state s, might move to state t.

This forms a natural model of dynamic logic or process algebra, again with an observability condition given by the topology.

The basic indexed structure over a relational space is called a relational bundle.

Definition. Let $(S, \twoheadrightarrow)$ be a relational space.

A **relational bundle** (X, f) on S consists of a relational space $(X, \twoheadrightarrow_X)$ and a continuous relational morphism $f : X \to S$.

A **relational bundle morphism** $h : (X, f) \to (Y, g)$ over S is a continuous relational morphism $h : X \to Y$ which satisfies $gh = f$.

The category of relational bundles on S and relational bundle morphisms over S we denote $\mathrm{RBn}(S)$.

The connection between relational bundles and the kind of indexed structures discussed in the introduction is straightforward: for $s \in S$ we define the fibre $X_s = f^{\leftarrow}\{s\}$ and for $s \twoheadrightarrow t$ in S we define $\twoheadrightarrow_{st} = \twoheadrightarrow_X \cap (X_s \times X_t)$.

One of the examples considered in [7] was a simple model of branching time, represented as a subset of $\mathbb{R}^2$. This example generalises as follows:

Example 3. Let X be a topological space, let $Y \subseteq X \times \mathbb{R}$, and let $f : Y \to \mathbb{R}$ be the second projection. Define $\twoheadrightarrow \subseteq Y \times Y$ by $y \twoheadrightarrow y'$ iff $f(y) \leqslant f(y')$ and there is a continuous map $\sigma : (f(y), f(y')) \to Y$ such that

$$\begin{aligned} f(\sigma(s)) &= s && (f(y) \leqslant s \leqslant f(y')) \\ \sigma(f(y)) &= y \\ \sigma(f(y')) &= y' \end{aligned}$$

Then (Y, f) is a relational bundle on $(\mathbb{R}, \leqslant)$.

We can see the paths $\sigma^{\rightarrow}(I)$ for open intervals I as "world lines," tracing the path of a particle over the time interval I. An alternative construction of a similar example is by gluing paths together:

Example 4. Let J be a set, and for each $j \in J$ let $I_j = (a_j, b_j)$ be an open interval in $\mathbb{R}$. For each pair $j, k \in J$ let V_{jk} be an open subset of $I_j \cap I_k$ such that

$$\begin{aligned} V_{jj} &= I_j \\ V_{jk} &= V_{kj} \\ V_{jk} \cap V_{kl} &\subseteq V_{jl}. \end{aligned}$$

Now let $Y = (\coprod_{j \in J} I_j)/{\sim}$ where

$$\iota_j(x) \sim \iota_k(y) \iff x, y \in V_{jk} \ \& \ x = y$$

with the quotient topology, and the relation defined by $y \twoheadrightarrow y'$ iff there exist $j_1, \ldots, j_n \in J$ and $x_0, \ldots, x_n \in \mathbb{R}$ such that

$$\begin{aligned}
\iota_{j_1}(x_0) &\sim y \\
x_p &\leqslant x_{p+1} && (0 \leqslant p \leqslant n-1) \\
x_p &\in V_{j_p j_{p+1}} && (1 \leqslant p \leqslant n-1) \\
\iota_{j_n}(x_n) &\sim y'.
\end{aligned}$$

Then the projection $f : \iota_j(x) \mapsto x$ is a relational bundle on $(\mathbb{R}, \leqslant)$.

The differences between this example and the last are subtle. The most obvious difference is that the first example could include a "two dimensional" subset $U \times I \subseteq Y$ for some non-discrete $U \subseteq Z$. Next, Example 3 could include a path with an end-point, while all the paths in Example 4 are built from open intervals. Finally, the branching behaviour of the two is different: because the glued patches V_{jk} are open, each of I_j and I_k has a least point above V_{jk}, which cannot be Hausdorff separated; whereas (provided X is Hausdorff) Example 3 is Hausdorff.

2.2 Relational Sections

We now come to what can be seen as the fundamental concept of this paper: that of "relational section." These plays the same role in the theory of relational sheaves as local sections do in standard sheaf theory. With this definition, the development of the theory is more or less inevitable, so it is worth examining the motivation of this concept.

If $x \in X$, then a relational section σ through x (i.e. satisfying $\sigma(f(x)) = x$) can be thought of as a possible world line of the individual x. It traces the development of x over some open set $U \ni f(x)$, in a way which is continuous and preserves the relation.

Definition. Let (X, f) be a relational bundle on $(S, \twoheadrightarrow)$, and $U \in \mathfrak{O}(S)$.

A **relational section** of f over U is a continuous function $\sigma : U \to X$ satisfying

$$\forall s \in U.\ f\sigma(s) = s$$
$$\forall s, t \in U.\ s \twoheadrightarrow t \ \Rightarrow\ \sigma(s) \twoheadrightarrow_X \sigma(t)$$

Let $\Gamma(f)(U)$ be the set of all relational sections of f over U; extend this to a presheaf by

$$\Gamma(f)^U_V(\sigma) = \sigma\big|_V$$

for $V \subseteq U$.

In Example 3 the relational sections over *connected* open sets are simply the partial sections in the usual sense, because of the way the relation is defined. Relational sections over disconnected open sets can always be extended to sections over connected open sets, which is not true for partial sections in the usual sense.

In Example 4 the relational sections are defined by sequences of I_j's in the same way that the relation is defined. Again, every relational section over a disconnected open set can be extended to one over an interval.

As expected from standard sheaf theory, the collection of relational sections of a relational bundle forms a presheaf, and this construction is functorial.

Lemma 1. *The map* $\Gamma : \mathrm{RBn}(S) \to \mathrm{Set}^{\mathfrak{O}(S)^{\mathrm{op}}}$ *is a functor.*

Proof. Let $h : (Y, g) \to (X, f)$ in $\mathrm{RBn}(S)$. Define $\Gamma(h) : \Gamma(g) \to \Gamma(f)$ by

$$\Gamma(h)_U(\sigma) = h\sigma$$

Then $h\sigma : U \to X$ is continuous, $fh\sigma(s) = g\sigma(s) = s$ and $s \twoheadrightarrow t \;\Rightarrow\; \sigma(s) \twoheadrightarrow_Y \sigma(t) \;\Rightarrow\; h\sigma(s) \twoheadrightarrow_X h\sigma(t)$ so $h\sigma$ is a relational section of f over U.

The functoriality of Γ is immediate. □

2.3 Relational Fibres

The next step in the development of the theory is the construction of a left adjoint to Γ. As in standard sheaf theory, this is defined by considering the local behaviour of a presheaf at each point, and showing that the collection of "germs" so defined has a natural topology. To this we need only add that it also has a natural relation, to get the whole of the relational-bundle structure.

We start by recalling some standard definitions of sheaf theory. See [2] or [15] for details.

Definition. Let S be a topological space, $F : \mathfrak{O}(S)^{\mathrm{op}} \to \mathrm{Set}$ a presheaf on S, and $s \in S$.

The **fibre** of F over s is

$$\begin{aligned} F_s &= \operatorname*{colim}_{U \ni s} F(U) \\ &= \{(U, a) | s \in U \in \mathfrak{O}(S) \;\&\; a \in F(U)\} / {\sim_s} \end{aligned}$$

where

$$(U, a) \sim_s (V, b) \iff \exists W \ni s.\; W \subseteq U \cap V \;\&\; F^U_W(a) = F^V_W(b)$$

The **germs** of F over s are the $\sim_s$-equivalence classes $[U,a]_s$.

The **projection** $L(F) : \coprod_{s\in S} F_s \to S$ is defined by

$$L(F)([U,a]_s) = s$$

The **unit** η_F is defined (for $U \in \mathfrak{O}(S)$, $a \in F(U)$ and $s \in U$) by

$$(\eta_F)_U(a)(s) = [U,a]_s$$

The **topology** on $\coprod_{s\in S} F_s$ is the finest topology such that all the maps $(\eta_F)_U(a) : U \to \coprod_{s\in S} F_s$ are continuous. In other words, Z is open iff

$$\forall U \in \mathfrak{O}(S).\ \forall a \in F(U).\ \{s \in U | [U,a]_s \in Z\} \in \mathfrak{O}(S) \tag{3}$$

The relation on the fibre space is defined by saying that two germs are related if their base points are related and they can be extended to a common element. This is exactly like the definition of the relation in Examples 3 and 4.

Definition. Let $(S, \twoheadrightarrow)$ be a relational space and $F : \mathfrak{O}(S)^{\mathrm{op}} \to \mathrm{Set}$ a presheaf on S.

The **relation** $\twoheadrightarrow_F \subseteq \coprod_{s\in S} F_s \times \coprod_{s\in S} F_s$ is defined by

$$[U,a]_s \twoheadrightarrow_F [V,b]_t \iff s \twoheadrightarrow t \ \&\ \exists W \ni s,t.\ \exists c \in F(W). \\ (U,a) \sim_s (W,c) \ \&\ (V,b) \sim_t (W,c) \tag{4}$$

Lemma 2. *The projection* $L(F) : \coprod_{s\in S} F_s \to S$ *is a relational bundle on* S.

Proof. That $L(F)$ preserves the relation is immediate from (4); continuity is standard. □

Lemma 3. *The unit* η_F *is a presheaf morphism* $F \to \Gamma L(F)$ *over* S.

Proof. That $(\eta_F)_U(a)$ is a local section of $L(F)$ over U, natural in U, is standard; we have only to show that it preserves the relation. But if $s \twoheadrightarrow t$ then $(U,a) \sim_s (U,a)$ and $(U,a) \sim_t (U,a)$ so $(\eta_F)_U(a)(s) \twoheadrightarrow_F (\eta_F)_U(a)(t)$. □

Finally we show that the relational bundle we have constructed is universal, so we have the desired adjunction.

Theorem 1. *The unit* η_F *is universal from* F *to* Γ, *i.e.* L *extends uniquely to a functor* $L : \mathrm{Set}^{\mathfrak{O}(S)^{\mathrm{op}}} \to \mathrm{RBn}(S)$ *left adjoint to* $\Gamma : \mathrm{RBn}(S) \to \mathrm{Set}^{\mathfrak{O}(S)^{\mathrm{op}}}$.

Proof. Let (X, f) be a relational bundle on S and $g : F \to \Gamma(f)$ in $\mathrm{Set}^{\mathfrak{O}(S)^{\mathrm{op}}}$.

Define $g^\sharp : L(F) \to f$ by

$$g^\sharp[U, a]_s = g_U(a)(s)$$

If $(U, a) \sim_s (V, b)$ then there is a $W \ni s$ such that $F^U_W(a) = F^V_W(b)$ so

$$g_U(a)(s) = g_W(F^U_W(a))(s) = g_W(F^V_W(b))(s) = g_V(b)(s)$$

i.e. $g^\sharp$ is well-defined.

If $[U, a]_s \; R^F \; [V, b]_t$ then $s \twoheadrightarrow t$ and there are $W \ni s, t$ and $c \in F(W)$ such that $(U, a) \sim_s (W, c)$ and $(V, b) \sim_t (W, c)$ so

$$g_U(a)(s) = g_W(c)(s) \; S \; g_W(c)(t) = g_V(b)(t)$$

i.e. $g^\sharp$ preserves the relation.

If $Z \in \mathfrak{O}(X)$, $U \in \mathfrak{O}(S)$ and $a \in F(U)$ then

$$\begin{aligned} \{s \in U | [U, a]_s \in (g^\sharp)^{\leftarrow}(Z)\} &= \{s \in U | g_U(a)(s) \in Z\} \\ &= (g_U(a))^{\leftarrow}(Z) \end{aligned}$$

which is open by continuity of $g_U(a)$, so $g^\sharp$ is continuous.

Now if $U \in \mathfrak{O}(S)$, $a \in F(U)$ and $s \in U$ then

$$\begin{aligned} (\Gamma(g^\sharp)\eta_F)_U(a)(s) &= g^\sharp[U, a]_s \\ &= g_U(a)(s) \end{aligned}$$

so $\Gamma(g^\sharp)\eta_F = g$; conversely if $\Gamma(h)\eta_F = g$ then

$$\begin{aligned} h[U, a]_s &= (h\eta_F)_U(a)(s) \\ &= (\Gamma(h)\eta_F)_U(a)(s) \\ &= g_U(a)(s) \\ &= g^\sharp[U, a]_s \end{aligned}$$

i.e. $h = g^\sharp$. $\square$

We finish this section by calculating explicit definitions of the functor L and the counit.

Corollary 4. *The functor L is defined, for $g : F \to G$, by*

$$L(g)[U, a]_s = [U, g_U(a)]_s$$

The counit $\epsilon_f : L\Gamma(f) \to f$ is defined by

$$\epsilon_f[U, \sigma]_s = \sigma(s)$$

Proof.

$$\begin{aligned}
L(g)[U,a]_s &= (\eta_G g)^\sharp [U,a]_s \\
&= (\eta_G)_U g_U(a)(s) \\
&= [U, g_U(a)]_s \\
\epsilon_f[U,\sigma]_s &= 1^\sharp_{\Gamma(f)}[U,\sigma]_s \\
&= \sigma(s)
\end{aligned}$$

□

3 Relational Sheaves

3.1 Relational Sheaves as Presheaves

The relational sheaves are those presheaves which can be described equivalently as relational bundles. In other words, they are those objects for which the adjunction of Theorem 1 restricts to an equivalence of categories. Just as in standard sheaf theory, we characterise them by a monopresheaf condition and a patching condition; the unusual point is that these conditions require *different* notions of cover.

The presheaf of relational sections of a relational bundle satisfies the usual monopresheaf condition of sheaf theory, for precisely the usual reason. However, if we try to patch together relational sections, the patched function will not, in general, preserve the relation, as preserving the relation is not a local property. We get round this by using a restricted notion of cover, which we call a "relational cover."

Definition. Let $(S, \twoheadrightarrow)$ be a relational space, and $U \in \mathfrak{O}(S)$.

A **relational cover** of U is a family of open sets $(U_j | j \in J)$ such that

$$\bigcup_j U_j = U$$

$$\forall s, t \in U.\ s \twoheadrightarrow t \ \Rightarrow\ \exists j \in J.\ s, t \in U_j$$

A **relational sheaf** F on $(S, \twoheadrightarrow)$ is a monopresheaf in the usual sense, which satisfies the patching condition for relational covers. In other words, for all $U \in \mathfrak{O}$:

- if $(U_j | j \in J)$ is a cover of U in the usual sense, and $a, b \in F(U)$ satisfy $\forall j \in J.\ F^U_{U_j}(a) = F^U_{U_j}(b)$ then $a = b$;

- if $(U_j|j \in J)$ is a relational cover of U, and $(a_j|j \in J)$ is a family satisfying $\forall j \in J.\ a_j \in F(U_j)$ and $\forall j,k \in J.\ F^{U_j}_{U_j \cap U_k}(a_j) = F^{U_k}_{U_j \cap U_k}(a_k)$ then there exists $a \in F(U)$ such that $\forall j \in J.\ F^{U}_{U_j}(a) = a_j$.

The category of relational sheaves on S and natural transformations between them we denote $\mathrm{RSh}(S)$.

The aim of this definition is to characterise the presheaves of the form $\Gamma(f)$: the next lemma shows that these presheaves do indeed satisfy the definition.

Lemma 5. *For any relational bundle (X, f) on X, the presheaf $\Gamma(f)$ is a relational sheaf on X.*

Proof. The monopresheaf condition is obvious.

Let $(U_j|j \in J)$ be a relational cover of U, and $\sigma_j : U_j \to Y$ a relational section of f over U_j for each $j \in J$; assume $\sigma_j\big|_{U_j \cap U_k} = \sigma_k\big|_{U_j \cap U_k}$ for all $j, k \in J$.

Define $\sigma : U \to X$ by

$$\sigma(s) = \sigma_j(s) \quad \text{for some } j \text{ such that } s \in U_j$$

Then σ is well defined because the σ_j agree on overlaps, it is continuous because the σ_j are, and $f\sigma(s) = f\sigma_j(s) = s$. Finally, if $s \twoheadrightarrow t$ then there is some j such that $s, t \in U_j$, so

$$\sigma(s) = \sigma_j(s) \twoheadrightarrow_X \sigma_j(t) = \sigma(t)$$

Therefore $\sigma \in \Gamma(f)(U)$, so $\Gamma(f)$ is a relational sheaf. $\square$

Although Lemma 5 gives us plenty of examples of relational sheaves, it is instructive to consider some more concrete examples.

Example 5. Any standard sheaf on a relational space is a relational sheaf.

This is immediate from the fact that any relational cover is a cover in the usual sense. We therefore look for examples of relational sheaves which are not sheaves.

Example 6. Let $(Z, \twoheadrightarrow_Z)$ and $(S, \twoheadrightarrow_S)$ be relational spaces, and for $U \in \mathfrak{O}(S)$ let $F(U)$ be the set of continuous relational functions from U to Z. Then F (with the usual restriction maps) is a relational sheaf, but not a sheaf.

As usual, we can add local conditions such as differentiability or holomorphism.

Example 7. Let S be an open subset of $\mathbb{R}^n$ and $\twoheadrightarrow$ a partial order on S. For $U \subseteq S$, let $F(U)$ be the set of monotone differentiable functions from U to $(\mathbb{R}, \leqslant)$. Then F is a relational sheaf, but not a sheaf.

The next proposition shows that, in fact, all relational sheaves are isomorphic to ones described in Lemma 5. Furthermore, it shows that the adjunction of Theorem 1 is an equivalence when restricted to relational sheaves.

Proposition 6. *If* $F : \mathfrak{O}(S)^{\mathrm{op}} \to \mathrm{Set}$ *is a relational sheaf on* S *then* $\eta_F : F \to \Gamma L(F)$ *is an isomorphism.*

Proof. Let $U \in \mathfrak{O}(S)$ and $\sigma \in \Gamma L(F)(U)$. Let

$$J = \{(V, a) | V \subseteq U, a \in F(V) \ \&\ \forall s \in V.\ [V, a]_s = \sigma(s)\}$$

If $s \in U$ then $\sigma(s) = [V_0, a_0]_s$ for some V_0, a_0; let

$$W = \{[V_0, a_0]_t | t \in V_0\}$$

which is open in $L(F)$ (see the proof of lemma 7) so

$$\begin{aligned} V_1 &= \sigma^{\leftarrow}(W) \\ &= \{t \in V_0 \cap U | \sigma(t) = [V_0, a_0]_t\} \end{aligned}$$

is open, $(V_1, F^{V_0}_{V_1}(a_0)) \in J$ and $s \in V_1$.

If $s, t \in U$ and $s \twoheadrightarrow t$ then $\sigma(s) \twoheadrightarrow_F \sigma(t)$ so there is some V_0, a_0 such that $\sigma(s) = [V_0, a_0]_s$ and $\sigma(t) = [V_0, a_0]_t$. As before,

$$V_1 = \{u \in V_0 \cap U | \sigma(u) = [V_0, a_0]_u\}$$

is open, $(V_1, F^{V_0}_{V_1}(a_0)) \in J$ and $s, t \in V_1$.

If $s \in V \cap V'$ for some $(V, a), (V', a') \in J$, then

$$[V, a]_s = \sigma(s) = [V', a']_s$$

so there is some $W \ni s$ such that $W \subseteq V \cap V'$ and $F^V_W(a) = F^{V'}_W(a')$. Therefore

$$\big(W \big| W \subseteq V \cap V' \ \&\ F^V_W(a) = F^{V'}_W(a')\big)$$

is a cover of $V \cap V'$ (in the usual sense), and $F^V_{V \cap V'}(a) = F^{V'}_{V \cap V'}(a')$.

Therefore, $(V | (V, a) \in J)$ is a relational cover of U, and $(a | (V, a) \in J)$ a compatible family, which defines a unique $\alpha_U(\sigma) \in F(U)$.

Now $(\eta_F)_U \alpha_U(\sigma)(s) = [U, \alpha_U(\sigma)]_s = [V_1, a_1]_s$ for some $(V_1, a_1) \in J$; but this is equal to $\sigma(s)$.

Conversely $\alpha_U(\eta_F)_U(a) = \alpha_U(s \mapsto [U, a]_s) = a$. □

3.2 Relational Sheaves as Bundles

The results of the last section show that the category of relational sheaves is equivalent to some full subcategory of the category of relational bundles, namely the image of L. Our next task is to characterise which relational bundles are equivalent to relational sheaves, i.e. isomorphic to those in the image of L. Just as in standard sheaf theory, we use a local homeomorphism condition, but here the definition is more complex in two ways. Firstly, the local homeomorphisms about each point must respect the relation: this is clear from the definition of relational section in Section 2.2. Secondly, each related pair must have a local homeomorphism about them: this essentially comes from the definition of the relation on fibres in Section 2.3.

Definition. Let (X, f) be a relational bundle on S.
Then f is a **local relational homeomorphism** on S iff

- For all $x \in X$ there is an open set $Z \ni x$ in X such that $f^{\rightarrow}(Z)$ is open in S and $f\big|_Z : Z \to f^{\rightarrow}(Z)$ is a relational homeomorphism.
- For all $x, y \in X$ satisfying $x \twoheadrightarrow_X y$ there is an open set $Z \ni x, y$ in X such that $f^{\rightarrow}(Z)$ is open in S and $f\big|_Z : Z \to f^{\rightarrow}(Z)$ is a relational homeomorphism.

The category of local relational homeomorphisms on S and relational bundle morphisms between them we denote $\mathrm{LRH}(S)$.

The aim of this definition is to characterise the bundles of the form $L(F)$: the next lemma shows that these presheaves do indeed satisfy the definition.

Lemma 7. *If* $F : \mathfrak{O}(S)^{\mathrm{op}} \to \mathrm{Set}$ *is a presheaf then the bundle* $L(F)$ *is a local relational homeomorphism.*

Proof. Let $[U, a]_s \in \coprod_{s \in S} F_s$, and fix some U and a. Let

$$Z = \{[U, a]_t | t \in U\}$$

If $W \in \mathfrak{O}(S)$ and $b \in F(W)$ then

$$\begin{aligned}\{t \in W | [W, b]_t \in Z\} &= \{t \in W \cap U | (W, b) \sim_t (U, a)\} \\ &= \bigcup \{Z \in \mathfrak{O}(S) | Z \subseteq W \cap U \;\&\; F^W_Z(b) = F^U_Z(a)\} \\ &\in \mathfrak{O}(S)\end{aligned}$$

so Z is open. Define $f : U \to Z$ by

$$f(t) = [U, a]_t$$

then $fL(F)[U,a]_t = [U,a]_t$ and $L(F)f(t) = t$ so f is inverse to $L(F)\big|_Z$; and $t \twoheadrightarrow u \;\Rightarrow\; [U,a]_t \twoheadrightarrow_F [U,a]_u$ so f preserves the relation. If Z is an open subset of Z then

$$f^{\leftarrow}(Z) = \{t \in U | [U,a]_t \in Z\}$$

which is open by (3), so f is continuous.

Let $[W,c]_s, [W,c]_t \in \coprod_{s\in S} F_s$ where $s \twoheadrightarrow t$ (it is easy to see that every related pair is of this form). Let

$$Z = \{[W,c]_u | u \in W\}$$

As before, Z is open; define $f : W \to Z$ as before, and f is a local inverse to $L(F)$. □

Lemma 7 gives us plenty of examples; indeed, we shall see that all examples are essentially of this form. Nonetheless there are some standard concrete examples which we wish to consider.

Example 8. Let S be a relational space and $\alpha \cdot S = \coprod_{j\leqslant\alpha} S$ the disjoint union of α copies of S (with $\iota_j(s) \twoheadrightarrow \iota_k(t)$ iff $j = k$ and $s \twoheadrightarrow t$). Then the codiagonal $\nabla : \alpha \cdot S \to S$ defined by $\nabla(\iota_j(s)) = s$ is a local relational homeomorphism.

Example 9. Let S be a relational space and U an open subset of S (with the obvious relation). Then the inclusion $U \hookrightarrow S$ is a local relational homeomorphism.

Example 3 is not a local relational homeomorphism in general, but it is easy to see that Example 4 is: take the open set containing a point to be any path I_j containing it, and the open set containing a related pair to be the union of two such paths.

Finally we show that every local relational homeomorphism is equivalent to a relational sheaf, namely the one given by the adjunction.

Proposition 8. *If (X, f) is a local relational homeomorphism on S then $\epsilon_f : L\Gamma(f) \to f$ is an isomorphism.*

Proof. For any $x \in X$, there is an open set $Z \ni x$ such that $f^{\rightarrow}(Z)$ is open, and $f\big|_Z$ is a relational homeomorphism. Define

$$\alpha : X \to \coprod_{s\in S} \Gamma(f)_s$$
$$\alpha(x) = [f^{\rightarrow}(Z), (f\big|_Z)^{-1}]_{f(x)}$$

for some such Z. Then α is uniquely determined because if $Z' \ni x$ is another such open, then $f^{\to}(Z \cap Z') \ni f(x)$ and

$$(f\big|_Z)^{-1}\big|_{f^{\to}(Z\cap Z')} = (f\big|_{Z'})^{-1}\big|_{f^{\to}(Z\cap Z')}$$

so $(f^{\to}(Z), (f\big|_Z)^{-1}) \sim_{f(x)} (f^{\to}(Z'), (f\big|_{Z'})^{-1})$.

Next α is continuous because if W is open in $\Gamma(f)$ then

$$\begin{aligned}\alpha^{\leftarrow}(W) &= \{x \in X | \exists Z \ni x.\ [f^{\to}(Z), (f\big|_Z)^{-1}]_{f(x)} \in W\} \\ &= \bigcup_Z f^{\leftarrow}\{s \in f^{\to}(Z) | [f^{\to}(Z), (f\big|_Z)^{-1}]_s \in W\} \\ &\in \mathfrak{O}(S)\end{aligned}$$

Also α is relational because if $x \twoheadrightarrow_X x'$ there is an open set $Z \ni x, x'$ such that $f^{\to}(Z)$ is open, and $f\big|_Z$ is a relational homeomorphism. By the uniqueness of α,

$$\alpha(x) = [f^{\to}(Z), (f\big|_Z)^{-1}]_{f(x)} \twoheadrightarrow_{\Gamma(f)} [f^{\to}(Z), (f\big|_Z)^{-1}]_{f(x')} = \alpha(x')$$

Finally, $\epsilon_f \alpha(x) = (f\big|_Z)^{-1} f(x) = x$ and if $[U, \sigma]_s \in L\Gamma(f)(U)$ then $f^{\leftarrow}(U)$ is an open set containing $\sigma(s)$, so $\alpha\epsilon_f[U, \sigma]_s = [U, \sigma]_s$ by the uniqueness of α. □

We can summarise the relationship between relational sheaves and local relational homeomorphisms as follows:

Theorem 2. *The functors* $L : \mathrm{RSh}(S) \to \mathrm{LRH}(S)$ *and* $\Gamma : \mathrm{LRH}(S) \to \mathrm{RSh}(S)$ *define an equivalence of categories.*

Proof. Simply put together the results of Lemmas 5 and 7 and Propositions 6 and 8 □

4 Properties of Relational Sheaves

4.1 Regular Subsheaves

The results so far show that we have identified an interesting category of sheaf-like structures on a relational space, whose "local" or geometric properties are analogous to those of standard sheaves. Our next task is to investigate the logical properties of this category, and show that it has enough structure to model predicate modal logic.

The following observation shows that we can interpret predicate intuitionistic logic (and indeed, higher-order logic) in relational sheaves.

Proposition 9. *If S is a relational space, then* $\mathrm{RSh}(S)$ *is a quasitopos.*

Proof. The relational covers on a relational space S form a coverage [10] on $\mathfrak{O}(S)$, i.e. a Gröthendiek topology on $\mathfrak{O}(S)$ considered as a category. Let $\mathcal{E}$ be the topos of sheaves for this topology. Since every relational cover is a cover in the usual sense, the usual coverage on $\mathfrak{O}(S)$ defines a topology j on $\mathcal{E}$. The relational sheaves are those sheaves with respect to relational covers which are separated with respect to the usual covers. In other words, $\mathrm{RSh}(S)$ is the category of j-separated objects of $\mathcal{E}$. But such a category is always a quasitopos (see [16]). □

In a quasitopos, the key structure is that of the strong (or equivalently regular) subobjects. These are the subobjects classified by the subobject classifier; in the semantics of predicate logic, they are used to interpret predicates. It is therefore essential to our understanding of relational sheaves to identify them in more elementary terms.

The idea of the following definition is that proving membership of a subobject is easier than proving existence. To define a relational section by patching, we need to guarantee that the patched section will preserve the relation, and this is the point of relational covers. However, if the section already exists as an element of a superobject, we already know that it preserves the relation, so it is enough to use a cover in the usual sense.

Definition. Let $F : \mathfrak{O}(S)^{\mathrm{op}} \to \mathrm{Set}$ be a relational sheaf on S.

A **regular subsheaf** of F is a subpresheaf $\phi \subseteq F$ such that for all $U \in \mathfrak{O}(S)$, all $a \in F(U)$ and all covers $(U_j | j \in J)$ in the usual sense,

$$\forall j \in J.\ F^U_{U_j}(a) \in \phi_{U_j} \quad \Rightarrow \quad a \in \phi_U$$

We denote the poset of regular subsheaves of F (ordered by inclusion) by $\mathrm{Reg}(F)$.

The following lemma shows that this definition does identify the strong subobjects of relational sheaves, and therefore the regular subsheaves give a canonical representative for each equivalence class of strong monomorphisms.

Lemma 10. *The poset* $\mathrm{Reg}(F)$ *is isomorphic to the poset of strong subobjects of F in* $\mathrm{RSh}(S)$.

Proof. Referring to the proof of Proposition 9, it is enough to show that the regular subsheaves of F correspond to those monomorphisms of $\mathcal{E}$ (between relational sheaves) which are j-closed. But being j-closed is precisely the defining condition for regular subsheaves.

It remains only to show that a regular subsheaf of a relational sheaf is a relational sheaf. The monopresheaf property is immediate; for the patching property, let $(U_j|j \in J)$ be a relational cover of U, and $(a_j|j \in J)$ a compatible family for ϕ. Then $(a_j|j \in J)$ is certainly a compatible family for F, so there exists $a \in F(U)$ such that $F^U_{U_J}(a) = a_j$. Since every relational cover is a cover in the usual sense, it follows that $a \in \phi_U$. □

It turns out that the regular subsheaves have an even simpler description in terms of local relational homeomorphisms: they are just the open sets of the fibre space. The next two results demonstrate this.

Lemma 11. *Let (X, f) be a relational bundle on S, and $V \in \mathfrak{O}(X)$. Let $\Gamma(f\big|_V)$ be the relational sheaf defined from the restriction of f to V; then $\Gamma(f\big|_V)$ is a regular subsheaf of $\Gamma(f)$.*

Proof. For each $U \in \mathfrak{O}(S)$, the set $\Gamma(f\big|_V)(U)$ is certainly a subset of $\Gamma(f)(U)$, since it consists of those sections which factor through V. If $U' \subseteq U$ and $\sigma \in \Gamma(f\big|_V)(U)$ then $\sigma\big|_{U'}$ factors through V, so is in $\Gamma(f\big|_V)(U')$; therefore $\Gamma(f\big|_V)$ is a subpresheaf of $\Gamma(f)$.

Let $(U_j|j \in J)$ be a cover of U (in the usual sense), and $\sigma \in \Gamma(f)(U)$ a section of f over U satisfying $\sigma\big|_{U_j} \in \Gamma(f\big|_V)(U_j)$ for all $j \in J$. For any $x \in U$, there is some $j \in J$ such that $x \in U_j$, so $\sigma(x) = \sigma\big|_{U_j}(x) \in V$. Therefore $\sigma \in \Gamma(f\big|_V)(U)$. □

Theorem 3. *If (X, f) is a local relational homeomorphism on S, then the construction of Lemma 11 defines an isomorphism between $\mathfrak{O}(X)$ and $\mathrm{Reg}(\Gamma(f))$.*

Proof. If ϕ is a regular subsheaf of $\Gamma(f)$, define $\hat{\phi} \subseteq \coprod_{s \in S} \Gamma(f)_s$ by

$$[U, \sigma]_s \in \hat{\phi} \iff \exists U' \subseteq U.\ s \in U'\ \&\ \sigma\big|_{U'} \in \phi_{U'}$$

Then $\hat{\phi}$ is open, since if $U \in \mathfrak{O}(S)$ and $\sigma \in \Gamma(f)(U)$ then

$$\begin{aligned} \{s \in U | [U, \sigma]_s \in \hat{\phi}\} &= \{s \in U | \exists U' \ni s.\ U' \subseteq U\ \&\ \sigma\big|_{U'} \in \phi_{U'}\} \\ &= \bigcup \{U' \subseteq U | \sigma\big|_{U'} \in \phi_{U'}\} \\ &\in \mathfrak{O}(S) \end{aligned}$$

Now,

$$\begin{aligned} (\eta_{\Gamma(f)})_U(\sigma) \in \Gamma(L\Gamma(f)\big|_{\hat{\phi}}; U) &\iff \forall s \in U.\ [U, \sigma]_s \in \hat{\phi} \\ &\iff \forall s \in U.\ \exists U' \ni s.\ U' \subseteq U\ \&\ \sigma\big|_{U'} \in \phi_{U'} \\ &\iff \sigma \in \phi_U \end{aligned}$$

so $\phi_U = (\eta_{\Gamma(f)})^{\leftarrow}_U(\Gamma(L\Gamma(f)\big|_{\hat{\phi}})(U))$.

Also for any $V \in \mathfrak{P}$,

$$\begin{aligned} [U,\sigma]_s \in \widehat{\Gamma(f\big|_V)} &\iff \exists U' \ni s.\ U' \subseteq U \ \&\ \forall s' \in U'.\ \sigma(s') \in V \\ &\iff s \in \sigma^{\leftarrow}(V) \\ &\iff \epsilon_f[U,\sigma]_s \in V \end{aligned}$$

so $\widehat{\Gamma(f\big|_V)} = \epsilon^{\leftarrow}(V)$.

Therefore the result follows from theorem 2. □

4.2 Properties of Local Relational Homeomorphisms

Before going on to consider the modal structure of relational sheaves, we need a few basic properties of local relational homeomorphisms. The following result (together with the fact that every identity map is a local relational homeomorphism) shows that the relational spaces and local relational homeomorphisms form a category, which we will call LRH.

Lemma 12. *If $f : S \to T$ and $g : R \to S$ are local relational homeomorphisms, then their composition $fg : R \to T$ is a local relational homeomorphism.*

Proof. If $r \in R$ then there are open sets $U \ni r$ and $V \ni g(r)$ such that $g^{\rightarrow}(U)$ and $f^{\rightarrow}(V)$ are open and $g\big|_U$ and $f\big|_V$ are relational homeomorphisms. Then $U \cap g^{\leftarrow}(V) \ni r$, $(fg)^{\rightarrow}(U \cap g^{\leftarrow}(V)) = f^{\rightarrow}(g^{\rightarrow}(U) \cap V)$ which is open, and $(fg)\big|_{U\cap g^{\leftarrow}(V)} = (f\big|_{g^{\rightarrow}(U)\cap V})(g\big|_{U\cap g^{\leftarrow}(V)})$ which is a relational homeomorphism. The argument for $r \twoheadrightarrow r'$ is similar. □

The next lemma shows that any map between local relational homeomorphisms is itself a local relational homeomorphism, so in fact LRH(S) is the slice category LRH $/S$.

Lemma 13. *Let $f : S \to T$ be a local relational homeomorphism, and $g : R \to S$ a continuous relational function. If the composition fg is a local relational homeomorphism then so is g.*

Proof. If fg is a local relational homeomorphism and $r \in R$ then there are open sets $U \ni r$ and $V \ni g(r)$ such that $(fg)^{\rightarrow}(U)$ and $f^{\rightarrow}(V)$ are open and $(fg)\big|_U$ and $f\big|_V$ are relational homeomorphisms. Then $U \cap g^{\leftarrow}(V) \ni r$, $g^{\rightarrow}(U \cap g^{\leftarrow}(V)) = f^{\leftarrow}(fg)^{\rightarrow}(U \cap g^{\leftarrow}(V))$ which is open, and $g\big|_{U\cap g^{\leftarrow}(V)} = (f\big|_{g^{\rightarrow}(U)\cap V})^{-1}(fg)\big|_{U\cap g^{\leftarrow}(V)}$ which is a relational homeomorphism. The argument for $r \twoheadrightarrow r'$ is similar. □

The next lemma shows that the pullback of a local relational homeomorphism is a local relational homeomorphism. This means that we can calculate limits in LRH(S) in the standard way: in particular, products are just fibre products with the usual topology and the obvious relation.

Lemma 14. *Let $f : S \to T$ be a local relational homeomorphism, and $g : R \to T$ a continuous relational function. The pullback $f' : R \times_T S \to R$ of f along g (in the category of relational spaces and continuous relational functions) is a local relational homeomorphism.*

Proof. If $\langle r, s\rangle \in R \times_T S$ then there is an open set $U \ni s$ such that $f^{\to}(U)$ is open and $f\big|_U$ is a relational homeomorphism. Then $g^{\leftarrow}f^{\to}(U)$ is open and contains r, so $V = (R \times_T S) \cap (g^{\leftarrow}f^{\to}(U) \times U)$ is open and contains $\langle r, s\rangle$. Now $(f')^{\to}(V) = g^{\leftarrow}f^{\to}(U)$, which is open, and $(f'\big|_V)^{-1} = \langle 1, (f\big|_U)^{-1}g\rangle$ so $f'\big|_V$ is a relational homeomorphism. The argument for $\langle r, s\rangle \twoheadrightarrow \langle r', s'\rangle$ is similar. □

Next we give a result which shows that local relational homeomorphisms behave well with respect to "change of base."

Lemma 15. *Let $f : S \to T$ be a local relational homeomorphism, and $g : R \to T$ a continuous p-morphism. The pullback $g' : R \times_T S \to S$ of g along f (in the category of relational spaces and continuous relational functions) is a continuous p-morphism.*

Proof. If $\langle r, s\rangle \in R \times_T S$ and $s \twoheadrightarrow s' \in U \in \mathfrak{O}(S)$ then, since f is a local relational homeomorphism, there is an open set $V \ni s, s'$ such that $f^{\to}(V)$ is open and $f\big|_V$ is a relational homeomorphism. Now $g(r) = f(s) \twoheadrightarrow f(s') \in f^{\to}(U \cap V)$ so, since g is a continuous p-morphism, there is some $r' \in R$ such that $r \twoheadrightarrow r'$ and $g(r') \in f^{\to}(U \cap V)$. Then, since $s = (f\big|_V)^{-1}g(r) \twoheadrightarrow (f\big|_V)^{-1}g(r')$, we have $\langle r, s\rangle \twoheadrightarrow \langle r', (f\big|_V)^{-1}g(r')\rangle \in R \times_T S$, and $(f\big|_V)^{-1}g(r') \in U$. □

4.3 The Relation Predicate

In the internal language of sheaves, the basic predicate is equality. In the language of relational sheaves, equality does not give us enough expressive power: we expect the modal connectives to talk about related fibres, and we need a way to talk about related elements in those fibres. The details of a first-order modal language which allows us to do this are worked out in a separate paper [8]; here we investigate the mathematical properties of the predicate which says when two elements are related.

Definition. Let (X, f) and (Y, g) be local relational homeomorphisms on S, and let

$$\twoheadrightarrow_S = \{\langle s, t\rangle \in S \times S | s \twoheadrightarrow t\}$$

with the topology inherited from the product topology on $S \times S$.

The **tensor product** of f and g is $f \otimes g : X \otimes Y \to \twoheadrightarrow_S$, defined by

$$X \otimes Y = \{\langle x, y\rangle \in X \times Y | f(x) \twoheadrightarrow g(y)\}$$
$$(f \otimes g)\langle x, y\rangle = \langle f(x), g(y)\rangle$$

with the topology inherited from the product topology on $X \times Y$.

The **relation predicate** on f is

$$\twoheadrightarrow_f = \{\langle x, y\rangle \in X \otimes X | x \twoheadrightarrow_X y\}$$

The next lemma shows that the tensor product gives us an object in the topos of sheaves on $\twoheadrightarrow_S$, so we can safely use the constructs of geometric logic to reason about $f \otimes g$.

Lemma 16. *If (X, f) and (Y, g) are local relational homeomorphisms on S then $f \otimes g$ is a local homeomorphism on $\twoheadrightarrow_S$.*

Proof. The map $f \otimes g$ can be constructed using pullbacks. Let $\pi_1, \pi_2 : \twoheadrightarrow_S \to S$ be the projection maps, and let $\pi_1^*(f)$ and $\pi_2^*(g)$ be the pullbacks of f and g along π_1 and π_2 respectively. Then $X \otimes Y$ is the pullback of $\pi_1^*(f)$ and $\pi_2^*(g)$, and $f \otimes g$ is the diagonal of this pullback square. Since f and g are local homeomorphisms, so is $f \otimes g$. □

The next lemma shows that, over $\twoheadrightarrow_S$, the relation predicate defines a subsheaf of the tensor product $f \otimes f$. We can therefore express it as a predicate in the logic.

Proposition 17. *If (X, f) is a local relational homeomorphism on S then $\twoheadrightarrow_f$ is an open subset of $X \otimes X$.*

Proof. Let $\langle x, y\rangle \in \twoheadrightarrow_f$. Then $x \twoheadrightarrow_X y$ so there is some open set $U \ni x, y$ such that $f\big|_U$ is a relational homeomorphism. Now consider $U \times U \cap X \otimes X$. If $\langle x', y'\rangle \in U \times U \cap X \otimes X$ then $x', y' \in U$ and $f(x') \twoheadrightarrow_S f(y')$ so $x' \twoheadrightarrow_X y'$. Therefore $U \times U \cap X \otimes X \subseteq \twoheadrightarrow_f$; this shows that $\twoheadrightarrow_f$ is open. □

The relation predicate allows us to reason about elements of related fibres, while the equality predicate allows us to reason about elements of the same fibre. If the base point s is reflexive, i.e. $s \twoheadrightarrow s$, the fibre over s therefore carries both predicates, and we need to understand the relationship between them. The following lemma shows that in fact, they are identical.

Lemma 18. *If $f : X \to S$ is a local relational homeomorphism and $s \in S$ satisfies $s \twoheadrightarrow_S s$ then $\twoheadrightarrow_{ss}$ is the identity relation on $f^{\leftarrow}\{s\}$.*

Proof. If $x \in X$ such that $f(x) = s$ then there is some open set $U \ni x$ such that $f\big|_U$ is a relational homeomorphism. But $f(x) = s \twoheadrightarrow_S s = f(x)$ so $x \twoheadrightarrow_X x$.

If $x, y \in X$ such that $x \twoheadrightarrow_X y$ and $f(x) = f(y) = s$ then there is some open set $U \ni x, y$ such that $f\big|_U$ is a relational homeomorphism. But $f(x) = f(y)$ so $x = y$. □

If two base points s and t are mutually related, i.e. $s \twoheadrightarrow t$ and $t \twoheadrightarrow s$, then we have two relation predicates in the fibre of the tensor product. The following lemma shows that these two are in fact mutually inverse.

Lemma 19. *If $f : X \to S$ is a local relational homeomorphism, $x, y \in X$ satisfy $x \twoheadrightarrow_X y$ and $f(x) \twoheadleftarrow_S f(y)$, then $x \twoheadleftarrow_X y$.*

Proof. Since f is a local relational homeomorphism, there is some open set $U \ni x, y$ such that $f\big|_U$ is a relational homeomorphism. But $f(y) \twoheadrightarrow_S f(x)$ so $y \twoheadrightarrow_X x$. □

5 Conclusions

In this paper we have developed most of the basic theory of sheaves on a relational space, adapting the theory of sheaves on a topological space. We have shown that the category of relational sheaves has enough structure to model first-order modal logic as presented in [8], and even higher-order logic. In the author's opinion, the results so far justify interest in these structures, and encourage further investigation.

The importance of sheaf theory in mathematics arises not from its applications to logic, but from the study of important examples which occur in geometry, representation theory, ring theory and so on. In this paper we have discussed some interesting examples of relational sheaves on the real numbers, but we have seen no concrete examples on other spaces. Relational spaces like that of Example 2 are important in computer science, and investigation of sheaves over them might be of great interest.

One aspect of the theory of sheaves on a topological space whose analogue in the theory of relational sheaves is not discussed in this paper is change of base space along a continuous function. There are several kinds of morphism between relational spaces, including continuous relational functions, continuous p-morphisms, strict p-morphisms and so on. Since all of these are important to various applications, it is worth investigating which structures are preserved by change of base along each of them.

There is a third definition of sheaves on a topological space, equivalent to sheaves and local homeomorphisms: namely, Ω-valued sets. Preliminary investigation indicates that there is an analogue of Ω-valued sets on a relational space, and that the theory could be presented in those terms. Furthermore, it appears that this approach would be "point free" or locallic, generalising the theory of sheaves on a frame or locale.

On the logical side, the big open problem is the completeness of first-order modal logic over this class of models. The completeness theorem for first-order logic over sheaves on a topological space depends on deep results from topos theory (see [9] for details). Whether these results can be adapted to the relational case remains to be seen.

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