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# Semantics and Proof Theory of an Intuitionistic Modal Sequent Calculus

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### Abstract

We prove soundness and adequacy for an intuitionistic modal sequent calculus with the modal Heyting algebra semantics presented in Hilken [7]. We produce a cut-elimination for this calculus. For comparison a description of a corresponding classical modal logic in a sequent style is given along with its semantics.

# 1 Introduction

Modal logics have found many applications in computer science. In most cases the logics have been classical and have been used to descibe properties of relational structures. In other words the logics have been analysed relative to suitable Kripke relational semantics.

In investigating intuitionistic modal logic there are two main motivations. One is purely philosophical, that intuitionists can have suitable definitions of modal operators and valid proofs in acceptable calculi. The other is semantical, certain problems have arisen for which descriptions in a logical language require that the logic has modal connectives but is also intuitionistic.

The recent work of Hilken [7] provides a new synthesis of Kripke semantics and the topological semantics of intuitionistic logic. It establishes a correspondence between two semantic notions for intuitionistic modal logic - modal Heyting algebras and topological relational spaces. It gives a contravariant equivalence between a category of topological spaces with relations and a category of modal frames (frames with modal operators satisfying four axioms). A frame is a partially ordered set with finite meets, arbitrary joins satisfying the infinite distributivity law. Defining a modal Heyting algebra to be a Heyting algebra with modal operators satisfying the same four axioms as modal frames, it also gives an equivalence between the category of modal frames and a category of modal Heyting algebras.

Wijesekera [12] gives sequent calculus formulations for both intuitionistic and classical predicate modal logic. The intuitionistic calculus he proves is sound and complete with respect to his chosen semantics (which are different to ours.) Dropping his quantifier rules and making a few other slight alterations we arrive at the calculi presented here in Sections 2 and 3.

In Section 2 we review classical monomodal logic. We give a sequent calculus for a language with one box connective and one (interdefinable) diamond connective. The semantics are given in terms of relational structures and the modal Boolean algebras they generate in a similar fashion to [9]. In Section 3 a language for intuitionistic logic with one box and one diamond operator is set down along with a calculus. Modal Heyting algebras are described and used to give the semantics of the logic. The soundness and completeness results for the intuitionistic logic are proved in Section 4, although some of the details are in the appendix. Section 5 concerns the cut rule which is shown to be eliminable.

# 2 Classical Modal Logic

In this Section we introduce a classical modal language and a classical modal sequent calculus similar to that of Wijesekera [12]. In addition we describe the associated semantics. Although the material in this Section is of independent interest, the most important reason for including it is that the intuitionistic modal calculus (which is the main subject of this report) is derived from the classical calculus.

## 2.1 The Language

Throughout this report we will be using modal propositional languages. In Section 1 we are dealing with classical logic and our language will have the following symbols:

- Variables $P$, $Q$, $R$, $\dots$ ;
- Constants $\top$, $\bot$ ;
- Propositional connectives $\neg$, $\wedge$, $\vee$ ;

- A box connective $\Box$ ;
- A diamond connective $\Diamond$.

The $\Box$ and $\Diamond$ connectives are known as the modal connectives. The modal connectives extend the propositional language consisting of variables, constants and propositional connectives as given above. The language described above is monomodal - it has only one $\Box$ and one $\Diamond$ connective. It is also possible to consider polymodal languages in which many modal connectives are present but this is not done here.

The formulas of the language are built up recursively from the variables using the following rules.

- If $P$ is a variable then $P$ is a formula.
- The constants $\top$ and $\bot$ are formulas.
- If $\phi$ is a formula then $\neg\phi, \Box\phi, \Diamond\phi$ are all formulas.
- If $\phi$ and $\psi$ are formulas then so are $\phi \wedge \psi$ , $\phi \vee \psi$.

Formulas can be identified with the tree structure corresponding to their construction. Parentheses $($, $)$ will be used where appropriate to make the tree structure of formulas explicit. They are not considered to be part of the language.

Call the set of variables $VAR$ and the set of formulas $FORM$.

## 2.2 Proofs

Define a *bag* of formulas, $\Gamma$, to be a finite set of formulas together with a function $f_\Gamma : FORM \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers including 0. In other words a bag is is a finite collection of formulas with repetitions of the same formula allowed and counted. The *number of occurrences* of a formula $\phi$ is the value $f_\Gamma(\phi)$. We will use the notation $\Gamma, \phi$ for the bag of formulas obtained by adding one more occurrence of the formula $\phi$ to the bag $\Gamma$. With this convention we may write a bag of formulas as an unordered list possibly containing repetitions. We use the notation $\Gamma, \Delta$ to mean the bag of formulas consisting of all the occurrences of every formula from the bags $\Gamma$ and $\Delta$ - so $f_{\Gamma,\Delta}(\phi) = f_\Gamma(\phi) + f_\Delta(\phi)$. The notation $\Box\Gamma$ is used to indicate the bag of formulas consisting of each occurrence of each formula of the bag $\Gamma$ written with a $\Box$ connective in front of it - if $\phi$ is a formula with $f_\Gamma(\phi) = n$ then $f_{\Box\Gamma}(\Box\phi) = n$. Similarly for the bag $\Diamond\Gamma$, if $\phi$ is a formula with $f_\Gamma(\phi) = n$ then $f_{\Diamond\Gamma}(\Diamond\phi) = n$.

A *judgement* is a string of symbols $\Gamma \vdash \Delta$ where $\Gamma$ and $\Delta$ are bags of formulas. Judgements are also referred to as sequents. Judgements arise in proofs. A *proof* is a tree of judgements constructed using one of the proof rules below to generate leaves.

### 2.2.1 Proof Rules

| | |
|---|---|
| **Axiom** | $\Gamma, \phi \vdash \phi, \Delta$ |
| **Weakening** | $\dfrac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}$ |
| **Contraction-left** | $\dfrac{\Gamma, \phi, \phi \vdash \Delta}{\Gamma, \phi \vdash \Delta}$ |
| **Contraction-right** | $\dfrac{\Gamma \vdash \psi, \psi, \Delta}{\Gamma \vdash \psi, \Delta}$ |
| **$\neg$-left** | $\dfrac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$ |

**¬-right**
$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

**⊤-left**
$$\frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}$$

**⊤-right**
$$\Gamma \vdash \top, \Delta$$

**∧-left**
$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

**∧-right**
$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}$$

**⊥-left**
$$\Gamma, \bot \vdash \Delta$$

**⊥-right**
$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

**∨-left**
$$\frac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$$

**∨-right**
$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}$$

**◇-left**
$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma', \Box\Gamma, \Diamond\phi \vdash \Diamond\Delta, \Delta'}$$

**□-right**
$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma', \Box\Gamma \vdash \Box\phi, \Diamond\Delta, \Delta'}$$

**Cut**
$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma', \phi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

The *antecedent* of a judgement is the bag of formulas on the left-hand-side of the judgement symbol, $\vdash$. The *succedent* is the bag of formulas on the right-hand-side of the judgement. A *premise* of a given judgement is a judgement which occurs immediately above the given judgement in the proof which contains it. The rules above mean that every judgement has either zero, one or two premises. Within this report, proofs will also sometimes be called *derivations* and we will say that we can *derive* a certain judgement. Sometimes the words "there is a proof of" have been omitted before a judgement is written. It should be reasonably clear in such circumstances that what is meant is that there is a proof of the judgement. The notation $\Gamma \nvdash \Delta$ is also occasionally used, and in this case what is intended is that there can be no proof of the judgement $\Gamma \vdash \Delta$.

## 2.3 Semantics

We interpret the consequences of these judgements using modal Boolean algebras and transition structures. A *transition structure* $\mathcal{A} = (\mathcal{A}, \prec)$ is a structure consisting of a non-empty set $A$ and a binary relation $\prec$. The *modal Boolean algebra* generated by $\mathcal{A}$ is $(\wp A, \Box, \Diamond, \subseteq, \cap, \cup, \neg, A, \emptyset)$, a Boolean algebra with the operations $\neg$, $\cup$, $\cap$ defined as complementation,union and intersection, with partial order $\subseteq$ and top and bottom elements $A$ and $\emptyset$ respectively. The operation $\Box$ is defined by $\Box : \wp A \to \wp A;\ X \mapsto \Box X$ where

$$a \in \Box X \Leftrightarrow (\forall x)(x \prec a \Rightarrow x \in X) \quad .$$

Similarly the operation $\Diamond : \wp A \to \wp A;\ X \mapsto \Diamond X$ is defined by

$$a \in \Diamond X \Leftrightarrow (\exists x)(x \prec a \quad \& \quad x \in X) \quad .$$

It follows from these definitions that the maps $\Box$ and $\Diamond$ are monotone -
if $X \subseteq Y$ then $\Box X \subseteq \Box Y$ and $\Diamond X \subseteq \Diamond Y$.

Given a modal Boolean algebra we can recover the original transition structure from which it came. This is done using the following statement, which holds for all $a, b \in A$ :

$$b \prec a \quad \Leftrightarrow \quad (\forall X)(a \in \Box X \Rightarrow b \in X) \quad .$$

From now on we will take the word "structure" to mean transition structure. A *valuation* $\alpha$ on a structure $\mathcal{A}$ is an assignment $\alpha : \mathit{VAR} \to \wp A;\ P \mapsto \alpha(P)$. The pair $(\mathcal{A}, \alpha)$ is called a *valued structure.* Each such valuation has a natural extension to a function, $[\![\cdot]\!]_\alpha : \mathit{FORM} \to \wp A$, from formulas to modal Boolean algebras.

Let $(\mathcal{A}, \alpha)$ be a given valued structure. For each formula $\phi$ the subset $[\![\phi]\!]_\alpha$ of $A$ is defined by induction on the complexity of $\phi$ using the following clauses.

- (constants) $\quad [\![\top]\!]_\alpha = A \quad , \quad [\![\bot]\!]_\alpha = \emptyset$
- (variables) $\quad [\![P]\!]_\alpha = \alpha(P)$
- (formulas $\theta, \psi$) $\quad [\![\theta \vee \psi]\!]_\alpha = [\![\theta]\!]_\alpha \cup [\![\psi]\!]_\alpha$
- (formulas $\theta, \psi$) $\quad [\![\theta \wedge \psi]\!]_\alpha = [\![\theta]\!]_\alpha \cap [\![\psi]\!]_\alpha$
- (formulas $\phi$) $\quad [\![\Box\phi]\!]_\alpha = \Box([\![\phi]\!]_\alpha)$
- (formulas $\phi$) $\quad [\![\Diamond\phi]\!]_\alpha = \Diamond([\![\phi]\!]_\alpha)$.

The calculus given is sound and complete with respect to such Boolean algebras. See Collinson [1].

## 3 Intuitionistic Modal Logic

The formal system is now adapted to give an intuitionistic version of a modal sequent calculus. This is a system designed to prove propositions with no external notion of their truth or falsity. The language of this intuitionistic modal logic has an implication symbol, $\to$. Negation is defined in terms of implication, and the constant $\bot$, defining $\neg\phi$ as $\phi \to \bot$ for each formula $\phi$.

Sequents occurring in intuitionistic sequent proofs contain *at most* one formula on the right-hand-side, see [5]. In the calculus presented below all judgements have *exactly* one formula on the right-hand-side. The constant $\bot$ is used instead of an empty succedent, so that there are sequents of the form $\Gamma \vdash \bot$ rather than $\Gamma \vdash$ . The system given here is similar to that of Wijesekera [12].

With such logics the law of the excluded middle (the judgement $\vdash \phi \vee \neg\phi$) cannot be derived. Double negation ( $\vdash (\neg(\neg\phi)) \to \phi$) fails as does one of the de Morgan laws, the suspect case being $\neg(\phi \wedge \psi) \vdash \neg\phi \vee \neg\psi$.

The set of variables is named *VAR* as before. The formulas of the language are built up recursively using the following clauses:

- if $P$ is a variable then $P$ is a formula
- $\bot$ and $\top$ are formulas
- if $\phi$ is a formula then $\Box\phi$ and $\Diamond\phi$ are formulas
- if $\phi$ and $\psi$ are formulas then $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \to \psi$ are formulas.

The new set of formulas is called *FORM.* As in the classical case, parentheses are used in formulas to make the tree structure explicit.

## 3.1 Proofs

A *bag of formulas* is defined as before. The bags $\Box\Gamma$ and $\Diamond\Gamma$ and $\Gamma, \Delta$ and $\Gamma, \phi$ are defined as before. In addition we define $\phi^n$ to be the bag of formulas consisting of $n$ occurrences of the formula $\phi$. A *judgement* is a string of symbols $\Gamma \vdash \phi$ where $\Gamma$ is a bag of formulas and $\phi$ is a formula. The judgements arise in proofs. A *proof* is a tree of judgements generated using the proofs rules listed below.

### 3.1.1 Proof Rules

**Axiom**

$$\Gamma, \phi \vdash \phi$$

**Weakening**

$$\frac{\Gamma \vdash \phi}{\Gamma', \Gamma \vdash \phi}$$

**Contraction**

$$\frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}$$

**$\top$-left**

$$\frac{\Gamma \vdash \phi}{\Gamma, \top \vdash \phi}$$

**$\top$-right**

$$\Gamma \vdash \top$$

**$\wedge$-left**

$$\frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \wedge \psi \vdash \theta}$$

**$\wedge$-right**

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

**$\bot$-left**

$$\Gamma, \bot \vdash \phi$$

**$\vee$-left**

$$\frac{\Gamma, \phi \vdash \theta \qquad \Gamma, \psi \vdash \theta}{\Gamma, \phi \vee \psi \vdash \theta}$$

**$\vee$-right-1**

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}$$

**$\vee$-right-2**

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

**$\rightarrow$-right**

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

**$\rightarrow$-left**

$$\frac{\Gamma \vdash \phi \qquad \Gamma, \psi \vdash \theta}{\Gamma, \phi \rightarrow \psi \vdash \theta}$$

**$\Diamond$-left-1**

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma', \Box\Gamma, \Diamond\phi \vdash \Diamond\psi}$$

**$\Diamond$-left-2**

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma', \Box\Gamma, \Diamond\phi \vdash \psi}$$

**$\Box$-right**

$$\frac{\Gamma \vdash \phi}{\Gamma', \Box\Gamma \vdash \Box\phi}$$

**Cut**

$$\frac{\Gamma \vdash \phi \qquad \Gamma', \phi \vdash \psi}{\Gamma, \Gamma' \vdash \psi}$$

There are two important derived rules that will be used in the following sections. The first is a strengthened contraction rule. This simply combines several contractions, possibly on more that one formula, into one contraction. Suppose that we have bags of formulas $\Gamma, \Gamma', \Gamma''$ where $\Gamma'$ and $\Gamma''$ contain the same formulas but not necessarily in the same numbers, and suppose that we have a formula $\phi$ and a judgement $\Gamma, \Gamma', \Gamma'' \vdash \phi$ then the strengthened contraction rule allows us to make the proof

$$\frac{\Gamma, \Gamma', \Gamma'' \vdash \phi}{\Gamma, \Gamma' \vdash \phi} \qquad .$$

The second derived rule is the mix rule (sometimes also called multicut) which looks like

$$\frac{\Gamma \vdash \phi \qquad \Gamma', \phi^n \vdash \psi}{\Gamma, \Gamma' \vdash \psi}$$

where $\Gamma$ and $\Gamma'$ are bags of formulas and $\phi$ and $\psi$ are formulas and $n$ is any integer greater than 0.

Notice that given all the other rules of the formal system, the presence of the mix rule is equivalent to the presence of the cut rule. The cut rule is clearly just the special case, $n = 1$, of the mix rule. The mix rule is derived from the cut rule, for each $n$, by $n$ applications of the cut rule and some contractions.

In the proofs of soundness, completeness and cut elimination to follow one of the methods used most is to check cases of which sort of rule some given instance of a rule is. It is not necessary when checking cases to deal with derived rules since they can at all times be replaced by combinations of the rules of the formal system.

## 3.2 Modal Heyting Algebras

The following definition comes from Hilken [7]. Define a *Modal Heyting Algebra* to be a partially ordered set $(H, \leq)$ with two constants $\top$ and $\bot$, binary operations $\wedge$, $\vee$, $\rightarrow$ and unary operations $\neg$, $\Box$, $\Diamond$ all satisfying the following axioms for all $a$, $b$, $c$ in $H$.

1. $a \wedge b \leq a$ and $a \wedge b \leq b$
2. $c \leq a \quad \& \quad c \leq b \quad \Rightarrow \quad c \leq a \wedge b$
3. $a \leq a \vee b$ and $b \leq a \vee b$
4. $a \leq c \quad \& \quad b \leq c \quad \Rightarrow \quad a \vee b \leq c$
5. $a \wedge (a \rightarrow b) \leq b$
6. $a \wedge c \leq b \quad \Rightarrow \quad c \leq a \rightarrow b$
7. $\bot \leq a$
8. $a \leq \top$
9. $\top \leq \Box\top$
10. $\Box a \wedge \Box b \leq \Box(a \wedge b)$
11. $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$
12. $\Diamond\bot \leq \bot$
13. $a \leq b \quad \Rightarrow \quad \Box a \leq \Box b$

14. $a \leq b \quad \Rightarrow \quad \Diamond a \leq \Diamond b$

**Comments**

The elements $\top$ and $\bot$ are known as the top and bottom elements of the algebra. Although $\top$ is given here as a constant this is not necessary, it can be defined as $\bot \to \bot$. The axioms 1 and 2 make the operation $\wedge$ a greatest lower bound (g.l.b) - for any two element set $\{a, b\}$ the element $a \wedge b$ is the greatest element of the algebra which is less than or equal to both $a$ and $b$. The axioms 3 and 4 make the operation $\vee$ a least upper bound (l.u.b) - for any two element set $\{a, b\}$ the element $a \vee b$ is the least element which is greater than or equal to both $a$ and $b$. The operation $\to$ is called pseudocomplementation. Axioms 1-8 make $(H, \leq)$ a pseudocomplemented lattice or Heyting algebra.

The remaining axioms describe the properties which we require for the modal connectives (of the intuitionistic modal logic described). Axioms 13 and 14 stipulate that $\Box$ and $\Diamond$ must be monotonic. Axioms 9 through to 12 are the four axioms considered by Hilken [7].

### 3.2.1 Valuation

A *valuation* is an assignment $\alpha : VAR \to H$ where $H$ is the underlying set of some modal Heyting algebra. Such a valuation is extended to formulas with $[\![\cdot]\!]_\alpha : FORM \to H$ defined as follows.

- $[\![\bot]\!]_\alpha = \bot \quad , \quad [\![\top]\!]_\alpha = \top$
- $[\![P]\!]_\alpha = \alpha(P)$ for all variables P
- $[\![\phi \wedge \psi]\!]_\alpha = [\![\phi]\!]_\alpha \wedge [\![\psi]\!]_\alpha$
- $[\![\phi \vee \psi]\!]_\alpha = [\![\phi]\!]_\alpha \vee [\![\psi]\!]_\alpha$
- $[\![\phi \to \psi]\!]_\alpha = [\![\phi]\!]_\alpha \to [\![\psi]\!]_\alpha$
- $[\![\Box\phi]\!]_\alpha = \Box[\![\phi]\!]_\alpha$
- $[\![\Diamond\phi]\!]_\alpha = \Diamond[\![\phi]\!]_\alpha$

We drop the subscript $\alpha$ on the the valuation $[\![\cdot]\!]_\alpha$ when it is clear what the valuation is.

Some properties of modal Heyting algebras to be used later are given in the appendix.

# 4 Completeness

We prove soundness and completeness results for the intuitionistic sequent calculus with the semantics of the previous section.

## 4.1 Soundness

**Lemma 4.1** If $\phi_1, \ldots, \phi_n \vdash \phi$ is derivable then for every modal Heyting algebra $H$ and every valuation $\alpha : VAR \to H$,

$$[\![\phi_1]\!]_\alpha \wedge [\![\phi_2]\!]_\alpha \wedge \ldots \wedge [\![\phi_n]\!]_\alpha \leq [\![\phi]\!]_\alpha.$$

**Proof**

See the appendix.

## 4.2 Adequacy

We now give the adequacy result for this intuitionistic logic and the stated notion of modal Heyting algebra. We outline a proof consisting of the construction of a model from the syntax. The details of the proof can be found in the appendix.

**Lemma 4.2** Let $\phi_1, \ldots, \phi_n$ be a list of formulas and $\psi$ be a formula. If for every modal Heyting algebra $H$ and every valuation $\alpha : VAR \to H$ we have $[\![\phi_1]\!]_\alpha \wedge \ldots \wedge [\![\phi_n]\!]_\alpha \leq [\![\psi]\!]_\alpha$ then there is a proof of $\phi_1, \ldots, \phi_n \vdash \psi$.

Define the interderivabilty equivalence relation $\sim$ on $FORM$ by

$$\phi \sim \psi \qquad \Leftrightarrow \qquad \phi \vdash \psi \quad \text{and} \quad \psi \vdash \phi \quad \text{are provable}$$

Denote the equivalence class of any formula $\phi$ by $[\phi]$.
Define $\mathcal{L}'$ to be the set of equivalence classes of $FORM$.

$$\mathcal{L}' = \frac{FORM}{\sim}$$

**Lemma 4.3** $\mathcal{L}'$ is a modal Heyting algebra with the following definitions :

$$\begin{array}{rcl}
\bot & = & [\bot] \\
\top & = & [\top] \\
[\phi] \vee [\psi] & = & [\phi \vee \psi] \\
[\phi] \wedge [\psi] & = & [\phi \wedge \psi] \\
[\phi] \to [\psi] & = & [\phi \to \psi] \\
\Box[\phi] & = & [\Box\phi] \\
\Diamond[\phi] & = & [\Diamond\phi]
\end{array}$$

and the partial order $\leq$ given by

$$[\phi] \leq [\psi] \qquad \Leftrightarrow \qquad [\phi] \wedge [\psi] = [\phi] \ .$$

Define the relation $\geq$ on $\mathcal{L}'$ by $\quad b \geq a \;\Leftrightarrow\; a \leq b$.

**Proof**
See the appendix.

We now proceed to define a valuation $\alpha : VAR \to H$ by $\alpha(P) = [P]$ and extend in the usual way to a function $[\![\cdot]\!]_\alpha : FORM \to H$.

**Lemma 4.4** For each formula $\phi$,

$$[\![\phi]\!]_\alpha = [\phi] \ .$$

**Proof**
The proof is by induction on the complexity of formulas. See the appendix for the details.

We now prove the following lemma.

**Lemma 4.5** If $[\![\phi]\!] \leq [\![\psi]\!]$ then there is a proof of $\phi \vdash \psi$.

**Proof**
For the proof see the appendix.

Since $[\![\phi_1]\!] \wedge \ldots \wedge [\![\phi_n]\!] = [\![\phi_1 \wedge \ldots \wedge \phi_n]\!]$, we have shown that
if $[\![\phi_1]\!] \wedge \ldots \wedge [\![\phi_n]\!] \leq [\![\psi]\!]$ then there is a proof of $\phi_1 \wedge \ldots \wedge \phi_n \vdash \psi$. To complete the proof of adequacy we need to show there is a proof of $\phi_1, \ldots, \phi_n \vdash \psi$. This is achieved by the following lemma.

**Lemma 4.6** For any formulas $\phi_1$, $\phi_2$, ... ,$\phi_n$ the following hold

(i) We can construct a proof of $\phi_1, \ldots, \phi_n \vdash \phi_1 \wedge \ldots \wedge \phi_n$.

(ii) If we have a proof of $\phi_1 \wedge \ldots \wedge \phi_n \vdash \psi$. then we can construct a proof of $\phi_1, \ldots, \phi_n \vdash \psi$.

**Proof**

See the appendix.

For the reasons noted above this completes the proof of adequacy.

**Comment**

The proof of adequacy could have been shortened by defining the partial order in Lemma 4.3 to be

$$[\phi] \leq [\psi] \quad \Leftrightarrow \quad \phi \vdash \psi \ .$$

If we do this then we have to check the partial order is well-defined. However once this is done the proof that $\mathcal{L}'$ is a modal Heyting algebra becomes easier and the result of Lemma 4.5 follows immediately from Lemmas 4.3 and 4.4.

# 5 Cut Elimination

The intuitionistic modal sequent calculus presented is just as powerful with the cut rule removed. In other words, any judgement that can be proved using the cut rule in the intuitionistic formal system given can also be proved without using the cut rule.

**Theorem 5.1**

The cut rule is redundant in the intuitionistic modal sequent calculus.

This is a direct consequence of Proposition 5.1 below. There are several different ways of proving cut-elimination for a sequent calculus. Most important is the divide between semantic and purely syntactic cut-eliminations.

The semantic version of cut-elimination consists of checking that the reduced calculus, with the cut rule removed, is still adequate when the same valuations as before are applied. Notice that if the calculus with cut is sound then the system without cut is automatically sound also.

The method of proof given by the lemma and proposition to follow is a syntactic one. There is a choice to be made between proving cut-elimination directly or proving an equivalent, mix-elimination. This depends on whether the calculus is to use sequences, bags or sets of formulas. It is often easier to prove mix-elimination for the first two due to the presence of structural rules, but for sets no mix-rule exists since there can be only one occurrence of a formula in a set. The formula removed by a cut is known as the *cut formula* and the formula removed by a mix is known as the *mix formula*.

Syntactic cut-elimination and mix-elimination proofs usually involve two nested inductions, and the consideration of all cases of rules. Exactly how the induction proceeds and how the cases are collected together depends very much upon the exact form of the calculus and the efforts of the author to make the proof both concise and exhaustive. One of the inductions usually takes place upon some quantity measuring the complexity of formulas where complexity is related to the number of connectives appearing in formulas - in the cut-elimination to be given this will be called the degree. The sub-induction typically takes place on some quantity related to the number of judgements it takes to introduce the cut (mix) formula - in the proof given below this is the height of the proof. (In many examples, for instance Gentzen's original, the rank of a proof is used instead of the height, where the rank is defined to be the maximum number of judgements before the final application of the mix rule in which the mix formula appears. Gentzen proves mix-elimination rather than cut-elimination, see [5].)

We will proceed by proving the eliminability of the mix rule rather than the cut rule. As noted previously the presence of this derived rule is equivalent to the presence of the cut rule given the other rules of the formal system.

For each $n$, the bag of formulas containing $n$ occurrences of the formula $\phi$ and no others will be denoted by $\phi^n$. Notice that $\Box\phi^n$ and $\Diamond\phi^n$ can be written unambiguously.

Define the degree of a formula by:

- $\partial(\top) = \partial(\bot) = 1$
- $\partial(P) = 1 \quad$ if $P$ is a variable
- $\partial(\phi \wedge \psi) = \partial(\phi \vee \psi) = \partial(\phi \to \psi) = \max\ (\partial(\phi), \partial(\psi)) + 1$
- $\partial(\Box\phi) = \partial(\Diamond\phi) = \partial(\phi) + 1$ .

Define the degree of a mix rule to be the degree of the formula it removes.
The degree $d(\pi)$ of a proof $\pi$ is defined as the supremum of the degrees of its mix rules. So $d(\pi) = 0$ iff $\pi$ is mix (and therefore also cut) free.
The height $h(\pi)$ of a proof is that of its associated tree: if $\pi$ ends in a rule whose premises are proved by $\pi_1, \pi_2, \ldots, \pi_n$ with $n = 0, 1, 2$ then
$h(\pi) = \sup(h(\pi_i)) + 1$. A proof consisting of a single judgement has height 0.

**Proposition 5.1** If $\tau$ is a proof of a judgement of degree $d > 0$ then a proof $\rho$ of the same judgement can be constructed with lower degree.

**Proof**
By induction on $h(\tau)$. Let $R$ be the last rule of $\tau$ and $\pi$ and $\pi'$ be the subproofs yielding the premises of $R$. There are two cases: $R$ is not a mix of degree $d$ or $R$ is a mix of degree $d$. The base case of the induction, for $h(\tau) = 1$, follows from the proof of the second case and the fact that proofs of height 0 have degree 0.
**Case 1** If $R$ is not a mix of degree $d$ then at the inductive step the inductive hypothesis gives a proof $\varpi$ of degree less than $d$ to replace $\pi$, and a proof $\varpi'$ of degree less than $d$ to replace $\pi'$. To construct $\rho$ we simply append $R$ to $\varpi$ and $\varpi'$.
**Case2** $R$ is a mix of degree $d$. At the inductive step we assume that we have a proof $\tau$ of $\Gamma, \Gamma' \vdash \psi'$ with the final rule $R$ being a mix of degree $d$ removing the formula $\phi$. The premises of $R$ are assumed to be $\Gamma \vdash \phi$ proved by $\pi$ and $\Gamma', \phi^n \vdash \psi'$ proved by $\pi'$. The proof $\tau$ looks like

$$R\;\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\qquad\qquad\dfrac{\vdots}{\Gamma', \phi^n \vdash, \psi'}\right\}\pi'}{\Gamma, \Gamma' \vdash \psi'}$$

The inductive hypothesis allows us to replace $\pi$ and $\pi'$ by $\varpi$ and $\varpi'$ respectively, each of degree less than $d$. This situation is taken care of by Lemma 5.1. It provides the requisite proof $\rho$ of $\Gamma, \Gamma' \vdash \psi'$ of degree less than $d$. ■

**Lemma 5.1** Let $\phi$ be a formula of degree $d > 0$ and $\tau$ be a proof with degree $d$ of $\Gamma, \Gamma' \vdash \psi'$. Let the final rule $R$ of $\tau$ be a mix removing an occurrence of the formula $\phi$ from the antecedent of one its premises and $n$ occurrences of $\phi$ from the succedent of the other premise. Let the left premise of $R$ be $\Gamma \vdash \phi$ proved by $\pi$ a subproof of $\tau$ of degree less than $d$. Let the other premise of $R$ be the judgement $\Gamma', \phi^n \vdash \psi'$ proved by $\pi'$ a proof of degree less than $d$. Let the final rule of $\pi$ be $r$ and the final rule of $\pi'$ be $r'$. We can replace $\tau$ by a proof $\rho$ of $\Gamma, \Gamma' \vdash \psi'$ with degree less than $d$.

**Proof**

Assume we have $\tau$ as described above. Here is a picture of the proof $\tau$.

$$R\,\dfrac{\pi\left\{ r\,\dfrac{\vdots}{\Gamma\vdash\phi} \right. \qquad \left. \dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\,r' \right\}\pi'}{\Gamma,\Gamma'\vdash\psi'}$$

The proof is by induction on $h(\pi)+h(\pi')$. The induction hypothesis is :

for every pair of proofs $\mu$ proving $\Gamma_1\vdash\phi$ and $\nu$ proving $\Gamma'_1,\phi^m\vdash\psi'_1$, where $m>0$, if each is of degree less than $d$, and if $h(\mu)+h(\nu)<h(\pi)+h(\pi')$ then there is a proof of $\Gamma_1,\Gamma'_1\vdash\psi'_1$ of degree less than $d$.

The induction step is proved by considering all the possible combinations of which sorts of rules $r$ and $r'$ are. (This simultaneously proves the base case of the induction, $h(\tau)=1$, using the fact that any proof of height 0 has degree 0.)

Note that the strengthened contraction rule is often used at the end of the replacement proofs to get the right number of instances of each formula. Similarly, when modal rules are applied in the proofs which replace $\tau$ it is necessary to be very careful about applying their weakening part, in order to end up with the required number of instances of each formula.

In the following list of cases we do not need to consider when $r$ and $r'$ are mixes - it is sufficient to consider when they are cuts of degree less than $d$.

We consider all the possible cases of what $r$ and $r'$ may be - not all 1 - and construct $\rho$ according to the first case which applies. Here is a list of the main cases considered.

1. $\pi$ is $\bot$-left.
2. $\pi'$ is $\top$-right.
3. $\pi$ is an axiom.
4. $\pi'$ is an axiom .
5. $r$ is a left structural rule.
6. $r'$ is a right structural rule.
7. $r$ is $\top$-left.
8. $r'$ is $\top$-left.
9. $r$ is a left logical rule.
10. $r'$ is a right logical rule.
11. $r'$ is a left logical rule introducing a formula, either other than $\phi$, or $\phi$ when there are more than $n$ copies of $\phi$ in the left-hand-side of the right-hand premise of the mix.
12. $\pi'$ is $\bot$-left.
13. $\pi$ is $\top$-right.
14. $r$ is the $\Box$-right rule.
15. $r$ is the $\Diamond$-left-1 rule.
16. $r$ is the $\Diamond$-left-2 rule.
17. $r'$ is the $\Box$-right rule.

18. $r'$ is the $\Diamond$-left-1 rule.

19. $r'$ is the $\Diamond$-left-2 rule.

20. $r$ is a right logical rule and $r'$ is a left logical rule introducing $\phi$ when there are precisely $n$ copies of $\phi$ in the left-hand-side of the right-hand premise of the mix.

Now we begin to consider these cases.

1. $\pi$ is an instance of the $\perp$-left rule.
   Then $\rho$ is simply an instance of of the $\perp$-left rule.

2. $\pi'$ is an instance of the $\top$-right rule
   Then $\rho$ is simply an instance of of the $\top$-right rule.

3. $\pi$ is an axiom
   Then $\pi$ is of the form $\Gamma, \phi \vdash \phi$. The proof $\tau$ looks like

$$\dfrac{\Gamma, \phi \vdash \phi \qquad \left.\begin{array}{c}\vdots \\ \hline \Gamma', \phi^n \vdash \psi'\end{array}\right\}\pi'}{\Gamma, \Gamma', \phi \vdash \psi'}$$

   and this is replaced by $\rho$ as shown below

$$\dfrac{\dfrac{\left.\begin{array}{c}\vdots \\ \hline \Gamma', \phi^n \vdash \psi'\end{array}\right\}\pi'}{\Gamma, \Gamma', \phi^n \vdash \psi'}\ \text{Weakening}}{\Gamma, \Gamma', \phi \vdash \psi'}\ \text{Contractions}$$

4. $\pi'$ is axiom
   There are two subcases

   - $\pi'$ is of the form $\Gamma', \phi^n, \psi' \vdash \psi'$ .
     The proof $\tau$ given by

$$\dfrac{\pi\left\{\begin{array}{c}\vdots \\ \hline \Gamma \vdash \phi\end{array}\right. \qquad \Gamma', \phi^n, \psi' \vdash \psi'}{\Gamma, \Gamma', \psi' \vdash \psi'}$$

     is replaced by the axiom

$$\Gamma, \Gamma', \psi' \vdash \psi'$$

   - $\pi'$ is of the form $\Gamma', \phi^n \vdash \phi$. Then the proof

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \Gamma', \phi^n \vdash \phi}{\Gamma, \Gamma' \vdash \phi}$$

is replaced by the proof

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right.}{\Gamma', \Gamma \vdash \phi}\ \text{Weakening}$$

5. $r$ is a left structural rule.
   There are two subcases considered separately.

   - $r$ is contraction or weakening.
     Let the premise of $r$ be $\Gamma_1 \vdash \phi$ and let it be proved by $\pi_1$ - the obvious subproof of $\pi$ of degree less than $d$. The proof $\tau$ shown below

$$\dfrac{r \quad \dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi}\right.}{\Gamma \vdash \phi} \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma, \Gamma' \vdash \psi'}$$

     is replaced by $\rho$ given below

$$\dfrac{\nu'\left\{\dfrac{\vdots}{\Gamma_1, \Gamma' \vdash \psi'}\right.}{\Gamma, \Gamma' \vdash \psi'}\ r$$

     where $\nu'$ is the proof of degree less than $d$ which replaces $\nu$ under the induction hypothesis and $\nu$ is the proof

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma_1, \Gamma' \vdash \psi'}\ \text{Mix}$$

   - $r$ is an instance of the cut rule on a formula $\theta$ of degree less than $d$.
     Then $r$ has two premises and $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \theta}\right. \qquad \left.\dfrac{\vdots}{\theta, \Gamma_2 \vdash \phi}\right\}\pi_2}{\Gamma_1, \Gamma_2 \vdash \phi} \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma_1, \Gamma_2, \Gamma' \vdash \psi'}$$

This is replaced by

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \theta}\right. \qquad \left.\dfrac{\vdots}{\Gamma_2, \Gamma', \theta \vdash \psi'}\right\}\mu'}{\Gamma_1, \Gamma_2, \Gamma' \vdash \psi'}\ \text{Cut}$$

where $\mu'$ is the proof which replaces $\mu$ below under the induction hypothesis. The proof $\mu$ is

$$\dfrac{\pi_2\left\{\dfrac{\vdots}{\Gamma_2, \theta \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma_1, \Gamma_2, \Gamma' \vdash \psi'}\ \text{Mix}$$

6. $r'$ is a structural rule
   There are subcases

   - $r'$ is a contraction or weakening but not of $\phi$.

   $$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'_1, \phi^n \vdash \psi'}\right\}\pi'_1}{\Gamma', \phi^n \vdash \psi'}\ r'}{\Gamma, \Gamma' \vdash \psi'}$$

   which is replaced by

   $$\dfrac{\mu'\left\{\dfrac{\vdots}{\Gamma, \Gamma'_1 \vdash \psi'}\right.}{\Gamma, \Gamma' \vdash \psi'}\ r'$$

   where $\mu'$ is the proof which under the induction hypothesis replaces $\mu$ as given below

   $$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma'_1, \phi^n \vdash \psi'}\right\}\pi'_1}{\Gamma, \Gamma'_1 \vdash \psi'}\ \text{Mix}$$

   - $r'$ weakens to add some, but not all, instances of $\phi$ (to the right-hand premise of the Mix) and possibly other formulas too.
   The proof $\tau$ looks like

   $$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'_1, \phi^m \vdash \psi'}\right\}\pi'_1}{\Gamma', \phi^n \vdash \psi'}\ r'}{\Gamma, \Gamma' \vdash \psi'}$$

This is replaced by

$$\cfrac{\mu'\left\{\cfrac{\vdots}{\Gamma,\Gamma'_1 \vdash \psi'}\right.}{\Gamma,\Gamma' \vdash \psi'}\quad \text{Weakening}$$

where $\mu'$ is the replacement for $\mu$ under the induction hypothesis and $\mu$ is given by

$$\cfrac{\pi\left\{\cfrac{\vdots}{\Gamma\vdash\phi}\right. \qquad \left.\cfrac{\vdots}{\Gamma'_1,\phi^m\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'_1\vdash\psi'}\quad \text{Mix}$$

- $r'$ is a structural rule contracting $\phi$ (and possibly other formulas).
  The proof $\tau$ looks like

$$\cfrac{\pi\left\{\cfrac{\vdots}{\Gamma\vdash\phi}\right. \qquad \cfrac{\left.\cfrac{\vdots}{\Gamma',\phi^{n+1}\vdash\psi'}\right\}\pi'_1}{\Gamma',\phi^n\vdash\psi'}}{\Gamma,\Gamma'\vdash\psi'}$$

This is replaced by $\mu'$ the proof which under the induction hypothesis replaces the $\mu$ given below

$$\cfrac{\pi\left\{\cfrac{\vdots}{\Gamma\vdash\phi}\right. \qquad \left.\cfrac{\vdots}{\Gamma',\phi^{n+1}\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'\vdash\psi'}\quad \text{Mix}$$

- $r'$ weakens to add all of the instances of $\phi$ (and possibly other formulas).
  Then (for $\Gamma'_1$ containing no occurrences of $\phi$) $\tau$ looks like

$$\cfrac{\pi\left\{\cfrac{\vdots}{\Gamma\vdash\phi}\right. \qquad \cfrac{\left.\cfrac{\vdots}{\Gamma'_1\vdash\psi'}\right\}\pi'_1}{\Gamma',\phi^n\vdash\psi'}}{\Gamma,\Gamma'\vdash\psi'}$$

and this is replaced by

$$\cfrac{\left.\cfrac{\vdots}{\Gamma'_1\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'\vdash\psi'}\quad \text{Weakening}$$

- $r'$ is a cut of a formula $\theta$ of degree less than $d$.
  The proof $\tau$ looks like (for $n_1 + n_2 = n$)

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\pi'_1\left\{\dfrac{\vdots}{\Gamma'_1, \phi^{n_1} \vdash \theta} \qquad \dfrac{\vdots}{\theta, \Gamma'_2, \phi^{n_2} \vdash \psi'}\right\}\pi'_2}{\Gamma'_1, \Gamma'_2, \phi^{n} \vdash \psi'}}{\Gamma, \Gamma'_1, \Gamma'_2 \vdash \psi'}$$

  The construction of $\rho$ now is split into three cases.

  - If $n_1 = 0$ then $\rho$ is

$$\dfrac{\pi'_1\left\{\dfrac{\vdots}{\Gamma'_1 \vdash \theta} \qquad \dfrac{\vdots}{\Gamma, \Gamma'_2, \theta \vdash \psi'}\right\}\mu'}{\Gamma, \Gamma'_1, \Gamma'_2 \vdash \psi'}\ \text{Cut}$$

    where $\mu'$ is the proof which, under the induction hypothesis, replaces $\mu$ given below

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad \dfrac{\vdots}{\Gamma'_2, \phi^{n}, \theta \vdash \psi'}\right\}\pi'_2}{\theta, \Gamma, \Gamma'_2 \vdash \psi'}\ \text{Mix}$$

  - If $n_2 = 0$ then $\rho$ is constructed in a similar fashion to the last case except that $\mu$ is constructed from a mix on $\pi$ and $\pi'_1$ and $\rho$ comes from a mix with $\mu'$ as the left-hand-premise and $\pi'_2$ as the right-hand-premise.
  - If $n_1 \neq 0$ and $n_2 \neq 0$ then construct $\mu_1$ by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad \dfrac{\vdots}{\Gamma'_1, \phi^{n_1} \vdash \theta}\right\}\pi'_1}{\Gamma, \Gamma'_1 \vdash \theta}\ \text{Mix}$$

    and construct $\mu_2$ by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad \dfrac{\vdots}{\Gamma'_2, \phi^{n_2}, \theta \vdash \psi'}\right\}\pi'_2}{\theta, \Gamma, \Gamma'_2 \vdash \psi'}\ \text{Mix}$$

    and using the induction hypothesis, replace these by proofs $\mu'_1$ and $\mu'_2$ respectively, both of degree less than $d$. The proof $\rho$ is given by

$$\dfrac{\mu'_1\left\{\dfrac{\vdots}{\Gamma, \Gamma'_1 \vdash \theta} \qquad \dfrac{\vdots}{\theta, \Gamma, \Gamma'_2 \vdash \psi'}\right\}\mu'_2}{\dfrac{\Gamma, \Gamma, \Gamma'_1, \Gamma'_2 \vdash \psi'}{\Gamma, \Gamma'_1, \Gamma'_2 \vdash \psi'}}\ \text{Cut}$$

7. $r$ is $\top$-left

   Suppose we have $\tau$ given by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi}\right.}{\Gamma_1, \top \vdash \phi} \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma_1, \top, \Gamma' \vdash \psi'}$$

   then we can replace $\tau$ with

$$\dfrac{\mu'\left\{\dfrac{\vdots}{\Gamma_1, \Gamma' \vdash \psi'}\right.}{\Gamma_1, \top, \Gamma' \vdash \psi'}\ \top\text{-left}$$

   where $\mu'$ is the proof which under the induction hypothesis replaces the proof $\mu$ given by

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma_1, \Gamma' \vdash \psi'}\ \text{Mix}$$

8. $r'$ is $\top$-left.

   There are three subcases

   - The cut formula $\phi$ is not $\top$.

     The proof $\tau$ is of the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma_1', \phi^n \vdash \psi'}\right\}\pi_1'}{\Gamma_1', \top, \phi^n \vdash \psi'}}{\Gamma, \Gamma_1', \top \vdash \psi'}$$

     This is replaced by $\rho$ given by

$$\dfrac{\mu'\left\{\dfrac{\vdots}{\Gamma, \Gamma_1' \vdash \psi'}\right.}{\Gamma, \Gamma_1', \top \vdash \psi'}\ \top\text{-left}$$

     where $\mu'$ is the proof obtained from the induction hypothesis applied to $\mu$ below.

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma'_1,\phi^n\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'_1\vdash\psi'}\ \text{Mix}$$

- If $\phi$ is $\top$ and there is more than one occurrence of it in the left-hand-side of the right-hand premise of the mix rule then $\tau$ has the form (for some $n>1$)

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\top}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma',\top^{n-1}\vdash\psi'}\right\}\pi'_1}{\Gamma',\top^{n}\vdash\psi'}}{\Gamma,\Gamma'\vdash\psi'}$$

  and this is replaced by

$$\left.\dfrac{\vdots}{\Gamma,\Gamma'\vdash\psi'}\right\}\mu'$$

  where under the induction hypothesis $\mu'$ replaces $\mu$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\top}\right. \qquad \left.\dfrac{\vdots}{\Gamma',\top^{n-1}\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'\vdash\psi'}\ \text{Mix}$$

- If $\phi$ is $\top$ and there is exactly one occurrence of it in the left-hand-side of the right-hand premise of the mix rule then $\tau$ has the form (for some $n>1$)

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\top}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'\vdash\psi'}\right\}\pi'_1}{\Gamma',\top\vdash\psi'}}{\Gamma,\Gamma'\vdash\psi'}$$

  and this is replaced by

$$\dfrac{\left.\dfrac{\vdots}{\Gamma'\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma'\vdash\psi'}\ \text{Weakening}$$

9. $r$ is a left logical rule.

   Go through the cases of what $r$ may be.

- $r$ is $\wedge$-left.
  Then $\tau$ is

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma,\theta_1,\theta_2\vdash\phi}\right.}{\Gamma,\theta_1\wedge\theta_2\vdash\phi} \qquad \left.\dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\right\}\pi'}{\Gamma,\theta_1\wedge\theta_2,\Gamma'\vdash\psi'}$$

and is replaced by

$$\dfrac{\left.\dfrac{\vdots}{\Gamma',\Gamma,\theta_1,\theta_2\vdash\psi'}\right\}\mu'}{\Gamma',\Gamma,\theta_1\wedge\theta_2\vdash\psi'}\ \wedge\text{-left}$$

where $\mu'$ is the proof which replaces $\mu$ under the induction hypothesis and $\mu$ is the proof

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma,\theta_1,\theta_2\vdash\phi}\qquad\dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\right\}\pi'}{\Gamma,\Gamma',\theta_1,\theta_2\vdash\psi'}\ \text{Mix}$$

- $r$ is $\vee$-left.
  Then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma,\theta_1\vdash\phi}\qquad\dfrac{\vdots}{\Gamma,\theta_2\vdash\phi}\right\}\pi_2}{\Gamma,\theta_1\vee\theta_2\vdash\phi}\qquad\left.\dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\right\}\pi'}{\Gamma,\Gamma',\theta_1\vee\theta_2\vdash\psi'}$$

and is replaced by

$$\dfrac{\mu_1'\left\{\dfrac{\vdots}{\Gamma,\Gamma',\theta_1\vdash\psi'}\qquad\dfrac{\vdots}{\Gamma,\Gamma',\theta_2\vdash\psi'}\right\}\mu_2'}{\Gamma,\Gamma',\theta_1\vee\theta_2\vdash\psi'}\ \vee\text{-left}$$

where each of the $\mu_i'$ replaces $\mu_i$ given by

$$\dfrac{\pi_i\left\{\dfrac{\vdots}{\Gamma,\theta_i\vdash\phi}\qquad\dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\right\}\pi'}{\Gamma,\Gamma',\theta_i\vdash\psi'}\ \text{Mix}$$

- $r$ is $\to$-left.
  Then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma, \theta_2 \vdash \phi}\right\}\pi_2}{\Gamma, \theta_1 \to \theta_2 \vdash \phi} \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma, \Gamma', \theta_1 \to \theta_2 \vdash \psi'}$$

  and is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1}\right.}{\Gamma', \Gamma \vdash \theta_1} \qquad \left.\dfrac{\vdots}{\Gamma', \Gamma, \theta_2 \vdash \psi'}\right\}\mu'}{\Gamma', \Gamma, \theta_1 \to \theta_2 \vdash \psi'} \quad \to\text{-left}$$

  where $\mu'$ is the proof which replaces $\mu$ given by

$$\dfrac{\pi_2\left\{\dfrac{\vdots}{\Gamma, \theta_2 \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi'}\right\}\pi'}{\Gamma', \Gamma, \theta_2 \vdash \psi'} \quad \text{Mix}$$

10. $r'$ is a right logical rule. There are four subcases

    - $r'$ is $\vee$-right-1
      Then $\tau$ looks like

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi_1}\right\}\pi_1'}{\Gamma', \phi^n \vdash \psi_1 \vee \psi_2}}{\Gamma, \Gamma' \vdash \psi_1 \vee \psi_2}$$

      and is replaced by the proof $\rho$ given by

$$\dfrac{\mu'\left\{\dfrac{\vdots}{\Gamma, \Gamma' \vdash \psi_1}\right.}{\Gamma, \Gamma' \vdash \psi_1 \vee \psi_2} \quad \vee\text{-right-1}$$

      where $\mu'$ is the proof which replaces $\mu$ below under the induction hypothesis (applied to $\pi$ and $\pi_1'$).

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi_1}\right\}\pi'_1}{\Gamma, \Gamma' \vdash \psi_1}\ \text{Mix}$$

- $r'$ is $\vee$-right-2.
  This is handled in an almost identical fashion to the last subcase.
- $r'$ is $\wedge$-right.
  Then $\tau$ is of the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\pi'_1\left\{\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi_2}\right\}\pi'_2}{\Gamma', \phi^n \vdash \psi_1 \wedge \psi_2}}{\Gamma, \Gamma' \vdash \psi_1 \wedge \psi_2}$$

and this is replaced by $\rho$ as below

$$\dfrac{\mu'_1\left\{\dfrac{\vdots}{\Gamma, \Gamma' \vdash \psi_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma, \Gamma' \vdash \psi_2}\right\}\mu'_2}{\Gamma, \Gamma' \vdash \psi_1 \wedge \psi_2}\ \wedge\text{-right}$$

where each of the $\mu'_i$ under the inductive hypothesis replaces the proof $\mu_i$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n \vdash \psi_i}\right\}\pi'_i}{\Gamma, \Gamma' \vdash \psi_i}\ \text{Mix}$$

- $r'$ is $\rightarrow$-right
  Then $\tau$ is of the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma', \psi_1, \phi^n \vdash \psi_2}\right\}\pi'_1}{\Gamma', \phi^n \vdash \psi_1 \rightarrow \psi_2}}{\Gamma, \Gamma' \vdash \psi_1 \rightarrow \psi_2}$$

and is replaced by the proof $\rho$ given by

$$\dfrac{\mu'\left\{\dfrac{\vdots}{\Gamma, \Gamma', \psi_1 \vdash \psi_2}\right.}{\Gamma, \Gamma' \vdash \psi_1 \rightarrow \psi_2}\ \rightarrow\text{-right}$$

where $\mu'$ is the proof obtained from the induction hypothesis applied to $\mu$ as below

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma', \phi^n, \psi_1 \vdash \psi_2}\right\}\pi'_1}{\Gamma, \Gamma', \psi_1 \vdash \psi_2}\ \text{Mix}$$

11. $r'$ is a left logical rule not creating $\phi$, or creating $\phi$ when there are greater than $n$ occurrences of $\phi$ on the left-hand-side of the right-hand premise of the mix.
    Consider what $r'$ is.

    - $r'$ is $\wedge$-left.
      The proof $\tau$ has the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'_1, \theta_1, \theta_2, \phi^n \vdash \psi'}\right\}\pi'_1}{\Gamma'_1, \theta_1 \wedge \theta_2, \phi^n \vdash \psi'}}{\Gamma, \Gamma'_1, \theta_1 \wedge \theta_2 \vdash \psi'}$$

      and is replaced by

$$\dfrac{\mu\left\{\dfrac{\vdots}{\Gamma, \Gamma'_1, \theta_1, \theta_2 \vdash \psi'}\right.}{\Gamma, \Gamma'_1, \theta_1 \wedge \theta_2 \vdash \psi'}\ \wedge\text{-left}$$

      where $\mu'$ replaces $\mu$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \left.\dfrac{\vdots}{\Gamma_1, \theta_1, \theta_2, \phi^n \vdash \psi'}\right\}\pi'_1}{\Gamma, \Gamma'_1, \theta_1, \theta_2 \vdash \psi'}\ \text{Mix}$$

    - $r'$ is $\vee$-left.
      The proof $\tau$ has the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \phi}\right. \qquad \dfrac{\pi'_1\left\{\dfrac{\vdots}{\Gamma_1, \theta_1, \phi^n \vdash \psi'}\right. \qquad \left.\dfrac{\vdots}{\Gamma_1, \theta_2, \phi^n \vdash \psi'}\right\}\pi'_2}{\Gamma'_1, \theta_1 \vee \theta_2, \phi^n \vdash \psi'}}{\Gamma, \Gamma'_1, \theta_1 \vee \theta_2 \vdash \psi'}$$

      and is replaced by

$$\dfrac{\mu_1'\left\{\dfrac{\vdots}{\Gamma,\Gamma_1',\theta_1\vdash\psi'}\qquad\qquad\dfrac{\vdots}{\Gamma,\Gamma_1',\theta_2\vdash\psi'}\right\}\mu_2'}{\Gamma,\Gamma_1',\theta_1\vee\theta_2\vdash\psi'}\quad\vee\text{-left}$$

where $\mu_i'$ replaces $\mu$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\qquad\qquad\dfrac{\vdots}{\Gamma_1',\theta_i,\phi^n\vdash\psi'}\right\}\pi_i'}{\Gamma,\Gamma_1',\theta_i\vdash\psi'}\quad\text{Mix}$$

- $r'$ is $\rightarrow$-left
  The proof $\tau$ has the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\right.\qquad\qquad\dfrac{\pi_1'\left\{\dfrac{\vdots}{\Gamma_1',\phi^n\vdash\theta_1}\qquad\qquad\dfrac{\vdots}{\Gamma_1',\theta_2,\phi^n\vdash\psi'}\right\}\pi_2'}{\Gamma_1',\theta_1\rightarrow\theta_2,\phi^n\vdash\psi'}}{\Gamma,\Gamma_1',\theta_1\rightarrow\theta_2\vdash\psi'}$$

  and is replaced by

$$\dfrac{\mu_1'\left\{\dfrac{\vdots}{\Gamma,\Gamma_1'\vdash\theta_1}\qquad\qquad\dfrac{\vdots}{\Gamma,\Gamma_1',\theta_2\vdash\psi'}\right\}\mu_2'}{\Gamma,\Gamma_1',\theta_1\rightarrow\theta_2\vdash\psi'}\quad\rightarrow\text{-left}$$

  where $\mu_1'$ replaces $\mu_1$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\qquad\qquad\dfrac{\vdots}{\Gamma_1',\phi^n\vdash\theta_1}\right\}\pi_1'}{\Gamma,\Gamma_1'\vdash\theta_1}\quad\text{Mix}$$

  and $\mu_2'$ replaces $\mu_2$ given by

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\qquad\qquad\dfrac{\vdots}{\Gamma_1',\theta_2,\phi^n\vdash\psi'}\right\}\pi_2'}{\Gamma,\Gamma_1',\theta_2\vdash\psi'}\quad\text{Mix}$$

12. $\pi'$ is the $\perp$-left rule.
    This has two subcases

- If $\phi$ is not $\bot$ then $\rho$ is simply an instance of $\bot$-left.
- If $\phi$ is $\bot$ then $\tau$ has the form

$$\dfrac{\pi\left\{\begin{array}{c}\vdots\\ \hline \Gamma\vdash\bot\end{array}\right. \qquad \Gamma',\bot^{n-1},\bot\vdash\psi'}{\Gamma,\Gamma'\vdash\psi'}$$

  We proceed by considering what sort of rule $r$ is. The possibilities for $r$ not handled yet are

  - $r$ is a right logical rule. This is not possible since $\bot$ is not composed of other formulas.
  - $r$ is an instance of the $\top$-right rule. This is clearly not possible since then $\phi$ would be $\top$.
  - $r$ is the $\Box$-right rule. This is not possible by the form of $\phi$.
  - $r$ is the $\Diamond$-left-1 rule. This is not possible by the form of $\phi$.
  - $r$ is the $\Diamond$-left-2 rule. Then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\begin{array}{c}\vdots\\ \hline \Gamma_1,\theta\vdash\bot\end{array}\right.}{\Gamma_1'',\Box\Gamma_1,\Diamond\theta\vdash\bot} \qquad \Gamma',\bot^{n-1},\bot\vdash\psi'}{\Gamma_1'',\Gamma',\Box\Gamma_1,\Diamond\theta\vdash\psi'}$$

    and this can be replaced by

$$\dfrac{\pi_1\left\{\begin{array}{c}\vdots\\ \hline \Gamma_1,\theta\vdash\bot\end{array}\right.}{\Gamma_1'',\Gamma',\Box\Gamma_1,\Diamond\theta\vdash\psi'}\ \Diamond\text{-left-2}$$

13. $\pi$ is $\top$-right.
Then $\tau$ has the form

$$\dfrac{\Gamma\vdash\top \qquad \dfrac{\vdots}{\Gamma',\top^{n}\vdash\psi'}\,r'}{\Gamma,\Gamma'\vdash\psi'}$$

We construct $\rho$ case-by-case depending on the form of $r'$. The cases not yet handled are

- $r'$ is a left logical rule creating $\top$ and there are n copies of $\top$ in the left-hand-side of the right-hand premise of the mix. This is not possible since $\top$ contains no logical connectives.
- If $r'$ is a modal rule then $\top$ comes entirely from the weakening part of the rule. Let the proof of the premise of $r'$ be called $\pi_1'$. A proof $\rho$ is constructed by applying the same sort of modal rule as $r'$, but making sure to get the correct antecedent by adding the weakening part carefully. For example in the $\Box$-case if $\tau$ is

$$\dfrac{\Gamma\vdash\top \qquad \dfrac{\left.\begin{array}{c}\vdots\\ \hline \Gamma_1'\vdash\psi_1'\end{array}\right\}\pi_1'}{\top^{n},\Gamma_1'',\Box\Gamma_1'\vdash\Box\psi_1'}}{\Gamma,\Gamma_1'',\Box\Gamma_1'\vdash\Box\psi_1'}$$

then $\rho$ is

$$\dfrac{\left.\dfrac{\vdots}{\Gamma'_1 \vdash \psi'}\right\}\pi'_1}{\Gamma, \Gamma''_1, \Box\Gamma'_1 \vdash \Box\psi'_1}\ \Box\text{-right}$$

14. If $r$ is the $\Box$-rule then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right.}{\Gamma''_1, \Box\Gamma_1 \vdash \Box\phi_1} \qquad \dfrac{\vdots}{\Gamma', \Box\phi_1^n \vdash \psi'}\ r'}{\Gamma''_1, \Box\Gamma_1, \Gamma' \vdash \psi'}$$

We consider what the remaining possibilities for $r'$ are

- It is not possible for $r'$ to be a left-hand logical rule creating $\Box\phi_1$.
- $r'$ is a modal rule and at least n of the $\Box\phi_1$ come from the weakening part of the rule. Let the proof of the premise of $r'$ be called $\pi'_1$. Then $\rho$ is constructed by applying the same sort of rule as $r'$ to $\pi'_1$ but adding the weakening part carefully to get the correct left-hand-side. For example if $r'$ is the $\Box$ rule and the proof $\tau$ is

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right.}{\Gamma''_1, \Box\Gamma_1 \vdash \Box\phi_1} \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'_1 \vdash \psi'_1}\right\}\pi'_1}{\Box\phi_1^n, \Gamma'''_1, \Box\Gamma'_1 \vdash \Box\psi'_1}}{\Gamma''_1, \Box\Gamma_1, \Gamma'''_1, \Box\Gamma'_1 \vdash \Box\psi'_1}$$

  then $\rho$ is

$$\dfrac{\left.\dfrac{\vdots}{\Gamma'_1 \vdash \psi'}\right\}\pi'_1}{\Gamma''_1, \Box\Gamma_1, \Gamma'''_1, \Box\Gamma'_1 \vdash \Box\psi'_1}\ \Box\text{-right}$$

- $r'$ is a modal rule and less than $n$ of the $\Box\phi_1$ come from the weakening part of the rule. Then some of the $\phi_1$ are contained in the antecedent of the premise of $r'$. Suppose that this premise is $\Gamma'_1, \phi_1^m \vdash \psi'_1$ proved by $\pi'_1$. The proof $\rho$ is obtained by applying the mix rule to $\pi_1$ and $\pi'_1$ to get $\Gamma_1, \Gamma'_1 \vdash \psi'$ then applying the same sort of rule as $r'$ (weakening carefully) to get the correct left-hand-side.
  - $r'$ is $\Box$-right.
    Then $\tau$ has the form (where $n \geq m \geq 1$)

$$\dfrac{r\ \dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right.}{\Gamma''_1, \Box\Gamma_1 \vdash \Box\phi_1} \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma'_1, \phi_1^m \vdash \psi'_1}\right\}\pi'_1}{\Gamma'''_1, \Box\Gamma'_1, \Box\phi_1^n \vdash \Box\psi'_1}\ r'}{\Gamma''_1, \Box\Gamma_1, \Gamma'''_1, \Box\Gamma'_1 \vdash \Box\psi'_1}\ \text{Mix}$$

and this is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma_1', \phi_1^m \vdash \psi_1'}\right\}\pi_1'}{\Gamma_1, \Gamma_1' \vdash \psi_1'}\;\text{Mix}}{\Gamma_1'', \Gamma_1''', \Box\Gamma_1, \Box\Gamma_1' \vdash \Box\psi_1'}\;\Box\text{-right}$$

- $r'$ is $\Diamond$-left-1.
  Then $\tau$ has the form (where $n \geq m \geq 1$)

$$\dfrac{r\;\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right.}{\Gamma_1'', \Box\Gamma_1 \vdash \Box\phi_1} \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma_1', \phi_1^m, \theta \vdash \psi_1'}\right\}\pi_1'}{\Gamma_1''', \Box\Gamma_1', \Box\phi_1^n, \Diamond\theta \vdash \Diamond\psi_1'}\;r'}{\Gamma_1'', \Box\Gamma_1, \Gamma_1''', \Box\Gamma_1', \Diamond\theta \vdash \Diamond\psi_1'}\;\text{Mix}$$

and this is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma_1', \phi_1^m, \theta \vdash \psi_1'}\right\}\pi_1'}{\Gamma_1, \Gamma_1', \theta \vdash \psi_1'}\;\text{Mix}}{\Gamma_1'', \Gamma_1''', \Box\Gamma_1, \Box\Gamma_1', \Diamond\theta \vdash \Diamond\psi_1'}\;\Diamond\text{-left-1}$$

- $r'$ is $\Diamond$-left-2 .
  Then $\tau$ has the form (where $n \geq m \geq 1$)

$$\dfrac{r\;\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right.}{\Gamma_1'', \Box\Gamma_1 \vdash \Box\phi_1} \qquad \dfrac{\left.\dfrac{\vdots}{\Gamma_1', \phi_1^m, \theta \vdash \bot}\right\}\pi_1'}{\Gamma_1''', \Box\Gamma_1', \Box\phi_1^n, \Diamond\theta \vdash \psi'}\;r'}{\Gamma_1'', \Box\Gamma_1, \Gamma_1''', \Box\Gamma_1', \Diamond\theta \vdash \psi'}\;\text{Mix}$$

and this is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1 \vdash \phi_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma_1', \phi_1^m, \theta \vdash \bot}\right\}\pi_1'}{\Gamma_1, \Gamma_1', \theta \vdash \bot}\;\text{Mix}}{\Gamma_1'', \Gamma_1''', \Box\Gamma_1, \Box\Gamma_1', \Diamond\theta \vdash \psi'}\;\Diamond\text{-left-2}$$

15. If $r$ is the $\Diamond$-left-1 rule then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1, \theta \vdash \phi_1}\right.}{\Gamma_1'', \Box\Gamma_1, \Diamond\theta \vdash \Diamond\phi_1} \qquad \dfrac{\vdots}{\Gamma', \Diamond\phi_1^n \vdash \psi'}\;r'}{\Diamond\theta, \Gamma_1'', \Box\Gamma_1, \Gamma' \vdash \psi'}$$

We consider what the remaining possibilities for $r'$ are

- It is not possible for $r'$ to be a left-hand logical rule creating $\Diamond\phi_1$.
- $r'$ is a modal rule and at least $n$ of the $\Diamond\phi_1$ come from the weakening part of the rule. Let the proof of the premise of $r'$ be called $\pi_1'$. Then $\rho$ is constructed by applying the same sort of rule as $r'$ to $\pi_1'$ but adding the weakening part carefully to get the correct left-hand-side. For example if $r'$ is the $\Box$-rule and if $\tau$ is

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\phi_1}\right.}{\Gamma_1'',\Box\Gamma_1,\Diamond\theta\vdash\Diamond\phi_1}\qquad\dfrac{\left.\dfrac{\vdots}{\Gamma_1'\vdash\psi_1'}\right\}\pi_1'}{\Diamond\phi_1^n,\Gamma_1'',\Box\Gamma_1'\vdash\Box\psi_1'}r'}{\Gamma_1'',\Box\Gamma_1,\Diamond\theta,\Gamma_1''',\Box\Gamma_1\vdash\Box\psi_1'}$$

  then $\rho$ is

$$\dfrac{\left.\dfrac{\vdots}{\Gamma_1'\vdash\psi_1'}\right\}\pi_1'}{\Gamma_1'',\Box\Gamma_1,\Diamond\theta,\Gamma_1''',\Box\Gamma_1'\vdash\Box\psi_1'}\ \Box\text{-right}$$

- If $r'$ is a modal rule and less than $n$ of the $\Diamond\phi_1$ come from the weakening part of the rule. Then at least one of the occurrences of $\phi_1$ is contained in the premise of $r'$. Suppose that this premise is $\Gamma_1',\phi_1^m\vdash\psi_1'$ proved by $\pi_1'$. The proof $\rho$ is obtained by applying the mix rule to $\pi_1$ and $\pi_1'$ to get $\Gamma_1,\theta,\Gamma_1'\vdash\psi'$ then applying the same sort of rule as $r'$ (weakening carefully) to get $\Gamma_1'',\Box\Gamma_1,\Gamma_1''',\Box\Gamma_1',\Diamond\theta,\vdash\Diamond\psi_1'$. In particular the subcases are
  - $r'$ is $\Box$-right.
    This does not apply.
  - $r'$ is $\Diamond$-left-1.
    Then $\tau$ has the form (where $n\geq m\geq 1$)

$$\dfrac{r\ \dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\phi_1}\right.}{\Gamma_1'',\Box\Gamma_1,\Diamond\theta\vdash\Diamond\phi_1}\qquad\dfrac{\left.\dfrac{\vdots}{\Gamma_1',\phi_1^m\vdash\psi_1'}\right\}\pi_1'}{\Gamma_1''',\Box\Gamma_1',\Diamond\phi_1^n\vdash\Diamond\psi_1'}\ r'}{\Gamma_1'',\Box\Gamma_1,\Gamma_1''',\Box\Gamma_1',\Diamond\theta\vdash\Diamond\psi_1'}\ \text{Mix}$$

    and this is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\phi_1}\right.\qquad\left.\dfrac{\vdots}{\Gamma_1',\phi_1^m\vdash\psi_1'}\right\}\pi_1'}{\Gamma_1,\Gamma_1',\theta\vdash\psi_1'}\ \text{Mix}}{\Gamma_1'',\Gamma_1''',\Box\Gamma_1,\Box\Gamma_1',\Diamond\theta\vdash\Diamond\psi_1'}\ \Diamond\text{-left-1}$$

  - $r'$ is $\Diamond$-left-2 .
    Then $\tau$ has the form (where $n\geq m\geq 1$)

$$\dfrac{r\;\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\phi_1}\right.}{\Gamma''_1,\Box\Gamma_1,\Diamond\theta\vdash\Diamond\phi_1}\qquad\dfrac{\left.\dfrac{\vdots}{\Gamma'_1,\phi_1^m\vdash\bot}\right\}\pi'_1}{\Gamma'''_1,\Box\Gamma'_1,\Diamond\phi_1^n,\vdash\psi'}\;r'}{\Gamma''_1,\Box\Gamma_1,\Gamma'''_1,\Box\Gamma'_1,\Diamond\theta\vdash\psi'}\ \text{Mix}$$

and this is replaced by

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\phi_1}\qquad\dfrac{\vdots}{\Gamma'_1,\phi_1^m\vdash\bot}\right\}\pi'_1}{\Gamma_1,\Gamma'_1,\theta\vdash\bot}\ \text{Mix}}{\Gamma''_1,\Gamma'''_1,\Box\Gamma_1,\Box\Gamma'_1,\Diamond\theta\vdash\psi'}\ \Diamond\text{-left-2}$$

16. If $r$ is the $\Diamond$-left-2 rule.
then $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\bot}\right.\;r}{\Gamma''_1,\Box\Gamma_1,\Diamond\theta\vdash\phi}\qquad\dfrac{\vdots}{\Gamma',\phi^n\vdash\psi'}\;r'}{\Gamma''_1,\Box\Gamma_1,\Diamond\theta,\Gamma'\vdash\psi'}\ \text{Mix}$$

and this is replaced by

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma_1,\theta\vdash\bot}\right.}{\Gamma''_1,\Gamma',\Box\Gamma_1,\Diamond\theta\vdash\psi'}\ \Diamond\text{-left-2}$$

17. $r'$ is an instance of the $\Box$-right rule.

There two are subcases

- At least $n$ of the $\phi$ come from the weakening part of $r'$.
  Then $\tau$ has the form

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\phi}\qquad\dfrac{\left.\dfrac{\vdots}{\Gamma'_1\vdash\psi'_1}\right\}\pi'_1}{\phi^n,\Gamma'''_1,\Box\Gamma'_1\vdash\Box\psi'_1}\;r'\right.}{\Gamma,\Gamma'''_1,\Box\Gamma'_1\vdash\Box\psi'_1}\ \text{Mix}$$

  and $\rho$ is

$$\dfrac{\left.\begin{array}{c}\vdots\\ \hline \Gamma'_1 \vdash \psi'_1\end{array}\right\}\pi'_1}{\Gamma, \Gamma'''_1, \Box\Gamma'_1 \vdash \psi'_1}\ \Box\text{-right}$$

- $\phi$ appears in the premise of $r'$. Then $\phi$ is of the form $\Box\phi_1$ for some $\phi_1$ which appears in the premise. The only case of what $r$ is that has not yet been handled is that $r$ is a right logical rule. This is not possible because of the form of $\phi$.

18. $r'$ is an instance of the $\Diamond$-left-1 rule.

    There are three subcases

    - At least $n$ of the $\phi$ of the right-hand-premise of the mix come from the weakening part of $r'$. Suppose that $\pi'_1$ proves the premise $\Gamma'_1, \theta \vdash \psi'_1$ of the rule $r'$ and that $\pi'$ proves $\phi^n, \Gamma'''_1, \Box\Gamma'_1, \Diamond\theta \vdash \Diamond\psi'_1$. Suppose also that $\pi$ proves $\Gamma \vdash \phi$. Then $\rho$ is

$$\dfrac{\left.\begin{array}{c}\vdots\\ \hline \Gamma'_1, \theta \vdash \psi'_1\end{array}\right\}\pi'_1}{\Gamma, \Gamma'''_1, \Box\Gamma'_1, \Diamond\theta \vdash \Diamond\psi'_1}\ \Diamond\text{-left-2}$$

    - Some of the $\phi^n$ don't come from the weakening and $\phi$ is of the form $\Box\phi_1$ for some $\phi_1$ which appears in the premise. The only case of what $r$ is that has not yet been handled is that $r$ is a right logical rule. This is not possible because of the form of $\phi$.
    - One of the $\phi^n$ appears in the premise and $\phi$ is of the form $\Diamond\phi_1$ for some $\phi_1$ which appears in the premise. The only case of what $r$ is that has not yet been handled is that $r$ is a right logical rule. This is not possible because of the form of $\phi$.

19. $r'$ is an instance of the $\Diamond$-left-2 rule.

    There are three subcases

    - At least $n$ of the $\phi$ in the right-hand premise of the mix come from the weakening part of $r'$. Suppose that $\pi'_1$ proves the premise $\Gamma'_1, \theta \vdash \bot$ of the rule $r'$ and that $\pi'$ proves $\phi^n, \Gamma'''_1, \Box\Gamma'_1, \Diamond\theta \vdash \psi'$ Suppose also that $\pi$ proves $\Gamma \vdash \phi$. Then $\rho$ is

$$\dfrac{\left.\begin{array}{c}\vdots\\ \hline \Gamma'_1, \theta \vdash \bot\end{array}\right\}\pi'_1}{\Gamma, \Gamma'''_1, \Box\Gamma'_1, \Diamond\theta \vdash \psi'}\ \Diamond\text{-left-2}$$

    - Less than $n$ copies of $\phi$ come from the weakening part of $r'$ and $\phi$ is of the form $\Box\phi_1$ for some $\phi_1$ which appears in the premise. The only case of what $r$ is that has not yet been handled is that $r$ is a right logical rule. This is not possible because of the form of $\phi$.
    - Less than $n$ copies of $\phi$ come from the weakening part of $r'$ and $\phi$ is of the form $\Diamond\phi_1$ for some $\phi_1$ which appears in the premise. The only case of what $r$ is that has not yet been handled is that $r$ is a right logical rule. This is not possible because of the form of $\phi$.

20. $r$ is a right logical rule and $r'$ is a left logical rule introducing $\phi$ when there are exactly n copies of $\phi$ in the left-hand-side of the right-hand-premise of the mix.

Again the method is to run through the possible cases of what the rules $r$ and $r'$ are. However, now we have the additional constraint that $r$ introduces the same formula on the right as $r'$ does on the left.

**$\wedge$-case**

Suppose that $\tau$ is

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma\vdash\theta_1}\qquad\dfrac{\vdots}{\Gamma\vdash\theta_2}\right\}\pi_2}{\Gamma\vdash\theta_1\wedge\theta_2}\qquad\dfrac{\left.\dfrac{\vdots}{\Gamma',(\theta_1\wedge\theta_2)^{n-1},\theta_1,\theta_2\vdash\psi'}\right\}\pi'_1}{\Gamma',(\theta_1\wedge\theta_2)^{n}\vdash\psi'}}{\Gamma,\Gamma'\vdash\psi'}$$

There are two cases giving the replacement for this

- If n=1 then the replacement for $\tau$ is

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma\vdash\theta_1}\right.\qquad\dfrac{\pi_2\left\{\dfrac{\vdots}{\Gamma\vdash\theta_2}\qquad\dfrac{\vdots}{\Gamma',\theta_1,\theta_2\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma',\theta_1\vdash\psi'}\text{Mix}}{\dfrac{\Gamma,\Gamma,\Gamma'\vdash\psi'}{\Gamma,\Gamma'\vdash\psi'}}\text{Mix}$$

- If $n>1$ then the replacement for $\tau$ is

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma\vdash\theta_1}\right.\qquad\dfrac{\pi_2\left\{\dfrac{\vdots}{\Gamma\vdash\theta_2}\qquad\dfrac{\vdots}{\Gamma,\Gamma',\theta_1,\theta_2\vdash\psi'}\right\}\mu'}{\Gamma,\Gamma,\Gamma',\theta_1\vdash\psi'}\text{Mix}}{\dfrac{\Gamma,\Gamma,\Gamma,\Gamma'\vdash\psi'}{\Gamma,\Gamma'\vdash\psi'}}\text{Mix}$$

  where $\mu'$ is the proof which replaces $\mu$ below under the induction hypothesis

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma\vdash\theta_1\wedge\theta_2}\qquad\dfrac{\vdots}{\Gamma',(\theta_1\wedge\theta_2)^{n-1},\theta_1,\theta_2\vdash\psi'}\right\}\pi'_1}{\Gamma,\Gamma',\theta_1,\theta_2\vdash\psi'}$$

**$\vee$-1 case**

The proof $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1}\right.}{\Gamma \vdash \theta_1 \vee \theta_2} \qquad \dfrac{\pi_1'\left\{\dfrac{\vdots}{\Gamma', (\theta_1 \vee \theta_2)^{n-1}, \theta_1 \vdash \psi'} \qquad \dfrac{\vdots}{\Gamma', (\theta_1 \vee \theta_2)^{n-1}, \theta_2 \vdash \psi'}\right\}\pi_2'}{\Gamma', (\theta_1 \vee \theta_2)^{n-1}, \theta_1 \vee \theta_2 \vdash \psi'}}{\Gamma, \Gamma' \vdash \psi'}$$

There are two subcases again.

- If $n = 1$ then the replacement for $\tau$ simply changes $\tau$ to

$$\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1} \qquad \dfrac{\vdots}{\Gamma', \theta_1 \vdash \psi'}\right\}\pi_1'}{\Gamma, \Gamma' \vdash \psi'}\ \text{Mix}$$

- If $n > 1$ then construct $\mu$ as below

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1 \vee \theta_2} \qquad \dfrac{\vdots}{\Gamma', (\theta_1 \vee \theta_2)^{n-1}, \theta_1 \vdash \psi'}\right\}\pi_1'}{\Gamma, \Gamma', \theta_1 \vdash \psi'}\ \text{Mix}$$

  and replace it with $\mu'$ a proof of degree less than $d$ using the inductive hypothesis. Then $\rho$ is

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1} \qquad \dfrac{\vdots}{\Gamma, \Gamma', \theta_1 \vdash \psi'}\right\}\mu'}{\Gamma, \Gamma, \Gamma' \vdash \psi'}\ \text{Mix}}{\Gamma, \Gamma' \vdash \psi'}$$

**$\vee$-2 case**
This is done in exactly the same way as the last case except that $\mu$ is created by a mix on $\pi$ and $\pi_2'$ instead of on $\pi$ and $\pi_1'$.

**$\to$-case**
In this case $\tau$ has the form

$$\dfrac{\dfrac{\pi_1\left\{\dfrac{\vdots}{\Gamma, \theta_1 \vdash \theta_2}\right.}{\Gamma \vdash \theta_1 \to \theta_2} \qquad \dfrac{\pi_1'\left\{\dfrac{\vdots}{\Gamma, (\theta_1 \to \theta_2)^{n-1} \vdash \theta_1} \qquad \dfrac{\vdots}{\Gamma, (\theta_1 \to \theta_2)^{n-1}, \theta_2 \vdash \psi'}\right\}\pi_2'}{\Gamma', (\theta_1 \to \theta_2)^{n-1}, \theta_1 \to \theta_2 \vdash \psi'}}{\Gamma, \Gamma' \vdash \psi'}$$

and again we must consider separately the cases $n = 1$ and $n > 1$.

- If $n = 1$ then $\rho$ is

$$\dfrac{\text{Mix}\ \dfrac{\pi_1'\left\{\dfrac{\vdots}{\Gamma' \vdash \theta_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma, \theta_1 \vdash \theta_2}\right\}\pi_1}{\Gamma', \Gamma \vdash \theta_2} \qquad \left.\dfrac{\vdots}{\Gamma', \theta_2 \vdash \psi'}\right\}\pi_2'}{\dfrac{\Gamma', \Gamma', \Gamma \vdash \psi'}{\Gamma', \Gamma \vdash \psi'}}\ \text{Mix}$$

- If $n > 1$ then construct $\mu_1$

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1 \to \theta_2}\right. \qquad \left.\dfrac{\vdots}{\Gamma', (\theta_1 \to \theta_2)^{n-1} \vdash \theta_1}\right\}\pi_1'}{\Gamma, \Gamma' \vdash \theta_1}\ \text{Mix}$$

  and construct $\mu_2$

$$\dfrac{\pi\left\{\dfrac{\vdots}{\Gamma \vdash \theta_1 \to \theta_2}\right. \qquad \left.\dfrac{\vdots}{\Gamma', (\theta_1 \to \theta_2)^{n-1}, \theta_2 \vdash \psi'}\right\}\pi_2'}{\Gamma, \Gamma', \theta_2 \vdash \psi'}\ \text{Mix}$$

  Under the induction hypothesis each of the $\mu_i$ is replaced by a $\mu_i'$ of degree less than $d$. The proof $\rho$ is given by

$$\dfrac{\text{Mix}\ \dfrac{\mu_1'\left\{\dfrac{\vdots}{\Gamma, \Gamma' \vdash \theta_1}\right. \qquad \left.\dfrac{\vdots}{\Gamma, \theta_1 \vdash \theta_2}\right\}\pi_1}{\Gamma, \Gamma, \Gamma' \vdash \theta_2} \qquad \left.\dfrac{\vdots}{\Gamma, \Gamma', \theta_2 \vdash \psi'}\right\}\mu_2'}{\dfrac{\Gamma, \Gamma, \Gamma', \Gamma, \Gamma' \vdash \psi'}{\Gamma, \Gamma' \vdash \psi'}}\ \text{Mix}$$

This exhausts the list of cases for $r$ and $r'$ so the lemma is proved. ■

The lemma proves the proposition which yields the theorem.

**Comments**

The above proof is extremely long but the order in which the cases were proved was chosen in the attempt to keep the length down and to avoid repetition. The whole proof could have been made shorter by treating more of the cases together.

The mix rule is used instead of the cut rule because it makes things easier (or at least briefer) in case 6 of the lemma when $r'$ is a contraction on the formula to be cut.

The subformulas of a formula are defined inductively as follows.

- $\top$ is the only subformula of $\top$ and $\bot$ is the only subformula of $\bot$.

- For every variable $P$, $P$ is the only subformula.
- Each one of the formulas $\phi \to \psi$, $\phi \wedge \psi$, $\phi \vee \psi$ has subformulas itself plus $\phi$, $\psi$ and their subformulas.
- Each one of $\neg\phi$, $\Box\phi$, $\Diamond\phi$ has subformulas itself plus $\phi$ and the subformulas of $\phi$.

A corollary of the eliminability of the cut-rule is the subformula property. This is the assertion that every proof may be replaced by a proof in which all the formulas appearing in all the judgements (of the new proof) are subformulas of the formulas appearing in the final judgement of the proof.

The cut-elimination theorem allows us to prove decidability for the intuitionistic modal calculus. Call a judgement containing at most three occurrences of any formula in each of the antecedent and succedent a reduced judgement. In order to give a decision procedure it is useful to note the following two facts.

1. Any judgement is provable if and only if a reduced judgement containing the same formulas is provable.
2. There are only a finite number of reduced judgements all of whose formulas are subformulas of a given judgement.

The decision procedure then consists of working upwards to construct a proof from the bottom of the tree, starting with the reduced judgement corresponding to the given judgement, and searching through the finite number of judgements mentioned in 2, to either construct a proof or conclude that there can be no proof.

It may also be possible to give a cut-elimination proof for the classical sequent calculus presented. One way of approaching such a proof would be to modify the method above to suit the classical rules and connectives. There is an added complication in these types of classical cut-elimination, in that we now have to consider cases for when a formula on right-hand-side of the left-hand premise of a cut (or mix) is the cut (mix) formula and when it is not, since classical sequents may have more than one formula in their succedent.

# A

## A.1 Properties of Modal Heyting Algebras

We list here some properties of modal Heyting algebras. Proofs are not given for those which are standard for Heyting algebras.

Every modal Heyting algebra $H$ has the following properties for all $a$, $b$, $c$, $d$ in $H$.

(i) $a \wedge b = b \wedge a$

(ii) $a \vee b = b \vee a$

(iii) If $a \leq a'$ then $a \wedge b \leq a' \wedge b$

(iv) If $a = a'$ then $a \wedge b = a' \wedge b$.

(v) If $a \leq a'$ then $a \vee b \leq a' \vee b$.

(vi) If $a = a'$ then $a \vee b = a' \vee b$

(vii) $a \wedge a = a$

(viii) $a \vee a = a$

(ix) $a \wedge \bot = \bot$

(x) $a \vee \bot = a$

(xi) $a \wedge \top = a$

(xii) $a \vee \top = \top$

(xiii) If $b \leq a \to c$ then $a \wedge b \leq c$.

(xiv) If $a \leq b$ then $b \to c \leq a \to c$.

(xv) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(xvi) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(xvii) If $a \leq b$ and $a \wedge c \leq d$ then $a \wedge (b \to c) \leq d$

(xviii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(xix) $a \vee (b \vee c) = (a \vee b) \vee c$.

(xx) For any $a_1, a_2, \ldots, a_n, b$ in $H$ we have
$\Box a_1 \wedge \Box a_2 \wedge \ldots \wedge \Box a_n \leq \Box(a_1 \wedge a_2 \wedge \ldots \wedge a_n)$
[This is proved by induction on $n$ using axiom 10.]

(xxi) For any $a_1, a_2, \ldots, a_n, b$ in $H$ we have
$\Box a_1 \wedge \Box a_2 \wedge \ldots \wedge \Box a_n \wedge \Diamond b \leq \Diamond(a_1 \wedge a_2 \wedge \ldots \wedge a_n \wedge b)$.
[This is by proved using the last property and axiom 11 as follows.
$\Box a_1 \wedge \ldots \wedge \Box a_n \wedge \Diamond b \leq \Box(a_1 \wedge \ldots a_n) \wedge \Diamond b \leq \Diamond(a_1 \wedge \ldots a_n \wedge b)$.]

## A.2 Soundness

Here are the details of the proof of soundness for the intuitionistic calculus.

**Proof** The proof is by induction on the height of the proof of the judgement $\phi_1, \ldots, \phi_n \vdash \phi$. The base of the induction consists of checking the statement of the lemma for the axioms, $\top$-right and $\bot$-left rules. The induction step consists of proving for the remaining rules that if the statement of the lemma holds for the judgements above a rule then it also holds for the judgement below the rule. All the cases are checked below using the axioms of modal Heyting algebras and the properties derived above.

**Axioms** Every axiom is of the form $\phi_1, \dots, \phi_n, \phi \vdash \phi$. Notice that $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket = \llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket$ follows by induction from $\llbracket\theta \wedge \psi\rrbracket = \llbracket\theta\rrbracket \wedge \llbracket\psi\rrbracket$. Then by property (iv) and axiom 1 we have $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket = \llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket \wedge \leq \llbracket\phi\rrbracket$.

**$\bot$-left** Any instance of the rule looks like $\phi_1, \dots, \phi_n, \bot \vdash \psi$. By property (ix) and axiom 7, $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\bot\rrbracket = \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \bot = \bot \leq \llbracket\psi\rrbracket$.

**$\top$-right** An instance of this rule is of the form $\phi_1, \dots, \phi_n \vdash \top$. Axiom 8 gives $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \top = \llbracket\top\rrbracket$.

**$\top$-left** Suppose $\phi_1, \dots, \phi_n \vdash \phi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$. The $\top$-left rule proves $\phi_1, \dots, \phi_n, \top \vdash \phi$ and property (xi) gives $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\top\rrbracket = \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \top = \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$.

**Weakening** Suppose that $\phi_1, \dots, \phi_n \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\psi\rrbracket$. By the weakening rule we can prove $\phi'_1, \dots, \phi'_p, \phi_1, \dots, \phi_n \vdash \psi$ and from axiom 1 we have $\llbracket\phi'_1\rrbracket \wedge \dots \wedge \llbracket\phi'_p\rrbracket \wedge \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\psi\rrbracket$.

**Contraction** Suppose $\phi_1, \dots, \phi_n, \phi, \phi \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$. The left contraction rule allows us to derive $\phi_1, \dots, \phi_n, \phi \vdash \psi$ and properties (vii) and (iv) give $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket = \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$.

**Cut** Suppose $\phi_1, \dots, \phi_n \vdash \phi$ and $\phi'_1, \dots, \phi'_p, \phi \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$ and $\llbracket\phi'_1\rrbracket \wedge \dots \wedge \llbracket\phi'_p\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$. The cut rule gives a proof of $\phi_1, \dots, \phi_n, \phi'_1, \dots, \phi'_p \vdash \psi$. By property (iii) we have $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi'_1\rrbracket \wedge \dots \wedge \llbracket\phi'_p\rrbracket \leq \llbracket\phi\rrbracket \wedge \llbracket\phi'_1\rrbracket \wedge \dots \wedge \llbracket\phi'_p\rrbracket \leq \llbracket\psi\rrbracket$.

**$\wedge$-left** Suppose $\phi_1, \dots, \phi_n, \phi, \psi \vdash \theta$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket \leq \llbracket\theta\rrbracket$. The $\wedge$-left rule gives a proof of $\phi_1, \dots, \phi_n, \phi \wedge \psi \vdash \theta$. Since $\llbracket\phi \wedge \psi\rrbracket = \llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket$ we have $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi \wedge \psi\rrbracket = \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket \leq \llbracket\theta\rrbracket$ using (iv).

**$\wedge$-right** Suppose $\phi_1, \dots, \phi_n \vdash \phi$ and $\phi_1, \dots, \phi_n \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\psi\rrbracket$. The $\wedge$-right rule yields $\phi_1, \dots, \phi_n \vdash \phi \wedge \psi$. Axiom 2 and the definition of $\llbracket\cdot\rrbracket$ give $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket = \llbracket\phi \wedge \psi\rrbracket$.

**$\vee$-left** Suppose $\phi_1, \dots, \phi_n, \phi \vdash \theta$ and $\phi_1, \dots, \phi_n, \psi \vdash \theta$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\theta\rrbracket$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\psi\rrbracket \leq \llbracket\theta\rrbracket$. The $\vee$-left rule proves $\phi_1, \dots, \phi_n, \phi \vee \psi \vdash \theta$. The definition of $\llbracket\cdot\rrbracket$, the distributivity of property (xvi), properties (v) and (viii) give

$$\begin{array}{ll} & \llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \wedge (\llbracket\phi\rrbracket \vee \llbracket\psi\rrbracket) \\ = & \llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket \wedge (\llbracket\phi\rrbracket \vee \llbracket\psi\rrbracket) \\ = & (\llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket \wedge \llbracket\phi\rrbracket) \vee (\llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket \wedge \llbracket\psi\rrbracket) \\ \leq & \llbracket\theta\rrbracket \vee (\llbracket\phi_1 \wedge \dots \wedge \phi_n\rrbracket \wedge \llbracket\psi\rrbracket) \\ \leq & \llbracket\theta\rrbracket \vee \llbracket\theta\rrbracket \\ = & \llbracket\theta\rrbracket. \end{array}$$

**$\vee$-right-1** Suppose $\phi_1, \dots, \phi_n \vdash \phi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$. The $\vee$-right-1 rule yields $\phi_1, \dots, \phi_n \vdash \phi \vee \psi$. Axiom 3 and the definition of $\llbracket\cdot\rrbracket$ gives $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket \leq \llbracket\phi\rrbracket \vee \llbracket\psi\rrbracket = \llbracket\phi \vee \psi\rrbracket$.

**$\vee$-right-2** Suppose $\phi_1, \dots, \phi_n \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\psi\rrbracket$. The $\vee$-right-2 rule yields $\phi_1, \dots, \phi_n \vdash \phi \vee \psi$. Axiom 3 and the definition of $\llbracket\cdot\rrbracket$ gives $\llbracket\phi_1\rrbracket \wedge \dots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\psi\rrbracket \leq \llbracket\phi\rrbracket \vee \llbracket\psi\rrbracket = \llbracket\phi \vee \psi\rrbracket$.

**$\rightarrow$-right** Suppose $\phi_1, \ldots, \phi_n, \phi \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$. Then $\phi_1, \ldots, \phi_n \vdash \phi \rightarrow \psi$. Axiom 6 and the definition of $\llbracket\cdot\rrbracket$ gives
$\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket \rightarrow \llbracket\psi\rrbracket = \llbracket\phi \rightarrow \psi\rrbracket$.

**$\rightarrow$-left** Suppose $\phi_1, \ldots, \phi_n \vdash \phi$ and $\phi_1, \ldots, \phi_n, \psi \vdash \theta$ and
$\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$ and $\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\psi\rrbracket \leq \llbracket\theta\rrbracket$. The $\rightarrow$-left rule allows us to prove $\phi_1, \ldots, \phi_n, \phi \rightarrow \psi \vdash \theta$. Property (xvii) gives the required result since
$\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi \rightarrow \psi\rrbracket = \llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge (\llbracket\phi\rrbracket \rightarrow \llbracket\psi\rrbracket) \leq \llbracket\theta\rrbracket$.

**$\Box$-right** Suppose $\phi_1, \ldots, \phi_n \vdash \phi$ and $\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \leq \llbracket\phi\rrbracket$. The $\Box$-right rule gives $\phi'_1, \ldots, \phi'_p, \Box\phi_1, \ldots, \Box\phi_n \vdash \Box\phi$. Now by axioms 1, 13 and property (xx),

$$\begin{array}{rcl} \llbracket\phi'_1\rrbracket \wedge \ldots \wedge \llbracket\phi'_p\rrbracket \wedge \llbracket\Box\phi_1\rrbracket \wedge \ldots \wedge \llbracket\Box\phi_n\rrbracket & \leq & \llbracket\Box\phi_1\rrbracket \wedge \ldots \wedge \llbracket\Box\phi_n\rrbracket \\ & = & \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \\ & \leq & \Box(\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket) \\ & \leq & \Box\llbracket\phi\rrbracket. \end{array}$$

**$\Diamond$-left-1** Suppose $\phi_1, \ldots, \phi_n, \phi \vdash \psi$ and $\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$. The $\Diamond$-left rule gives $\phi'_1, \ldots, \phi'_p, \Box\phi_1, \ldots, \Box\phi_n, \Diamond\phi \vdash \Diamond\psi$. Now axioms 1, 14 and property (xxi) give,

$$\begin{array}{rl} & \llbracket\phi'_1\rrbracket \wedge \ldots \wedge \llbracket\phi'_p\rrbracket \wedge \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket \\ \leq & \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket \\ \leq & \Diamond(\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket) \\ \leq & \Diamond\llbracket\psi\rrbracket. \end{array}$$

**$\Diamond$-left-2** Suppose $\phi_1, \ldots, \phi_n, \phi \vdash \bot$ and $\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket \wedge \llbracket\phi\rrbracket \leq \bot$. Then we can construct a proof of $\phi'_1, \ldots, \phi'_p, \Box\phi_1, \ldots, \Box\phi_n, \Diamond\phi \vdash \psi$. By axiom 1,
$\llbracket\phi'_1\rrbracket \wedge \ldots \wedge \llbracket\phi'_p\rrbracket \wedge \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket \leq \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket$.
Now by property (xxi), axiom 14 and the hypothesis,
$\Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket \leq \Diamond(\llbracket\phi_1\rrbracket \wedge \ldots \wedge \llbracket\phi_n\rrbracket) \leq \Diamond\bot$ . Axiom 12 gives $\Diamond\bot \leq \bot$ and axiom 7 gives $\bot \leq \llbracket\psi\rrbracket$. Putting all this together with the transitivity of $\leq$ gives
$\llbracket\phi'_1\rrbracket \wedge \ldots \wedge \llbracket\phi'_p\rrbracket \wedge \Box\llbracket\phi_1\rrbracket \wedge \ldots \wedge \Box\llbracket\phi_n\rrbracket \wedge \Diamond\llbracket\phi\rrbracket \leq \llbracket\psi\rrbracket$.

This finishes the proof of soundness. ■

## A.3 Adequacy

**Proof** of Lemma 4.3 .
First we check that the operations are well-defined.

**$\vee$-case** Suppose $\theta_1 \sim \theta_2$ and $\psi_1 \sim \psi_2$. Then $\theta_1 \vdash \theta_2$, $\theta_2 \vdash \theta_1$, $\psi_1 \vdash \psi_2$, $\psi_2 \vdash \psi_1$ are provable. From these we construct the proof

$$\frac{\dfrac{\theta_1 \vdash \theta_2}{\theta_1 \vdash \theta_2 \vee \psi_2} \qquad \dfrac{\psi_1 \vdash \psi_2}{\psi_1 \vdash \theta_2 \vee \psi_2}}{\theta_1 \vee \psi_1 \vdash \theta_2 \vee \psi_2}$$

and interchanging 1 and 2 yields a proof of $\theta_2 \vee \psi \vdash \theta_2 \vee \psi_2$. Hence $[\theta_1] \vee [\psi_1] = [\theta_2] \vee [\psi_2]$.

**$\wedge$-case** Suppose $\theta_1 \sim \theta_2$ and $\psi_1 \sim \psi_2$. Then $\theta_1 \vdash \theta_2$, $\theta_2 \vdash \theta_1$, $\psi_1 \vdash \psi_2$, $\psi_2 \vdash \psi_1$ are provable. From these we can construct a proof

$$\frac{\dfrac{\dfrac{\theta_1 \vdash \theta_2}{\theta_1, \psi_1 \vdash \theta_2}}{\theta_1 \wedge \psi_1 \vdash \theta_2} \qquad \dfrac{\dfrac{\psi_1 \vdash \psi_2}{\theta_1, \psi_1 \vdash \psi_2}}{\theta_1 \wedge \psi_1 \vdash \psi_2}}{\theta_1 \wedge \psi_1 \vdash \theta_2 \wedge \psi_2}$$

and interchanging 1 and 2 in the above proof yields a proof of
$\theta_2 \wedge \psi \vdash \theta_2 \wedge \psi_2$. Hence $[\theta_1] \wedge [\psi_1] = [\theta_2] \wedge [\psi_2]$.

**$\to$-case** Suppose $\theta_1 \sim \theta_2$ and $\psi_1 \sim \psi_2$. Then $\theta_1 \vdash \theta_2$, $\theta_2 \vdash \theta_1$, $\psi_1 \vdash \psi_2$, $\psi_2 \vdash \psi_1$ are provable. From these we can construct the proof

$$\dfrac{\dfrac{\theta_2 \vdash \theta_1 \qquad \dfrac{\psi_1 \vdash \psi_2}{\theta_2, \psi_1 \vdash \psi_2}}{\theta_1 \to \psi_1, \theta_2 \vdash \psi_2}}{\theta_1 \to \psi_1 \vdash \theta_2 \to \psi_2}$$

and interchanging 1 and 2 in the above proof gives a proof of
$\theta_2 \to \psi \vdash \theta_2 \to \psi_2$.
Hence $[\theta_1] \to [\psi_1] = [\theta_2] \to [\psi_2]$.

**$\Box$-case** Suppose $[\theta_1] = [\theta_2]$. Then $\theta_1 \vdash \theta_2$ and $\theta_2 \vdash \theta_1$ have proofs. Then we can construct proofs

$$\dfrac{\theta_1 \vdash \theta_2}{\Box\theta_1 \vdash \Box\theta_2} \qquad\qquad \dfrac{\theta_2 \vdash \theta_1}{\Box\theta_2 \vdash \Box\theta_1}$$

so that $\Box[\theta_1] = \Box[\theta_2]$.

**$\Diamond$-case** Suppose $[\theta_1] = [\theta_2]$. Then $\theta_1 \vdash \theta_2$ and $\theta_2 \vdash \theta_1$ have proofs. Then we can construct proofs

$$\dfrac{\theta_1 \vdash \theta_2}{\Diamond\theta_1 \vdash \Diamond\theta_2} \qquad\qquad \dfrac{\theta_2 \vdash \theta_1}{\Diamond\theta_2 \vdash \Diamond\theta_1}$$

so that $\Diamond[\theta_1] = \Diamond[\theta_2]$.

We can conclude that all the operations are well-defined.
Now we check that $\mathcal{L}'$ satisfies the axioms for a modal Heyting algebra.
Let $a = [\phi]$, $b = [\psi]$ and $c = [\theta]$ be any three equivalence classes.

1. $a \wedge b \leq a \quad \Leftrightarrow \quad (a \wedge b) \wedge a = a \wedge b$.
   We can prove the right hand side with the two proofs constructed below

$$\dfrac{\phi \wedge \psi, \phi \vdash \phi \wedge \psi}{(\phi \wedge \psi) \wedge \phi \vdash \phi \wedge \psi} \qquad\qquad \dfrac{\phi \wedge \psi \vdash \phi \wedge \psi \qquad \dfrac{\phi, \psi \vdash \phi}{\phi \wedge \psi \vdash \phi}}{\phi \wedge \psi \vdash (\phi \wedge \psi) \wedge \phi}$$

   Similarly, $a \wedge b \leq b$.

2. $c \leq a \;\&\; c \leq b \quad \Rightarrow \quad c \leq a \wedge b$.
   Suppose that $c \leq a \quad \& \quad c \leq b$. Then $c \wedge a = c$ and $c \wedge b = c$ so that there are proofs of $\theta \wedge \phi \vdash \theta$ , $\theta \vdash \theta \wedge \phi$ ,$\theta \wedge \psi \vdash \theta$ , $\theta \vdash \theta \wedge \psi$. With these we can construct proofs

$$\dfrac{\theta, \phi \wedge \psi \vdash \theta}{\theta \wedge (\phi \wedge \psi) \vdash \theta}$$

$$\dfrac{\theta \vdash \theta \qquad \dfrac{\dfrac{\theta \vdash \theta \wedge \phi \qquad \dfrac{\theta, \phi \vdash \phi}{\theta \wedge \phi \vdash \phi}}{\theta \vdash \phi} \qquad \dfrac{\theta \vdash \theta \wedge \psi \qquad \dfrac{\theta, \psi \vdash \psi}{\theta \wedge \psi \vdash \psi}}{\theta \vdash \psi}}{\theta \vdash \phi \wedge \psi}}{\theta \vdash \theta \wedge (\phi \wedge \psi)}$$

   so that we have proofs of $\theta \wedge (\phi \wedge \psi) \vdash \theta$ and $\theta \vdash \theta \wedge (\phi \wedge \psi)$ and hence $(a \wedge b) \wedge c = c$ as required.

3. $a \leq a \vee b$ since we can construct proofs

$$\frac{\phi, \phi \vee \psi \vdash \phi}{\phi \wedge (\phi \vee \psi) \vdash \phi} \qquad\qquad \frac{\phi \vdash \phi \qquad \dfrac{\phi \vdash \phi}{\phi \vdash \phi \vee \psi}}{\phi \vdash \phi \wedge (\phi \vee \psi)}$$

   so that $a \wedge (a \vee b) = a$. Similarly $b \leq a \vee b$

4. Suppose that $c \geq a$ and $c \geq b$. Then $c \wedge a = a$ and $c \wedge b = b$ so that there are proofs of $\theta \wedge \phi \vdash \phi$, $\phi \vdash \theta \wedge \phi$, $\theta \wedge \psi \vdash \psi$ and $\psi \vdash \theta \wedge \psi$ from which we can construct proofs

$$\frac{\theta, \phi \vee \psi \vdash \phi \vee \psi}{\theta \wedge (\phi \vee \psi) \vdash \phi \vee \psi}$$

$$\dfrac{\dfrac{\dfrac{\phi \vdash \theta \wedge \phi \qquad \dfrac{\theta, \phi \vdash \theta}{\theta \wedge \phi \vdash \theta}}{\phi \vdash \theta} \qquad \dfrac{\psi \vdash \theta \wedge \psi \qquad \dfrac{\theta, \psi \vdash \theta}{\theta \wedge \psi \vdash \theta}}{\psi \vdash \theta}}{\phi \vee \psi \vdash \theta} \qquad \phi \vee \psi \vdash \phi \vee \psi}{\phi \vee \psi \vdash \theta \wedge (\phi \vee \psi)}$$

5. We can construct proofs

$$\frac{\phi \wedge (\phi \to \psi), \psi \vdash \phi \wedge (\phi \to \psi)}{(\phi \wedge (\phi \to \psi)) \wedge \psi \vdash \phi \wedge (\phi \to \psi)}$$

$$\dfrac{\dfrac{\dfrac{\phi \vdash \phi \qquad \phi, \psi \vdash \psi}{\phi, \phi \to \psi \vdash \psi}}{\phi \wedge (\phi \to \psi) \vdash \psi} \qquad \phi \wedge (\phi \to \psi) \vdash \phi \wedge (\phi \to \psi)}{\phi \wedge (\phi \to \psi) \vdash \psi \wedge (\phi \wedge (\phi \to \psi))}$$

   which gives the result $\phi \wedge (\phi \to \psi) \sim \psi \wedge (\phi \wedge (\phi \to \psi))$ so that $a \wedge (a \to b) \leq b$.

6. Suppose that $a \wedge c \leq b$. Then there are proofs of $(\phi \wedge \theta) \wedge \psi \vdash \phi \wedge \theta$ and $\phi \wedge \theta \vdash (\phi \wedge \theta) \wedge \psi$. Using these we can construct the following proofs

$$\frac{\theta, \phi \to \psi \vdash \theta}{\theta \wedge (\phi \to \psi) \vdash \theta}$$

$$\dfrac{\theta \vdash \theta \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\theta, \phi \vdash \phi \qquad \theta, \phi \vdash \theta}{\theta, \phi \vdash \phi \wedge \theta} \qquad \phi \wedge \theta \vdash (\phi \wedge \theta) \wedge \psi}{\theta, \phi \vdash (\phi \wedge \theta) \wedge \psi} \qquad \dfrac{\phi \wedge \theta, \psi \vdash \psi}{(\phi \wedge \theta) \wedge \psi \vdash \psi}}{\theta, \phi \vdash \psi}}{\theta \vdash \phi \to \psi}}{\theta \vdash \theta \wedge (\phi \to \psi)}$$

   So we have $\theta \wedge (\phi \to \psi) \sim \theta$. So $[\theta] \wedge ([\phi] \to [\psi]) = [\theta]$ and hence $c \leq a \to b$.

7. For any formula $\phi$ we can construct proofs

$$\dfrac{\bot, \phi \vdash \bot}{\bot \wedge \phi \vdash \bot} \qquad\qquad \bot \vdash \bot \wedge \phi$$

from which it follows that $\bot \leq a$.

8. For any formula $\phi$ we can construct proofs

$$\dfrac{\phi, \top \vdash \phi}{\phi \wedge \top \vdash \phi} \qquad\qquad \dfrac{\phi \vdash \phi \qquad \phi \vdash \top}{\phi \vdash \phi \wedge \top}$$

from which it follows that $a \leq \top$.

9. We can construct proofs

$$\top \wedge \Box\top \vdash \top \qquad\qquad \dfrac{\top \vdash \top \qquad \dfrac{\vdash \top}{\top \vdash \Box\top}}{\top \vdash \top \wedge \Box\top}$$

so that $[\top] = [\top] \wedge \Box[\top]$ and hence $\top \leq \Box\top$.

10. For any formulas $\phi$ ,$\psi$ we can construct proofs

$$\dfrac{\Box\phi \wedge \Box\psi, \Box(\phi \wedge \psi) \vdash \Box\phi \wedge \Box\psi}{\Box\phi \wedge \Box\psi \wedge \Box(\phi \wedge \psi) \vdash \Box\phi \wedge \Box\psi}$$

$$\dfrac{\Box\phi \wedge \Box\psi \vdash \Box\phi \wedge \Box\psi \qquad \dfrac{\dfrac{\dfrac{\phi, \psi \vdash \phi \qquad \phi, \psi \vdash \psi}{\phi, \psi \vdash \phi \wedge \psi}}{\Box\phi, \Box\psi \vdash \Box(\phi \wedge \psi)}}{\Box\phi \wedge \Box\psi \vdash \Box(\phi \wedge \psi)}}{\Box\phi \wedge \Box\psi \vdash \Box\phi \wedge \Box\psi \wedge \Box(\phi \wedge \psi)}$$

so $\Box\phi \wedge \Box\psi \sim \Box\phi \wedge \Box\psi \wedge \Box(\phi \wedge \psi)$ and hence $\Box a \wedge \Box b \leq \Box(a \wedge b)$.

11. For any formulas $\phi, \psi$ we can construct proofs

$$\dfrac{\Box\phi \wedge \Diamond\psi, \Diamond(\phi \wedge \psi) \vdash \Box\phi \wedge \Diamond\psi}{\Box\phi \wedge \Diamond\psi \wedge \Diamond(\phi \wedge \psi) \vdash \Box\phi \wedge \Diamond\psi}$$

$$\dfrac{\Box\phi \wedge \Diamond\psi \vdash \Box\phi \wedge \Diamond\psi \qquad \dfrac{\dfrac{\dfrac{\phi, \psi \vdash \phi \qquad \phi, \psi \vdash \psi}{\phi, \psi \vdash \phi \wedge \psi}}{\Box\phi, \Diamond\psi \vdash \Diamond(\phi \wedge \psi)}}{\Box\phi \wedge \Diamond\psi \vdash \Diamond(\phi \wedge \psi)}}{\Box\phi \wedge \Diamond\psi \vdash \Box\phi \wedge \Diamond\psi \wedge \Diamond(\phi \wedge \psi)}$$

so $\Box[\phi] \wedge \Diamond[\psi] \wedge \Diamond[\phi \wedge \psi] = \Box[\phi] \wedge \Diamond[\psi]$ and hence $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

12. We can construct proofs

$$\dfrac{\Diamond\bot, \bot \vdash \Diamond\bot}{\Diamond\bot \wedge \bot \vdash \Diamond\bot}$$

$$\dfrac{\Diamond\bot \vdash \Diamond\bot \qquad \dfrac{\bot \vdash \bot}{\Diamond\bot \vdash \bot}}{\Diamond\bot \vdash \Diamond\bot \wedge \bot}$$

from which it follows that $\Diamond[\bot] = \Diamond[\bot] \wedge [\bot]$ so we can conclude $\Diamond\bot \leq \bot$.

13. Suppose that $a \leq b$. Then $[\phi] \wedge [\psi] = [\phi]$ and so there are proofs of $\phi \wedge \psi \vdash \phi$ and $\phi \vdash \phi \wedge \psi$. From these we can construct proofs

$$\dfrac{\dfrac{\phi, \psi \vdash \phi}{\Box\phi, \Box\psi \vdash \Box\phi}}{\Box\phi \wedge \Box\psi \vdash \Box\phi}$$

$$\dfrac{\Box\phi \vdash \Box\phi \qquad \dfrac{\dfrac{\phi \vdash \phi \wedge \psi \qquad \dfrac{\phi, \psi \vdash \psi}{\phi \wedge \psi \vdash \psi}}{\phi \vdash \psi}}{\Box\phi \vdash \Box\psi}}{\Box\phi \vdash \Box\phi \wedge \Box\psi}$$

from which it follows that $\Box[\phi] = \Box[\phi] \wedge \Box[\psi]$ and as a consequence we have $\Box a \leq \Box b$.

14. Suppose that $a \leq b$. Then $[\phi] \wedge [\psi] = [\phi]$ and so there are proofs of $\phi \wedge \psi \vdash \phi$ and $\phi \vdash \phi \wedge \psi$. From these we can construct proofs

$$\dfrac{\Diamond\phi, \Diamond\psi \vdash \Diamond\phi}{\Diamond\phi \wedge \Diamond\psi \vdash \Diamond\phi}$$

$$\dfrac{\Diamond\phi \vdash \Diamond\phi \qquad \dfrac{\dfrac{\phi \vdash \phi \wedge \psi \qquad \dfrac{\phi, \psi \vdash \psi}{\phi \wedge \psi \vdash \psi}}{\phi \vdash \psi}}{\Diamond\phi \vdash \Diamond\psi}}{\Diamond\phi \vdash \Diamond\phi \wedge \Diamond\psi}$$

from which it follows that $\Diamond[\phi] \wedge \Diamond[\psi] = \Diamond[\phi]$ and hence that $\Diamond a \leq \Diamond b$.

This completes the proof that $\mathcal{L}'$ is a modal Heyting algebra. ■

**Proof** of Lemma 4.4

The proof is by induction on the complexity of formulas.

- $[\![\bot]\!] = \bot = [\bot] \quad , \quad [\![\top]\!] = \top = [\top]$
- $[\![P]\!] = \alpha(P) = [P] \qquad$ for every variable $P$.
- Suppose that $[\![\psi]\!] = [\psi]$ and $[\![\theta]\!] = [\theta]$.
  Then $[\![\psi \wedge \theta]\!] = [\![\psi]\!] \wedge [\![\theta]\!] = [\psi] \wedge [\theta] = [\psi \wedge \theta]$ as required.
- Suppose that $[\![\psi]\!] = [\psi]$ and $[\![\theta]\!] = [\theta]$.
  Then $[\![\psi \vee \theta]\!] = [\![\psi]\!] \vee [\![\theta]\!] = [\psi] \vee [\theta] = [\psi \vee \theta]$ as required.
- Suppose that $[\![\psi]\!] = [\psi]$ and $[\![\theta]\!] = [\theta]$.
  Then $[\![\psi \to \theta]\!] = [\![\psi]\!] \to [\![\theta]\!] = [\psi] \to [\theta] = [\psi \to \theta]$ as required.
- Suppose that $[\![\psi]\!] = [\psi]$.
  Then $[\![\Box\psi]\!] = \Box[\![\psi]\!] = \Box[\psi] = [\Box\psi]$ as required.
- Suppose that $[\![\psi]\!] = [\psi]$.
  Then $[\![\Diamond\psi]\!] = \Diamond[\![\psi]\!] = \Diamond[\psi] = [\Diamond\psi]$ as required.

This completes the induction. ■

**Proof** of Lemma 4.5.

Suppose that $[\![\phi]\!] \leq [\![\psi]\!]$. Then by the previous lemma $[\phi] \leq [\psi]$ so that $[\phi] \wedge [\psi] = [\phi]$ and hence there are proofs of $\phi \wedge \psi \vdash \phi$ and $\phi \vdash \phi \wedge \psi$. From these we can construct the proof

$$\dfrac{\phi \vdash \phi \wedge \psi \qquad \dfrac{\phi, \psi \vdash \psi}{\phi \wedge \psi \vdash \psi}}{\phi \vdash \psi}$$

yielding a proof of the judgement $\phi \vdash \psi$. ■

**Proof** of Lemma 4.6.

(i) This is proved by induction on $n$. The case $n = 1$ holds trivially. Suppose that the statement holds for $n = k$. We prove the statement for $n = k + 1$ with the proof

$$\dfrac{\dfrac{\phi_1, \ldots, \phi_k \vdash \phi_1 \wedge \ldots \wedge \phi_k}{\phi_1, \ldots, \phi_k, \phi_{k+1} \vdash \phi_1 \wedge \ldots \wedge \phi_k} \qquad \phi_1, \ldots, \phi_k, \phi_{k+1} \vdash \phi_{k+1}}{\phi_1, \ldots, \phi_k, \phi_{k+1} \vdash \phi_1 \wedge \ldots \wedge \phi_k \wedge \phi_{k+1}}$$

so we can prove $\phi_1, \ldots, \phi_k, \phi_{k+1} \vdash \phi_1 \wedge \ldots \wedge \phi_k \wedge \phi_{k+1}$. By induction (i) holds for all $n$.

(ii) By (i) there is a proof of $\phi_1, \ldots, \phi_n \vdash \phi_1 \wedge \ldots \wedge \phi_n$ so supposing that $\phi_1 \wedge \ldots \wedge \phi_n \vdash \psi$ we can construct a proof of $\phi_1, \ldots, \phi_n \vdash \psi$ with the cut

$$\dfrac{\phi_1, \ldots, \phi_n \vdash \phi_1 \wedge \ldots \wedge \phi_n \qquad \phi_1 \wedge \ldots \wedge \phi_n \vdash \theta}{\phi_1, \ldots, \phi_n \vdash \theta}$$

■

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