University of Manchester
Department of Computer Science

CS3282

Digital Communications ’05-’06

Section 5: Detection of Binary Signals in AWGN

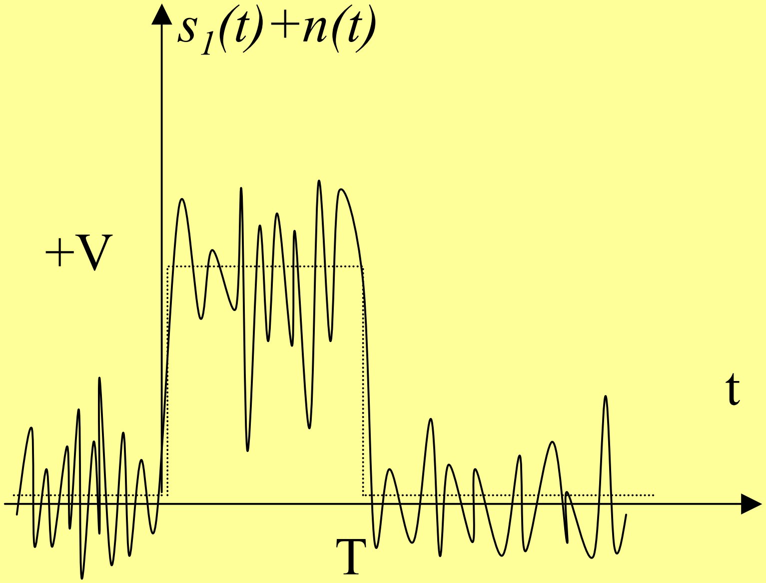
- Consider transmission of single bit using binary signalling.
- At receiver in absence of noise, let symbols be $s_1(t)$ and $s_0(t)$.
- If received signal is affected by AWGN n(t), we receive:

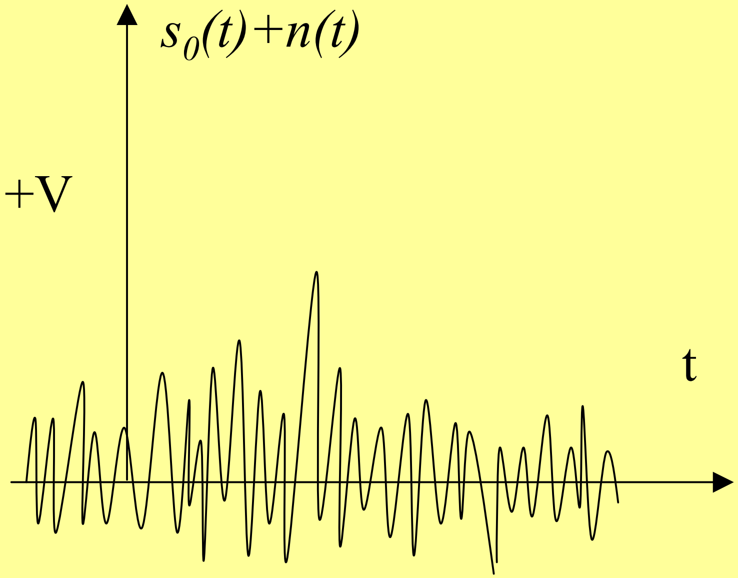
$$r(t) = s_i(t) + n(t) : i=1 \text{ or } 0$$

- Assume n(t) has '2-sided' PSD $N_0/2$ Watts/Hz.
- Power & bandwidth of noise n(t) is not specified, just PSD.
- Simple example is 'unipolar' signalling with 'rectangular' pulse shape.
- Alternative is to receive 'bipolar' signalling with a rect pulse shape.
- For unipolar signalling, with AWGN,

received signal $r(t)$ will be as shown for '*1*' & '*0*':

Bipolar signalling with AWGN





- Restrict analysis to single bit to begin with.
- For unipolar signalling, strategy for detecting *1* or *0* would be to sample $r(t)$ at $t=T/2$, & compare it with a 'threshold' $+V/2$.
- If $r(T/2) > +V/2$, likely $s_1(t)$ otherwise $s_0(t)$.
- For rect pulses, no matter whether we sample in middle, at beginning or at end .
- Wherever we sample, risk that noise $n(t)$ will cause $s_1(t)$ to be mistaken for $s_0(t)$ or vice-versa.
- Choice of threshold half way between *zero* & $+V$ is good when probabilities of $s_0(t)$ & $s_1(t)$ are known to be equal, i.e. 0.5.
- This is commonly the case.

Exercise 5.1: For 0 & 1 volt binary symbols probabilities are *0.1* & *0.9* rather than *0.5* & *0.5*. Variance of noise at detector is *1/16*. What is bit-error prob P_B if threshold $\gamma = 0.5$?
Can we reduce P_B by choosing a different value of γ ?

Solution: $\sigma = 1/4 = 0.25$. With $\gamma = 0.5$,

$$P_B = 0.1Q(0.5/\sigma) + 0.9Q(0.5/\sigma) = Q(0.5/\sigma) \text{ as usual.}$$

Now $d\,Q(z)/dz = -(1/\sqrt{(2\pi)})\exp(-z^2/2)$ and

$$P_B = prob(0)*Q(\gamma/\sigma) + prob(1)*Q((1-\gamma)/\sigma)$$

$$dP_B/d\gamma = (-1/(\sigma\sqrt{(2\pi)}))\,[prob(0)\exp(-\gamma^2/(2\sigma^2)) - prob(1)\exp(-(1-\gamma)^2/(2\sigma^2))\,)\,]$$

$dP_B/d\gamma = 0$ if $prob(0)\exp(-\gamma^2/(2\sigma^2)) = prob(1)\exp(-(1-\gamma)^2/(2\sigma^2))\,)$

$$\text{i.e. } \log_e(prob(0)) - \gamma^2/(2\sigma^2) = \log_e(prob(1)) - (1-\gamma)^2/(2\sigma^2)$$

$$\therefore\ \gamma_{best} = 0.5 - \sigma^2 \log_e(\,prob(1\ volt)\,/\,prob(0)\,)$$

Exercise 5.1 (continued)

So we can find a value of γ that gives a lower value of P_B as derived on previous slide. Its value is:

$$\gamma_{best} = 0.5 - \sigma^2 \log_e(prob(1) / prob(0)) = 0.5 - [\log_e(9)]/16$$
$$= 0.36$$

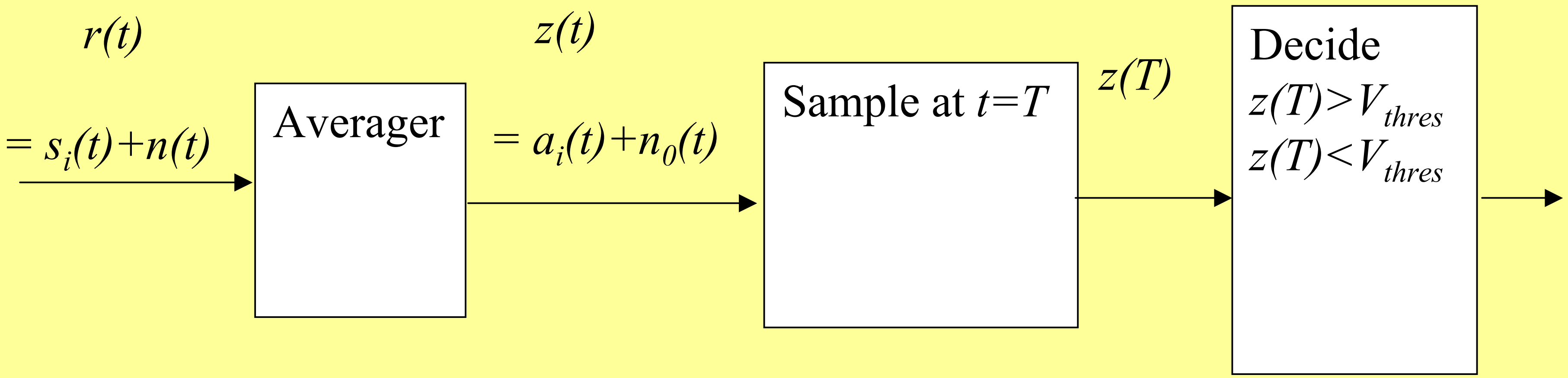
$$P_B = prob(0) * Q(\gamma / \sigma) + prob(1) * Q((1-\gamma) / \sigma)$$

$$= 0.1\, Q(0.36/0.25) + 0.9\, Q(0.64/0.25) =$$
$$0.1Q(1.44) + 0.9Q(2.56) = \ldots$$

Can use trial & error with $Q(z)$ curve, or a simple MATLAB program to check this. Remember that $Q(z) = 0.5 * erfc(z / (\sqrt{2}))$

```
% Investigate effect of choice of threshold on unipolar signalling
% when probabilities of ones and zeros are not equal.
prob1=0.9;  prob0=0.1;   % Voltages are 1 and 0
nstdev = 1/4 ;           % Standard dev of AWGN
pemin = 1;               % Becomes minimum bit-error probability
bestgamma=0;             % Becomes best thrshold between 0 & 1
for i=10:90
   gamma=i/100;
   pe = prob0*0.5*erfc((gamma/nstdev)/1.414);
   pe = pe + prob1*0.5*erfc(((1-gamma)/nstdev)/1.414);
   disp(sprintf('gamma=%f pe=%g',gamma,pe));
   if pe<pemin pemin = pe; bestgamma=gamma; end
end;
disp(sprintf('bestgamma=%f',bestgamma));
disp(sprintf('pemin=%g',pemin));
```

Improved detectn process for unipolar rect symbols $s_1(t)$ or $s_0(t)$:



$r(t)$

$= s_i(t)+n(t)$

Averager

$z(t)$

$= a_i(t)+n_0(t)$

Sample at $t=T$

$z(T)$

Decide

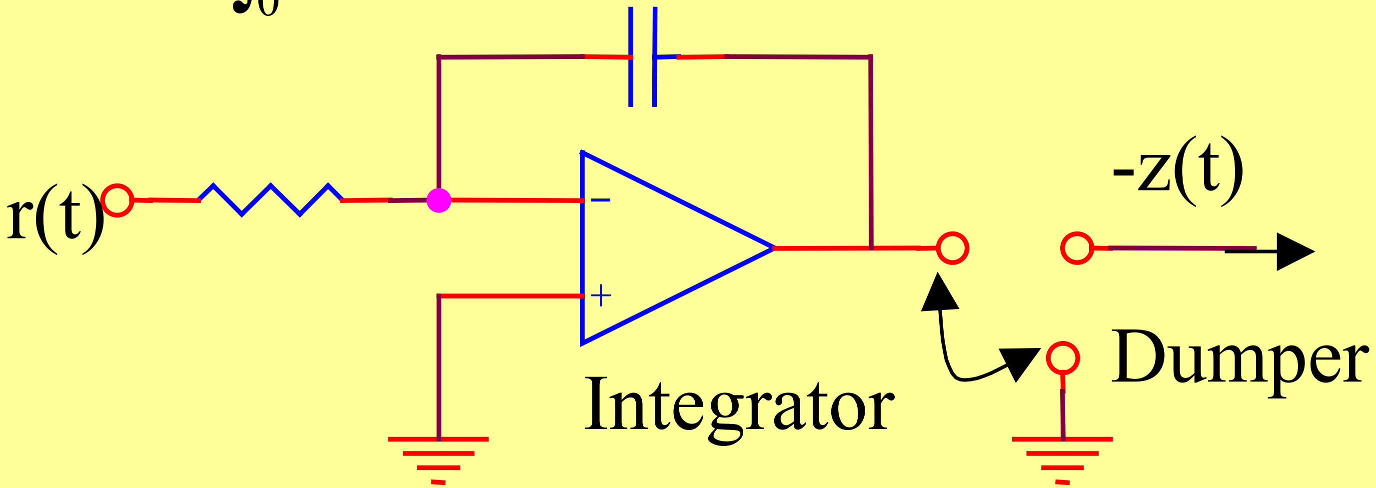
$z(T)>V_{thres}$

$z(T)<V_{thres}$

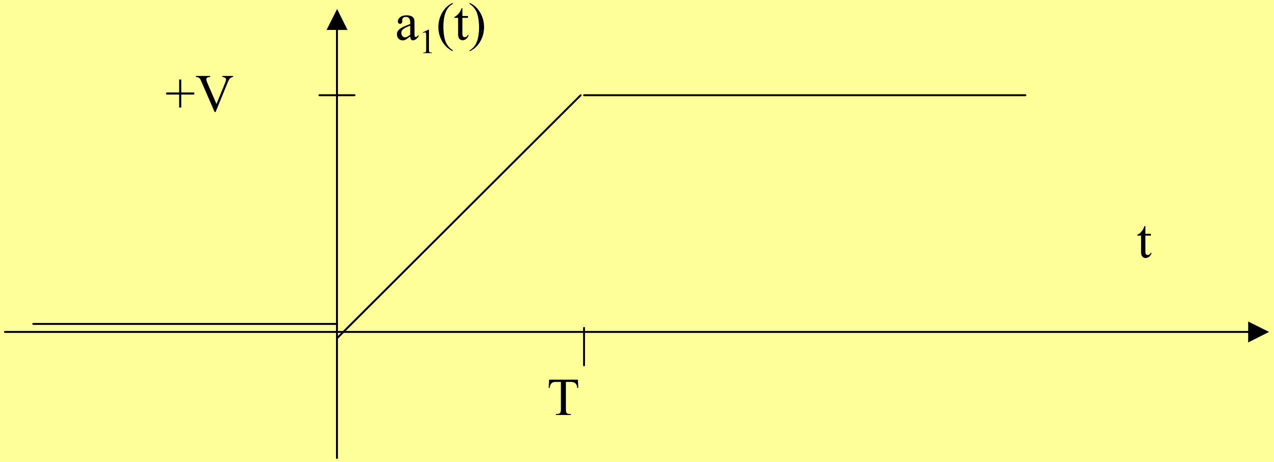
Response of averager to $s_i(t)$ is $a_i(t)$, & response to $n(t)$ is $n_0(t)$. Averager (or smoother) could ideally produce an output

$$z(t) = (1/T)\int_0^t r(\tau)\,d\tau$$

To generate $\int_0^t r(\tau)d\tau$



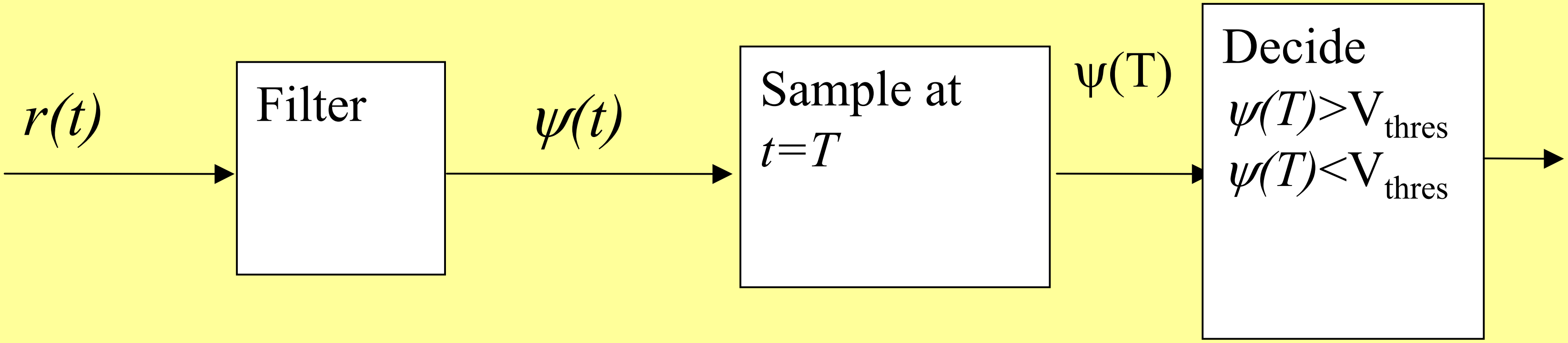
- Force integrator output to start at zero volts at t=0.
  (A switch dumps charge on capacitor.)
- 'Integrate and dump' circuit.
- For unipolar signalling with rect pulse, response $a_0(t)$ of averager to $s_0(t)$ without noise would be zero.
- Response $a_1(t)$ to $s_1(t)$ would be as shown on next slide.
- Starts at zero & reaches $+V$ at $t=T$.
- Remains at $+V$ until we apply the charge dumper again.



- Instead of clean $a_1(t)$, averager produces $z(t) = a_1(t) + n_0(t)$ where $n_0(t)$ is due to $n(t)$.
- Averager reduces noise.
- Hence chances of noise causing wrong decision reduced.
- Sample at t=T since averaging must be allowed to finish.
- After z(t) sampled, charge dumping sets averager output to zero,
- Circuit then able to receive another pulse.

Matched filter' method for detecting rectangular pulse shape.

- Use a filter rather than an averager.
- Denote filter response to *r(t)* as $\psi(t)$ (ψ *is* PSI)
- $a_i(t)$ & $n_0(t)$ now different from what they were with averager.



r(t)

Filter

$\psi(t)$

Sample at

$t=T$

$\psi(T)$

Decide

$\psi(T)>V_{thres}$

$\psi(T)<V_{thres}$

- Low-pass filter is sort of averager, but there are differences.
- Let frequency-response of filter be H(*f*) which is FT of *h(t)*.
- Assume that the impulse-response is:

Filter’s impulse resp:

$$h(t) = \begin{cases} 1/T : 0 \le t \le T \\ 0 : otherwise \end{cases}$$

Reason for this impulse-response will be made clear later.
In response to r(t), filter's output is:

$$\psi(t) = \int_{-\infty}^{\infty} h(\tau) r(t-\tau) d\tau$$

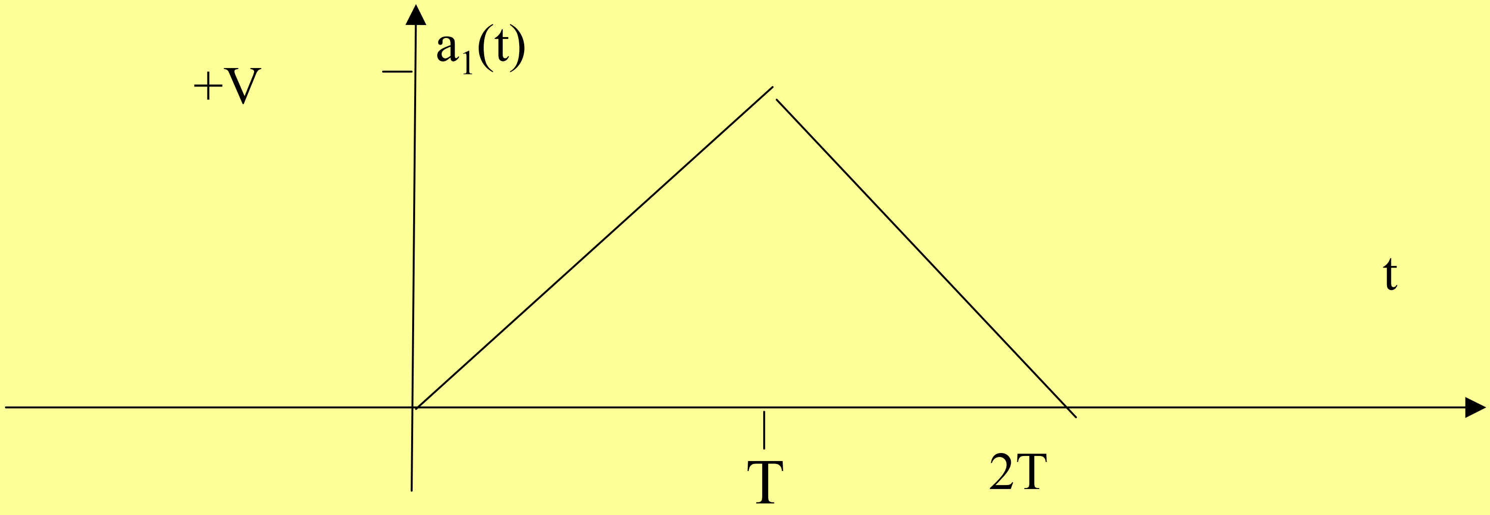
In this case: $\psi(t) = (1/T)\int_0^T r(t-\tau)d\tau$

and at t=T: $\psi(T) = (1/T)\int_0^T r(T-\tau)d\tau$

Substituting t=T-τ which means that dτ = -dt, we obtain:

$$\psi(T) = (1/T)\int_T^0 -r(t)dt = (1/T)\int_0^T r(t)dt$$

- Identical to value of z(T) we obtained using the averager.
- Another way of generating z(T) for a single pulse.
- Uses a filter rather than an averaging circuit.
- Note $\psi(t)$ equals z(t) only at t=T, but this is only point we need.
- Response of filter to $s_1(t)$ is as follows:



Filter output decays to zero at t=2T without 'charge dumper'.

Estimating bit-error rate when ‘ I & D’ or ‘MF’ is employed.

- Bit-error rate will be the same for both approaches.
- Analyze matched filter approach:
- Consider 2-sided PSD of $n_0(t)$, where $n_0(t)$ is filter's response to $n(t)$.

$$PSD_0(f) = PSD(f)\ |H(f)|^2$$

where $PSD(f)$ is the ‘2-sided’ PSD of noise $n(t)$.

- Power of $n_0(t)$ is therefore:

$$\int_{-\infty}^{\infty} PSD_0(f)\,df = \int_{-\infty}^{\infty} PSD(f)\,|H((f))|^2\,df$$

$$= 0.5N_0 \int_{-\infty}^{\infty} \left|H((f))\right|^2 df = 0.5N_0 / T$$

since PSD(f)=0.5N_0 constant for all f, and, by Parseval’s Theorem,

$$\int_{-\infty}^{\infty} \left|H((f))\right|^2 df = \int_{-\infty}^{\infty} \left|h(t)\right|^2 dt = \int_0^T (1/T)^2\,dt = 1/T$$

- When AWGN applied to filter, output is coloured Gaussian noise.

To summarise:

- If white Gaussian noise $n(t)$ with 2-sided PSD $N_0/2$ Watts/Hz were applied to a filter with impulse-response:

$$h(t) = \begin{cases} 1/T : 0 \leq t \leq T \\ 0 : otherwise \end{cases}$$

output would be Gaussian noise $n_0(t)$ of power $0.5N_0/T$ Watts.

$\therefore$ if $r(t) = s_i(t) + n(t)$, with $i=1$ or 0, applied to same filter, output is:

$$\psi(t) = a_i(t)+n_0(t)$$

where $a_i(t)$ is filter's response to $s_i(t)$ for $i = 1$ or 0.

- If we sample $\psi(t)$ at $t=T$, $\psi(T) = a_i(T)+n_0(T)$ is equal to $z(T)$ as obtained by sampling output $z(t)$ from an averaging circuit at $t=T$.

- In case of unipolar signalling with +*V* & *zero* valued *rect* pulses, we know that $a_i(T)$ is +*V* or *zero*, but what can we say about $n_0(T)$?
- It is a voltage obtained by sampling a random signal which may cause a bit-error depending on how large it is.
- Cannot say what this voltage will be exactly so we cannot say definitely that it will or will not cause an error.
- Can estimate probability of it being large enough to cause an error
- Use properties of statistical process that describes production of this voltage ; i.e. a statistical process which is Gaussian with zero mean & variance $0.5N_0/T$.
- Subject to certain conditions, if average value of a random voltage signal is zero, its <u>power</u> equals the <u>variance</u> (σ^2) of a statistical process that describes it.

- Effect of "I&D" or MF is to produce $a_i(T) + n_0(T)$ where $n_0(T)$ is random variable of variance $0.5N_0/T$.
- If $a_i(T)$ is $+V$ & $a_0(T)=0$, & we take threshold at $+V/2$, probability of bit-error is:

  $P_b =$ '$prob(n_0(T) < -V/2)$ when $s_1(t)$ received'

  $+$ '$prob(n_0(T) > V/2)$ when $s_0(t)$ is received'.
- No correlation between noise & bit-stream. Therefore

  $P_b = prob(n_0(T) < -V/2) * prob$ (transmitting 1)

  $+$ '$prob(n_0(T) > V/2) * prob$(transmitting 0)
- If '1' & '0' equally likely,

  $$P_b = 0.5\ [\ prob(n_0(T) < -V/2) + prob(n_0(T) > +V/2)\ ].$$
- For AWGN with mean=0: prob($n_0(T) < -V/2$) = prob($n_0(T) > +V/2$)
- Therefore for unipolar rect signalling,

  $$P_b = \text{prob}(\ n_0(T) > +V/2\ )$$

- If $n_0(t)$ had power equal to 1 Watt, variance $\sigma_0^2 = 1$ then P_b would be equal to $Q(V/2)$;
- However, noise power is $0.5N_0/T$, so $\sigma_0^2 = 0.5N_0/T$.
- Standard deviation σ_0 is square root of this value.
- Therefore:

$$P_b = Q\left(\frac{V/2}{\sqrt{0.5N_0/T}}\right)$$

Example 5.2A:

Receive 1 volt & 0 volt rect binary symbols at 100 Baud.

Transmission distorted by AWGN with zero mean & PSD:

N_0=0.00025 Watts/Hz

Estimate bit-error probability with an "I&D" or matched filter.

Assume equal occurrence of 1's & 0's & appropriate threshold

Solution:

Consider output from averager or MF . Noise has variance.

$$\sigma_0^{\,2} = N_0/(2T) = 0.00025\times 0.02 = 1.25\times 10^{-2}$$

Therefore, $\underline{\sigma_0=0.112.}$

Pulses unaffected by averager,. Decision threshold = 0.5Volts.

Bit-error probability is:

$$P(\text{"0"})\times P(n_0(T)>(1-0.5))+P(\text{"1"})\times P(n_0(T)<(0-0.5))$$
$$=0.5\times Q\left(\frac{0.5}{\sigma_0}\right)+0.5\times Q\left(\frac{0.5}{\sigma_0}\right)$$
$$=Q(4.46)$$
$$\approx \underline{\underline{3.5\times 10^{-6}}}$$

Bit error-rate is one bit in 200,000.

(About one character wrong in a 6-page document)

Interesting to compare this with what would be obtained without matched filter or averager.

But under assumption that n(t) is AWGN, its power is infinite. Therefore $P_b = Q(0)$ = 0.5, the worst we can get. Not 1??

- We get more sensible result when we realise that receiver 'front-end' has filter to restrict bandwidth of received signal.

- Assume that noise band-width is restricted to $\pm B$ Hz, for some value of B, by a 'front-end' filter

- Noise power now restricted to $N_0 B$ Watts.

- So variance at decision stage of detector would also be $N_0 B$.

- Now consider the following example:

Example 5.2B:

Receive 1 volt & 0 volt rect binary symbols at 100 Baud distorted by zero mean AWGN with N_0=0.00025 Watts/Hz.

Estimate bit-error probability without a matched filter or averager.

Ideal low-pass filter $H(f)$ employed with bandwidth ±500 Hz.

Assume equal occurrence of 1 & 0, with appropriate threshold.

Compare with previous example.

Solution:

Filtered noise has power = 0.00025 x 500 = 0.125 Watt.

Standard deviation of noise is therefore 0.35

Decision on basis of 0 & 1 Volt pulses; threshold = 0.5 Volts.

Bit-error probability is prob of noise sample exceeding 0.5 Volts with "0" or being less than –0.5 Volts with "1".

i.e. $Q(0.5/0.35) = Q(1.42) = 8\text{x}10^{-2}$

One bit in 12.5 (About one character wrong every one or two.)

Example 5.3A:

Repeat previous example with rect pulses changed to:

- **a.** +3 Volts for logic “1” and +2 Volts for “0”.
- **b.** +0.5 Volts for “1” and –0.5 Volts for “0”.

Assume equal prob of *1*& *0* and a ‘half-way’ threshold.

Solution:

Although the decision thresholds change, (2.5 V for (a) and 0 V for (b) the solutions are identical.

Example 5.4: Receiver receives AMI coded data with ±1 volt & 0 volt rect binary symbols at 100 Baud.

Distorted by zero mean AWGN with N_0=0.00025 Watts/Hz.

I&D circuit of MF employed.

Estimate bit-error probability assuming equal occurrence of 1 & 0 and appropriate threshold.

Compare with what would be obtained if "I&D" or MF replaced by ideal low-pass filter band-limiting from –500Hz to +500Hz. Assume that this bandwidth is wide enough to avoid imposing significant changes on shape of the rect pulses.

Solution: Same bit-error rate

<u>Example 5.5</u>: Due to malfunction, dc level drifts so that rect pulses become 1.2 & 0.2 volts rather than 1 & 0 volts. How is bit-error probability affected if threshold remains at 0.5 ?

<u>Solution</u>: With the "I & D" circuit, the error probability is:

$$P("0")\times P(n_0(T)>0.3)+P("1")\times P(n_0(T)<-0.7)$$

$$=0.5\times Q\left(\frac{0.3}{0.112}\right)+0.5\times Q\left(\frac{0.7}{0.112}\right)$$

$$=0.5Q(2.68)+0.5Q(6.25)$$

$$\approx 1.75\times 10^{-3}+1\times 10^{-10}$$

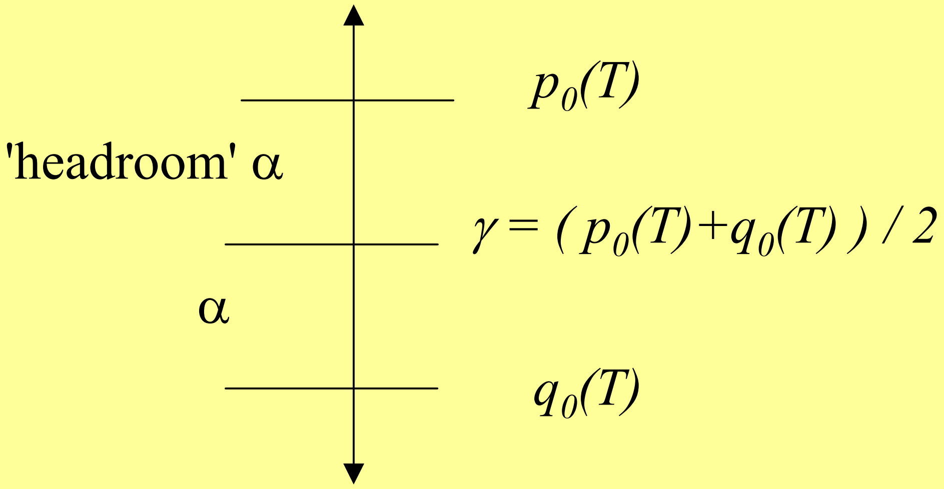
$$\approx \underline{\underline{1.75\times 10^{-3}}}$$

Matched filter for arbitrary signal shapes

- Consider signalling with $p(t)$ for '1' & $q(t)$ for '0'.
- Received with zero mean AWGN of 2-sided PSD $N_0/2$ Watts/Hz.
- Received signal, $r(t)$, passed thro' filter to produce $z(t)$.
- Response of filter to $p(t)$, $q(t)$ & $n(t)$ is $p_0(t)$, $q_0(t)$ & $n_0(t)$.
- Detector chooses sampling point, $t=T$, & sets threshold :

$$\gamma = \left(p_0(T) + q_0(T) \right) / 2.$$

- Assume that each pulse starts at $t=0$ & ends at $t = T$.
- Assuming $p_0(T) > q_0(T)$: when $z(T) \geq \gamma$ detector delivers '1'
  otherwise '0'.
- Correct unless $n_0(T) > \alpha$ for $q(t)$
  or $n_0(T) < -\alpha$ for $p(t)$
  where α is the "headroom" between γ & $p_0(T)$ or $q_0(T)$.



$p_0(T)$

'headroom' α

$\gamma = (p_0(T)+q_0(T)) / 2$

α

$q_0(T)$

$$\alpha = p_0(T) - \gamma \quad = \quad \gamma - q_0(T) \quad = \quad (p_0(T) \; - q_0(T))/2$$

- Assuming equal prob for 1 & 0,

$$P_B = 0.5Q(\alpha/\sigma_0) \; + \; 0.5Q(-\alpha/\sigma_0) \; = \; Q(|\alpha/\sigma_0|)$$

where σ_0 is standard deviation of filtered noise $n_0(t)$.

- H(*f*) could be any low-pass filter, but for rect pulses a MF or I&D is better.
- To generalise to pulses of any shape, make H((*f*)) a matched filter “tuned” to difference in shape between *p(t)* & *q(t)*.

A matched filter maximises $|\alpha/\sigma_0| = (p_0(T) - q_0(T)/(2\sigma_0)|$
This minimises the bit-error probability $P_B = Q(|\alpha/\sigma_0|)$.

If $p(t)$ & $q(t)$ have Fourier transform $P(f)$ & $Q(f)$, then :

$$\alpha = 0.5(p_0(T) - q_0(T)) = 0.5\int_{-\infty}^{\infty} H(f)(\ P(f) - Q(f)\ )e^{j2\pi fT}\,df$$

$$Noise\ Power = 0.5N_0\int_{-\infty}^{\infty} |H(f)|^2\,df$$

To minimise P_B, we would like to maximize:

$$|\alpha/\sigma_0|^2 = \frac{\left|0.5\int_{-\infty}^{\infty} H(f)(\ P(f) - Q(f)\ )e^{2\pi jfT}\,df\ \right|^2}{0.5N_0\int_{-\infty}^{\infty} |H(f)|^2\,df}$$

Study using Schwartz inequality

Schwartz’s inequality for complex *x(t)* & *y(t)*

$$\left|\int x(t)y(t)dt\right|^2 \leq \int |\mathrm{x(t)}|^2 dt \int |\mathrm{y(t)}|^2 dt$$

Equality if $x(t) = k\,y^*(t)$ for any constant k

Proof: Not required.

Apply to expression for $|\alpha/\sigma_0|^2$, taking *t* as *f*, *x(t)* as *H(f)*

& y(t) as $(P(f)\text{-}Q(f))e^{2\pi j f T}$

$$|\alpha/\sigma_0|^2 \le \frac{\int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} \left|(P(f) - Q(f) e^{2\pi jfT}\right|^2 df}{2N_0 \int_{-\infty}^{\infty} |H(f)|^2 df}$$

$$\therefore |\alpha/\sigma_0|^2 \le \frac{1}{2N_0} \int_{-\infty}^{\infty} \left|(P(f) - Q(f) e^{2\pi jfT}\right|^2 df = \frac{1}{2N_0} \int_{-\infty}^{\infty} |P(f) - Q(f)|^2 df$$

For equality, to give max $|\alpha/\sigma_0|^2$, we must have

$$H(f) = k(\ P^*(f) - Q^*(f)\ )e^{-2\pi jfT} \quad \text{for some constant } k$$

Taking inverse FT gives impulse resp h(t) of filter required to maximise $|\alpha/\sigma_0|^2$. Call this the "*matched filter*".

Maximum value of $|\alpha/\sigma_0|^2$ is:

$$\frac{1}{2N_0}\int_{-\infty}^{\infty}\left|P(f)-Q(f)\right|^2 df \;=\; \frac{1}{2N_0}\int_{-\infty}^{\infty}\left|p(t)-q(t)\right|^2 dt \;=\; \frac{1}{2N_0}E_d$$

by Parseval's theorem.

E_d is 'energy of the difference signal p(t)-q(t) :

$$E_d=\int_{-\infty}^{\infty}|\,P(f)-Q(f)\,|^2\;df=\int_{-\infty}^{\infty}|\,p(t)-q(t)\,|^2\;dt$$

With matched filter, we get min possible bit-error prob, which is

$$Q\left(\sqrt{E_d/(2N_0)}\right)$$

- If FT of p(t) is *P(f)*, then FT of *p(-t)* is *P*(f)*.
- Similarly for *q(t)*.
- Complex conj in freq-domain ⇒ time-reversal in time-domain.
- Also, FT of *p(t-T)* is *P(f)*$e^{-2j\pi fT}$.
- Multiplying FT spectrum by $e^{-2j\pi fT}$ gives delay of *T* seconds.

- Matched filter H(*f*) for *p(t)* & *q(t)* has impulse response *h(t)* equal to *p(t)-q(t)* modified in 3 ways:
  - ((i) reversed in time,
  - ((ii) delayed by *T* seconds
  - ((iii) multiplied by any constant k.

- This result tells us that a matched filter for case where *p(t)* is *rect* & *q(t)* is zero, has impulse-response given earlier.
- We can now design matched filters for other symbol shapes.

Correlation Detector

Ideal matched filter for p(t) & q(t) at 1/T Baud has:

$$h(t) = k(p(T-t) - q(T-t)) \quad \textit{for any } k$$

When received signal is r(t), output from the matched filter is:

$$z(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau = \int_0^t r(\tau)h(t-\tau)d\tau$$

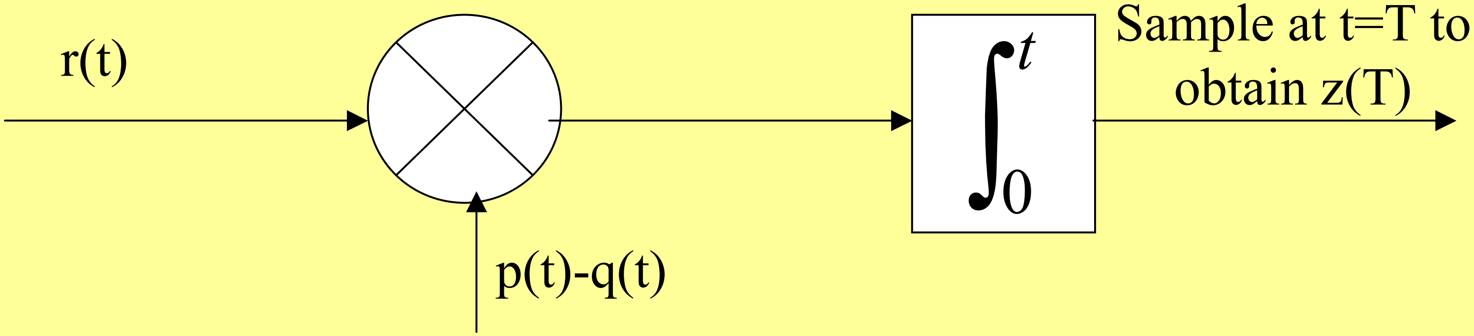
if filter is causal; i.e. if h(t-τ) =0 for t < τ and r(t) starts at t=0.

This is cross-correlation at delay t seconds between r(t) and h(-t):

$$\therefore z(t) = \int_0^t r(\tau)k(p(T-(t-\tau)) - q(T-(t-\tau)))d\tau$$

$$\therefore \ \mathrm{z(T)} = \int_0^{\mathrm{T}} \mathrm{kr}(\tau)(\mathrm{p}(\tau) - q(\tau))d\tau \ = \int_0^{\mathrm{T}} \mathrm{kr(t)(p}(t) - q(t))dt$$

z(T) can be produced in a different way:



Sample at t=T to obtain z(T)

- This is a “correlation detector”
- Often more practical & easier to realise than a matched filter.
- Similar to a “coherent detector”.
- Output depends on energy of symbol difference relative to the noise power.
- Actual shape of p(t)-q(t) is now not of primary importance.

To summarise this section so far,

- Receive p(t) or q(t) corrupted by AWGN n(t) of 2-sided PSD N_0 /2 Watts/Hz.
- Matched filter H((f)) maximises $|\alpha/\sigma_0|$ when $\alpha = (p_0(t) - q_0(t) / 2$.
- $p_0(t)$, $q_0(t)$ & $n_0(t)$ are responses to p(t), q(t) & n(t).
- σ_0 is standard deviation of $n_0(t)$ & symbol rate is 1/T.
- Correlation detector is alternative way of calculating matched filter output required at decision points.
- With binary transmission with p(t) & q(t) & defining a threshold $\gamma = (p_0(T)+q_0(T))/2$ (equi-prob 1 & 0) bit-error prob = $Q(|\alpha/\sigma_0|)$.
- $Q(z/\sigma_0)$ is prob of Gaussian variable of zero mean & variance $\sigma_0{}^2$ being greater than z.
- If E_d denotes energy of difference between p(t) & q(t), bit-error probability with a matched filter is:

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

- This formula requires only E_d and N_0.
- Actual shapes of *p(t)* & *q(t)* do not matter as long as we use a matched filter.
- When *p(t)* is rectangular of height A & duration *T*, & *q(t)*=0, this formula gives us

$$P_B = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

Exercise 5.7: A binary signal with NRZ +1 & 0 volt rect symbols p(t) & q(t) & a symbol rate of 1/T Baud is corrupted by AWGN with 2-sided PSD $N_0 /2 = 1\text{x}10^{-3}$ Watts/ Hz.

If received signal detected with matched filter, what is max bit-rate that can be sent with a bit-error rate less than 1 bit in 1000?

Estimate error-rate with no MF if noise bandwidth B=10kHz?

Solution:

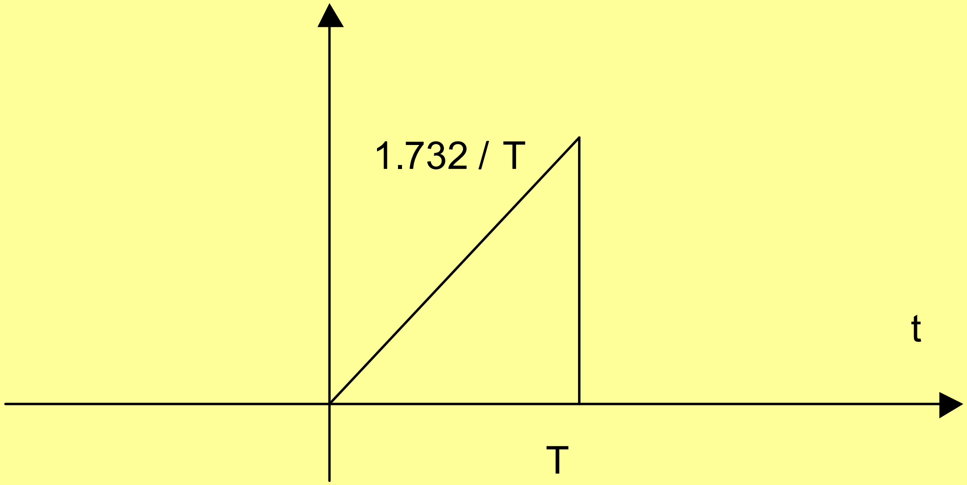
$$P_B = Q\left(\sqrt{\frac{1\times T}{4\times 10^{-3}}}\right) \leq 0.001$$

$$\sqrt{250T} \geq 3.1$$

$$\frac{1}{T} \leq 26Hz = Maximum\ bit-rate\ with\ matched\ \ filter$$

With no MF, $P_B = Q(1/(2\sigma))$ with $\sigma^2 = 0.2$.

Exercise 5.8: Consider again Exercise 5.7 where instead of being a rectangular symbol, p(t) is a triangular symbol of height √3 T and duration T as shown below. Sketch the impulse-response of the matched filter and calculate the minimum bit-rate achievable using this matched filter.



Example 5.9: In a binary digital communication system, the signal component obtained from a correlation receiver is expected to be $p_0(T) = 1$ volt for "1" & $q_0(T) = -1$ volt for "0".

Assume equal probability of 1 & 0 & a threshold $\gamma=0$ volts.

If the noise at the correlator output has unit variance, find the bit-error probability & the corresponding bit-error rate(BER).

Solution: $P_B = Q(1/1) = Q(1) = 0.18.$

Error rate = 1 bit in 5.5 bits.

Easy because we are told σ_0^2.

Exercise 5.10: How would bit-error probability obtained in problem 5.9 be affected if the threshold γ were inappropriately chosen to be 0.2 volts instead of zero.

Solution: $P_B = 0.5Q(1.2/1) + 0.5\,Q(0.8/1) = 0.19.$

Bit-error rate (BER) 1 bit in 5.

Note that $\int_{-\infty}^{\infty} |\,p(t)\,|^2\,dt - \int_{-\infty}^{\infty} |\,q(t)\,|^2\,dt$

i.e. difference between energy of p(t) & energy of q(t) is not necessarily equal to E_d since we cannot say that $\int_{-\infty}^{\infty} p(t)q(t)dt$ is always equal to zero.

If this expression is zero, p(t) and q(t) are *orthogonal*.

Unipolar signalling is orthogonal since the product of p(t) & q(t) is zero for all t because one of the symbols is zero for all t.

Bipolar signalling is not orthogonal.

Average Energy per bit

- The higher the power of the transmitted signal conveying a sequence of symbols, the higher the voltages of the symbols, the better will they be seen above the noise & therefore the lower the bit-error probablty.
- If average power of signal is P Watts & bit-rate is 1/T b/s,
  "average energy per bit", E_b, is P divided by (1/T).
- Units are Joules/bit.
- Watts (Joules/second) divided by bits/second, become Joules/bit.
- Useful to know how much energy each bit will cost us to send.
- Different for different signalling techniques.
- Required power of transmission is E_b times the bit-rate.
- Do not confuse E_b with E_d (energy of difference signal).

Unipolar signalling

- Let p(t) be shaped pulse & q(t) = 0 for 0 ≤ t ≤ T. This is unipolar signalling. We have shown that when a matched filter is employed,

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

with $E_d = \int_{-\infty}^{\infty} [p(t) - q(t)]^2\, dt = \int_0^T [p(t)]^2\, dt = energy\ of\ \ p(t)$

- Assuming equi-prob 1 & 0, for unipolar signalling, $E_b = E_d/2$.

$$\therefore \quad P_B = \mathrm{Q}(\sqrt{\mathrm{E}_b / N_0}\,)$$

Exercise 5.11: Let p(t) = A & q(t) = 0 for $0 \le t \le T$.
This is NRZ unipolar signalling with rectangular pulses.
Using the formula just established, show that $P_B = Q\left(\sqrt{A^2 T/(2N_0)}\right)$,
assuming equal probability of ones & zeros.

Solution: E_b = Average of $\{A^2T$ and $0\} = A^2T/2$.

Therefore $P_B = Q\left(\sqrt{A^2 T/(2N_0)}\right)$..

This agrees with earlier results

Bipolar signalling

- Let p(t) = s(t) & q(t) = -s(t) for $0 \le t \le T$.
- Binary signalling with anti-podal signals

  i.e. two signals which are negative of each other.
- Signal s(t) may have any shape. When a matched filter is employed with impulse-response 2s(T-t); i.e. matched to p(t)-q(t):

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

with $E_d = \int_{-\infty}^{\infty}\left[p(t)-q(t)\right]^2 dt = 4\int_0^T \left[p(t)\right]^2 dt = 4 \times energy\ of\ \ p(t) = 4E_b$

since energy of p(t) & q(t) are equal.

- In terms of E_b & N_0 :

$$P_B = \mathrm{Q}(\sqrt{2\,\mathrm{E_b} / N_0}\ )$$

Exercise 5.12: Let p(t) = A and q(t) = -A for $0 \le t \le T$. This is bipolar signalling with antipodal rectangular NRZ pulses. Calculate P_B assuming equal numbers of 1s and 0s.

Solution: Average energy per bit, E_b, = A^2T because energy of a rect symbol of height ±A & duration T seconds is A^2T.

$$\therefore \quad P_B = \mathrm{Q}(\sqrt{2\,\mathrm{E_b}/N_0}\,) = \mathrm{Q}\,(\sqrt{2\mathrm{A}^2T/N_0}\,)$$

To do this another way, note that:

$$E_d = \int_{-\infty}^{\infty}\left[p(t)-q(t)\right]^2 dt = \int_0^T \left[p(t)-q(t)\right]^2 dt = 4A^2T$$

$$\therefore P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{4A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

Comments about uni-polar & bipolar signaling:

- Consider which is more efficient in terms of having lower bit-error probability for a given average energy per bit.
- Consider graph of P_B against $10 \log_{10}(E_b/N_0)$ for matched filter reception of (i) unipolar & (ii) bipolar base-band signalling.
- Horizontal axis tells us how much greater, in dBs, the ‘average energy per bit’ is than the PSD N_0 of the noise.
- It is ratio of E_b in Joules/bit to N_0 in Watts/Hz.
- If noise bandwidth is 0 to B Hz, & its power is σ_0^2, then $N_0 = \sigma_0^2 / B$.
- You can draw this graph with the aid of a graph of Q(z) against z.
- When $E_b = N_0$, (0 dB), $P_B \approx 0.8\text{x}10^{-1}$ for bipolar & 0.2 (BER = 1 in 5) for unipolar.
- When $E_b = 10N_0$, (10 dB), $P_B \approx 0.3\text{x}10^{-5}$ for bipolar & 10^{-3} for unipolar.
- Bit-error probability of 10^{-3} means a BER of 1 in 1000.

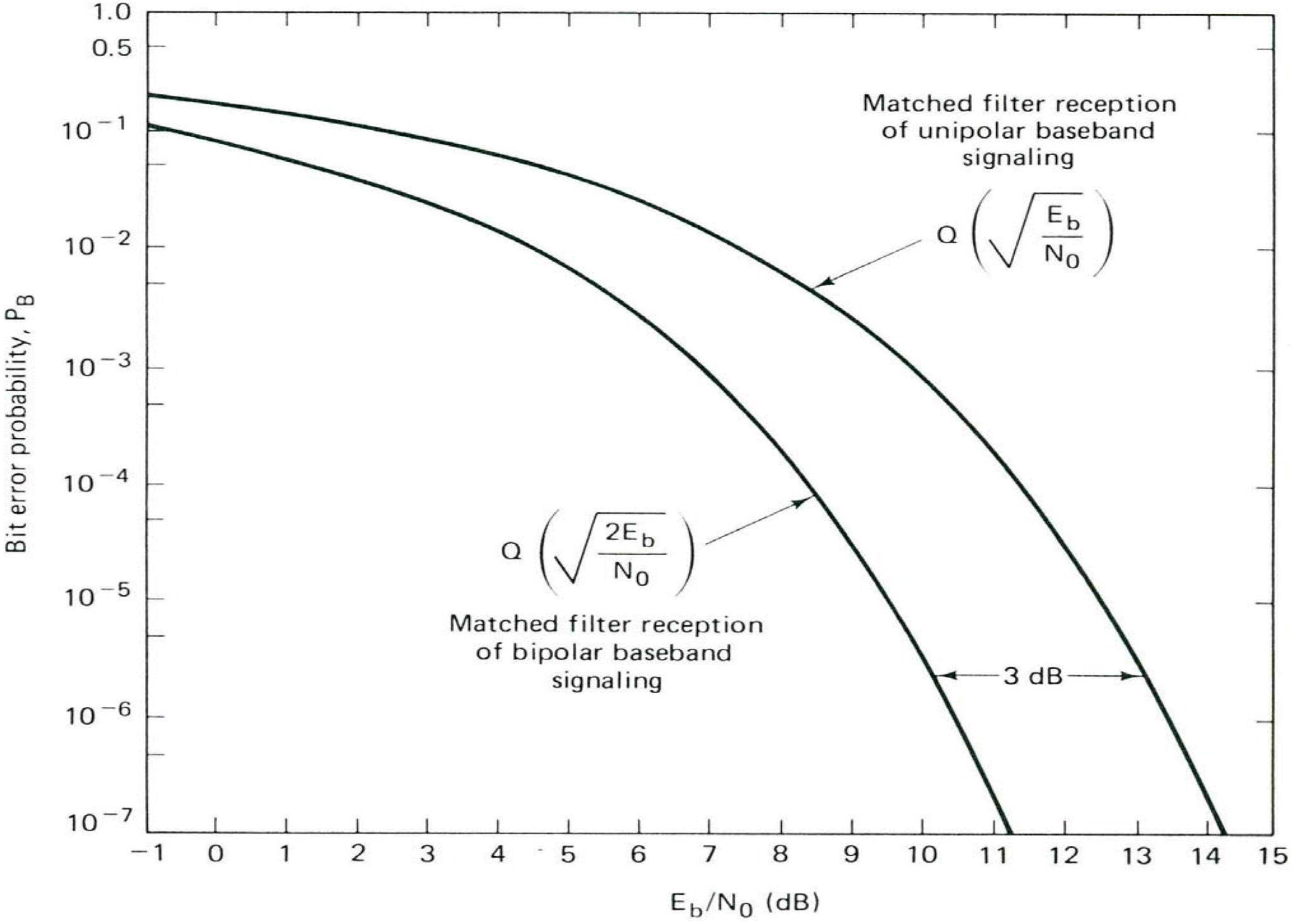


Figure 2.30 Bit error performance of unipolar and bipolar signaling.

- For given average energy / bit, bipolar better than unipolar.
- For same rect symbol height A, $E_b = A^2$ for bipolar
  & $A^2/2$ for unipolar (i.e less).
- For same E_b, reduce height for bipolar to A / 1.414.
- For same BER reduce height for bipolar to A/2.
  (reduces power by 3dB in comparison to unipolar).
- If $P_B=10^{-4}$, i.e. BER of 1 in 1000,
  ratio E_b/N_0 must be 8.2dB for bipolar & 11.2dB for unipolar.
- If N_0 is same, unipolar must have E_b 3dB higher than bipolar to achieve same BER.
- To have the same error probability

$$Q\left(\sqrt{2\,E_b(bipolar)/N_0}\right) = Q\left(\sqrt{E_b(unipolar)/N_0}\right)$$

which means that $E_b(\text{unipolar}) = 2E_b(\text{bipolar})$.

- This factor of two is an energy difference of 3dB.

- These results will stand us in good stead when we look at modulated data transmission over a radio channel.

- Unipolar signalling is “base-band orthogonal signalling”.
- Bipolar signalling is “base-band antipodal signalling”.

- Results obtained for matched filter (or correlation) detection of **bipolar** & **unipolar** signalling will be same for

- **coherently detected single carrier modulated antipodal signalling (e.g. binary PSK)**

- **orthogonal signalling (e.g. binary ASK),**

respectively.

Exercise 5.13: (Exam question) Binary digital communication system transmits equally likely symbols p(t) & q(t) at 1/T Baud where p(t) = At/T for $0 \le t \le T$ & q(t) = 0.

AWGN with 2-sided PSD $N_0/2 = 10^{-15}$ Watts/Hz is introduced.

Symbols detected by a matched filter detector.

a) Show that the optimal threshold $\gamma = A^2T/6$

b) Give impulse-response of matched filter & show that bit-error probability is

$$Q\left(\sqrt{A^2T/(6N_0)}\right)$$

c) If A = 0.2mV find the maximum bit-rate possible such that P_B does not exceed 10^{-3}.

d) Is this orthogonal signalling?

Exercise 5.14:

How would your answer to Exercise 5.13 be affected if a correlation detector rather than a matched filter were to be used.

Give a block diagram of the detector.

Exercise 5.15: A binary digital communication system transmits equally likely symbols p(t) & q(t) at 1/T Baud where:

$$p(t) = \begin{cases} +\mathrm{A} & : 0 \le \mathrm{t} \le \mathrm{T}/2 \\ -\mathrm{A} & : \mathrm{T}/2 < \mathrm{t} \le \mathrm{T} \end{cases} \text{ and } q(t) = \begin{cases} -\mathrm{A} & : 0 \le \mathrm{t} \le \mathrm{T}/2 \\ +\mathrm{A} & : \mathrm{T}/2 < \mathrm{t} \le \mathrm{T} \end{cases}$$

a) What is name for this form of coding & what are its advantages and disadvantages?

The signal is corrupted by AWGN with 2-sided PSD o $N_0/2$ WHz^{-1} is added. Symbols detected by a matched filter detector.

b) Give impulse-response of the matched filter required and determine the optimal threshold.

c) Derive expression for $P_{B,}$ in terms of E_b & N_0.

d) Compare this expression with what was obtained for bipolar signalling using rectangular symbols of height A & duration T.