Topological Duality for Intuitionistic Modal Algebras II: p-Frames

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Abstract

This paper introduces the category of p-frames and p-frame morphisms, as a localic or "point-free" analogue of the category of relational spaces and continuous p-morphisms. The contravariant adjunction between modal frames and relational spaces factorises into an adjunction between modal frames and p-frames, followed by a contravariant adjunction between p-frames and relational spaces. Under certain continuity conditions, the adjunction between modal frames and p-frames restricts to an equivalence, giving a representation theorem for modal frames. Under certain continuity and compactness conditions, the adjunction between p-frames and relational spaces restricts to a duality, giving a representation theorem for p-frames. These results can be used to derive soundness and completeness results for intuitionistic modal logic.

1 Introduction

This paper is the second in a series investigating topological structures arising from the study of modal logic. In the first paper [6], categories MDLat of modal distributive lattices, MFrm of modal frames and RelSp of relational spaces are defined, and used to give a sound and complete semantics of intuitionistic propositional modal logic. Modal distributive lattices and modal frames are defined by adding to the usual structure (of distributive lattices and frames respectively) a pair of operators $\Box$ and $\Diamond$ satisfying certain axioms; and relational spaces are simply topological spaces with a binary relation on the points. These categories are linked by a series of adjunctions:

$$\mathsf{MDLat} \underset{\mathcal{U}}{\overset{\mathcal{I}}{\rightleftarrows}} \mathsf{MFrm} \underset{\mathcal{O}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{RelSp}^{\mathrm{op}} \tag{1}$$

generalising the adjunctions between distributive lattices, frames (or locales) and topological spaces studied in [7]. The functor $\mathcal{I}\colon \mathsf{MDLat} \to \mathsf{MFrm}$ lifts $\Box$ and $\Diamond$ from a modal distributive lattice to its frame of ideals, and the contravariant functor $\mathcal{O}\colon \mathsf{RelSp} \to \mathsf{MFrm}$ defines $\Box$ and $\Diamond$ on the frame of open sets of a relational space by a generalisation of the Kripke semantics. These constructions are summarised in Section 2.1.

The relationship between modal distributive lattices and modal frames is a straightforward generalisation of the non-modal case: the modal operators $\Box$ and $\Diamond$ lift continuously to ideals to give a left adjoint to the forgetful functor, and the modal frames which arise in this way can be characterised by continuity and compactness conditions. The link between modal frames and relational spaces is more complex: although the definition of the modal connectives on the open sets of a relational space is fairly natural, the adjoint to this construction involves points which are not the usual frame points, and which have a complex inductive definition. This makes the construction hard to analyse and the resulting spaces almost impossible to characterise in an intrinsic way.

There is a suspicion that the complexity of the adjunction between MFrm and RelSp arises from the fact that the modal frame point construction is doing two different things at once: first, it is replacing the frame with a topological space; second, it is replacing the modal connectives with a relation. It might make the construction simpler (or at least make the reason for the complexity clearer) if it could be factorised into two steps, so that the existence of points was separate from the Kripke semantics. The fact that the modal frame points are not the same as the usual frame points shows that the two constructions are not independent, and that a modal structure on a space will not provide the right intermediate between modal frames and relational spaces. Instead we put a relational structure on a frame.

Putting a relational structure on a frame is straightforward: a binary relation on a space corresponds (via the frame/space duality) to a quotient on the coproduct of the frame with itself. In Section 2.2 we show how to represent such a structure as a binary function (written as the infix operator $\circledcirc$) from the given frame to another, satisfying various axioms. This structure (which we call a *relational frame*) allows us to define the modal operation $\Box$ naturally using the right adjoint $\boxdot$ to the map $(\_ \circledcirc \top)$, in a way which agrees with the topological version of the Kripke semantics. In special cases, the operation $\Diamond$ can be defined in a similar way using a left adjoint to the same map, but in general this does not agree with the topological Kripke semantics, and indeed this left adjoint need not exist in the topological examples. Similar problems arise when we try to define p-morphisms between relational frames: there seems to be no frame-theoretic construction which agrees with the definition of continuous p-morphism given in [6].

Our answer to these problems is to add extra structure to our relational frames. In Section 2.2 we define a *p-frame* as a relational frame with an operation $\diamondtimes$ satisfying certain axioms. This generalises both the topological examples and the examples with a left adjoint to the map $(\_ \circleddash \top)$. It also allows us to define p-morphisms in a natural way, giving us the category pFrm of p-frames and p-frame morphisms, and the functor $\mathcal{Q}\colon \mathsf{RelSp} \to \mathsf{pFrm}$ which takes a relational space to its p-frame of open sets, and a continuous p-morphism to its inverse image map. In Section 2.3 we show that the definition of $\Box$ and $\Diamond$ on a p-frame gives us a functor $\mathcal{V}\colon \mathsf{pFrm} \to \mathsf{MFrm}$, which maps a p-frame to a modal frame on the same underlying frame. The composition of $\mathcal{V}$ with $\mathcal{Q}$ is equal to the functor $\mathcal{O}$ of (1); in other words, $\mathcal{O}$ factorises through pFrm.

The aim of this paper is to study the category pFrm and its relationship with MFrm and RelSp via the functors $\mathcal{V}$ and $\mathcal{Q}$. In particular, we construct adjoints to $\mathcal{V}$ and $\mathcal{Q}$ (thus factorising $\mathcal{P}$) and study when these adjunctions restrict to equivalences. We deduce a "pointless" version of Kripke semantics for intuitionistic modal logic, for which the axiomatisation given in [6] is sound and complete. This semantics can be generalised to logical calculi for which $\Diamond\bot \neq \bot$ (see [1] and [4] for examples) by weakening the conditions on $\diamondtimes$ (we call the resulting structure a *pre-p-frame*) without losing completeness.

Overview of Paper

Section 2 defines the main structures and categories studied in the paper. Section 2.1 summarises the definitions and results of [6], on which this paper is based. Section 2.2 defines relational frames, pre-p-frames and p-frames, gives several constructions of examples, and defines the functor from relational spaces to p-frames. Section 2.3 defines the functor from p-frames to modal frames.

Section 3 defines the adjunction between modal frames and p-frames, and shows that it restricts to an equivalence. The universal pre-p-frame is defined from a pre-modal frame using a coverage construction. This does not, in general, give the universal p-frame from a modal frame: we indicate how this can be constructed as a quotient, but do not spell out the details.

Section 4 studies special properties of the coverage construction of Section 3 which hold under certain continuity conditions. Section 4.1 shows that the continuity of $\boxdot$ and $\diamondtimes$ follows from that of $\Box$ and $\Diamond$ respectively. Section 4.2 shows that the $\Box$ and $\Diamond$ defined from $\boxdot$ and $\diamondtimes$ are the same as the originals, provided these are continuous. Section 4.3 shows that if $\Diamond$ is continuous and the original pre-modal frame is actually a modal frame, then the constructed pre-p-frame is actually a p-frame. Section 4.4 shows that if $\Box$ is cocontinuous, then the $\Box$ and $\Diamond$ defined from $\boxdot$ and $\Diamonddot$ are the same as the originals.

Section 5 defines the contravariant adjunction between p-frames and relational spaces, and shows that it restricts to a duality. The points of a p-frame are certain pairs consisting of a frame point and a frame element, similar to the modal-frame points of [6].

Section 6 shows that, under certain continuity and compactness conditions, a p-frame has enough points. In other words, p-frames satisfying these conditions can be represented concretely as relational spaces. From this we deduce a completeness theorem for intuitionistic modal logic.

2 Basic Definitions

2.1 Modal Frames and Relational Spaces

In this section we summarise the basic definitions of [6]. For further details, motivation and examples, we refer the reader to the earlier paper.

A *modal frame* is a frame (or locale) A together with two monotone operators $\Box, \Diamond \colon A \to A$ satisfying

$$\top \leqslant \Box\top \tag{2}$$

$$\Box a \wedge \Box b \leqslant \Box(a \wedge b) \tag{3}$$

$$\Box a \wedge \Diamond b \leqslant \Diamond(a \wedge b) \tag{4}$$

$$\Diamond\bot \leqslant \bot. \tag{5}$$

We will use the term *pre-modal frame* for the same structure satisfying axioms (2–4) but not necessarily (5).

A *modal frame morphism* f between modal frames is a frame morphism satisfying

$$f(\Box a) \leqslant \Box f(a) \tag{6}$$

$$f(\Diamond a) \leqslant \Diamond f(a). \tag{7}$$

The category of modal frames and modal frame morphisms is denoted MFrm, and the category of pre-modal frames and modal frame morphisms is denoted pMFrm.

Modal frames correspond closely to the syntax of intuitionistic modal logic. Indeed, we can always construct a modal frame by taking the frame of ideals of the Lindenbaum algebra of the language, and extending $\Box$ and $\Diamond$ to ideals in an obvious way. This construction is described in detail in [6], and the modal frames which arise in this way are characterised by the following properties:

- The underlying frame is *spectral* (its compact elements form a meet-closed basis);
- $\Box$ and $\Diamond$ are *compact* (they map compact elements to compact elements);
- $\Box$ and $\Diamond$ are *continuous* (they preserve directed joins).

We call modal frames which satisfy these three conditions *modally spectral.*

A *relational space* is a topological space $(X, \mathfrak{O})$ together with a binary relation $R \subseteq X \times X$. Note that we impose no further conditions on this structure; in particular, there need be no relationship between the relation and the topology.

A *continuous p-morphism* between two relational spaces $(X, \mathfrak{O}, R)$ and $(Y, \mathfrak{P}, S)$ is a continuous function $f\colon X \to Y$ satisfying

$$x \; R \; x' \;\Rightarrow\; f(x) \; S \; f(x') \tag{8}$$

$$f(x) \; S \; y \;\&\; y \in U \in \mathfrak{P} \;\Rightarrow\; \exists x'.\; x \; R \; x' \;\&\; f(x') \in U. \tag{9}$$

The category of relational spaces and continuous p-morphisms is denoted RelSp.

Relational spaces generalise Kripke models of modal logic by adding a topology on the set of worlds. There are examples throughout mathematics, several of which are important to modal logic. The real numbers with their usual (metric) topology and their usual ($\leqslant$) relation form a natural model of real time temporal logic. A domain model of the state of a computer system with the Scott topology and the input-output relation of a program is a natural model of program logic.

The categories MFrm and RelSp are connected by a contravariant functor $\mathcal{O}$, which generalises the Kripke semantics of modal logic. This functor is defined as follows:

$$\mathcal{O}\colon \mathsf{RelSp} \to \mathsf{MFrm} \tag{10}$$

$$\mathcal{O}(X, \mathfrak{O}, R) = (\mathfrak{O}, \Box^R, \Diamond^R) \tag{11}$$

$$\mathcal{O}(f) = f^{\leftarrow} \tag{12}$$

where

$$\Box^R(U) = \{x \in X | \forall y \in X.\; x \; R \; y \;\Rightarrow\; y \in U\}^\circ \tag{13}$$

$$\Diamond^R(U) = \{x \in X | \exists y \in X.\; x \; R \; y \;\&\; y \in U\}^\circ \tag{14}$$

The subject of the paper [6] is the contravariant adjoint to $\mathcal{O}$ on the right, its restriction to a duality between subcategories of MFrm and RelSp, and its application to the semantics of intuitionistic modal logic.

2.2 Relational Frames and p-Frames

In this section we study notions of binary relation on a frame, in order to define an intermediate between modal frames and relational spaces. Our approach is to take existing spatial structures, and dualise them via the familiar space/frame duality.

A binary relation on a space X is a subset R of the product $X \times X$, which can be represented as a pair of continuous functions $f, g\colon R \to X$ which are jointly strong monic. The dual of this notion (on a frame A) is that of a pair of frame morphisms $p, q\colon A \to B$ which are jointly strong epi; this is equivalent to the condition that every element $b \in B$ satisfies

$$b = \bigvee \{p(a_1) \wedge q(a_2) | a_1, a_2 \in A \ \&\ p(a_1) \wedge q(a_2) \leqslant b\}. \tag{15}$$

In order to simplify our notation, we prefer to combine the maps p and q into a single (infix) operator $\circleddash\colon A \times A \to B$ defined by $a_1 \circleddash a_2 = p(a_1) \wedge q(a_2)$. The original maps can be recovered from the relations $p(a) = a \circleddash \top$ and $q(a) = \top \circleddash a$. This allows us to axiomatise the resulting structure as follows:

Definition. A *relational frame* $(A, B, \circleddash)$ is a pair of frames A, B and a monotone map $\circleddash\colon A \times A \to B$ satisfying:

$$\top \circleddash \top = \top \tag{16}$$
$$(a \wedge b) \circleddash (c \wedge d) = a \circleddash c \wedge b \circleddash d \tag{17}$$
$$a \circleddash \left(\bigvee S\right) = \bigvee \{a \circleddash b | b \in S\} \tag{18}$$
$$\left(\bigvee S\right) \circleddash b = \bigvee \{a \circleddash b | a \in S\} \tag{19}$$
$$b = \bigvee \{a_1 \circleddash a_2 | a_1 \circleddash a_2 \leqslant b\} \tag{20}$$

Axioms (16–19) ensure that $\_ \circleddash \top$ and $\top \circleddash \_$ are frame morphisms, and axiom (20) is (15) rewritten in terms of $\circleddash$.

As well as the obvious topological examples, we can construct relational frames by taking the joint image of a pair of frame morphisms as follows:

Example 1. Let $f, g\colon A \to C$ be frame morphisms, and let B be the set of $c \in C$ which satisfy

$$c = \bigvee \{f(a_1) \wedge g(a_2) | f(a_1) \wedge g(a_2) \leqslant c\}. \tag{21}$$

This is a subframe of C; define $\owedge \colon A \times A \to B$ by

$$a \owedge b = f(a) \wedge g(b). \tag{22}$$

Then $(A, B, \owedge)$ is a relational frame.

In fact, every relational frame arises in this way: just take $C = B$, $f(a) = a \owedge \top$ and $g(a) = \top \owedge a$.

The notion of relational frame is adequate for studying relational morphisms and the semantics of $\Box$. However, p-morphisms and the semantics of $\Diamond$ depend on taking the interior of the direct image of an open set along the relation, which cannot (in general) be described in purely frame-theoretic terms. We therefore add a new operation $\Diamonddot$ to the structure, which is intended to represent this direct image.

Definition. A *pre-p-frame* $(A, B, \owedge, \Diamonddot)$ is a relational frame $(A, B, \owedge)$ together with a monotone map $\Diamonddot \colon B \to A$ satisfying

$$a \wedge \Diamonddot b \leqslant \Diamonddot(a \owedge \top \wedge b). \tag{23}$$

A *p-frame* is a pre-p-frame which satisfies

$$\Diamonddot(a \owedge \top) \leqslant a \tag{24}$$

$$\Diamonddot(b \vee \Diamonddot b \owedge \top) \leqslant \Diamonddot b. \tag{25}$$

An *open* p-frame is a p-frame which satisfies

$$b \leqslant \Diamonddot b \owedge \top. \tag{26}$$

Note that (25) follows from (24) and (26), that the Frobenius identity

$$a \wedge \Diamonddot b = \Diamonddot(a \owedge \top \wedge b)$$

follows from (23) and (24), and that (24) and (26) are the triangle identities for $\Diamonddot$ to be left adjoint to $\_ \owedge \top \colon A \to B$. We deduce the following:

Lemma 1. *Let $(A, B, \owedge)$ be a relational frame. Then $\_ \owedge \top \colon A \to B$ is an open morphism with left adjoint $\Diamonddot \colon B \to A$, if and only if $(A, B, \owedge, \Diamonddot)$ is an open p-frame.*

This means that we can construct examples of open p-frames in exactly the same way as in Example 1:

Example 2. Let $f, g \colon A \to C$ be frame morphisms, such that f is open with left adjoint $f_!$. Define B and $\owedge$ as in Example 1, and define $\Diamonddot \colon B \to A$ by $\Diamonddot b = f_!(b)$. Then $(A, B, \owedge, \Diamonddot)$ is an open p-frame.

As before, every open p-frame arises in this way. However, relational spaces need not have $\_ \otimes \top$ an open map, unless the inverse image of an open set along the relation is open. In the paper [6], natural models of real-time temporal logic are studied which do not satisfy this condition, and do not have open p-frames. The next lemma shows that all relational spaces do have p-frames even if they are not open. We call any p-frame which is equivalent to the p-frame of a relational space *spatial*.

Lemma 2. *Let $(X, \mathfrak{O}, R)$ be a relational space, and let $\mathfrak{O}_R$ be the topology which R inherits as a subset of $X \times X$. In other words, $W \subseteq R$ is open in $\mathfrak{O}_R$ if and only if for all $\langle x, y\rangle \in W$ there exist $U, V \in \mathfrak{O}$ such that*

$$\langle x, y\rangle \in (U \times V) \cap R \subseteq W$$

Define $\otimes_R$ and $\diamond_R$ by

$$U \otimes_R V = (U \times V) \cap R \tag{27}$$

$$\diamond_R W = \{x \in X | \exists y \in X.\ \langle x, y\rangle \in W\}^\circ \tag{28}$$

Then $(\mathfrak{O}, \mathfrak{O}_R, \otimes_R, \diamond_R)$ is a p-frame.

Proof. To show (23), let $U \in \mathfrak{O}, W \in \mathfrak{O}_R$ and $x \in U \cap \diamond_R W$. By the definition of $\diamond_R$, there is some $y \in X$ such that $x\ R\ y$ and $\langle x, y\rangle \in W$, and by the definition of $\otimes_R$, $\langle x, y\rangle \in U \otimes_R \top$.

This shows that $U \cap \diamond_R W \subseteq \{x \in X | \exists y.\ x\ R\ y\ \&\ \langle x, y\rangle \in U \otimes_R \top \cap W\}$; but the left hand side is open, so we deduce that $U \cap \diamond_R W \subseteq \diamond_R(U \otimes_R \top \cap W)$.

To show (24), let $U \in \mathfrak{O}$ and $x \in \diamond_R(U \otimes_R \top)$. By the definition of $\diamond_R$, there is some $y \in X$ such that $x\ R\ y$ and $\langle x, y\rangle \in U \otimes_R \top$, and by the definition of $\otimes_R$, $x \in U$. Therefore $\diamond_R(U \otimes_R \top) \subseteq U$.

To show (25), let $W \in \mathfrak{O}_R$ and let $x \in \diamond_R(W \cup \diamond_R W \otimes_R X)$. By the definition of $\diamond_R$, there is some $y \in X$ such that $x\ R\ y$ and $\langle x, y\rangle \in W \cup \diamond_R W \otimes_R X$. If $\langle x, y\rangle \in \diamond_R W \otimes_R X$ then $x \in \diamond_R W$ so, again by the definition of $\diamond_R$, there is some $y' \in X$ such that $x\ R\ y'$ and $\langle x, y'\rangle \in W$; otherwise $\langle x, y\rangle \in W$. Either way, $\exists y.\ x\ R\ y\ \&\ \langle x, y\rangle \in W$.

This shows that $\diamond_R(W \cup \diamond_R W \otimes_R X) \subseteq \{x \in X | \exists y.\ x\ R\ y\ \&\ \langle x, y\rangle \in W\}$; but the first set is open, so we can deduce that $\diamond_R(W \cup \diamond_R W \otimes_R X) \subseteq \diamond W$. □

Further examples of p-frames arise from the following lemma, which shows that we can always construct a p-frame from a relational frame in a universal way. The definition of $\diamond$ given in (29) can be thought of as similar to the formal definition of Σ in terms of Π often used in type theory, and the definition of $\exists$ in terms of $\forall$ in higher-order logic. However, unlike those cases, $\diamonddot$ is not uniquely determined by the rest of the structure, and spatial p-frames need not satisfy (29).

Lemma 3. *Let* $(A, B, \oslash)$ *be a relational frame, and let* $\boxdot$ *be the right adjoint to* $\_ \oslash \top \colon A \to B$*. Define* $\diamonddot$ *by*

$$\diamonddot b = \bigwedge_{c \in A} \big(\boxdot(b \to c \oslash \top) \to c\big) \tag{29}$$

then $(A, B, \oslash, \diamonddot)$ *is a p-frame, and for any other operation* $\diamonddot'$ *making* $(A, B, \oslash, \diamonddot')$ *a p-frame,* $\diamonddot' \leqslant \diamonddot$*.*

Proof. To show (23), note that $\boxdot$ preserves $\wedge$, so for any $c \in A$,

$$\begin{aligned} a \wedge \diamonddot b \wedge \boxdot(a \oslash \top \wedge b \to c \oslash \top) &\leqslant \boxdot(a \oslash \top) \wedge \boxdot(a \oslash \top \wedge b \to c \oslash \top) \wedge \diamonddot b \\ &\leqslant \boxdot(b \to c \oslash \top) \wedge \bigwedge_{c' \in A} \big(\boxdot(b \to c' \oslash \top) \to c'\big) \\ &\leqslant c \end{aligned}$$

so $a \wedge \diamonddot b \leqslant \bigwedge_{c \in A} \big(\boxdot(a \oslash \top \wedge b \to c \oslash \top) \to c\big) = \diamonddot(a \oslash \top \wedge b)$.

To show (24), take $c = a$ so

$$\diamonddot(a \oslash \top) = \bigwedge_{c \in A} \big(\boxdot(a \oslash \top \to c \oslash \top) \to c\big) \leqslant a$$

To show (25), note from (23) that $\diamonddot a \wedge \boxdot c \leqslant \diamonddot(a \wedge \boxdot c \oslash \top) \leqslant \diamonddot(a \wedge c)$ so for any $c \in A$,

$$\begin{aligned} \diamonddot(b \vee \diamonddot b \oslash \top) \wedge \boxdot(b \to c \oslash \top) &\leqslant \diamonddot((b \vee \diamonddot b \oslash \top) \wedge (b \to c \oslash \top)) \\ &\leqslant \diamonddot(c \oslash \top \vee \diamonddot(c \oslash \top) \oslash \top) \\ &\leqslant \diamonddot(c \oslash \top) \leqslant c \end{aligned}$$

so $\diamonddot(b \vee \diamonddot b \oslash \top) \leqslant \bigwedge_{c \in A} \big(\boxdot(b \to c \oslash \top) \to c\big) = \diamonddot b$.

To show universality, let $\diamonddot'$ be an operation satisfying (23–25), let $b \in B$ and $c \in A$. Then $\diamonddot' b \wedge \boxdot(b \to c \oslash \top) \leqslant \diamonddot'(c \oslash \top) \leqslant c$, so

$$\diamonddot' b \leqslant \bigwedge_{c \in A} \big(\boxdot(b \to c \oslash \top) \to c\big).$$

□

A continuous relational function between relational spaces is a continuous function which preserves the relation. Translating this into frame-theoretic terms gives us the notion of relational frame morphism, defined as follows:

Definition. A *relational frame morphism* $(f, g)\colon (A, B, \circleddash) \to (A', B', \circleddash')$ is a pair of frame morphisms

$$f\colon A \to A'$$
$$g\colon B \to B'$$

such that

$$g(a \circleddash b) = f(a) \circleddash' f(b). \tag{30}$$

Note that, by axiom (20), g is uniquely determined by f, so we can write $f\colon (A, B, \circleddash) \to (A', B', \circleddash')$ without ambiguity.

A *p-frame morphism* $(f, g)\colon (A, B, \circleddash, \diamondsuit) \to (A', B', \circleddash', \diamondsuit')$ is a relational frame morphism which satisfies

$$f(\diamondsuit a) \leqslant \diamondsuit' g(a). \tag{31}$$

The following lemma shows that examples of relational frame morphisms and p-frame morphisms arise from continuous relational functions and continuous p-morphisms respectively, by taking the inverse-image map.

Lemma 4. *Let $f\colon (X, \mathfrak{O}, R) \to (Y, \mathfrak{P}, S)$ be a continuous relational function. Define $f^R\colon \mathfrak{P}_S \to \mathfrak{O}_R$ by*

$$\langle x, y\rangle \in f^R(W) \iff \langle f(x), f(y)\rangle \in W. \tag{32}$$

Then $(f^{\leftarrow}, f^R)\colon (\mathfrak{P}, \mathfrak{P}_S, \circleddash_S) \to (\mathfrak{O}, \mathfrak{O}_R, \circleddash_R)$ is a relational frame morphism. Furthermore, if f is a continuous p-morphism, then

$$(f^{\leftarrow}, f^R)\colon (\mathfrak{P}, \mathfrak{P}_S, \circleddash_S, \diamondsuit_S) \to (\mathfrak{O}, \mathfrak{O}_R, \circleddash_R, \diamondsuit_R)$$

is a p-frame morphism.

Proof. The relational frame morphism condition (30) is immediate. To prove (31), let $x \in f^{\leftarrow}(\diamondsuit_S W)$. Then $f(x) \in \diamondsuit_S W$ so there is some $y \in Y$ such that $f(x)\ S\ y$ and $\langle f(x), y\rangle \in W$. Since $W \in \mathfrak{P}_S$, there exist $U, V \in \mathfrak{P}$ such that $\langle f(x), y\rangle \in (U \times V) \cap S \subseteq W$. By the p-morphism property (9), there is some $x' \in X$ such that $x\ R\ x'$ and $f(x') \in V$, so $\langle x, x'\rangle \in f^R((U \times V) \cap S) \subseteq f^R(W)$. Since $f^{\leftarrow}(\diamondsuit_S W)$ is open, this shows that $f^{\leftarrow}(\diamondsuit_S W) \subseteq \diamondsuit_R f^R(W)$. □

We denote the category of relational frames and relational frame morphisms by RFrm, the category of pre-p-frames and p-frame morphisms by ppFrm, and the category of p-frames and p-frame morphisms by pFrm.

The first step in the factorisation of $\mathcal{O}$ is now complete. We record this as follows:

Lemma 5. *The constructions of Lemmas 2 and 4 define a contravariant functor*

$$\mathcal{Q}\colon \mathsf{RelSp} \to \mathsf{pFrm} \tag{33}$$
$$\mathcal{Q}(X, \mathfrak{O}, R) = (\mathfrak{O}, \mathfrak{O}_R, \obslash_R, \diamonddot_R) \tag{34}$$
$$\mathcal{Q}(f) = (f^{\leftarrow}, f^R). \tag{35}$$

2.3 p-Frames and Modal Frames

In order to complete our factorisation of the functor $\mathcal{O}$, we need to be able to turn a p-frame into a modal frame. In other words, we need to define the modal connectives $\Box$ and $\Diamond$ on the underlying frame. We motivate the definitions by considering Lemma 2. There, the frame B is a topology on the set of related pairs $\langle x, y\rangle$. If we think of x as this world or state and y as the next, an element of B is a binary condition on the pair. A condition on the next world alone is therefore of the form $\top \obslash a$ for some $a \in A$. Considering the Kripke semantics of $\Box$ and $\Diamond$ (which only talk about the next world) therefore leads us to the following definition.

Definition. Let $(A, B, \obslash, \diamonddot)$ be a (pre-) p-frame. The map $\_ \obslash \top\colon A \to B$ preserves all joins, so it has a right adjoint. Let $\boxdot\colon B \to A$ be this right adjoint, so

$$\boxdot b = \bigvee \{a \in A | a \obslash \top \leqslant b\}. \tag{36}$$

Define $\Box_B, \Diamond_B\colon A \to A$ by

$$\Box_B a = \boxdot(\top \obslash a) \tag{37}$$
$$\Diamond_B a = \diamonddot(\top \obslash a) \tag{38}$$

The following lemma shows that this construction does indeed give us a modal frame.

Lemma 6. *If $(A, B, \obslash, \diamonddot)$ is a pre-p-frame then $(A, \Box_B, \Diamond_B)$ is a pre-modal frame. Furthermore, if $(A, B, \obslash, \diamonddot)$ is a p-frame then $(A, \Box_B, \Diamond_B)$ is a modal frame.*

Proof. Axioms (2) and (3) follow from the fact that $\boxdot$ and $\top \obslash \_$ preserve finite meets. Axiom (4) follows from (23) as follows:

$$\begin{aligned}\Box_B a \wedge \Diamond_B b = \boxdot(\top \obslash a) \wedge \diamonddot(\top \obslash b) &\leqslant \diamonddot(\boxdot(\top \obslash a) \obslash \top \wedge \top \obslash b)\\ &\leqslant \diamonddot(\top \obslash a \wedge \top \obslash b) = \diamonddot(\top \obslash (a \wedge b)) = \Diamond_B(a \wedge b).\end{aligned}$$

Axiom (5) follows from (24) as follows:

$$\Diamond_B \bot = \diamonddot(\top \obslash \bot) = \diamonddot \bot = \diamonddot(\bot \obslash \top) \leqslant \bot. \qquad \Box$$

The next lemma shows that this construction extends to morphisms in the obvious way.

Lemma 7. *If* $(f, g) \colon (A, B, \obslash, \Diamonddot) \to (A', B', \obslash', \Diamonddot')$ *is a p-frame morphism then* $f \colon (A, \Box_B, \Diamond_B) \to (A', \Box_{B'}, \Diamond_{B'})$ *is a modal frame morphism.*

Proof. Axiom (6) is proved as follows:

$$\begin{aligned} f(\Box_B a) \obslash' \top = f(\boxdot(\top \obslash a)) \obslash' f(\top) &= g(\boxdot(\top \obslash a) \obslash \top) \\ &\leqslant g(\top \obslash a) = f(\top) \obslash' f(a) = \top \obslash' f(a) \end{aligned}$$

so by adjointness,

$$f(\Box_B a) \leqslant \boxdot'(\top \obslash f(a)) = \Box_{B'} f(a)$$

Axiom (7) is proved as follows:

$$f(\Diamond_B a) = f(\Diamonddot(\top \obslash a)) \leqslant \Diamonddot' g(\top \obslash a) = \Diamonddot'(\top \obslash' f(a)) = \Diamond_{B'} f(a). \qquad \Box$$

This completes our factorisation of $\mathcal{O}$ as $\mathcal{V} \circ \mathcal{Q}$. We record this as follows:

Lemma 8. *This construction defines functors* $\mathcal{U} \colon \mathsf{ppFrm} \to \mathsf{pMFrm}$ *and* $\mathcal{V} \colon \mathsf{pFrm} \to \mathsf{MFrm}$*; The two functors* $\mathcal{O}$ *and* $\mathcal{V} \circ \mathcal{Q} \colon \mathsf{RelSp} \to \mathsf{MFrm}$ *are equal.*

3 The Adjunction between Modal Frames and p-Frames

Our main aim in this paper is to factorise the adjoint to $\mathcal{O}$ into two functors, one adjoint to $\mathcal{V}$ and the other adjoint to $\mathcal{Q}$. In fact we start by constructing a left adjoint to $\mathcal{U}$ (which is why we introduced pre-modal frames and pre-p-frames) because the construction is much more straightforward. In particular, this construction allows us to keep the same underlying frame A, so we only have to construct the frame B.

The axioms of a p-frame are supposed to capture the idea that B is a quotient of the coproduct of A with itself. The standard construction of the coproduct of frames (see [7, §II.2.12]) uses a coverage C on the product $A \times A$ to generate the coproduct as the frame of C-ideals. If the coverage C is extended by adding more covers, the resulting frame of C-ideals will be a quotient of the coproduct, as required. The coverage we need to generate the universal pre-p-frame from a pre-modal frame is as follows.

Definition. Let $(A, \Box, \Diamond)$ be a pre-modal frame. Define a coverage $C^\Box$ on $A \times A$ by

- If $S \subseteq A$ and $b \in A$ then $\{(a, b) | a \in S\}$ covers $(\bigvee S, b)$;
- If $a \in A$ and $S \subseteq A$ then $\{(a, b) | b \in S\}$ covers $(a, \bigvee S)$;
- If $a, b, c \in A$ and $a \leqslant \Box c$ then $\{(a, b \wedge c)\}$ covers (a, b).

The first two clauses in the definition of $C^\Box$ are taken from the usual definition of the coproduct of frames. The third clause is intended to capture the idea that, for a related pair of worlds, if $\Box c$ is true in the first, then c must be true in the second.

The next lemma establishes that this is indeed a coverage, so we can form the frame of ideals in the usual way.

Lemma 9. *$C^\Box$ is a coverage (in the sense of [7, §II 2.11]) on $A \times A$.*

Proof. We have only to check that the three clauses are stable under $\wedge$. But this is straightforward. $\square$

In order to get a pre-p-frame, we use the coverage $C^\Box$ to define the frame B, the tensor $\circledcirc$ and the operation $\Diamond\!\!\!\!\cdot\,$ as follows.

Definition. Let B be the set of $C^\Box$-ideals, and for $a, b \in A$ let $a \circledcirc b$ be the least $C^\Box$ ideal containing (a, b). Define $\diamond\!\!\!\cdot : B \to A$ by

$$\diamond\!\!\!\cdot\, I = \bigvee \{a \wedge \Diamond b | (a, b) \in I\}. \tag{39}$$

The next three lemmas establish that this construction has the required properties, and gives us the adjoint to $\mathcal{U}$.

Lemma 10. *$(A, B, \circledcirc, \diamond\!\!\!\cdot\,)$ is a pre-p-frame.*

Proof. Axioms (16), (17) and (20) are immediate from the definitions, and axioms (18) and (19) follow from the first two clauses in the definition of $C^\Box$.

Axiom (23) is proved as follows. If $(b, c) \in I$ then $(a \wedge b, c) \in a \circledcirc \top \wedge I$ so

$$\begin{aligned} a \wedge \diamond\!\!\!\cdot\, I &= a \wedge \bigvee \{b \wedge \Diamond c | (b, c) \in I\} \\ &= \bigvee \{a \wedge b \wedge \Diamond c | (b, c) \in I\} \\ &\leqslant \diamond\!\!\!\cdot\,(a \circledcirc \top \wedge I). \end{aligned}$$ $\square$

Next we show that the identity map is the unit.

Lemma 11. *The identity map* $1\colon (A, \Box, \Diamond) \to (A, \Box_B, \Diamond_B)$ *is a modal frame morphism.*

Proof. To show (6) note that $(\Box a, a) \leqslant (\top, a) \in \top \ogreaterthan a$ and $\{(\Box a, a)\}$ covers $(\Box a, \top)$, so $(\Box a, \top) \in \top \ogreaterthan a$. Therefore $\Box a \ogreaterthan \top \subseteq \top \ogreaterthan a$, and by adjointness, $\Box a \leqslant \boxcircle(\top \ogreaterthan a)$.

To show (7) note that $(\top, a) \in \top \ogreaterthan a$ so $\Diamond a \leqslant \bigvee\{b \wedge \Diamond c | (b, c) \in \top \ogreaterthan a\}$. $\Box$

Finally we show that the construction is universal.

Lemma 12. *If* $(A', B', \ogreaterthan', \diamonddiamond')$ *is a pre-p-frame and* $f\colon (A, \Box, \Diamond) \to (A', \Box_{B'}, \Diamond_{B'})$ *then* $f\colon (A, B, \ogreaterthan, \diamonddiamond) \to (A', B', \ogreaterthan', \diamonddiamond')$.

Proof. We use the universal property of $C^{\Box}$-ideals to define $g\colon B \to B'$ satisfying $g(a \ogreaterthan b) = f(a) \ogreaterthan' f(b)$; we have only to show that the map $(a, b) \mapsto f(a) \ogreaterthan' f(b)$ takes covers to joins. The first two clauses of $C^{\Box}$ are immediate, and for the third clause, note that

$$f(\Box c) \ogreaterthan' \top \leqslant \Box_{B'} f(c) \ogreaterthan' \top = \boxcircle'(\top \ogreaterthan' f(c)) \ogreaterthan' \top \leqslant \top \ogreaterthan' f(c)$$

so if $a \leqslant \Box c$ then

$$\begin{aligned} f(a) \ogreaterthan' f(b) = f(a \wedge \Box c) \ogreaterthan' f(b) &= f(a) \ogreaterthan' f(b) \wedge f(\Box c) \ogreaterthan' \top \\ &= f(a) \ogreaterthan' f(b) \wedge \top \ogreaterthan' f(c) = f(a) \ogreaterthan' f(b \wedge c). \end{aligned}$$

This shows that (f, g) is a relational frame morphism. To show that it is a p-frame morphism, note that if $(a, b) \in I$ then

$$\begin{aligned} f(a \wedge \Diamond b) \leqslant f(a) \wedge \Diamond_{B'} f(b) &= f(a) \wedge \diamonddiamond'(\top \ogreaterthan' f(b)) \\ &\leqslant \diamonddiamond'(f(a) \ogreaterthan' \top \wedge \top \ogreaterthan' f(b)) = \diamonddiamond'(f(a) \ogreaterthan' f(b)) \end{aligned}$$

so

$$\begin{aligned} f(\diamonddiamond I) &= \bigvee\{f(a \wedge \Diamond b) | (a, b) \in I\} \\ &\leqslant \bigvee\{\diamonddiamond'(f(a) \ogreaterthan' f(b)) | (a, b) \in I\} \\ &\leqslant \diamonddiamond' \bigvee\{f(a) \ogreaterthan' f(b) | (a, b) \in I\} \\ &= \diamonddiamond' g(I). \end{aligned}$$

$\Box$

Theorem 1. *The mapping* $(A, \Box, \Diamond) \mapsto (A, B, \ogreaterthan, \diamonddiamond)$ *defines a left adjoint to* $\mathcal{U}\colon \mathsf{ppFrm} \to \mathsf{pMFrm}$. *Furthermore, this adjunction restricts to an equivalence between those pre-modal frames which can be constructed from pre-p-frames, and those pre-p-frames which can be constructed from pre-modal frames.*

Proof. The only thing left to prove for the adjunction is the uniqueness of the arrow given by universality. But this is immediate because g is determined by f.

For the equivalence it is enough to notice that both the unit and counit are identity maps (therefore both epi and monic), so the triangle identities are isomorphisms. $\square$

Having constructed the adjoint to $\mathcal{U}\colon \mathsf{ppFrm} \to \mathsf{pMFrm}$, we could go on to construct an adjoint to $\mathcal{V}\colon \mathsf{pFrm} \to \mathsf{MFrm}$. This would be reasonably straightforward, by quotienting the construction used above by a suitable equivalence relation, in order to impose the axioms (24) and (25). However, the resulting construction would change the underlying frame A, and so be useless for completeness results. Instead of pursuing the details of this construction, in the next section we consider some reasonable conditions under which the existing construction already satisfies the axioms, and so the equivalence relation is not needed.

4 The Continuous Case

As discussed in Section 2.1, the modal frames which arise as the ideal completions of modal distributive lattices are modally spectral, and in particular, $\Box$ and $\Diamond$ are continuous. It turns out that continuity conditions like this allow us to deduce several useful properties of the construction of Section 3, leading to a completeness theorem for intuitionistic modal logic.

Definition. A *continuously* (pre-) modal frame is a (pre-) modal frame in which $\Box$ and $\Diamond$ preserve directed joins.

A *continuously* (pre-) p-frame is a (pre-) p-frame in which $\boxdot$ and $\diamonddot$ preserve directed joins.

This terminology is a little awkward, but it allows us to reserve terms such as 'continuous modal frame' for modal frames whose underlying frame is continuous, i.e. locally compact.

All the proofs in this section work the same way: we present a particular subset of $A \times A$; we show under some continuity condition that it is a $C^{\Box}$-ideal; we deduce that the pre-p-frame satisfies a particular axiom. It is possible that with greater ingenuity in the construction of the subsets, the axioms could be proved without the continuity assumptions. However the explicit construction of $C^{\Box}$-ideals is quite delicate, and the Author's attempts to generalise these results have not been successful.

4.1 The Continuity of $\diamonddot$ and $\boxdot$

It is easy to see (since $\olessthan$ is continuous in each argument) that whenever $(A, B, \olessthan, \diamonddot)$ is a (pre-) p-frame such that $\diamonddot$ and $\boxdot$ are continuous, the (pre-) modal frame $(A, \Box_B, \Diamond_B)$ constructed from it has continuous $\Box_B$ and $\Diamond_B$. Our first results in this section show that the adjoint functor satisfies the converse property, so the adjunction restricts to the continuous case.

Lemma 13. *If $\Diamond$ is continuous, and D is a directed set of $C^{\Box}$-ideals, then the set*

$$I_D = \{(a, b) | \forall e.\ a \wedge \Diamond(b \vee e) \leqslant \bigvee D_e^{\Diamond}\} \tag{40}$$

$$\text{where} \quad D_e^{\Diamond} = \{a' \wedge \Diamond(b' \vee e) | (a', b') \in \bigcup D\} \tag{41}$$

is a $C^{\Box}$-ideal.

Proof. We show that I_D is downwards closed, and closed under covers.

If $(a', b') \leqslant (a, b) \in I_D$ then for any $e \in A$,

$$a' \wedge \Diamond(b' \vee e) \leqslant a \wedge \Diamond(b \vee e) \leqslant \bigvee D_e^{\Diamond},$$

so $(a', b') \in I_D$.

If $S \subseteq A$ and for all $a \in S$, $(a, b) \in I_D$ then for any $e \in A$,

$$\bigvee S \wedge \Diamond(b \vee e) = \bigvee \{a \wedge \Diamond(b \vee e) | a \in S\} \leqslant \bigvee D_e^{\Diamond},$$

so $(\bigvee S, b) \in I_D$.

The case of $(a, \bigvee S)$ for $S \subseteq A$ we split into three: S empty, S a pair, and S directed. For the case of S empty, we need to show that $(a, \bot) \in I_D$ for all $a \in A$. But since D is nonempty $(a, \bot) \in \bigcup D$, so for any $e \in A$, $a \wedge \Diamond(\bot \vee e) \leqslant \bigvee D_e^{\Diamond}$.

If $(a, b) \in I_D$ and $(a, c) \in I_D$ then for any $e \in A$,

$$\begin{aligned}
a \wedge \Diamond(b \vee c \vee e) &\leqslant a \wedge \bigvee D_{c \vee e}^{\Diamond} \\
&= \bigvee \{a \wedge a' \wedge \Diamond(b' \vee c \vee e) | (a', b') \in \bigcup D\} \\
&\leqslant \bigvee \{a' \wedge \bigvee D_{b' \vee e}^{\Diamond} | (a', b') \in \bigcup D\} \\
&= \bigvee \{a' \wedge a'' \wedge \Diamond(b'' \vee b' \vee e) | (a', b'), (a'', b'') \in \bigcup D\};
\end{aligned}$$

but since D is directed, for all $(a', b') \in I' \in D$ and $(a'', b'') \in I'' \in D$ there is some $I \in D$ such that $I' \cup I'' \subseteq I$, so $(a' \wedge a'', b' \vee b'') \in I \in D$. Therefore

$$\begin{aligned}
&\bigvee \{a' \wedge a'' \wedge \Diamond(b'' \vee b' \vee e) | (a', b'), (a'', b'') \in \bigcup D\} \\
&\qquad\qquad \leqslant \bigvee \{a \wedge \Diamond(b \vee e) | (a, b) \in \bigcup D\}
\end{aligned}$$

so $(a, b \vee c) \in I_D$.

If S is directed and for all $b \in S$, $(a, b) \in I_D$ then for any $e \in A$,

$$\begin{aligned} a \wedge \diamond(\bigvee S \vee e) &= a \wedge \diamond(\bigvee \{b \vee e | b \in S\}) \\ &= \bigvee \{a \wedge \diamond(b \vee e) | b \in S\} \\ &\leqslant \bigvee D_e^{\diamond} \end{aligned}$$

so $(a, \bigvee S) \in I_D$.

If $(a, b \wedge c) \in I_D$ and $a \leqslant \Box c$ then for any $e \in A$,

$$\begin{aligned} a \wedge \diamond(b \vee e) = a \wedge \Box c \wedge \diamond(b \vee e) \leqslant a \wedge \diamond(c \wedge (b \vee e)) \leqslant a \wedge \diamond((c \wedge b) \vee e) \\ \leqslant \bigvee D_e^{\diamond} \end{aligned}$$

so $(a, b) \in I_D$. $\Box$

The construction of I_D allows us to show that $\diamondsuit$ preserves directed joins:

Lemma 14. *If I_D is a $C^{\Box}$-ideal then*

$$\diamondsuit \bigvee D = \bigvee \{\diamondsuit I | I \in D\}. \tag{42}$$

Proof. First we show that $\bigvee D \leqslant I_D$. If $I \in D$ and $(a, b) \in I$ then for all e, $a \wedge \diamond(b \vee e) \in D_e^{\diamond}$, so $I \subseteq I_D$. Since I_D is a $C^{\Box}$-ideal, this means that $\bigvee D \leqslant I_D$.

Next we show that $\diamondsuit I_D \subseteq \bigvee D_{\perp}^{\diamond}$. If $(a, b) \in I_D$ then $a \wedge \diamond b = a \wedge \diamond(b \vee \perp) \leqslant \bigvee D_{\perp}^{\diamond}$ so $\diamondsuit I_D \subseteq \bigvee D_{\perp}^{\diamond}$ by (39).

Finally we show that $\bigvee D_{\perp}^{\diamond} \leqslant \bigvee \{\diamondsuit I | I \in D\}$. If $I \in D$ and $(a, b) \in I$ then $a \wedge \diamond(b \vee \perp) = a \wedge \diamond b \leqslant \diamondsuit I$ so $\bigvee D_{\perp}^{\diamond} \leqslant \bigvee \{\diamondsuit I | I \in D\}$.

Putting this together, we have $\diamondsuit \bigvee D \leqslant \diamondsuit I_D \leqslant \bigvee D_{\perp}^{\diamond} \leqslant \bigvee \{\diamondsuit I | I \in D\}$. The other comparison is immediate. $\Box$

Next we do the same thing for $\Box$:

Lemma 15. *If $\Box$ is continuous, and D is a directed set of $C^{\Box}$-ideals, then the set*

$$J_D = \{(a, b) | \forall e.\ a \wedge \Box(b \vee e) \leqslant \bigvee D_e^{\Box}\} \tag{43}$$

$$\textit{where} \quad D_e^{\Box} = \{a' \wedge \Box(b' \vee e) | (a', b') \in \bigcup D\} \tag{44}$$

is a $C^{\Box}$-ideal.

Proof. This is exactly like the proof of Lemma 13, with $\Box$ substituted for $\Diamond$. $\square$

The construction of J_D allows us to show that $\boxdot$ preserves directed joins:

Lemma 16. *If J_D is a $C^\Box$-ideal then*

$$\boxdot \bigvee D = \bigvee \{\boxdot I | I \in D\}. \tag{45}$$

Proof. First we show that $\bigvee D \leqslant J_D$. If $I \in D$ and $(a, b) \in I$ then for all e, $a \wedge \Box(b \vee e) \in D_e^\Box$, so $I \subseteq J_D$. Since J_D is a $C^\Box$-ideal, this means that $\bigvee D \leqslant J_D$.

Next we show that $\boxdot J_D \subseteq \bigvee D_\perp^\Box$. If $a \ogreaterthan \top \leqslant J_D$ then $(a, \top) \in J_D$ so $a = a \wedge \Box(\top \vee \perp) \leqslant \bigvee D_\perp^\Box$; therefore $\boxdot J_D \subseteq \bigvee D_\perp^\Box$ by (36).

Finally we show that $\bigvee D_\perp^\Box \leqslant \bigvee \{\boxdot I | I \in D\}$. If $I \in D$ and $(a, b) \in I$ then $(a \wedge \Box b, b) \in I$ so $(a \wedge \Box b, \top) \in I$ so $a \wedge \Box b \in \boxdot I$; therefore $\bigvee D_\perp^\Box \leqslant \bigvee \{\boxdot I | I \in D\}$.

Putting this together, we have $\boxdot \bigvee D \leqslant \boxdot J_D \leqslant \bigvee D_\perp^\Box \leqslant \bigvee \{\boxdot I | I \in D\}$. The other comparison is immediate. $\square$

Putting together Lemmas 13–16 we deduce that the adjunction of Theorem 1 restricts to an adjunction between continuously pre-modal frames and continuously pre-p-frames.

4.2 Equivalence for Pre-Modal Frames

If we start with a pre-modal frame, construct the pre-p-frame of Lemma 10, and from this construct the pre-modal frame of Lemma 6, we end up with a new pre-modal structure on the same frame. It is of great interest to know when the new pre-modal structure is the same as the original, as this gives a representation of such pre-modal frames as pre-p-frames. In this section we show that continuity of the modal connectives is sufficient for this.

Lemma 17. *If $\Diamond$ is continuous and $a \in A$ then the set*

$$I_a = \{(b, c) | \forall e.\ b \wedge \Diamond(c \vee e) \leqslant \Diamond(a \vee e)\} \tag{46}$$

is a $C^\Box$-ideal.

Proof. We show that I_a is downwards closed, and closed under covers.

If $(b', c') \leqslant (b, c) \in I_a$ then for any $e \in A$, $b' \wedge \Diamond(c' \vee e) \leqslant b \wedge \Diamond(c \vee e) \leqslant \Diamond(a \vee e)$, so $(b', c') \in I_a$.

If $S \subseteq A$ and for all $b \in S$, $(b, c) \in I_a$ then for any $e \in A$,

$$\bigvee S \wedge \Diamond(c \vee e) = \bigvee\{b \wedge \Diamond(c \vee e) | b \in S\} \leqslant \Diamond(a \vee e),$$

so $(\bigvee S, c) \in I_a$.

The case of $(b, \bigvee S)$ for $S \subseteq A$ we split into three: S empty, S a pair, and S directed. If S is empty, for any $e \in A$, $b \wedge \Diamond(\bot \vee e) \leqslant \Diamond e \leqslant \Diamond(a \vee e)$, so $(b, \bot) \in I_a$.

If $(b, c) \in I_a$ and $(b, d) \in I_a$ then for any $e \in A$,

$$b \wedge \Diamond(c \vee d \vee e) \leqslant b \wedge \Diamond(a \vee d \vee e) \leqslant \Diamond(a \vee e),$$

so $(b, c \vee d) \in I_a$.

If S is directed and for all $c \in S$, $(b, c) \in I_a$ then for any $e \in A$,

$$\begin{aligned} b \wedge \Diamond(\bigvee S \vee e) &= b \wedge \Diamond(\bigvee\{c \vee e | c \in S\}) \\ &= \bigvee\{b \wedge \Diamond(c \vee e) | c \in S\} \\ &\leqslant \Diamond(a \vee e) \end{aligned}$$

so $(b, \bigvee S) \in I_a$.

If $(b, c \wedge d) \in I_a$ and $b \leqslant \Box d$ then for any $e \in A$,

$$\begin{aligned} b \wedge \Diamond(c \vee e) = b \wedge \Box d \wedge \Diamond(c \vee e) \leqslant b \wedge \Diamond(d \wedge (c \vee e)) \leqslant b \wedge \Diamond((d \wedge c) \vee e) \\ \leqslant \Diamond(a \vee e) \end{aligned}$$

so $(b, c) \in I_a$. $\Box$

The construction of I_a allows us to show that the $\Diamond_B$ operator given by (38) is the same as the original $\Diamond$:

Lemma 18. *If I_a is a $C^{\Box}$-ideal then*

$$\diamonddot(\top \ogreaterthan a) = \Diamond a \tag{47}$$

Proof. Since $\top \wedge \Diamond(a \vee e) \leqslant \Diamond(a \vee e)$, we know that $(\top, a) \in I_a$, so $\top \ogreaterthan a \subseteq I_a$. Therefore

$$\begin{aligned} \diamonddot(\top \ogreaterthan a) &\leqslant \diamonddot I_a \\ &= \bigvee\{b \wedge \Diamond c | (b, c) \in I_a\} \\ &= \bigvee\{b \wedge \Diamond c | \forall e.\ b \wedge \Diamond(c \vee e) \leqslant \Diamond(a \vee e)\} \\ &\leqslant \Diamond a \end{aligned}$$

The other comparison is part of Lemma 11. $\Box$

Next we do the same for $\Box$:

Lemma 19. *If $\Box$ is continuous and $a \in A$ then the set*

$$J_a = \{(b,c) | \forall e.\ b \wedge \Box(c \vee e) \leqslant \Box(a \vee e)\} \tag{48}$$

is a $C^{\Box}$-ideal.

Proof. This is exactly like the proof of Lemma 17, with $\Box$ substituted for $\Diamond$. ☐

The construction of J_a allows us to show that the $\Box_B$ operator given by (37) is the same as the original $\Box$:

Lemma 20. *If J_a is a $C^{\Box}$-ideal then*

$$\boxdot(\top \oslash a) = \Box a \tag{49}$$

Proof. Since $\top \wedge \Box(a \vee e) \leqslant \Box(a \vee e)$, we know that $(\top, a) \in J_a$, so $\top \oslash a \subseteq J_a$. Therefore

$$\begin{aligned}
\boxdot(\top \oslash a) &\leqslant \boxdot J_a \\
&= \bigvee\{b \in A | (b, \top) \in J_a\} \qquad \text{by (36)} \\
&= \bigvee\{b \in A | \forall e.\ b \wedge \Box(\top \vee e) \leqslant \Box(a \vee e)\} \\
&\leqslant \Box a
\end{aligned}$$

The other comparison is part of Lemma 11. ☐

Putting together Lemmas 17–20 we deduce that the adjunction of Theorem 1 restricts to an equivalence of categories between continuously pre-modal frames and certain pre-p-frames. To summarise:

Theorem 2. *The category of continuously pre-modal frames and modal frame morphisms is equivalent to a full coreflective subcategory of the category of continuously pre-p-frames and p-frame morphisms.*

4.3 Equivalence for Modal Frames

We have seen in Lemma 6 that the pre-modal frame corresponding to a p-frame is in fact a modal frame. We now show that continuity is a sufficient condition for the converse.

Lemma 21. *If $\Diamond$ is continuous and $a \in A$ then the set*

$$K_a = \{(b, c) | \forall e.\ b \wedge \Diamond(c \vee e) \leqslant a \vee \Diamond e\} \tag{50}$$

is a $C^{\Box}$-ideal.

Proof. This is almost exactly like the proof of Lemma 17; it is left to the reader to make the trivial alterations. □

The construction of K_a allows us to show that the pre-p-frame of $C^{\Box}$-ideals constructed from a modal frame satisfies axiom (24):

Lemma 22. *If K_a is a $C^{\Box}$-ideal and $\Diamond \bot = \bot$ then*

$$\diamondsuit(a \oslash \top) \leqslant a \tag{51}$$

Proof. Since $a \wedge \Diamond(\top \vee e) \leqslant a \leqslant a \vee \Diamond e$, we know that $(a, \top) \in K_a$, so $a \oslash \top \subseteq K_a$. Therefore

$$\begin{aligned} \diamondsuit(a \oslash \top) &\leqslant \diamondsuit K_a \\ &= \bigvee \{b \wedge \Diamond c | (b, c) \in K_a\} \\ &= \bigvee \{b \wedge \Diamond c | \forall e.\ b \wedge \Diamond(c \vee e) \leqslant a \vee \Diamond e\} \\ &\leqslant a \vee \Diamond \bot = a \end{aligned}$$

□

Lemma 23. *If $\Diamond$ is continuous and I is a $C^{\Box}$-ideal then the set*

$$I' = \{(a, b) | \forall e.\ a \wedge \Diamond(b \vee e) \leqslant \bigvee I_e^{\Diamond}\} \tag{52}$$

$$\text{where} \quad I_e^{\Diamond} = \{a' \wedge \Diamond(b' \vee e) | (a', b') \in I\} \tag{53}$$

is a $C^{\Box}$-ideal.

Proof. This is just the case $D = \{I\}$ of Lemma 13. □

The construction of I' allows us to deduce that the pre-p-frame of $C^{\Box}$-ideals satisfies axiom (25):

Lemma 24. *If I' is a $C^{\Box}$-ideal then*

$$\diamondsuit(I \vee \diamondsuit I \oslash \top) \leqslant \diamondsuit I \tag{54}$$

Proof. First we show that $I \vee \Diamond\mkern-13mu\circ\, I \ogreaterthan \top \subseteq I'$. If $(a, b) \in I$ then for any e, $a \wedge \Diamond\mkern-13mu\circ\,(b \vee e) \in I_e^{\Diamond}$ so $(a, b) \in I'$. Also

$$a \wedge \Diamond b \wedge \Diamond(\top \vee e) \leqslant a \wedge \Diamond b \leqslant a \wedge \Diamond(b \vee e) \leqslant \bigvee I_e^{\Diamond},$$

so $(a \wedge \Diamond b, \top) \in I'$, and $(a \wedge \Diamond b) \ogreaterthan \top \subseteq I'$. Therefore

$$\begin{aligned} I \vee \Diamond\mkern-13mu\circ\, I \ogreaterthan \top &= I \vee \left(\bigvee\{a \wedge \Diamond b | (a, b) \in I\} \ogreaterthan \top\right) \\ &= I \vee \bigvee\{(a \wedge \Diamond b) \ogreaterthan \top | (a, b) \in I\} \\ &\subseteq I'. \end{aligned}$$

Next we show that $\Diamond\mkern-13mu\circ\, I' \leqslant \Diamond\mkern-13mu\circ\, I$. If $(a, b) \in I'$ then

$$a \wedge \Diamond b \leqslant a \wedge \Diamond(b \vee \bot) \leqslant \bigvee I_{\bot}^{\Diamond} = \Diamond\mkern-13mu\circ\, I$$

so

$$\Diamond\mkern-13mu\circ\, I' = \bigvee\{a \wedge \Diamond b | (a, b) \in I'\} \leqslant \Diamond\mkern-13mu\circ\, I,$$

Therefore $\Diamond\mkern-13mu\circ\,(I \vee \Diamond\mkern-13mu\circ\, I \ogreaterthan \top) \leqslant \Diamond\mkern-13mu\circ\, I' \leqslant \Diamond\mkern-13mu\circ\, I$. $\square$

Putting together Lemmas 21–24 we deduce that the $C^{\Box}$-ideal construction takes continuously modal frames to p-frames. Together with earlier results, we deduce the following, which can be interpreted as a completeness theorem for intuitionistic modal logic.

Theorem 3. *The category of continuously modal frames and modal frame morphisms is equivalent to a full coreflective subcategory of the category of continuously p-frames and p-frame morphisms.*

4.4 Cocontinuity and Equivalence

Just to show that continuity of $\Box$ or $\Diamond$ is not a necessary condition for this sort of result, our next construction uses *co-continuity* of $\Box$. Since $\Box$ always preserves $\wedge$ and $\top$, this means that $\Box$ preserves all meets:

Lemma 25. *If $\Box$ preserves all meets and $a \in A$ then the set*

$$L_a = \{(b, c) | b \leqslant \Box(c \to a)\} \tag{55}$$

is a $C^{\Box}$-ideal.

Proof. We show that L_a is downwards closed, and closed under covers.

If $(b', c') \leqslant (b, c) \in L_a$ then $b' \leqslant b \leqslant \Box(c \to a) \leqslant \Box(c' \to a)$, so $(b', c') \in L_a$.

If $S \subseteq A$ and for all $b \in S$, $(b, c) \in L_a$ then $\bigvee S \leqslant \Box(c \to a)$ so $(\bigvee S, c) \in L_a$.

If for all $c \in S$, $(b, c) \in L_a$ then

$$\begin{aligned} b &\leqslant \bigwedge\{\Box(c \to a) | c \in S\} \\ &\leqslant \Box \bigwedge\{c \to a | c \in S\} \\ &\leqslant \Box\left(\bigvee S \to a\right) \end{aligned}$$

so $(b, \bigvee S) \in L_a$.

If $(b, c \wedge d) \in L_a$ and $b \leqslant \Box d$ then

$$b = b \wedge \Box d \leqslant \Box((c \wedge d) \to a) \wedge \Box d \leqslant \Box(c \to a)$$

so $(b, c) \in L_a$. $\Box$

The construction of L_a allows us to show that *both* $\Box_B$ and $\Diamond_B$ agree with the originals:

Lemma 26. *If L_a is a $C^{\Box}$-ideal then*

$$\Diamonddot(\top \ogreaterthan a) = \Diamond a \tag{56}$$

$$\boxcircle(\top \ogreaterthan a) = \Box a \tag{57}$$

Proof. Since $\Box(a \to a) = \Box\top = \top$, we know that $(\top, a) \in L_a$, so $\top \ogreaterthan a \subseteq L_a$. Therefore

$$\begin{aligned} \Diamonddot(\top \ogreaterthan a) &\leqslant \Diamonddot L_a \\ &= \bigvee\{b \wedge \Diamond c | (b, c) \in L_a\} \\ &= \bigvee\{b \wedge \Diamond c | b \leqslant \Box(c \to a)\} \\ &\leqslant \bigvee\{\Box(c \to a) \wedge \Diamond c | c \in A\} \\ &\leqslant \Diamond a. \end{aligned}$$

Similarly,

$$\begin{aligned} \boxcircle(\top \ogreaterthan a) &\leqslant \boxcircle L_a \\ &= \bigvee\{b \in A | (b, \top) \in L_a\} \\ &= \bigvee\{b \in A | b \leqslant \Box a\} \\ &\leqslant \Box a. \end{aligned}$$

The other comparisons are proved in Lemma 11. $\Box$

Once again we can deduce an equivalence theorem, this time for pre-modal frames with co-continuous $\Box$.

Theorem 4. *The category of co-continuous pre-modal frames and modal frame morphisms is equivalent to a full subcategory of* ppFrm.

5 The Adjunction Between p-Frames and Relational Spaces

Our next task in the factorisation of the adjunction is the construction of a contravariant adjoint to $\mathcal{Q}\colon \mathsf{RelSp} \to \mathsf{pFrm}$, on the right. In other words, we need to construct a relational space from a p-frame, in a universal way. The core of the construction is (as might be expected) the definition of the appropriate notion of point for the space. This is similar to the construction of modal frame points in [6], in that a point consists of two parts: a frame point p and an element of the frame which represents the interior of the set of unrelated points. The difference is that the second component is an element of the frame B, rather than A.

The reader might have hoped that this finer analysis of the adjunction would have removed the need for such a complex construction. The problem is that in a p-frame, $\diamondsuit$ (which represents the interior of the direct image along the relation) is separate from B (which represents the relation itself). This separation is necessary because the point-sensitive operation of taking the interior of the direct image cannot be directly represented frame-theoretically. The element b captures that part of the structure which is missing from the frame point.

In a relational space, two points x and y are related precisely when the pair $\langle x, y\rangle$ is in the relation R. In order to put this into frame-theoretic terms, we need an operation of pairing for points, which we define as follows:

Definition. Let $(A, B, \otimes, \diamondsuit)$ be a p-frame, and let $p, q\colon A \to \mathbf{2}$ be points of A. We define the *pair* $p \otimes q\colon B \to \mathbf{2}$ by

$$(p \otimes q)(b) = \bigvee\{p(a_1) \wedge q(a_2) | a_1 \otimes a_2 \leqslant b\}. \tag{58}$$

Note that $p \otimes q$ need not be a point of B; if it is, we say p is *related* to q.

The following lemma shows that every point of B is of the form $p \otimes q$.

Lemma 27. *Let* $(A, B, \otimes, \diamondsuit)$ *be a p-frame, and* r *a point of* B. *If* $p = r \circ (\_ \otimes \top)$ *and* $q = r \circ (\top \otimes \_)$ *then* $p \otimes q = r$, *and thus* p *is related to* q.

Proof. For any $b \in B$,

$$\begin{aligned}(p \otimes q)(b) &= \bigvee\{r(a_1 \otimes \top) \wedge r(\top \otimes a_2)|a_1 \otimes a_2 \leqslant b\} \\ &= \bigvee\{r(a_1 \otimes a_2)|a_1 \otimes a_2 \leqslant b\} \\ &= r(\bigvee\{a_1 \otimes a_2|a_1 \otimes a_2 \leqslant b\} \\ &= r(b)\end{aligned}$$

□

This allows us to define the p-frame points as follows:

Definition. Let $(A, B, \otimes, \diamond)$ be a p-frame. A *pre-point* of A is a pair (p, b) where $p\colon A \to \mathbf{2}$ is a frame point and $b \in B$ satisfies

$$p(\diamond b) = 0. \tag{59}$$

Two pre-points are *related*, $(p, b)\ R_A\ (q, c)$ if and only if

$$p \otimes q \text{ is a } \textit{point} \text{ of } B, \tag{60}$$

$$(p \otimes q)(b) = 0. \tag{61}$$

The set of *p-frame points* $\mathbb{P}_A$ is the *largest* set P of pre-points which satisfies

$$(p, b) \in P \ \&\ a \not\leqslant b \ \Rightarrow\ \exists(q, c) \in P.\ (p, b)\ R_A\ (q, c)\ \&\ (p \otimes q)(a) = 1. \tag{62}$$

It is easy to see that any union of sets of pre-points which satisfy (62) itself satisfies (62), so the set $\mathbb{P}_A$ is well defined. The relation on modal frame points is just R_A, so it remains only to define the topology.

Definition. The **unit** $\phi_A\colon A \to \mathcal{P}(\mathbb{P}_A)$ is defined by

$$\phi_A(a) = \{(p, b) \in \mathbb{P}_A | p(a) = 1\}. \tag{63}$$

The **topology** $\mathfrak{O}_A$ on $\mathbb{P}_A$ is the image of ϕ_A.

The first step in proving adjointness is defining the unit.

Lemma 28. *The map* $\phi_A\colon (A, B, \otimes, \diamond) \to (\mathfrak{O}_A, \mathfrak{O}_{R_A}, \otimes_{R_A}, \diamond_{R_A})$ *is a p-frame morphism.*

Proof. That ϕ_A is a frame morphism is standard. We define $\psi_A\colon B \to \mathfrak{O}_{R_A}$ by

$$\psi_A(b) = \bigcup\{\phi_A(a_1) \times \phi_A(a_2) \cap R_A | a_1 \otimes a_2 \leqslant b\}. \tag{64}$$

Then if $(p, b)\ R_A\ (q, c)$,

$$\begin{aligned}\langle (p,b),(q,c)\rangle \in \psi_A(d) &\iff \exists a_1, a_2 \in A.\ a_1 \otimes a_2 \leqslant d \ \&\ p(a_1) = q(a_2) = 1\\ &\iff (p \otimes q)(d) = 1\end{aligned}$$

but $p \otimes q$ is a frame point, so ψ_A is a frame morphism. Therefore (ϕ_A, ψ_A) is a relational frame morphism.

If $(p, b) \in \phi_A(\diamond d)$ then $p(\diamond d) = 1$ but $p(\diamond b) = 0$ so $d \not\leqslant b$. By (62), there is some $(q, c) \in \mathbb{P}_A$ such that $(p, b)\ R_A\ (q, c)$ and $(p \otimes q)(d) = 1$, i.e. $\langle (p, b), (q, c)\rangle \in \psi_A(d)$. Therefore

$$\phi_A(\diamond d) \subseteq \{(p, b) \in \mathbb{P}_A | \exists (q, c) \in \mathbb{P}_A.\ \langle (p, b), (q, c)\rangle \in \psi_A(d)\}$$

but $\phi_A(\diamond d)$ is open, so is a subset of $\diamond_{R_A} \psi_A(d)$. Therefore (ϕ_A, ψ_A) is a p-frame morphism. □

Next we define the arrow which must exist for universality:

Definition. Let $(X, \mathfrak{O}, R)$ be a relational space, and $(f, g)\colon (A, B, \otimes, \diamond) \to (\mathfrak{O}, \mathfrak{O}_R, \otimes_R, \diamond_R)$ a p-frame morphism. Define $f^\sharp\colon X \to \mathbb{P}_A$ by $f^\sharp(x) = (p_x, b_x)$ where

$$p_x(a) = 1 \text{ iff } x \in f(a) \tag{65}$$

$$b_x = \bigvee\{b \in B | \forall y \in X.\ x\ R\ y \ \Rightarrow\ \langle x, y\rangle \notin g(b)\}. \tag{66}$$

Lemma 29. *For every $x \in X$, the pair (p_x, b_x) is a pre-point.*

Proof. If $p_x(\diamond b_x) = 1$ then $x \in f(\diamond b_x) \subseteq \diamond_R g(b_x)$ so by (28) there is some $y \in X$ such that $\langle x, y\rangle \in g(b_x)$. But

$$g(b_x) = \bigcup\{g(b) | \forall y \in X.\ x\ R\ y \ \Rightarrow\ \langle x, y\rangle \notin g(b)\}$$

which gives a contradiction. Therefore $p_x(\diamond b_x) = 0$, and (p_x, b_x) is a pre-point. □

Lemma 30. *If $x\ R\ y$ then $\langle x, y\rangle \in g(b) \iff (p_x \otimes p_y)(b) = 1$. It follows that $(p_x, b_x)\ R_A\ (p_y, b_y)$.*

Proof. Let $x\ R\ y$. From the definitions,

$$\begin{aligned} g(b) &= \bigcup\{f(a_1) \times f(a_2) \cap R | a_1 \otimes a_2 \leqslant b\}\\ (p_x \otimes p_y)(b) &= \bigvee\{p_x(a_1) \wedge p_y(a_2) | a_1 \otimes a_2 \leqslant b\}\end{aligned}$$

so $\langle x, y\rangle \in g(b)$ if and only if $\exists a_1, a_2.\ a_1 \otimes a_2 \leqslant b \ \&\ x \in f(a_1) \ \&\ y \in f(a_2)$, if and only if $\exists a_1, a_2.\ a_1 \otimes a_2 \leqslant b \ \&\ p_x(a_1) = 1 \ \&\ p_y(a_2) = 1$, if and only if $(p_x \otimes p_y)(b) = 1$.

It follows that $p_x \otimes p_y$ is a point of B, because g is a frame morphism.

Finally, if $(p_x \otimes p_y)(b_x) = 1$ then $\langle x, y\rangle \in g(b_x)$ which, as in Lemma 29, gives a contradiction. Therefore $(p_x \otimes p_y)(b_x) = 0$ and $(p_x, b_x)\ R_A\ (p_y, b_y)$. □

Lemma 31. *For every $x \in X$, the pair (p_x, b_x) is a p-frame point.*

Proof. We will show that the image of $f^\sharp$ satisfies (62), so is a subset of $\mathbb{P}_A$. If $(p_x, b_x) = f^\sharp(x)$ and $a \nleqslant b_x$, then by (66), there is some $y \in X$ such that $\langle x, y\rangle \in g(a)$. But then $x\ R\ y$ so $(p_x, b_x)\ R_A\ (p_y, b_y)$ (which is in the image of $f^\sharp$), and $(p_x \otimes p_y)(a) = 1$ by Lemma 30. □

Lemma 32. *The map $f^\sharp \colon (X, \mathfrak{O}, R) \to (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ is a continuous p-morphism which satisfies $(f^\sharp)^{\leftarrow}\phi_A = f$.*

Proof. Continuity is standard, and the relation property is Lemma 30, so it remains to prove the p-morphism property (9). If $(p_x, b_x)\ R_A\ (q, c)$ and $(q, c) \in \phi_A(a)$ then $q(a) = 1$ so $(p_x \otimes q)(\top \otimes a) = 1$, but $(p_x \otimes q)(b_x) = 0$ so $\top \otimes a \nleqslant b_x$. By (66), there is some $y \in X$ such that $\langle x, y\rangle \in g(\top \otimes a) = X \otimes_R f(a)$; then $x\ R\ y$ and $y \in f(a)$, so $(p_y, b_y) \in \phi_A(a)$.

For the last part, $x \in (f^\sharp)^{\leftarrow}\phi_A(a)$ iff $f^\sharp(x) \in \phi_A(a)$ iff $p_x(a) = 1$ iff $x \in f(a)$, so $(f^\sharp)^{\leftarrow}\phi_A = f$. □

Lemma 33. *The map $f^\sharp$ is the unique continuous p-morphism $(X, \mathfrak{O}, R) \to (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ which satisfies $(f^\sharp)^{\leftarrow}\phi_A = f$.*

Proof. Let $h \colon (X, \mathfrak{O}, R) \to (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ satisfy $h^{\leftarrow}\phi_A = f$, and let $(q_x, c_x) = h(x)$ for each $x \in X$.

Then $p_x(a) = 1$ iff $x \in f(a) = h^{\leftarrow}\phi_A(a)$ iff $h(x) \in \phi_A(a)$ iff $q_x(a) = 1$, so $q_x = p_x$.

If $b_x \nleqslant c_x$ then, by (62), there is some $(q, c) \in \mathbb{P}_A$ such that $(q_x, c_x)\ R_A\ (q, c)$ and $(q_x \otimes q)(b_x) = 1$. By (58) there are $a_1, a_2 \in A$ such that $a_1 \otimes a_2 \leqslant b_x$ and $q_x(a_1) = 1$ and $q(a_2) = 1$ i.e. $(q, c) \in \phi_A(a_2)$. But h is a continuous p-morphism, so by (9), there is some $y \in X$ such that $x\ R\ y$ and $h(y) \in \phi_A(a_2)$ i.e. $q_y(a_2) = 1$. Now, by the first part of the proof, $(p_x \otimes p_y)(b_x) = (q_x \otimes q_y)(b_x) \geqslant q_x(a_1) \wedge q_y(a_2) = 1$ but by Lemma 30, $(p_x, b_x)\ R_A\ (p_y, b_y)$ so $(p_x \otimes p_y)(b_x) = 0$ which is a contradiction. Therefore $b_x \leqslant c_x$.

If $c_x \nleqslant b_x$ then by (66), there is some $y \in X$ such that $x\ R\ y$ and $\langle x, y\rangle \in g(c_x)$ i.e. $(p_x \otimes p_y)(c_x) = 1$. But by Lemma 30, $(q_x, c_x)\ R_A\ (q_y, c_y)$ so $(p_x \otimes p_y)(c_x) = (q_x \otimes q_y)(c_x) = 0$, which is a contradiction. Therefore $c_x \leqslant b_x$.

Putting this all together, $f^\sharp(x) = (p_x, b_x) = (q_x, c_x) = h(x)$ for all x, so $h = f^\sharp$. □

The main theorem of this section now follows.

Theorem 5. *The p-frame point construction* $(A, B, \circleddash, \diamondsuit) \mapsto (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ *defines a contravariant adjoint on the right to* $\mathcal{Q}\colon \mathsf{RelSp} \to \mathsf{pFrm}$*. The adjoint functor* $\mathcal{P}\colon \mathsf{pFrm} \to \mathsf{RelSp}$ *is defined on objects and arrows by*

$$\mathcal{P}(A, B, \circleddash, \diamondsuit) = (\mathbb{P}_A, \mathfrak{O}_A, R_A) \tag{67}$$

$$\mathcal{P}(f, g)(p, b) = (p \circ f, \bigvee\{c \in B | g(c) \leqslant b\} \tag{68}$$

and the unit $\zeta_X\colon (X, \mathcal{O}, R) \to (\mathbb{P}_{\mathfrak{O}}, \mathfrak{O}_{\mathfrak{O}}, R_{\mathfrak{O}})$ *is defined by* $\zeta_X(x) = (p_x, W_x)$ *where*

$$p_x(U) = 1 \text{ iff } x \in U \tag{69}$$

$$W_x = \{\langle x', y\rangle \in R | x \neq x'\}^{\circ}. \tag{70}$$

Proof. Lemmas 28–33 show that we have constructed an adjoint to $\mathcal{Q}$. The functor and unit can be constructed by universality in the usual way. □

Corollary 34. *The adjunction of Theorem 5 restricts to a duality between those relational spaces which can be constructed from p-frames, and those p-frames which can be constructed from relational spaces.*

Proof. This is equivalent to showing that the two triangle identities are isomorphisms. One of these is immediate, since ϕ_A is always surjective, thus epi; but the triangle identity says that it is split monic. The other triangle identity says that $\mathcal{P}(\phi_A)\zeta_{\mathcal{P}(A)} = 1_{\mathbb{P}_A}$; we will show that $\zeta_{\mathcal{P}(A)}\mathcal{P}(\phi_A) = 1_{\mathbb{P}_{\mathfrak{O}_A}}$.

Let $(p, W) \in \mathbb{P}_{\mathfrak{O}_A}$, and let

$$x = \mathcal{P}(\phi_A)(p, W) = (p \circ \phi_A, \bigvee\{b \in B | \psi_A(b) \subseteq W\})$$

so $\zeta_{\mathcal{P}(A)}(x) = (p_x, W_x)$ where p_x and W_x are defined by (69) and (70) respectively.

Now, $p_x\phi_A(a) = 1$ iff $x \in \phi_A(a)$ iff $p\phi_A(a) = 1$ so $p = p_x$.

To show that $W = W_x$ we use the fact that the topology $\mathfrak{O}_{R_A}$ has the image of ψ_A as a basis.

$$\begin{aligned} \psi_A(b) \subseteq W_x &\iff \forall (q, c) \in \mathbb{P}_A.\ x\ R_A\ (q, c) \ \Rightarrow\ \langle x, (q, c)\rangle \notin \psi_A(b) \\ &\iff \forall (q, c) \in \mathbb{P}_A.\ x\ R_A\ (q, c) \ \Rightarrow\ (p \circ \phi_A \circleddash q)(b) = 0 \end{aligned}$$

If $\psi_A(b) \subseteq W$ then $b \leqslant \bigvee\{b \in B | \psi_A(b) \subseteq W\}$ so $x\ R_A\ (q, c) \ \Rightarrow\ (p \circ \phi_A \circleddash q)(b) = 0$ therefore $\psi_A(b) \subseteq W_x$.

Conversely, if $\psi_A(b) \nsubseteq W$ then $b \nleqslant \bigvee\{b \in B | \psi_A(b) \subseteq W\}$, so by (62) there is a point $(q, c) \in \mathbb{P}_A$ such that $x\ R_A\ (q, c)$ and $(p \circ \phi_A \circleddash q)(b) = 1$ i.e. $\psi_A b \nsubseteq W_x$. Therefore $W_x = W$. □

6 The Existence of Points

Corollary 34 shows that certain p-frames can be represented concretely as relational spaces. However, the conditions for this are essentially useless: they require us to know the space before we start. If we wish to prove a completeness theorem, we need to show that the p-frames constructed from Lindenbaum algebras of languages are of this form, which boils down to showing that these p-frames have *enough points*. This is the aim of this section.

The p-frame points defined in Section 5 are essentially inductive in character. Condition (62) implies that any putative point (p, b) requires the existence of many related points (q, c) which in turn require the existence of points related to them, and so on. In order to construct a particular point therefore, we need to construct a whole tree of related points. The proof given in this section achieves this by first constructing the frame points $p, q, \ldots$ and then using them to define the elements $b, c, \ldots$. An essential step in this construction is the problem of, given p, finding q such that $p \otimes q$ is a point of B. The first step in the proof is to define a frame where these points $p \otimes q$ live.

Definition. Let $(A, B, \otimes)$ be a relational frame, and p a point of A. We define the element $a_0 \in A$, the nucleus j_p on B and the frame B/p as follows:

- Let $a_0 = \bigvee\{a | p(a) = 0\}$;
- For each $a \in A$ such that $p(a) = 1$, let $j_a(b) = a \otimes \top \to b$;
- Let $j_p = \bigvee\{j_a \in N(B) | p(a) = 1\}$ (join taken in $N(B)$);
- Let $B/p = B_{j_p}$.

Lemma 35. *The points of B/p are in bijective correspondence with points r of B which satisfy $p \leqslant r \circ (\_ \otimes \top)$. Furthermore, $r \circ (\_ \otimes \top) = p$ iff $r(a_0 \otimes \top) = 0$.*

Proof. For any nucleus j on a frame B, the points of B_j are equivalent to points r of B which satisfy $r \circ j \leqslant r$, so we have only to show that $r \circ j_p \leqslant r$ iff $p \leqslant r \circ (\_ \otimes \top)$.

If $r \circ j_p \leqslant r$ and $p(a) = 1$ then $r(a \otimes \top) \geqslant rj_p(a \otimes \top) \geqslant rj_a(a \otimes \top) = 1$. Therefore $p \leqslant r \circ (\_ \otimes \top)$.

If $p \leqslant r \circ (\_ \otimes \top)$ and $p(a) = 1$ then $r(a \otimes \top) = 1$ so $rj_a(b) = rj_a(b) \wedge r(a \otimes \top) = r(a \otimes \top \to b \wedge a \otimes \top) \leqslant r(b)$, therefore $j_a \leqslant r_* r$. Since $r_* r$ is a nucleus, this means that $j_p \leqslant r_* r$, so $rj_p \leqslant r$.

If $r(a_0 \otimes \top) = 0$ and $p(a) = 0$ then $a \leqslant a_0$ so $r(a \otimes \top) \leqslant r(a_0 \otimes \top) = 0$. Therefore $p \geqslant r(\_ \otimes \top)$; the converse follows from the fact that $p(a_0) = 0$. $\square$

One of the main conditions used in this section is that for every point p of A, the frame B/p has enough points. It is useful to have some terminology for relational frames with this property.

Definition. A relational frame $(A, B, \circleddash)$ has *enough points along* $\circleddash$ if for each point p of A, the frame B/p has enough points.

The next lemma gives us plenty of p-frames with this property.

Lemma 36. *If $(A, B, \circleddash)$ is a relational frame such that B is stably locally compact and $\boxdot$ is continuous, then it has enough points along $\circleddash$.*

Proof. Let $b, c \in B/p$ such that $b \nleqslant c$. Define j_a as above, and let

$$T = \uparrow\{a \circleddash \top | p(a) = 1\} = \{b \in B | p(\boxdot b) = 1\}.$$

Since $\boxdot$ is continuous and preserves finite meets, T is a Scott-open filter; since B is locally compact, we can choose a Scott-open filter S containing b and disjoint from c; since B is stably locally compact, $T \wedge S$ is a Scott-open filter. If $c \in T \wedge S$ then there exist $a \in A$ and $b' \in S$ such that $p(a) = 1$ and $a \circleddash \top \wedge b' \leqslant c$, so $b' \leqslant a \circleddash \top \to c = j_a(c) \leqslant j_p(c) = c$, contradicting $c \notin S$.

Therefore $c \notin T \wedge S$ and we can choose a point r of B such that $r(c) = 0$, and for all $b' \in S \wedge T$, $r(b') = 1$. Now, if $p(a) = 1$ then $a \circleddash \top \in T$ so $r(a \circleddash \top) = 1$. Therefore $p \leqslant r \circ (\_ \circleddash \top)$, so r is a point of B/p by Lemma 35; and $r(c) = 0$ and $r(b) = 1$. $\square$

The following is a technical lemma which is used to show that certain elements of B are not identified by j_p.

Lemma 37. *Let $(A, B, \circleddash, \diamondplus)$ be a (pre-) p-frame such that $\diamondplus$ is continuous. If p is a point of A, j_p is as defined above and $c \in B$ satisfies $p(\diamondplus c) = 0$ then $p(\diamondplus j_p(c)) = 0$.*

Proof. First we give a more explicit construction of j_p by transfinite induction, as follows.

$$\begin{aligned} j_1(c) &= \bigvee\{j_a(c) | p(a) = 1\} && \text{(Note that this join is directed)} \\ j_{\alpha+1}(c) &= j_1(j_\alpha(c)) \\ j_\omega(c) &= \bigvee_{\alpha<\omega} j_\alpha(c) && \text{(For } \omega \text{ a limit ordinal)} \end{aligned}$$

This induction must reach a limit since B is a set: then $j_p = j_\beta$ where β is this limit.

Now we show by transfinite induction that if $p(\diamondsuit c) = 0$ then $p(\diamondsuit j_\alpha(c)) = 0$ for all c and for all ordinals α, and in particular for $\alpha = \beta$.

If $p(a) = 1$ then

$$\begin{aligned} p(\diamondsuit j_a(c)) &= p(a \wedge \diamondsuit j_a(c)) \\ &\leqslant p(\diamondsuit(a \oslash \top \wedge (a \oslash \top \to c))) \\ &\leqslant p(\diamondsuit c) \\ &= 0. \end{aligned}$$

Base case:

$$\begin{aligned} p(\diamondsuit j_1(c)) &= p\big(\diamondsuit \bigvee \{j_a(c) | p(a) = 1\}\big) \\ &= \bigvee \{p(\diamondsuit j_a(c)) | p(a) = 1\} \qquad \text{by continuity} \\ &= 0. \end{aligned}$$

Successor case: by the inductive hypothesis, $p(\diamondsuit j_\alpha(c)) = 0$ so

$$\begin{aligned} p(\diamondsuit j_{\alpha+1}(c)) &= p(\diamondsuit j_1(j_\alpha(c))) \\ &= 0 \qquad \text{by the base case.} \end{aligned}$$

Limit case:

$$\begin{aligned} p(\diamondsuit j_\omega(c)) &= p\big(\diamondsuit \bigvee_{\alpha<\omega} j_\alpha(c)\big) \\ &= \bigvee_{\alpha<\omega} p(\diamondsuit j_\alpha(c)) \qquad \text{by continuity} \\ &= 0 \qquad \text{by inductive hypothesis.} \end{aligned}$$

□

The next lemma can be thought of as the inductive step in the generation of a tree of points.

Lemma 38. *Let $(A, B, \oslash, \diamondsuit)$ be a p-frame such that $\diamondsuit$ is continuous, and let p be a point of A such that B/p has enough points. If $b \in B$ satisfies $p(\diamondsuit b) = 1$, then there exists a point q of A such that $p \oslash q$ is a point of B and $(p \oslash q)(b) = 1$.*

Proof. First note that $p(\diamondsuit(a_0 \oslash \top)) \leqslant p(a_0) = 0$ by (24), so $p(\diamondsuit j_p(a_0 \oslash \top)) = 0$ by Lemma 37. If $j_p(b) \leqslant j_p(a_0 \oslash \top)$ then

$$1 = p(\diamondsuit b) \leqslant p(\diamondsuit j_p(b)) \leqslant p(\diamondsuit j_p(a_0 \oslash \top)) = 0$$

which is a contradiction; therefore we can choose a point r of B/p such that $r(j_p(b)) = 1$ and $r(j_p(a_0 \oslash \top)) = 0$. Now rj_p is a point of B, and by Lemma 35, $rj_p \circ (\_ \oslash \top) = p$, so letting $q = rj_p \circ (\top \oslash \_)$ gives $rj_p = p \oslash q$, and hence $(p \oslash q)(b) = r(j_p(b)) = 1$. □

We are now ready to prove the essential 'existence of points' theorem, which allows us to construct p-frame points from frame points.

Lemma 39. *Let $(A, B, \circledcirc, \diamondsuit)$ be a p-frame with enough points along $\circledcirc$ such that $\diamondsuit$ continuous. For each point p of A there exists $b \in B$ such that (p, b) is a p-frame point.*

Proof. For each $c \in B$ such that $p(\diamondsuit c) = 1$, we can choose (by Lemma 38) a point q_c of A such that $p \circledcirc q_c$ is a point of B and $(p \circledcirc q_c)(c) = 1$. Let $b = \bigvee S$ where

$$S = \{d \in B | \forall c \in B.\ p(\diamondsuit c) = 1 \ \Rightarrow \ (p \circledcirc q_c)(d) = 0\}.$$

Now $b \in S$ since $p \circledcirc q_c$ preserves joins, so b is the greatest element of S, so

$$p(\diamondsuit b) = p(\diamondsuit \max_{d \in S} d) = \max_{d \in S} p(\diamondsuit d);$$

and if $p(\diamondsuit d) = 1$ then $(p \circledcirc q_d)(d) = 1$ so $d \notin S$. Therefore $p(\diamondsuit b) = 0$, i.e. (p, b) is a pre-point.

To show that (p, b) is a point, let P be the set of all pre-points which can be constructed from frame points in this way; we will show that P satisfies (62) so is a subset of the set of p-frame points.

If $(p, b) \in P$ and $d \not\leq b$ then $d \notin S$ so there is some $c \in B$ such that $p(\diamondsuit c) = 1$ and $(p \circledcirc q_c)(d) = 1$. By the above process, we can find $e \in B$ such that $(q_c, e) \in P$; we will show that $(p, b)\ R_B\ (q_c, e)$.

We know that $p \circledcirc q_c$ is a point of B, and

$$q_c(b) = \bigvee \{q_c(d') | \forall c' \in B.\ p(\diamondsuit c') = 1 \ \Rightarrow \ (p \circledcirc q'_c)(d') = 0\} = 0.$$

Therefore (p, b) is a p-frame point as required. $\square$

Just to show how far we have got in the proof of the duality theorem, we prove the following:

Proposition 40. *If $(A, B, \circledcirc, \diamondsuit)$ is a p-frame which has enough points along $\circledcirc$, such that $\diamondsuit$ is continuous, and such that A has enough points, then the unit $\phi_A \colon A \to \mathfrak{O}_A$ is a frame isomorphism.*

Proof. The unit ϕ_A is always surjective, so we have only to show that it reflects order. If $a_1 \not\leq a_0$ then since A has enough points, there is a frame point p of A such that $p(a_1) = 1$ and $p(a_0) = 0$. By Lemma 39, there is some b such that (p, b) is a p-frame point. But then $(p, b) \in \phi_A(a_1)$ and $(p, b) \notin \phi_A(a_0)$ so $\phi_A(a_1) \not\subseteq \phi_A(a_0)$. $\square$

The next lemma is an 'existence of points' theorem for related pairs of p-frame points.

Lemma 41. *Let* $(A, B, \oslash, \diamondplus)$ *be a p-frame with enough points along* $\oslash$ *such that* $\diamondplus$ *continuous. For each point* $p \oslash q$ *of* B *there exist* $b, c \in B$ *such that* (p, b) *and* (q, c) *are p-frame points and* $(p, b)\ R_A\ (q, c)$.

Proof. By Lemma 39 there exist $b_0, c \in B$ such that $(p, b_0), (q, c)$ are p-frame points. Let $b = \bigvee S$ where

$$S = \{d \in B | d \leqslant b_0\ \&\ (p \oslash q)(d) = 0\}.$$

Now $b \leqslant b_0$ so $p(\diamondplus b) = 0$ i.e. (p, b) is a pre-point, and $(p \oslash q)(b) = 0$ so $(p, b)\ R_A\ (q, c)$. We have only to show that (p, b) is a p-frame point.

If $d \nleqslant b$ then $d \notin S$ so either $d \nleqslant b_0$ or $(p \oslash q)(d) = 1$. In the first case, since (p, b_0) is a p-frame point there is some (q', c') such that $(p, b_0)\ R_A\ (q', c')$ and $(p \oslash q')(d) = 1$, but $q'(b) \leqslant q'(b_0) = 0$ so $(p, b)\ R_A\ (q', c')$. In the second case, $(p, b)\ R_A\ (q, c)$ and $(p \oslash q)(d) = 1$. □

We can now get a bit further in the proof of the duality theorem:

Proposition 42. *If* $(A, B, \oslash, \diamondplus)$ *is a p-frame which has enough points along* $\oslash$*, such that* $\diamondplus$ *is continuous, and such that* A *and* B *have enough points, then the unit* $\phi_A \colon B \to \mathfrak{O}_{R_A}$ *is a relational frame isomorphism.*

Proof. Proposition 40 shows that ϕ_A is a frame isomorphism, and ψ_A (defined by (64)) is always surjective, so we have only to show that it reflects order. If $b_1 \nleqslant b_0$ then since B has enough points, there is a frame point $p \oslash q$ of B such that $(p \oslash q)(b_1) = 1$ and $(p \oslash q)(b_0) = 0$. By Lemma 41, there are some $b, c \in B$ such that (p, b) and (q, c) are p-frame points and $(q, b)\ R_A\ (q, c)$. But then $\langle (p, b), (q, c) \rangle \in \psi_A(b_1)$ and $\langle (p, b), (q, c) \rangle \notin \psi_A(b_0)$ so $\psi_A(b_1) \nsubseteq \psi_A(b_0)$. □

In order to show that the unit is a p-frame isomorphism, we have to show that $\diamondplus$ is equivalent to $\diamondplus_{R_A}$. For this we need a variant of Lemma 38 which puts stronger conditions on the related point q; to prove which we seem to need stronger conditions on the original point p.

Lemma 43. *Let* $(A, B, \oslash, \diamondplus)$ *be a p-frame such that* $\diamondplus$ *is continuous, and let* $a \in A$ *and* $b \in B$*. If* p *is a point of* A *which is* maximal *such that* $p(a) = 1$ *and* $p(\diamondplus b) = 0$*, and if* B/p *has enough points, then for all* $c \in B$ *such that* $p(\diamondplus c) = 1$*, there exists a point* q *of* A *such that* $p \oslash q$ *is a point of* B*,* $(p \oslash q)(c) = 1$ *and* $(p \oslash q)(b) = 0$.

Proof. First note that $p(\diamondplus(b \vee \diamondplus b \oslash \top)) \leqslant p(\diamondplus b) = 0$ by (25), so $p(\diamondplus j_p(b \vee \diamondplus b \oslash \top)) = 0$ by Lemma 37. If $j_p(c) \leqslant j_p(b \vee \diamondplus b \oslash \top)$ then

$$1 = p(\diamondplus c) \leqslant p(\diamondplus j_p(c)) \leqslant p(\diamondplus j_p(b \vee \diamondplus b \oslash \top)) = 0$$

which is a contradiction; therefore we can choose a point r of B/p such that $r(j_p(c)) = 1$ and $r(j_p(b \vee \diamond b \otimes \top)) = 0$. Now $rj_p \circ (\_ \otimes \top)$ is a point of A, and by Lemma 35, $rj_p \circ (\_ \otimes \top) \geqslant p$, but $rj_p(a \otimes \top) \geqslant p(a) = 1$ and $rj_p(\diamond b \otimes \top) = 0$, so $rj_p \circ (\_ \otimes \top) = p$ by maximality of p.

Finally, letting $q = rj_p \circ (\top \otimes \_)$ gives $rj_p = p \otimes q$, and hence $(p \otimes q)(c) = r(j_p(c)) = 1$ and $(p \otimes q)(b) = r(j_p(b)) = 0$. □

We can now prove the last of the 'existence of points' lemmas.

Lemma 44. *Let $(A, B, \otimes, \diamond)$ be a p-frame with enough points along $\otimes$, such that A has enough points, and such that $\diamond$ is continuous. If $a \in A$ and $d \in B$ such that $a \nleqslant \diamond d$, then there is a p-frame point (p, b) such that $p(a) = 1$ and $b \geqslant d$.*

Proof. Consider the set of points p of A such that $p(a) = 1$ and $p(\diamond d) = 0$: since A has enough points, this set is non-empty, and it is closed under directed joins, so we can use Zorn's lemma to choose a point p of A which is *maximal* such that $p(a) = 1$ and $p(\diamond d) = 0$. For each $c \in B$ such that $p(\diamond c) = 1$, we can choose (by Lemma 43) a point q_c of A such that $p \otimes q_c$ is a point of B and $(p \otimes q_c)(c) = 1$ and $(p \otimes q_c)(d) = 0$.

As in Lemma 39, let $b = \bigvee S$ where

$$S = \{e \in B | \forall c \in B.\ p(\diamond c) = 1 \ \Rightarrow\ (p \otimes q_c)(e) = 0\}$$

so (p, b) is a pre-point. Also $d \in S$ since $(p \otimes q_c)(d) = 0$, so $b \geqslant d$.

If $e \nleqslant b$ then $e \notin S$ so there is some $c \in B$ such that $p(\diamond c) = 1$ and $(p \otimes q_c)(e) = 1$. By Lemma 39, we can find $b' \in B$ such that (q_c, b') is a p-frame point; as in Lemma 39, $(p, b)\ R_B\ (q_c, b')$. □

Finally we have the isomorphism required for duality.

Proposition 45. *If $(A, B, \otimes, \diamond)$ is a p-frame which has enough points along $\otimes$, such that $\diamond$ is continuous, and such that A and B have enough points, then the unit $\phi_A \colon B \to \mathfrak{O}_{R_A}$ is a p-frame isomorphism.*

Proof. Proposition 42 shows that ϕ_A is a relational frame isomorphism, and it is always the case that $\phi_A(\diamond b) \subseteq \diamond_{R_A}\psi_A(b)$ so we have only to show the converse. If $\phi_A(a) \nsubseteq \phi_A(\diamond b)$ then by Proposition 40 $a \nsubseteq \diamond b$ so by Lemma 44 there is some p-frame point (p, c) such that $p(a) = 1$ and $b \leqslant c$. Then $(p, c) \in \phi_A(a)$, but if $(p, c)\ R_A\ (q, d)$ then $(p \otimes q)(b) \leqslant (p \otimes q)(c) = 0$ so $\langle (p, c), (q, d) \rangle \notin \psi_A(b)$, therefore $(p, c) \notin \diamond_{R_A}\psi_A(b)$. □

We summarise the results of this section in the following theorem.

Theorem 6. *The category of p-frames with enough points in A, B and along ⧁, and with continuous ⟐ is dual to a full subcategory of the category of relational spaces.*

If we define a *continuous p-frame* as a p-frame $(A, B, ⧁, ⟐)$ such that A is locally compact, B is stably locally compact, and $\boxdot$ and ⟐ are continuous, then Lemma 36 gives us the following corollary.

Corollary 46. *The category of continuous p-frames is dual to a full subcategory of the category of relational spaces.*

7 Conclusions

The aim of this paper, as presented in the introduction, was to factorise the contravariant adjunction and duality between relational spaces and modal frames through some intermediate category, in order to separate the question of the existence of points from the Kripke semantics of the modal connectives. We have introduced the category of p-frames, and used it to give such a factorisation, and hence a finer analysis of the adjunction and duality. We have found reasonable continuity conditions under which modal frames can be represented as p-frames, and reasonable continuity and compactness conditions under which p-frames can be represented as relational spaces. Indeed, the paper provides condsiderable evidence to support the proposal that p-frames are the "point-free" version of relational spaces.

The link between modal frames and p-frames is fairly straightforward, at least in the continuous case. Although the construction of the universal (pre-) p-frame on a (pre-) modal frame by the method of coverages is fairly complex, we have shown that the properties of the resulting p-frame can often be analysed by constructing particular $C^{\Box}$-ideals. It remains to be seen whether this technique can be extended beyond the continuous and co-continuous cases.

The link between p-frames and relational spaces is perhaps disappointing, in that the construction of a space from a p-frame is just as complex as the original construction of a space from a modal frame. However, the structure of p-frames, with the operation ⟐ almost independent of ⧁, goes some way to explain the need for this complexity. Indeed, it is rather suprising that the frame and both operations can be represented by a single relation on a space.

The representation of modal frames as p-frames allows us to give a semantics of modal logic which makes no mention of worlds, but which can nonetheless be considered a generalisation of the Kripke semantics. It remains to be seen whether this point-free semantics can be extended to properties such as reflexivity, symmetry, transitivity, confluence etc. which are easily characterised in classical modal logic, and can be defined categorically.

As a side effect of the mathematical development presented in this paper, we have introduced pre-modal frames, and shown that they can be represented (at least in the continuous case) as pre-p-frames. This allows us to extend our semantics of modal logic to calculi which do not satisfy $\Diamond\bot = \bot$, which have been considered in the literature. We have not presented any concrete examples of pre-p-frames (other than p-frames) but it seems likely that relational spaces with a designated set of "inconsistent points" (similar to the "inconsistent worlds" of [1]) could be used to generate such examples.

The structure of a p-frame $(A, B, \circledgtr, \diamonddot)$ suggests the possibility of a logical calculus with *two* classes of propositions, corresponding to the two frames A and B. The link between p-frames and relational spaces gives a Kripke-like semantics for such logics, where the A-propositions are interpreted over worlds in the usual way, and the B-propositions are interpreted over *related pairs* of worlds. Although this does not seem a very useful extension to propositional modal logic, in the case of first-order modal logic there is a natural predicate "corresponds to", satisfied by corresponding elements in related worlds, which is of precisely this form. This possibility will be developed in a later paper.

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