Topological Duality for Intuitionistic Modal Algebras

Barnaby P. Hilken

August 20, 2001

Abstract

This paper describes a generalisation of the adjunction and duality between topological spaces and frames, inspired by the Kripke semantics of modal logic. A *relational space* is defined as a topological space with a binary relation on the points, and a *modal frame* is defined as a frame with two operations $\Box$ and $\Diamond$ satisfying various axioms. The frame of open sets of a relational space has a natural modal structure; with appropriate morphisms, this extends to a contravariant functor from the category of relational spaces to the category of modal frames. This paper defines an adjoint to this functor, and shows that the adjunction restricts to a duality between the subcategories in the image of the adjunction.

The paper goes on to consider those modal frames which are freely generated from modal distributive lattices, and in particular, those arising from the Lindenbaum algebras of intuitionistic propositional modal languages. These *spectral modal frames* are constructed by defining a modal structure on the frame of ideals, and characterised (up to isomorphism) by exactness and compactness conditions of the modal connectives. Finally, they are shown to be equivalent to the frames of open sets of relational spaces; from this a completeness theorem for intuitionistic modal logic follows.

1 Introduction

The contravariant adjunction and duality between topological spaces and frames (or locales) underlies important results in lattice theory, representation theory, topos theory and logic; for a full exposition, see [?]. In particular, the completeness theorem for intuitionistic propositional logic follows immediately from the fact that the spectral (or coherent) frames "have enough points"—they are equivalent to the frames of open sets of their spaces of points. The aim of this paper is to develop a new duality theory which bears the same relationship to modal logic as the above does to propositional logic.

The usual (Kripke) models of modal logic consist of a set of "worlds" with a binary relation of "accessibility;" the semantics of the modal connectives is then given by

$$\begin{aligned} \Box(\phi) &= \{x | \forall y.\ x\ R\ y \ \Rightarrow\ y \in \phi\} \\ \Diamond(\phi) &= \{x | \exists y.\ x\ R\ y\ \&\ y \in \phi\}. \end{aligned}$$

We combine the two notions of model by considering sets carrying both a topology and a relation. In traditional modal-logic terms, this can be thought of as putting a topology on the set of worlds, which says how similar (or hard to distinguish) worlds are; the usual models can be recovered by taking the topology to be discrete. Following the topological semantics of intuitionistic propositional logic, we interpret propositions as open sets in the model. Since the definitions of $\Box$ and $\Diamond$ given above do not generally yield open sets, we take their interior as the definition of $\Box$ and $\Diamond$ in our new model.

In mathematical terms, what we have described above is a pair of unary operations on the frame of open sets of any topological space with a binary relation. The first result of this paper is that, with suitable definitions of morphisms, this construction yields a contravariant functor from the category of *relational spaces* to the category of *modal frames.* The thesis of this work is that these modal frames are the algebraic version of topological spaces carrying a relation in the same way that frames are the algebraic version of plain topological spaces.

The main result of the paper is the construction of a contravariant adjoint (on the right) to the functor described above. This corresponds in the non-modal case to the construction of the space of points of a frame, and in the discrete case to the definition of worlds as maximal consistent sets. The difficulty of this construction is that the usual frame points do not give a space with the right properties, and so we define certain *modal frame points.* This gives the required adjoint, and it is then reasonably straightforward to show that the adjunction restricts to a duality (or contravariant equivalence) between those relational spaces which are isomorphic to the space of points of a modal frame, and those modal frames which are isomorphic to the frame of opens of a relational space.

In the second part of the paper we consider those modal frames which arise as the completions of *modal distributive lattices*, defined as distributive lattices with $\Box$ and $\Diamond$ satisfying the same axioms as modal frames. These arise naturally as the Lindenbaum algebras of intuitionistic propositional modal languages, satisfying the same axioms. Here the results are more straightforward: the modal connectives on a distributive lattice lift to modal connectives on its frame of ideals, and this defines a left adjoint to the forgetful functor. We characterise the modal frames which arise in this way, in terms of continuity and compactness properties, and show that they are always in the scope of the duality. From this result, the completeness of a certain version of intuitionistic modal logic is an easy corollary.

Note

In this paper, a *frame* is a partially ordered set with all joins, distributing over finite meets; a frame morphism is a monotone function which preserves finite meets and all joins. The author hopes that those readers who are used to the term "Kripke frame" will not suffer greatly from culture shock.

Some notation: if X is a topological space and $S \subseteq X$ we write S° for the topological interior of S; if $f : Y \to X$ we write $f^\leftarrow$ for the inverse-image map of f; we write $\mathbf{2}$ for the two-element frame $\{0, 1\}$.

2 The Adjunction

2.1 Relational Spaces

The basic concrete structures which form the subject of this paper are simply topological spaces with a binary relation on the points. We do not impose any extra conditions relating the two structures; this is partly because we don't need to, and partly because there seems to be no natural condition which includes all the important examples. We therefore define:

Definition. A **relational space** $(X, \mathfrak{O}, R)$ is a topological space $(X, \mathfrak{O})$ together with a binary relation

$$R \subseteq X \times X.$$

This definition is so general that there are innumerable examples, of widely varying character. We mention the following for their importance as models of certain modal logics: they motivate much of the work in this paper.

Example 1. The real numbers $\mathbb{R}$ with their usual (metric) topology and their usual $\leqslant$ ordering form a topological relational structure. This is a natural model of time, familiar from Newtonian physics.

Example 2. As is a simple model of branching time, consider

$$X = \{(x, y) \in \mathbb{R}^2 | x \leqslant 0 \ \&\ y = 0 \text{ or } x \geqslant 0 \ \&\ y = \pm x\}$$

with the topology inherited from $\mathbb{R}^2$ and the order

$$(x, y) \leqslant (x', y') \iff x \leqslant x' \ \&\ yy' \geqslant 0.$$

This forms a branching path, with one branch point at $x = 0$.

Example 3. Let S be a domain-theoretic model of the state of a computer system, with the Scott topology. Let two states be related if they are the initial and final states of a possible "run" of the system. This is a natural model of program logic or dynamic logic; for a similar approach see [**?**] and [**?**].

Example 4. Let $(X, \leqslant)$ be a partial order, and $R \subseteq X \times X$: such structures have been considered as models of intuitionistic modal logic by several authors; for example [**?**], [**?**], [**?**], [**?**], [**?**] and [**?**]. Take any topology for which $\leqslant$ is the specialisation order (such as the upper interval, Scott or Alexandrov topology). This gives a relational space, which is equivalent to the original structure as a model of intuitionistic modal logic. However, the choice between several inequivalent topologies shows that some information is lost when considering partial-order models.

Other examples of interest are ordered spaces (see [**?**]) and topological tolerance spaces (see [**?**] and [**?**]).

The most obvious morphisms between relational spaces are continuous functions which preserve the relation, but they are not the most useful from our point of view. We define

Definition. A **continuous relational function**

$$f : (X, \mathfrak{O}, R) \to (Y, \mathfrak{P}, S)$$

is a continuous function $f : (X, \mathfrak{O}) \to (Y, \mathfrak{P})$ which satisfies

$$x \ R \ x' \ \Rightarrow \ f(x) \ S \ f(x'). \tag{1}$$

A **continuous p-morphism** is a continuous relational function which satisfies

$$f(x) \ S \ y \ \&\ y \in U \in \mathfrak{P} \ \Rightarrow \ \exists x'.\ x \ R \ x' \ \&\ f(x') \in U. \tag{2}$$

This terminology is slightly confusing, as a continuous p-morphism need not be a p-morphism in the usual sense of a function which preserves the relation and satisfies

$$f(x)\; S\; y \;\Rightarrow\; \exists x'.\; x\; R\; x' \;\&\; f(x') = y. \tag{3}$$

Instead, a continuous p-morphism gets arbitrarily close to satisfying (3), by taking $U \ni y$ to be arbitrarily small. If the topology is discrete, we can take $U = \{y\}$, and recover (3).

It is easy to see that relational spaces and continuous p-morphisms form a category, which we will call RelSp. It is this category which is the subject of this paper.

2.2 Modal Frames

We wish to study the algebraic structure of the open sets of relational spaces. The open sets of a topological space form a frame (see [**?**]); the extra structure which we use to study the relation is a pair of operators on the frame, whose definition is inspired by modal logic. This leads to the following algebraic structure.

Definition. A **modal frame** $(A, \leqslant, \Box, \Diamond)$ is a frame $(A, \leqslant)$ together with a pair of monotone maps $\Box, \Diamond : A \to A$ satisfying

$$\top \leqslant \Box(\top) \tag{4}$$

$$\Box(a) \wedge \Box(b) \leqslant \Box(a \wedge b) \tag{5}$$

$$\Diamond(a) \wedge \Box(b) \leqslant \Diamond(a \wedge b) \tag{6}$$

$$\Diamond(\bot) \leqslant \bot. \tag{7}$$

There are two important classes of examples. The first is the construction of a modal frame from a relational space, by defining $\Box$ and $\Diamond$ on open sets: it is this construction which motivates the choice of axioms. The second is the extension of modal connectives on a distributive lattice to its frame of ideals: this can be applied to the Lindenbaum algebra of an intuitionistic propositional modal calculus. This example is discussed in greater detail in Section 3.

Example 5. Let $(X, \mathfrak{O}, R)$ be a relational space, and define $\Box^R, \Diamond^R : \mathfrak{O} \to \mathfrak{O}$ by

$$\Box^R(U) = \{x \in X | \forall x' \in X.\; x\; R\; x' \;\Rightarrow\; x' \in U\}^\circ \tag{8}$$

$$\Diamond^R(U) = \{x \in X | \exists x' \in X.\; x\; R\; x' \;\&\; x' \in U\}^\circ. \tag{9}$$

Then $(\mathfrak{O}, \subseteq, \Box^R, \Diamond^R)$ is a modal frame.

Example 6. Let $(D, \leqslant)$ be a distributive lattice, and $\Box, \Diamond : D \to D$ a pair of monotone maps satisfying (4–7). Let $\mathrm{Idl}(D)$ be the set of ideals of D, and define $\Box', \Diamond' : \mathrm{Idl}(D) \to \mathrm{Idl}(D)$ by

$$\Box'(I) = \{a \in D | \exists b \in I.\ a \leqslant \Box b\} \tag{10}$$
$$\Diamond'(I) = \{a \in D | \exists b \in I.\ a \leqslant \Diamond b\}. \tag{11}$$

Then $(\mathrm{Idl}(D), \subseteq, \Box', \Diamond')$ is a modal frame.

Perhaps the most obvious kind of morphism between modal frames is a function which preserves all the structure; in fact, the morphisms which arise in concrete examples to not preserve $\Box$ and $\Diamond$ strictly, but have a comparison as in the following definition.

Definition. A **modal frame morphism**

$$f : (A, \leqslant^A, \Box^A, \Diamond^A) \to (B, \leqslant^B, \Box^B, \Diamond^B)$$

is a frame morphism $f : (A, \leqslant^A) \to (B, \leqslant^B)$ which satisfies

$$f(\Box^A(a)) \leqslant^B \Box^B(f(a)) \tag{12}$$
$$f(\Diamond^A(a)) \leqslant^B \Diamond^B(f(a)). \tag{13}$$

f is **strict** if (12,13) are equalities.

It is easy to see that modal frames and modal frame morphisms form a category, which we will call MFrm.

It should come as no surprise that the construction of Example 5 extends to a contravariant functor between our two categories, by taking a continuous p-morphism to its inverse image map. We record this as:

Proposition 1. *If $f : (X, \mathfrak{O}, R) \to (Y, \mathfrak{P}, S)$ is a continuous relational morphism, then $f^{\leftarrow} : (\mathfrak{P}, \subseteq, \Box^S, \Diamond^S) \to (\mathfrak{O}, \subseteq, \Box^R, \Diamond^R)$ is a modal frame morphism.*

The map $\mathcal{O} : \mathrm{RelSp} \to \mathrm{MFrm}$ defined by

$$\mathcal{O}(X, \mathfrak{O}, R) = (\mathfrak{O}, \subseteq, \Box^R, \Diamond^R)$$
$$\mathcal{O}(f) = f^{\leftarrow}$$

is a contravariant functor.

Proof. That $f^{\leftarrow}$ is a frame morphism is standard; we need to show that $f^{\leftarrow}(\Box^S U) \subseteq \Box^R f^{\leftarrow}(U)$ and $f^{\leftarrow}(\Diamond^S U) \subseteq \Diamond^R f^{\leftarrow}(U)$.

Let $x \in f^{\leftarrow}(\Box^S U)$ so $f(x) \in \Box^S U$. If $x \; R \; x'$ then $f(x) \; S \; f(x')$ by (1), so $f(x') \in U$, i.e. $x' \in f^{\leftarrow}(U)$. We have shown that

$$f^{\leftarrow}(\Box^S U) \subseteq \{x | \forall x'. \; x \; R \; x' \;\Rightarrow\; x' \in f^{\leftarrow}(U)\}$$

but $f^{\leftarrow}(\Box^S U)$ is open, so it is a subset of $\Box^R f^{\leftarrow}(U)$.

Let $x \in f^{\leftarrow}(\Diamond^S U)$ so $f(x) \in \Diamond^S U$. Then there is some $y \in Y$ such that $f(x) \; S \; y$ and $y \in U$. By (2), there is some $x' \in X$ such that $x \; R \; x'$ and $f(x') \in U$, i.e. $x' \in f^{\leftarrow}(U)$. We have shown that

$$f^{\leftarrow}(\Diamond^S U) \subseteq \{x | \exists x'. \; x \; R \; x' \;\&\; x' \in f^{\leftarrow}(U)\}$$

but $f^{\leftarrow}(\Diamond^S U)$ is open, so it is a subset of $\Diamond^R f^{\leftarrow}(U)$.

That $\mathcal{O}$ is a functor is standard. □

Our first aim in this paper is the construction of an adjoint to $\mathcal{O}$.

2.3 Modal Frame Points

In classical modal logic (see [**?**]), $\Box$ can be used to define a binary relation on the usual points (the maximal consistent sets), in such a way that it can be recovered from the relation. In contrast to this, $\Box$ and $\Diamond$ cannot be used to define a relation on frame points which adequately captures their behaviour (except in special cases). Instead, we define a new class of points, called *modal frame points*, which carry a natural relation, and give us the adjoint we seek.

The modal frame points are certain pairs, consisting of a frame point and an element of the frame. The frame point can be thought of as describing which open sets this point inhabits; the element can be thought of as the greatest open set disjoint from all the related points, i.e. the interior of the set of unrelated points.

Definition. Let $(A, \leqslant, \Box, \Diamond)$ be a modal frame.
A **pre-point** of $(A, \leqslant, \Box, \Diamond)$ is a pair (p, a) where $p : A \to \mathbf{2}$ is a frame point and $a \in A$ satisfies

$$p(\Diamond a) = 0. \tag{14}$$

Two pre-points are **related**, $(p, a) \; R_A \; (q, b)$ if and only if

$$q(a) = 0 \tag{15}$$
$$\forall c \in A. \; p(\Box c) \leqslant q(c). \tag{16}$$

The set of **modal frame points** $\mathbb{P}_A$ is the *largest* set P of pre-points which satisfies

$$(p, a) \in P \ \&\ c \nleqslant a \ \Rightarrow\ \exists (q, b) \in P.\ (p, a)\ R_A\ (q, b)\ \&\ q(c) = 1. \tag{17}$$

Condition (17) captures the property that a is the interior of the set of unrelated points: if c is not a subset of a, then it must contain some related point. It is easy to see that any union of sets of pre-points which satisfy (17) itself satisfies (17), so the set $\mathbb{P}_A$ is well defined. The relation on modal frame points is just R_A, so it remains only to define the topology.

Definition. The **unit** $\phi_A : A \to \mathcal{P}(\mathbb{P}_A)$ is defined by

$$\phi_A(b) = \{(p, a) \in \mathbb{P}_A | p(b) = 1\}. \tag{18}$$

The **topology** $\mathfrak{O}_A$ on $\mathbb{P}_A$ is the image of ϕ_A.

That the image of ϕ_A is a topology (i.e. a subframe of $\mathcal{P}(\mathbb{P}_A)$) is exactly like the corresponding proof for frame points.

Lemma 2. *The unit*

$$\phi_A : (A, \leqslant, \Box, \Diamond) \to (\mathfrak{O}_A, \subseteq, \Box^{R_A}, \Diamond^{R_A})$$

is a modal frame morphism.

Proof. That ϕ is a frame morphism is standard, so we have only to prove (12) and (13).

Let $(p, a) \in \phi_A(\Box c)$ so $p(\Box c) = 1$. If $(p, a)\ R_A\ (q, b)$ then $q(c) \geqslant p(\Box c) = 1$ (by 16) so $(q, b) \in \phi_A(c)$. Therefore

$$\phi_A(\Box c) \subseteq \{x | \forall y.\ x\ R_A\ y\ \Rightarrow\ y \in \phi_A(c)\}$$

but $\phi_A(\Box c)$ is open, so $\phi_A(\Box c) \subseteq \Box \phi_A(c)$.

Let $(p, a) \in \phi_A(\Diamond c)$ so $p(\Diamond c) = 1$. Then $c \nleqslant a$ so (by 17) there is some (q, b) such that $(p, a)\ R_A\ (q, b)$ and $q(c) = 1$ so $(q, b) \in \phi_A(c)$. Therefore

$$\phi_A(\Diamond c) \subseteq \{x | \exists y.\ x\ R_A\ y\ \&\ y \in \phi_A(c)\}$$

but $\phi_A(\Diamond c)$ is open, so $\phi_A(\Diamond c) \subseteq \Diamond \phi_A(c)$. □

2.4 The Adjunction

The first big result of this paper is the contravariant adjunction between modal frames and relational spaces. This section is devoted to the proof of this result, which we formalise as follows.

Theorem 1. *The map $(A, \leqslant, \Box, \Diamond) \mapsto (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ is the object part of a contravariant functor adjoint to $\mathcal{O}$ on the right, with unit ϕ_A on the frame side.*

The first step in the proof is the construction of the arrows arising from the universal property.

Definition. Let $(A, \leqslant, \Box, \Diamond)$ be a modal frame, $(X, \mathcal{O}, R)$ a relational space, and $f : (A, \leqslant, \Box, \Diamond) \to (\mathcal{O}, \subseteq, \Box^R, \Diamond^R)$ a modal frame morphism.

Define $f^\sharp : X \to \mathbb{P}_A$ by

$$f^\sharp(x) = (p_x, a_x) \text{ where } \quad p_x(b) = 1 \text{ iff } x \in f(b) \qquad (19)$$
$$a_x = \bigvee \{c \in A | \forall y \in f(c).\ x \not R\ y\}.$$

We have to show that (p_x, a_x) is a modal frame point; this is the result of the next three lemmas.

Lemma 3. *For all $x \in X$, $f^\sharp(x)$ is a pre-point.*

Proof. If $p_x(\Diamond a_x) = 1$ then $x \in f(\Diamond a_x) \subseteq \Diamond^R f(a_x)$ so there is a $y \in X$ such that $x\ R\ y$ and $y \in f(a_x)$. But $f(a_x) = \bigcup\{f(c) | \forall y \in f(c).\ x \not R\ y\}$, which contradicts $x\ R\ y$. Therefore $p_x(\Diamond a_x) = 0$. □

Lemma 4. *If $x, y \in X$ such that $x\ R\ y$ then $f^\sharp(x)\ R_A\ f^\sharp(y)$.*

Proof. If $p_y(a_x) = 1$ then $y \in f(a_x) = \bigcup\{f(c) | \forall y' \in f(c).\ x \not R\ y\}$, contradicting $x\ R\ y$. Therefore $p_y(a_x) = 0$.

If $p_x(\Box b) = 1$ then $x \in f(\Box b) \subseteq \Box^R f(b)$, so (by definition of $\Box^R$) $y \in f(b)$, i.e. $p_y(b) = 1$. Therefore $p_x(\Box b) \leqslant p_y(b)$. □

Lemma 5. *If $x \in X$ then $f^\sharp(x) \in \mathbb{P}_A$.*

Proof. Let $P = \{f^\sharp(x') | x' \in X\}$; we will show that P satisfies (17), so $x \in P \subseteq \mathbb{P}_A$.

If $(p_x, a_x) \in P$ and $b \not\leqslant a_x = \bigvee\{c | \forall y \in f(c).\ x \not R\ y\}$ then there is $y \in f(b)$ such that $x\ R\ y$. But then $(p_y, a_y) = f^\sharp(y) \in P$, $p_y(b) = 1$, and $(p_x, a_x)\ R_A\ (p_y, a_y)$ by Lemma 4. □

Lemma 4 shows that $f^\sharp$ preserves the relation; in order to show that it is a continuous p-morphism, it is necessary to prove continuity and property (2). This is done in the next two lemmas.

Lemma 6. *For all $a \in A$, $(f^\sharp)^{\leftarrow}(\phi_A(a)) = f(a)$, and therefore $f^\sharp$ is continuous.*

Proof. $x \in (f^\sharp)^{\leftarrow}(\phi_A(a))$ iff $f^\sharp(x) \in \phi_A(a)$ iff $p_x(a) = 1$ iff $x \in f(a)$. Therefore $(f^\sharp)^{\leftarrow}(\phi_A(a)) = f(a)$. □

Lemma 7. *If $x \in X$ and $(p, a) \in \phi_A(b) \in \mathfrak{O}_A$ such that $f^\sharp(x)\ R_A\ (p, a)$, then there exists $y \in X$ such that $x\ R\ y$ and $f^\sharp(y) \in \phi_A(b)$.*

Proof. Since $p(a_x) = 0$ and $p(b) = 1$, we know that $b \nleqslant a_x = \bigvee\{c | \forall y \in f(c).\ x \not R\ y\}$, so there is a $y \in f(b)$ such that $x\ R\ y$. But then $f^\sharp(y) \in \phi_A(b)$ by Lemma 6. □

Finally, we have to show that $f^\sharp$ is the unique arrow with the requisite property.

Lemma 8. *If $g : (X, \mathcal{O}, R) \to (\mathbb{P}_A, \mathfrak{O}_A, R_A)$ satisfies $g^{\leftarrow}\phi_A = f$, then $g = f^\sharp$.*

Proof. For all $x \in X$, let $(q_x, b_x) = g(x)$; we will show that $p_x = q_x$ and $a_x = b_x$.

$p_x(c) = 1$ iff $x \in f(c)$ iff $x \in g^{\leftarrow}\phi_A(c)$ iff $g(x) \in \phi_A(c)$ iff $q_x(c) = 1$, so $p_x = q_x$.

If $a_x \nleqslant b_x$ then (from the definition of a_x) there is a $c \nleqslant b_x$ such that $\forall y \in f(c).\ x \not R\ y$. Since $(q_x, b_x) \in \mathbb{P}_A$, there is $(q, b) \in \mathbb{P}_A$ such that $(q_x, b_x)\ R_A\ (q, b)$ and $q(c) = 1$, i.e. $(q, b) \in \phi_A(c)$. Since g satisfies (2), there is some y such that $x\ R\ y$ and $g(y) \in \phi_A(c)$, i.e. $y \in f(c)$, which is a contradiction. Therefore $a_x \leqslant b_x$.

If $b_x \nleqslant a_x$ then there is a $y \in f(b_x)$ such that $x\ R\ y$. Then $g(x)\ R_A\ g(y)$ so $q_y(b_x) = 0$ i.e. $y \notin f(b_x)$, which is a contradiction. Therefore $b_x \leqslant a_x$. □

This completes the proof of Theorem 1.

We conclude this section by presenting explicitly the functor from modal frames to relational spaces, and the unit on the topological side.

Corollary 9. *The contravariant functor $\mathcal{F}$: MFrm $\to$ RelSp adjoint to $\mathcal{O}$ is defined on arrows $f : A \to B$ by*

$$\mathcal{F}(f)(p, b) = (p \circ f, \bigvee\{a | f(a) \leqslant b\}).$$

The topological unit ψ of the adjunction is defined by

$$\psi_X(x) = (p_x, a_x) \text{ where } \quad \begin{aligned} p_x(U) &= 1 \text{ iff } x \in U \\ a_x &= \{y | x \not R\ y\}^\circ. \end{aligned} \tag{20}$$

2.5 The Duality Theorem

If we have a relational space, take its modal frame of open sets, and construct the relational space of modal frame points, it may not be isomorphic to the space we started with. Similarly, if we take a modal frame, construct the space of modal frame points, and look at the modal frame of opens, this may not be isomorphic to the original frame. The spaces and frames where such isomorphisms do hold (more precisely, where the unit of the adjunction is iso) are therefore of particular interest. In this section we show that, just as with the adjunction between topological spaces and frames, the unit ϕ_A is iso precisely when A is (isomorphic to) the modal frame of opens of a relational space, and the unit ψ_X is iso precisely when X is (isomorphic to) the relational space of points of a modal frame.

It is useful to have terms for frames and spaces which are in the scope of the duality.

Definition. A modal frame A is **modally spatial** if ϕ_A is an isomorphism. A relational space X is **modally localic** if ψ_X is an isomorphism.

Proposition 10. *If X is a relational space, then $\mathcal{O}(X)$ is modally spatial.*

Proof. Since the topology on $\mathcal{F}(A)$ is the image of ϕ_A, ϕ_A is necessarily onto. But $\phi_{\mathcal{O}(X)}$ is split monic by the triangle identities, so is an isomorphism. □

Proposition 11. *If A is a modal frame, then $\mathcal{F}(A)$ is modally localic.*

Proof. By the triangle identities, $\mathcal{F}(\phi_A)\psi_{\mathcal{F}(A)} = 1_{\mathcal{F}(A)}$; we will show that $\psi_{\mathcal{F}(A)}\mathcal{F}(\phi_A) = 1_{\mathcal{F}\mathcal{O}\mathcal{F}(A)}$.

Let $(p, U) \in \mathcal{F}\mathcal{O}\mathcal{F}(A)$. Let

$$x = \mathcal{F}(\phi_A)(p, U) = (p\phi_A, \bigvee\{a|\phi_A(a) \subseteq U\})$$

so $\psi_X(x) = (p_x, a_x)$ as in (20).

$p_x\phi_A(a) = 1$ iff $x \in \phi_A(a)$ iff $p\phi_A(a) = 1$, so $p = p_x$.

Note that $\phi_A(a) \subseteq U$ iff $a \leqslant \bigvee\{a|\phi_A(a) \subseteq U\}$, and

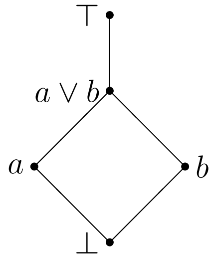
$$\begin{aligned} \phi_A(a) \subseteq a_x &\iff \phi_A(a) \subseteq \{y|x \not{R}_A\ y\} \\ &\iff \forall(q, b) \in \mathbb{P}_A.\ x\ R_A\ (q, b) \ \Rightarrow\ q(a) = 0. \end{aligned}$$

If $a \leqslant \bigvee\{a|\phi_A(a) \subseteq U\}$ and $x\ R_A\ (q, b)$ then $q(\bigvee\{a|\phi_A(a) \subseteq U\}) = 0$ so $q(a) = 0$. Therefore $U \subseteq a_x$.

If $a \not\leqslant \bigvee\{a|\phi_A(a) \subseteq U\}$ then by (17), there is some $(q, b) \in \mathbb{P}_A$ such that $x\ R_A\ (q, p)$ and $q(a) = 1$. Therefore $a_x \subseteq U$. □

In the non-modal case, topological spaces which are modally localic are sober, which implies T_0. We now present an example to show that this separation axiom fails in the modal case.

Example 7. Let A be the frame



with the modality

$$\begin{aligned} \Box(x) &= x \\ \Diamond(a \vee b) &= \top \\ \Diamond(x) &= x \qquad (\text{if } x \neq a \vee b). \end{aligned}$$

This frame has three frame points

$$\begin{aligned} p_\top &= \{\top\} \\ p_a &= \{a, a \vee b, \top\} \\ p_b &= \{b, a \vee b, \top\} \end{aligned}$$

and five modal frame points

$$\begin{aligned} t_a &= (p_a, b) \\ t_b &= (p_b, a) \\ s_a &= (p_\top, b) \\ s_b &= (p_\top, a) \\ s_{a \vee b} &= (p_\top, \bot). \end{aligned}$$

Since they have the same frame point, s_a, s_b and $s_{a \vee b}$ are not separated, so $\mathcal{F}(A)$ is not T_0.

3 The Spectral Case

In this section we apply the theory we have developed to intuitionistic modal logic, and derive soundness and completeness theorems. This is a natural generalisation of the application of frame theory to intuitionistic logic, and the results are completely analogous. As a generalisation of the Kripke semantics of classical modal logic, it is less obvious: if we start with a classical modal algebra, the construction below gives a space with a non-trivial topology. This provides new and interesting models of classical modal logic, generalising the Stone-space models of classical logic.

3.1 Modal Distributive Lattices

Rather than bother with the details of the syntax of modal logic, we will assume that the reader is familiar with the Lindenbaum construction of an algebra from a propositional calculus, and work with distributive lattices. Formalising Example 6, we define

Definition. A **modal distributive lattice** $(A, \leqslant, \Box, \Diamond)$ is a distributive lattice $(A, \leqslant)$ together with a pair of monotone maps $\Box, \Diamond : A \to A$ satisfying

$$\top \leqslant \Box(\top) \tag{21}$$
$$\Box(a) \wedge \Box(b) \leqslant \Box(a \wedge b) \tag{22}$$
$$\Diamond(a) \wedge \Box(b) \leqslant \Diamond(a \wedge b) \tag{23}$$
$$\Diamond(\bot) \leqslant \bot. \tag{24}$$

A **modal Heyting algebra** is a modal distributive lattice with Heyting implication.

The axioms (21–24) are not the only sets of axioms which have been proposed for intuitionistic modal logic, and we refer the reader to [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] for some alternatives. Nonetheless, in the author's opinion, the very concrete nature of the relational spaces as models, and the many natural examples of them throughout mathematics, make this approach particularly compelling. The axioms are also general enough to include most of the intuitionistic modal logics considered in the literature.

Definition. A **modal distributive-lattice morphism**

$$f : (A, \leqslant^A, \Box^A, \Diamond^A) \to (B, \leqslant^B, \Box^B, \Diamond^B)$$

is a distributive-lattice morphism $f : (A, \leqslant^A) \to (B, \leqslant^B)$ which satisfies

$$f(\Box^A(a)) \leqslant^B \Box^B(f(a)) \tag{25}$$
$$f(\Diamond^A(a)) \leqslant^B \Diamond^B(f(a)). \tag{26}$$

The morphism f is **strict** if (25,26) are equalities.

It is easy to see that modal distributive lattices and modal distributive-lattice morphisms form a category, which we will call MDLat. It is also immediate that there is a forgetful functor

$$\mathcal{J} : \text{MFrm} \to \text{MDLat} \tag{27}$$

since every modal frame is a modal distributive lattice, and similarly for morphisms. Our next result shows that the construction of Example 6 is not arbitrary, but in fact defines the left adjoint to $\mathcal{J}$.

Proposition 12. *The forgetful functor* $\mathcal{J}$: MFrm → MDLat *has a left adjoint*

$$\mathcal{I} : \text{MDLat} \to \text{MFrm} \tag{28}$$

defined by $\mathcal{I}(A) = (\text{Idl}(A), \subseteq, \Box', \Diamond')$ *where*

$$\Box'(I) = \downarrow\{\Box a | a \in I\} \tag{29}$$

$$\Diamond'(I) = \downarrow\{\Diamond a | a \in I\}. \tag{30}$$

The unit of the adjunction is $a \mapsto \downarrow a$, *and is strict.*

Proof. That $\Box'(I)$ and $\Diamond'(I)$ are ideals satisfying (4–7) is a straightforward exercise: each axiom of modal frames follows from the corresponding axiom of modal distributive lattices. $\mathcal{I}(A)$ is therefore a modal frame.

That the unit is a strict modal distributive-lattice morphism is also straightforward: we need only check that $\Box'\downarrow a = \downarrow\Box a$ and $\Diamond'\downarrow a = \downarrow\Diamond a$.

For the universal property, let $(B, \leqslant, \Box^B, \Diamond^B)$ be a modal frame, and $f : A \to B$ a modal distributive lattice morphism. Define $f^\sharp : \text{Idl}(A) \to B$ by

$$f^\sharp(I) = \bigvee\{f(a)|a \in I\}.$$

It is standard that this is the unique frame morphism satisfying $f^\sharp(\downarrow a) = f$. Now, for any $a \in I$,

$$f(\Box a) \leqslant \Box^B f(a) \leqslant \Box^B \bigvee\{f(a)|a \in I\}$$

so

$$f^\sharp(\Box' I) \leqslant \bigvee\{f(\Box a)|a \in I\} \leqslant \Box^B \bigvee\{f(a)|a \in I\}$$

i.e. $f^\sharp(\Box' I) \leqslant \Box^B f^\sharp(I)$ and similarly $f^\sharp(\Diamond' I) \leqslant \Diamond^B f^\sharp(I)$. Therefore $f^\sharp$ is a modal frame morphism. ∎

3.2 Modally Spectral Frames

Recall (from [**?**]) the following definitions and results about a frame $(A, \leqslant)$:

> An element $a \in A$ is **compact** (or **finite**) if for all directed $D \subseteq A$, if $\bigvee D \geqslant a$ then there is some $d \in D$ such that $d \geqslant a$. We write $K(A)$ for the set of compact elements of A.
>
> A is **spectral** (or **coherent**) if finite meets of compact elements are compact, and for all $a \in A$,
>
> $$a = \bigvee \{b \in K(A) | b \leqslant a\}. \tag{31}$$
>
> If A is spectral then $K(A)$ is a sub distributive-lattice of A, and A is isomorphic to $\mathrm{Idl}(K(A))$. The isomorphism is given by
>
> $$a \mapsto \{b \in K(A) | b \leqslant a\}. \tag{32}$$
>
> If B is a distributive lattice, then $\mathrm{Idl}(B)$ is spectral, and B is isomorphic to $K(\mathrm{Idl}(B))$. The isomorphism is given by
>
> $$b \mapsto {\downarrow} b. \tag{33}$$

Definition. $\Box$ or $\Diamond$ is **continuous** if it preserves directed joins.
$\Box$ or $\Diamond$ is **compact** if it takes compact elements to compact elements.
A is **modally spectral** if it is spectral and $\Box$ and $\Diamond$ are both continuous and compact.

Proposition 13. *If B is a modal distributive lattice, then $\mathcal{I}(B)$ is modally spectral.*

Proof. That $\mathrm{Idl}(B)$ is spectral is standard. If $D \subseteq \mathrm{Idl}(B)$ is directed, then $\bigvee D = \bigcup D$ so

$$\begin{aligned} \Box'(\bigvee D) = \Box'(\bigcup D) &= \{a | \exists I \in D.\ \exists b \in I.\ a \leqslant \Box b\} \\ &= \bigcup_{I \in D} \{a | \exists b \in I.\ a \leqslant \Box b\} \\ &= \bigvee_{I \in D} \Box' I \end{aligned}$$

so $\Box'$ is continuous, and $\Diamond'$ similarly.

If I is compact in $\mathrm{Idl}(B)$ then $I = \downarrow(b)$ for some $b \in B$ by (33), and

$$\begin{aligned}\Box'{\downarrow}(b) &= \{a | \exists c \leqslant b.\ a \leqslant \Box c\} \\ &= \downarrow(\Box b)\end{aligned}$$

so $\Box'$ is compact, and $\Diamond'$ similarly. Therefore $(\mathrm{Idl}(B), \subseteq, \Box', \Diamond')$ is modally spectral. ☐

Proposition 14. *If A is modally spectral frame, then (32) defines an isomorphism of modal frames with* $\mathrm{Idl}(K(A))$.

Proof. That (32) is an isomorphism of frames is standard; that $K(A)$ is a modal distributive lattice is immediate from the fact that $\Box$ and $\Diamond$ are compact. We have only to show that the isomorphism takes $\Box$ to $\Box'$ and $\Diamond$ to $\Diamond'$.

Every element of A is expressible as a directed join of compact elements by (31), so

$$\begin{aligned}\Box a &= \Box \bigvee \{b \in K(A) | b \leqslant a\} \\ &= \bigvee \{\Box b | b \in K(A)\ \&\ b \leqslant a\}\end{aligned}$$

but this set is directed, which means for compact c that $c \leqslant \Box a$ iff there is $b \in K(A)$ such that $b \leqslant a$ and $c \leqslant \Box b$. Therefore

$$\begin{aligned}\Box a \mapsto &\{c \in K(A) | c \leqslant \Box a\} \\ &= \{c \in K(A) | \exists b \in K(A).\ b \leqslant a\ \&\ c \leqslant \Box b\} \\ &= \Box' \{c \in K(A) | c \leqslant a\}.\end{aligned}$$

Similarly, $\Diamond a \mapsto \Diamond' \{c \in K(A) | c \leqslant a\}$, so (32) is an isomorphism of modal frames. ☐

3.3 Existence of Points

Recall the following definitions and results about a spectral frame $(A, \leqslant)$:

> A set $S \subseteq A$ is **Scott-open** if for all directed $D \subseteq A$, if $\bigvee D \in S$ then there is some $d \in D$ such that $d \in S$.
>
> If $b \not\leqslant a \in A$ then there is a Scott-open filter containing b but not a.
>
> If $F \subseteq A$ is a Scott-open filter and $a \in A$ is compact, then $F \wedge a = \{b \wedge a | b \in F\}$ is a Scott-open filter.
>
> A Scott-open filter not containing a can be expanded to a completely prime filter (i.e. a point) not containing a.

Some notation is useful: if $(A, \leqslant, \Box, \Diamond)$ is a modal frame, let

$$p_\Box = \{a | p(\Box a) = 1\}$$
$$p_\Diamond = \{a | p(\Diamond a) = 1\}.$$

Lemma 15. *If $\Box$ is continuous and p is a frame point then $p_\Box$ is a Scott-open filter.*

Proof. $\top \in p_\Box$, since $p(\Box\top) = p(\top) = 1$.

If $a, b \in p_\Box$ then $p(\Box(a \wedge b)) = p(\Box a \wedge \Box b) = 1$ so $a \wedge b \in p_\Box$.

If D is directed and $\bigvee D \in p_\Box$ then $\bigvee\{p(\Box d) | d \in D\} = p(\Box \bigvee D) = 1$ so there is some $d \in D$ such that $p(\Box d) = 1$ i.e. $d \in p_\Box$. □

Lemma 16. *If A is modally spectral then for any pre-point (p, a) and any $b \in p_\Diamond$ there is a frame point q satisfying $q(a) = 0$, $q(b) = 1$ and $p \circ \Box \leqslant q$.*

Proof. Let $S = \{c \in K(A) | c \leqslant b\}$, so S is directed and $\bigvee S = b$. Then

$$\begin{aligned} 1 = p(\Diamond b) &= p(\Diamond \bigvee S) \\ &= \bigvee\{p(\Diamond c) | c \in S\} \end{aligned}$$

so there is some $c \in S$ such that $p(\Diamond c) = 1$. By Lemma 15 and the compactness of c, $p_\Box \wedge c$ is a Scott-open filter.

If $d \in p_\Box$ then $p(\Diamond(d \wedge c)) \geqslant p(\Box d \wedge \Diamond c) = 1$ so $a \notin p_\Box \wedge c$; expand $p_\Box \wedge c$ to a point q such that $q(a) = 0$. Then $q(b) \geqslant q(c) = 1$ and $p \circ \Box \leqslant q$. □

Lemma 17. *If A is modally spectral then for every pre-point (p, a), there exists $b \geqslant a$ such that (p, b) is a modal frame point.*

Proof. For each $c \in p_\Diamond$ there is some frame point q_c satisfying $p \circ \Box \leqslant q_c$, $q_c(c) = 1$ and $q_c(a) = 0$ by Lemma 16.

Let $b = \bigvee S$ where

$$S = \{d | \forall c \in p_\Diamond.\ q_c(d) = 0\}. \tag{34}$$

Then $\bigvee S \in S$ because q_c preserves joins, so

$$p(\Diamond b) = \bigvee\{p(\Diamond d) | \forall c \in p_\Diamond.\ q_c(d) = 0\}.$$

But if $p(\Diamond d) = 1$ then $q_d(d) = 1$ so $d \notin S$. Therefore $p(\Diamond b) = 0$, and (p, b) is a pre-point.

To show that (p, b) is a point, let P be the set of pre-points (p, b) which can be constructed from pre-points (p, a) by the above process. We will show that P satisfies (17), so is a subset of $\mathbb{P}_A$.

If $(p, b) \in P$ and $d \not\leqslant b$ then $d \notin S$, so there is some $c \in p_\Diamond$ such that $q_c(d) = 1$. Since $q_c(\Diamond\bot) = q_c(\bot) = 0$, we can use the above construction to find $e \geqslant \bot$ such that $(q_c, e) \in P$. We will show that $(p, b)\ R_A\ (q_c, e)$.

We know that $p \circ \Box \leqslant q_c$. Also,

$$\begin{aligned} q_c(b) &= q_c(\bigvee S) \\ &= \bigvee\{q_c(d') | \forall c' \in p_\Diamond.\ q_{c'}(d') = 0\} \\ &= 0 \end{aligned}$$

so (p, b) is a point.

Finally, $q_c(a) = 0$ for all c, so $a \in S$ and $b \geqslant a$. $\Box$

Lemma 18. *If A is modally spectral then ϕ_A is one-one.*

Proof. If $b \not\leqslant a$ then there is some frame point p such that $p(b) = 1$ and $p(a) = 0$, and by Lemma 17 there is some $c \geqslant \bot$ such that (p, c) is a modal frame point. Then $(p, c) \in \phi_A(b)$ and $(p, c) \notin \phi_A(a)$ so $\phi_A(b) \not\subseteq \phi_A(a)$. $\Box$

Lemma 19. *If A is modally spectral then $\Diamond^{R_A}\phi_A(a) \subseteq \phi_A(\Diamond a)$.*

Proof. If $b \not\leqslant \Diamond a$ then there is some frame point p such that $p(b) = 1$ and $p(\Diamond a) = 0$, and by Lemma 17 there is some $c \geqslant a$ such that (p, c) is a modal frame point. Then $(p, c) \in \phi_A(b)$ and if $(p, c)\ R_A\ (q, d)$ then $q(a) \leqslant q(c) = 0$ so $(q, d) \notin \phi_A(a)$; so $\phi_A(b) \not\subseteq \Diamond^{R_A}\phi_A(a)$.

Therefore if $\phi_A(b) \subseteq \Diamond^{R_A}\phi_A(a)$ then $b \leqslant \Diamond a$ so $\phi_A(b) \subseteq \phi_A(\Diamond a)$, and the result follows because every open set is of the form $\phi_A(b)$. $\Box$

Lemma 20. *If A is modally spectral then $\Box^{R_A}\phi_A(a) \subseteq \phi_A(\Box a)$.*

Proof. If $b \not\leqslant \Box a$ then there is some frame point p such that $p(b) = 1$ and $p(\Box a) = 0$, and by Lemma 17 there is some $c \geqslant \bot$ such that (p, c) is a modal frame point. Now, $p_\Box$ is a Scott-open filter not containing a, so there is some frame point q_0 such that $p(\Box x) = 1 \ \Rightarrow\ q_0(x) = 1$ and $q_0(a) = 0$.

Let $c' = \bigvee S$ where

$$S = \{d \leqslant c | q_0(d) = 0\}$$

then $p(\Diamond c') \leqslant p(\Diamond c) = 0$, so (p, c') is a pre-point.

By Lemma 17, there is some e_0 such that (q_0, e_0) is a modal frame point. Then $q_0(c') = \bigvee\{q_0(d) | d \leqslant c \ \&\ q_0(d) = 0\} = 0$ and if $p(\Box x) = 1$ then $q_0(x) = 1$, so $(p, c')\ R_A\ (q_0, e_0)$.

We will show that (p, c') is a modal frame point. If $d \not\leqslant c'$ then $d \notin S$, so either $d \not\leqslant c$ or $q_0(d) = 1$. But if $q_0(d) = 1$, then (q_0, e_0) shows that $(p, c') \in \mathbb{P}_A$.

If $d \not\leqslant c$ then, because $(p, c) \in \mathbb{P}_A$, there is some (q, e) such that $(p, c)\ R_A\ (q, e)$ and $q(d) = 1$. But then $q(c') \leqslant q(c) = 0$, so $(p, c')\ R_A\ (q, e)$.

Now, $(p, c')\ R_A\ (q_0, e_0)$ and $(q_0, e_0) \notin \phi_A(a)$, so $(p, c') \notin \Box^{R_A}\phi_A(a)$; but $(p, c') \in \phi_A(b)$, so $\phi_A(b) \not\subseteq \Box^{R_A}\phi_A(a)$.

Therefore if $\phi_A(b) \subseteq \Box^{R_A}\phi_A(a)$ then $b \leqslant \Box a$ so $\phi_A(b) \subseteq \phi_A(\Box a)$, and the result follows because every open set is of the form $\phi_A(b)$. □

Our main result, that modally spectral frames are modally spatial, follows.

Theorem 2. *If A is modally spectral then ϕ_A is iso.*

Proof. ϕ_A is onto by definition of the topology $\mathfrak{O}_A$, and one-one by Lemma 18, so it is certainly a frame isomorphism; by Lemmas 19 and 20, it preserves $\Box$ and $\Diamond$. □

¿From this theorem, we can deduce that the category of modal distributive lattices is dual to a subcategory of the category of relational spaces. The problem of characterising precisely which relational spaces arise in this way in terms of topological and relational properties appears not to be straightforward.

4 Conclusions

The results of this paper provide a promising new approach to modal logic. The relational spaces are a natural class of mathematical structures, with no arbitrary or restrictive conditions, and plenty of familiar concrete examples. The semantics generalise and unify the topological semantics of intuitionistic logic and the Kripke semantics of modal logic in a straightforward way. The resulting intuitionistic modal logic is completely axiomatised by four straightforward axioms. Finally, the duality theorem shows that the results fit into a wider mathematical context, and we can expect more results in this area. It is worth considering how this theory can be applied and extended.

Two important potential applications are mentioned as examples in the paper: temporal logic based on the real numbers, and dynamic logic based on Scott domains. In both these applications the interpretation of propositions as open sets enforces a natural "observability" condition on the logic. In the case of temporal logic, it means that any proposition true at time t must be true over some interval $(t - \delta, t + \delta)$ containing t. In the case of dynamic logic, it means that any proposition true of an infinite computation must become true at some finite stage in that computation. This combination of modal logic with observability conditions has great potential for application to domain theory, concurrency theory, and many other areas of computer science.

Two areas of modal logic not mentioned in this paper are proof theory and correspondence theory. The four modal axioms presented in the paper provide a good starting point for the study of the syntax, and preliminary investigation indicates that there is a sequent-calculus formulation of the logic with cut-elimination. The correspondence theory of topological modal logic seems to be a much harder problem. Even reflexivity cannot be characterised by modal properties: on the real numbers, the reflexive ordering $\leqslant$ and the strict ordering $<$ give rise to the same modal connectives $\Box$ and $\Diamond$, so are indistinguishable in the logic. There is a subtle interaction between the relation and the topology which needs further investigation.

¿From the locale-theoretic point of view, there are several obvious lines for further development. The existence of points theorem for modally spectral frames is clearly not the best result we could hope for, and it seems likely that it can be generalised to the case where the underlying frame is compact regular, or even stably locally compact. It is less clear whether the continuity condition on $\Box$ (which is closely related to whether the direct image $R^{\rightarrow}(K)$ of a compact set is compact) can be weakened to some more local property.

The duality presented in this paper extends the methods of frame theory in a significant and potentially useful way. Although there is much still to be done, this promises to be the basis of an important new technique in logic and theoretical computer science.