A First Order Modal Logic and its Sheaf Models

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Abstract: We present a new way of formulating first order modal logic which circumvents the usual difficulties associated with variables changing their reference on moving between states. This formulation allows a very general notion of model (sheaf models). The key idea is the introduction of syntax for describing relations between individuals in related states. This adds an extra degree of expressiveness to the logic, and also appears to provide a means of describing the dynamic behaviour of computational systems in a way somewhat different from traditional program logics.

1 Introduction

1.1 First order modal logic

The fundamental difficulty in formulating a first order modal logic is the shifting reference of variables. Let ϕ be a proposition with a free variable x. Then x ranges over individuals in a state (we use the terminology of transition relations between states, rather than accessibility relations between worlds). However, in the proposition $\Box\phi$, whose assertion we read as "in all related states, ϕ holds", x has now shifted its reference to individuals in related states. With this shift, substitution may become compromised, so that $(\Box\phi)[t/x]$ and $\Box(\phi[t/x])$ may not both be defined or be equal. Notice also that the range of quantification shifts. In $\forall x.\Box\phi$ the quantification is over the set of individuals in the present state, whereas in $\Box(\forall x.\phi)$ the quantification is over sets of individuals in related states.

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Formulations of first order modal logics have met these difficulties in several ways: The usual method is to restrict the range of models. Models where individuals in a state are also present in related states (i.e. there is an inclusion between these sets of individuals) have been proposed (e.g. [29]). The usual treatment is even more restrictive, but accommodates quantification better: the condition imposed on models is that the same set of individuals is present at each state (called 'constant domain' models, [10]). Some authors (e.g. [3]) also introduce a calculus of explicit substitutions into the logic.

These solutions are unsatisfactory. Firstly, there is no logical apparatus to treat the shifting reference of variables, neither any notation nor any inference rules. Secondly, the restrictions on models are draconian: modal logic is meant to be a logic of changing, dynamic situations, yet at its heart we impose a constancy on the sets of individuals in models. Notice also that the logic admits a language for describing subsets of individuals, namely the language of propositions, but there is no language for describing relations between individuals in related states; such relations are also just subsets - subsets of products of sets of individuals.

It was whilst examining logics of sheaves, and with logics of computation in mind, that the first author devised an alternative way of formulating modal logics that circumvents these difficulties. This formulation allows a very general notion of model (sheaf models), and we do not need any of the above restrictions on sets of individuals. The key idea that makes this possible is the introduction of another syntax class to provide a language for describing relations between individuals in related states. We call these (provisionally) *transition propositions*. Not only does this circumvent previous difficulties, but also it adds an extra degree of expressiveness to the logic. Moreover, it appears to provide a means of describing the dynamic behaviour of computational systems in a way somewhat different from traditional program logics (a point we return to later).

We begin with an informal explanation of the new logic. For ordinary propositions ϕ whose free variables are drawn from a (typed) context $\Gamma = x_1 : X_1, \ldots, x_n : X_n$, we write $[\Gamma]\phi$. Transition propositions ϕ have variables drawn from *two* contexts Γ and Δ and we write

$$[\Gamma][\Delta]\phi$$

for a transition proposition in context. Such a ϕ relates individuals in the present state – ranged over by the variables of Γ – to individuals in a related state – ranged over by the variables of Δ.

One form of transition proposition is given by

$$[\Gamma][\Delta]\, t \twoheadrightarrow_X u$$

where t is a term of type X in context Γ and u a term, also of type X, in context Δ. We read this as "t may become u" (note that $\twoheadrightarrow$ here is a connective for building transition propositions, and is distinct from the relation present in models which we also denote by $\twoheadrightarrow$; the notation suggests the meaning). We introduce quantifiers which link variables in related states. For example, for transition proposition $[\Gamma][\Delta, x : X]\,\phi$, we form

$$[\Gamma][\Delta]\,\forall x \twoheadleftarrow_X t.\ \phi$$

whose assertion we read (informally) as:

> "for all x that t may become in the related state, ϕ holds".

These 'bounded' quantifiers are one form of modality. The $\Box$ and $\Diamond$ modal operators are also present, but are not dual complements as the logic is intuitionistic. Using $\Box$ and $\Diamond$ we may assert, for example,

$$[\Gamma]\,\Diamond(\forall x \twoheadleftarrow_X t.\ \phi)$$

where ϕ is a transition proposition in context $[\Gamma][x : X]$. This is read as

> "there is a related state, such that for all x that t may become in this related state, ϕ holds".

Those familiar with logics for reasoning about programs in imperative (state-based) programming languages may recognise some of the ideas in this new logic as reminiscent of 'values of variables' as they change through a computation. In particular, a notation for the value of a variable before and after execution is often introduced into such logics. For example, Jones [12] uses the notation $\overleftharpoon{x}$ to denote the value of a variable x before execution of a program, the value after is simply x. We may then write 'specifications' i.e. descriptions of program behaviour, such as

$$\overleftharpoon{x} \geq 0 \;\Rightarrow\; x \geq \overleftharpoon{x}.$$

In the modal logic, we may interpret this as

$$x \geq 0 \;\Rightarrow\; \Box(\forall y \twoheadleftarrow x.\ y \geq x)$$

that is "if $x \geq 0$, then in all final states, for all y that x may become, $y \geq x$". At the moment this is no more than an analogy, but it suggests that the modal logic we present may well encompass some forms of program logics.

1.2 Sheaf models

Later in the paper we introduce a general construction of models for first order intuitionistic modal logic. This construction draws upon three well-established approaches to logic:

1. Topological models of intuitionistic logic [11],
2. Relational models of modal logic [13], [14],
3. Sheaf models of first order logic [5], [25], [15].

The first two of these combine to provide models for propositional intuitionistic modal logics, and have been studied in [8]. The starting point is the simple notion, called a *relational space* in [8], of a set S together with two independent structures on it: a topology $\mathcal{O}(S)$ and a binary relation, which we write as $\twoheadrightarrow \subseteq S \times S$.

Two examples of relational spaces indicate the kinds of systems we are interested in describing: (1) The relational space consisting of the real numbers with their metric topology and the $\leqslant$ relation. (2) Let S be a domain-theoretic model of the state of a computer system, with the Scott topology. Let two states be related if they are the initial and final states of a possible 'run' of the system. This is a natural model of the dynamic logic of computer programs.

Maps between relational spaces are not simply continuous functions which preserve the relations, but are an adaptation to topological spaces of the notion of *p-morphisms* – a form of 'simulation' between relational spaces. Propositional modal logics are introduced as certain partially ordered sets: a *modal frame* $(A, \leqslant, \Box, \Diamond)$ is a frame $(A, \leqslant)$ (i.e. a partially ordered set with all joins, and finite meets, joins distributing over finite meets) together with a pair of monotone maps $\Box, \Diamond : A \to A$ satisfying

$$\begin{aligned} \top &\leqslant \Box(\top) \\ \Box(a) \wedge \Box(b) &\leqslant \Box(a \wedge b) \\ \Diamond(a) \wedge \Box(b) &\leqslant \Diamond(a \wedge b) \\ \Diamond(\bot) &\leqslant \bot. \end{aligned}$$

A morphism of modal frames is a frame morphism f laxly preserving the modal operators:

$$\begin{aligned} f(\Box(a)) &\leqslant \Box(f(a)) \\ f(\Diamond(a)) &\leqslant \Diamond(f(a)) \end{aligned}$$

Hilken [8] extends the contravariant adjunction between topological spaces and frames (see, for example, [11]) to the case of relational spaces and modal frames, and also considers cases where this restricts to a duality (a contravariant equivalence of categories). This analysis not only introduces a class of propositional intuitionistic modal logics but also gives a mathematical setting for their topological semantics.

This present paper is one of a series in which we extend the development of topological models of intuitionistic modal logic to sheaf models and to first order logics (another paper in the series is [9]).

We begin by setting out the logic in Section 2 and then describe the sheaf models in Sections 3 and 4.

2 A first order intuitionistic modal logic

We present a first order intuitionistic modal logic. For conciseness we present only the rules that illustrate the modal structure of the logic. The omitted logical connectives are handled using standard rules and standard semantics (see e.g. [15]) and are discussed below. Moreover, the types are restricted to first order – higher order types (function types) are not included although the semantics could be extended to cope with them (see [15]).

We present the logic as a sequent calculus. The rules for the modalities $\Box$ and $\Diamond$ are an adaptation of those for classical modal logic. They may be found in, for example, [29] and [2].

2.1 Signature

The logic is built from a *signature* of symbols consisting of

1. a set of sorts (type symbols X),
2. a set of operations (function symbols $f : X_1 \times \cdots \times X_n \to X$),
3. a set of relations (predicate symbols $r : X_1 \times \cdots \times X_n$).

2.2 Variables, Contexts and Terms

Judgements:

- $[\Gamma]$ means that Γ is a well-formed context of the form $x_1 : X_1, \ldots, x_n : X_n$, i.e. $x_1, \ldots, x_n$ are distinct variables, and $X_1, \ldots, X_n$ are types.

- $[\Gamma][\Delta]$ means $[\Gamma]$ and $[\Delta]$ and that the variables of Γ and Δ are disjoint[1].

- $[\Gamma]\, t : X$ means $[\Gamma]$ and that t is a well-formed term of type X whose variables are drawn from Γ.

Proof rules for constructing terms, propositions and transition propositions: the rules of 'projection' for each form of entailment $\vdash$, for example,

$$x_1 : X_1, \ldots, x_n : X_n \vdash x_i : X_i \quad \text{for } i = 1, \ldots, n,$$

and the rules for constructing terms and predicates from elements of the signature, are all standard and are omitted for conciseness.

2.3 Propositions

We present the rules for propositional formation, and propositional entailment. We omit, for conciseness, the standard rules for the connectives of intuitionistic logic with equality, i.e. the rules for $\top$ (true), $\bot$ (false), $\wedge$ (and), $\vee$ (or), $\Rightarrow$ (implies), $=$ (equality) and the two quantifiers $\forall$ and $\exists$.

Judgements:

- $[\Gamma]\, \phi$ means $[\Gamma]$ and that ϕ is a well-formed proposition with variables drawn from Γ.

- $[\Gamma]\, \Phi \vdash \psi$ means that Φ is a list of propositions $\phi_1, \ldots, \phi_n$ satisfying $[\Gamma]\, \phi_1$ up to $[\Gamma]\, \phi_n$, that $[\Gamma]\, \psi$, and that ψ follows from the hypotheses Φ.

Formation rules:

$$\frac{[\Gamma][]\, \phi}{[\Gamma]\, \Box\phi} \qquad \frac{[\Gamma][]\, \phi}{[\Gamma]\, \Diamond\phi}$$

Proof rules:

$$\frac{[\Gamma][]\, \Phi \vdash \psi}{[\Gamma]\, \Box\Phi \vdash \Box\psi} \qquad \frac{[\Gamma][]\, \Phi, \psi \vdash \theta}{[\Gamma]\, \Box\Phi, \Diamond\psi \vdash \Diamond\theta} \qquad \frac{[\Gamma][]\, \Phi, \psi \vdash \bot}{[\Gamma]\, \Box\Phi, \Diamond\psi \vdash \bot}$$

where, if $\Phi = \phi_1, \ldots, \phi_n$ then $\Box\Phi = \Box\phi_1, \ldots, \Box\phi_n$. For simplicity, we have written the rules as distinguishing the rightmost item in a context, but mean any item to be so distinguished.

[1] Disjointness is possibly not necessary here, but certainly aids readability.

2.4 Transition propositions

We introduce the syntax class of transition propositions. Variables in transition propositions are drawn from a pair of contexts Γ and Δ and we write

$$[\Gamma][\Delta]\phi$$

for a transition proposition in context. Such a ϕ relates individuals in the present state – ranged over by the variables of Γ – to individuals in a related state – ranged over by the variables of Δ. The relevant rules are given below.

Again, for conciseness, we omit the standard rules for connectives and quantification. However, in this case, unlike the case of propositions, we restrict these to geometric logic. Thus the connectives for transition propositions are $\top$ (true), $\bot$ (false), $\wedge$ (and), $\vee$ (or) and equality. The single propositional quantifier $\exists$ we admit (and include in the rules). The restriction to geometric logic arises from the semantics where we need the logic to be preserved through pullback along continuous maps between spaces. Such logics are geometric. We are not clear as to whether the restriction to a geometric logic is a natural restriction, nor how it affects the utility of the logic.

Judgements:

- $[\Gamma][\Delta]\,\phi$ means $[\Gamma][\Delta]$ and that ϕ is a well-formed transition proposition with variables of the present state drawn from Γ and variables of a related state drawn from Δ.
- $[\Gamma][\Delta]\,\Phi \vdash \psi$ means that Φ is a list of transition propositions $\phi_1, \ldots, \phi_n$ satisfying $[\Gamma][\Delta]\,\phi_1$ up to $[\Gamma][\Delta]\,\phi_n$, that $[\Gamma][\Delta]\,\psi$, and that ψ follows from the hypotheses Φ.

Formation rules:

$$\frac{[\Delta]\,\phi \quad [\Gamma][\Delta]}{[\Gamma][\Delta]\,\phi} \qquad \frac{[\Gamma][\Delta, x : X]\,\phi}{[\Gamma][\Delta]\,\exists x : X.\ \phi}$$

$$\frac{[\Gamma]\,t : X \quad [\Delta]\,u : X \quad [\Gamma][\Delta]}{[\Gamma][\Delta]\,t \twoheadrightarrow_X u} \qquad \frac{[\Gamma][\Delta, x : X]\,\phi \quad [\Gamma]\,t : X}{[\Gamma][\Delta]\,\forall x \twoheadleftarrow_X t.\ \phi}$$

Proof rules:

$$\frac{[\Delta]\,\Phi \vdash \psi \quad [\Gamma][\Delta]}{[\Gamma][\Delta]\,\Phi \vdash \psi}$$

$$\begin{array}{c} [\Gamma][\Delta, x : X]\,\Phi, \psi \vdash \theta \quad [\Gamma][\Delta]\,\Phi, \theta \\ \hline\hline [\Gamma][\Delta]\,\Phi, \exists x : X.\ \psi \vdash \theta \end{array}$$

$$\begin{array}{c} [\Gamma][\Delta, x : X]\,\Phi[x/x'] \vdash \psi[x/x'] \quad [\Gamma][\Delta, x : X, x' : X]\,\Phi, \psi \\ \hline\hline [\Gamma][\Delta, x : X, x' : X]\,\Phi, x =_X x' \vdash \psi \end{array}$$

$$\begin{array}{c} [\Gamma][\Delta, x : X]\,\Phi, t \twoheadrightarrow_X x \vdash \psi \quad [\Gamma][\Delta]\Phi \\ \hline\hline [\Gamma][\Delta]\,\Phi \vdash \forall x \twoheadleftarrow_X t.\ \psi \end{array}$$

Note: although universal quantification is not admitted in geometric logic, we do include a form of bounded quantification as $[\Gamma][\Delta]\,\forall x \twoheadleftarrow_X t.\ \phi$ which is necessary to describe relations between individuals. A similar bounded existential quantifier could be included $[\Gamma][\Delta]\,\exists x \twoheadleftarrow_X t.\ \phi$ (and would make the treatment of quantifiers more uniform). Such a bounded existential quantifier is, however, subsumed under the general existential quantification.

3 Sheaves

The models of the logic that we are concerned with are sheaves. Sheaf models of logics are described in [5] (see also [25], [15]). For the purposes of this paper, it is more convenient to use an equivalent formulation in terms of local homeomorphisms.

Definition. A *local homeomorphism* from topological space T to space S is a continuous function $f : T \to S$ such that, for all $x \in T$, there is an open neighbourhood U of x, such that $f(U)$ is open and $f|_U$ is a homeomorphism.

Definition. A map from local homeomorphism $p_1 : T_1 \to S$ to local homeomorphism $p_2 : T_2 \to S$ (both over the same space S) is a continuous function $f : T_1 \to T_2$ such that $p_1 = f; p_2$.

A map between local homeomorphisms is itself necessarily a local homeomorphism.

The equivalence between the category LH/S of local homeomorphisms over space S and the category $\mathsf{Sh}(S)$ of sheaves over base S is standard material (see, for example, [11]). The category $\mathsf{Sh}(S)$, and thus LH/S, is a topos.

We extend this to relational spaces:

Definition. A *relational space* $(S, \twoheadrightarrow_S)$ is a topological space $(S, \mathcal{O}(S))$ together with a binary relation $\twoheadrightarrow_S \subseteq S \times S$.

Definition. A *continuous relational function* $f : (T, \twoheadrightarrow_T) \to (S, \twoheadrightarrow_S)$ is a continuous function $f : T \to S$ which satisfies

$$x \twoheadrightarrow_T x' \;\Rightarrow\; f(x) \twoheadrightarrow_S f(x').$$

A *continuous p-morphism* is a continuous relational function which satisfies

$$f(x) \twoheadrightarrow_S y \;\&\; y \in U \in \mathcal{O}(S) \;\Rightarrow\; \exists x'.\; x \twoheadrightarrow_T x' \;\&\; f(x') \in U.$$

Note: a continuous p-morphism need not be a p-morphism in the usual sense: the definition is an adaptation of the concept to topological spaces.

Definition. A *local relational homeomorphism* from relational space $(T, \twoheadrightarrow_T)$ to relational space $(S, \twoheadrightarrow_S)$ is a continuous relational function $f : T \to S$ such that

1. For all $x \in T$, there is an open neighbourhood U of x, such that $f(U)$ is open and $f|_U$ is both a homeomorphism and a relational isomorphism, and

2. For all x,y in T such that $x \twoheadrightarrow_T y$, there is an open set U in T such that $x \in U$ and $y \in U$, and such that $f(U)$ is open and $f|_U$ is both a homeomorphism and a relational isomorphism.

Definition. A map from local relational homeomorphism $p_1 : T_1 \to S$ to local relational homeomorphism $p_2 : T_2 \to S$ (both over the same space S) is a continuous relational function $f : T_1 \to T_2$ such that $p_1 = f; p_2$.

Again, a map between local relational homeomorphisms is itself necessarily a local relational homeomorphism. Continuous p-morphisms are not used here – they appear as 'change of base' maps, which are not present in this account.

The category of local relational homeomorphisms over relational space $(S, \twoheadrightarrow_S)$, $\mathsf{LRelH}/(S, \twoheadrightarrow_S)$, is equivalent to a category of 'relational sheaves' over $(S, \twoheadrightarrow_S)$ [9]. It is not, however, a topos. It is a quasitopos, i.e. it has all finite limits, finite colimits and a subobject classifier for regular monos. Regular monos in $\mathsf{LRelH}/(S, \twoheadrightarrow_S)$ correspond to open subsets of the source space. The pullback of a local relational homeomorphism along a continuous relational function is itself a local relational homeomorphism. These, and other, results about local relational homeomorphisms and relational sheaves may be found in [9].

4 Sheaf models

Before defining the semantics we set out some mathematical preliminaries.

Firstly, it is customary when using fibred descriptions to abbreviate notation: using T and $p : T \to S$ interchangeably, when p is understood. We shall adopt this convention, and use other abbreviations where convenient. For example, using S and $(S, \twoheadrightarrow_S)$ interchangeably, when $\twoheadrightarrow_S$ is understood.

Now, let $(S, \twoheadrightarrow_S)$ be a relational space and

- let LRelH/S be the quasitopos of local relational homeomorphisms over S,
- let $\twoheadrightarrow_S = \{\langle s, t\rangle \in S \times S \mid s \twoheadrightarrow_S t\}$, with product topology,
- $\pi_1, \pi_2 : (\twoheadrightarrow_S) \to S$ be the projection maps. These are continuous functions (as we use the product topology),
- Let $\mathsf{LH}/\twoheadrightarrow_S$ be the topos of local homeomorphisms over the space $\twoheadrightarrow_S$.

Let f^* be the functor defined by pulling back along f. The functors $\pi_1^*, \pi_2^* : \mathsf{LRelH}/S \to \mathsf{LH}/\twoheadrightarrow_S$ are the inverse image of geometric morphisms and, hence, preserve $\top$, $\bot$, $\wedge$, $\vee$, $\exists$ and equality.

Define the bifunctor $\otimes : \mathsf{LRelH}/S \times \mathsf{LRelH}/S \to \mathsf{LH}/\twoheadrightarrow_S$ by $p \otimes q = \pi_1^*(p) \times \pi_2^*(q)$. The following diagram, where each square is a pullback, may help:

$$
\begin{tikzcd}
T \otimes T' \arrow[r] \arrow[d] & T \otimes 1 \arrow[r] \arrow[d] & T \arrow[d, "p"] \\
1 \otimes T' \arrow[r] \arrow[d] & \twoheadrightarrow_S \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & S \\
T' \arrow[r, "q"] & S &
\end{tikzcd}
$$

For $p : T \to S$ in LRelH/S, define $\twoheadrightarrow_p \subseteq p \otimes p$ by $\twoheadrightarrow_p = \{\langle y, z\rangle \in T \times T \mid y \twoheadrightarrow_T x\}$. This set is open, so defines a regular subobject.

For $p \in \mathsf{LRelH}/S$ and $q \in \mathsf{LH}/\twoheadrightarrow_S$, we have

$$\mathsf{id}_q \times (\mathsf{id}_p \otimes !_p) : q \times (p \otimes p) \to q \times (p \otimes 1_S)$$

in $\mathsf{LH}/\twoheadrightarrow_S$ and $\top \times (\twoheadrightarrow_p) \subseteq q \times (p \otimes p)$ in $\mathsf{LH}/\twoheadrightarrow_S$. Thus, $\phi \mapsto (\mathsf{id}_q \times (\mathsf{id}_p \otimes !_p))^*(\phi) \wedge (\top \times \twoheadrightarrow_p) : Sub(q \times (p \otimes 1_S)) \to Sub(q \times (p \otimes p))$ in $\mathsf{LH}/\twoheadrightarrow_S$.

This map has a right adjoint:

$$\forall_{\twoheadrightarrow_p} : Sub(q \times (p \otimes p)) \to Sub(q \times (p \otimes 1_S)).$$

4.1 Semantics

We begin by fixing a relational space $(S, \twoheadrightarrow_S)$, and define the semantics by induction on the derivations, first choosing interpretations for the elements of the signature, and then showing how to extend this to the logic.

A *valuation* V of a signature is:

1. For each sort X, a local relational homeomorphism $V(X)$ over $(S, \twoheadrightarrow_S)$,
2. For each operation, $f : X_1 \times \cdots \times X_n \to X$, a map of local relational homeomorphisms over $(S, \twoheadrightarrow_S)$, $V(f) : V(X_1) \times \cdots \times V(X_n) \to V(X)$,
3. For each predicate symbol, $r : X_1 \times \cdots \times X_n$, a regular subobject $V(r) \subseteq V(X_1) \times \cdots \times V(X_n)$.

We extend a valuation V to an interpretation $[\![-]\!]$ of the logic as follows:

Contexts: A context $x_1 : X_1, \ldots, x_n : X_n$ is interpreted as the local relational homeomorphism defined by the product $[\![X_1]\!] \times \cdots \times [\![X_n]\!]$ of local relational homeomorphisms over $(S, \twoheadrightarrow_S)$ i.e. a product in the category LRelH/S which is a pullback in LRelH. Notice that the empty context is interpreted as the terminal object 1 in LRelH/S which is the identity $\mathsf{id} : (S, \twoheadrightarrow_S) \to (S, \twoheadrightarrow_S)$.

Terms: Interpret a term $[x_1 : X_1, \ldots, x_n : X_n]t : X$ as a map of local relational homeomorphisms $[\![t]\!] : [\![X_1]\!] \times \cdots \times [\![X_n]\!] \to [\![X]\!]$.

Propositions: Interpret a proposition $[\Gamma]\phi$ as a regular subobject of $[\![\Gamma]\!]$ in LRelH/S.

Transition propositions: Interpret a transition proposition $[\Gamma][\Delta]\phi$ as a subobject of $[\![\Gamma]\!] \otimes [\![\Delta]\!]$ in $\mathsf{LH}/\twoheadrightarrow_S$. This captures that fact that transition propositions determine a relation between individuals in two states, the two states themselves being related by the relation $\twoheadrightarrow_S$.

Each of these is defined by induction on the derivations of well-formedness as follows:

For the formation rules for the modalities $\Box$ and $\Diamond$:

$$\frac{[\Gamma][]\,\phi}{[\Gamma]\,\Box\phi} \qquad \frac{[\Gamma][]\,\phi}{[\Gamma]\,\Diamond\phi}$$

Let $[\![\phi]\!] \subseteq [\![\Gamma]\!] \otimes 1_S$ in $\mathsf{LH}/\twoheadrightarrow_S$. Then define:

$$
\begin{aligned}
[\![\Box\phi]\!] &= \{x \in T \mid \forall s \in S.\ p(x) \twoheadrightarrow s \ \Rightarrow\ \langle x, s\rangle \in [\![\phi]\!]\}^\circ \\
[\![\Diamond\phi]\!] &= \{x \in T \mid \exists s \in S.\ p(x) \twoheadrightarrow s \ \&\ \langle x, s\rangle \in [\![\phi]\!]\}^\circ.
\end{aligned}
$$

where $[\![\Gamma]\!] = p : T \to S$.

This follows the standard Kripke interpretation of the modal operators:

$$
\begin{aligned}
\Box\phi &= \{x | \forall y.\ x \twoheadrightarrow y \ \Rightarrow\ y \in \phi\} \\
\Diamond\phi &= \{x | \exists y.\ x \twoheadrightarrow y \ \&\ y \in \phi\}.
\end{aligned}
$$

We follow [8] in adapting these definitions to topology by taking interiors to form open sets.

We deal with each of the following rules in turn.

$$
\frac{[\Delta]\,\phi \quad [\Gamma][\Delta]}{[\Gamma][\Delta]\,\phi}
\qquad
\frac{[\Gamma][\Delta, x : X]\,\phi}{[\Gamma][\Delta]\,\exists x : X.\ \phi}
$$

$$
\frac{[\Gamma]\,t : X \quad [\Delta]\,u : X \quad [\Gamma][\Delta]}{[\Gamma][\Delta]\,t \twoheadrightarrow_X u}
\qquad
\frac{[\Gamma][\Delta, x : X]\,\phi \quad [\Gamma]\,t : X}{[\Gamma][\Delta]\,\forall x \twoheadleftarrow_X t.\ \phi}
$$

For the first rule, let $[\![\phi]\!] \subseteq [\![\Delta]\!]$ in LRelH/S, then $[\![\phi]\!] = [\![\Gamma]\!] \otimes [\![\phi]\!]$ in $\mathsf{LH}/\twoheadrightarrow_S$.

At this point there is a question of the 'coherence' of this definition, i.e. is the semantics we define independent of the derivation chosen? This 'weakening' rule allows multiple derivations of the same sequent. However, the coherence follows from the fact that we are dealing with geometric logic for transition propositions, and geometric logic is preserved under pullbacks of continuous functions.

For the second rule, we use the standard definition of existential quantification as a left adjoint.

For the third rule, let $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$ in LRelH/S and $[\![u]\!] : [\![\Delta]\!] \to [\![X]\!]$ in LRelH/S, then $[\![t \twoheadrightarrow_X u]\!] = (t \otimes u)^*(\twoheadrightarrow_{[\![X]\!]})$.

For the fourth rule, let $[\![\phi]\!] \subseteq [\![\Gamma]\!] \otimes ([\![\Delta]\!] \times [\![X]\!])$ in $\mathsf{LH}/\twoheadrightarrow_S$, and $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$ in LRelH/S. Then $[\![\forall x \twoheadleftarrow_X t.\ \phi]\!] = \langle 1, [\![t]\!] \otimes !\rangle^* \forall_{\twoheadrightarrow_{[\![X]\!]}} f^*([\![\phi]\!])$. Here $\forall_{\twoheadrightarrow_{[\![X]\!]}}$ is the right adjoint defined in the preamble, and f is the composition of the interchange map $([\![\Gamma]\!] \otimes [\![\Delta]\!]) \times ([\![X]\!] \otimes [\![X]\!]) \to ([\![\Gamma]\!] \times [\![X]\!]) \otimes ([\![\Delta]\!] \times [\![X]\!])$ with $\pi \otimes \mathsf{id} : ([\![\Gamma]\!] \times [\![X]\!]) \otimes ([\![\Delta]\!] \times [\![X]\!]) \to [\![\Gamma]\!] \otimes ([\![\Delta]\!] \times [\![X]\!])$.

This completes the definition of the semantics.

Soundness of this semantics follows from standard techniques. For example the proof rules for the quantifiers are in adjoint form, and they are defined in the model as adjoints. The proof rules for modality are standard in intuitionistic logic, and the semantics is given in terms of the standard Kripke definitions.

As for completeness: There are good reasons for believing that a form of sheaf models will provide a complete semantics (see [1] for a topos-theoretic completeness proof of a first order logic). However, a completeness proof for this logic will, we believe, be a technical *tour-de-force*.

5 Conclusions

We have presented a new form of first order modal logic and defined its sheaf semantics. Two aspects need considerable further work, both are mentioned above. These are the completeness of the semantics, and the role of the logic in describing the behaviour of computational systems, in particular, as a general form of program logic.

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