# University of Manchester CS3282: Digital Communications Jan-Jun 2006 Section 2: Notes on the Fourier Transform

In Comms, analogue FT often expressed in terms of $f$ rather than ω (radians/second).

For a while, refer to the FT of x(t) as X(($f$)).

<u>FT</u>: -

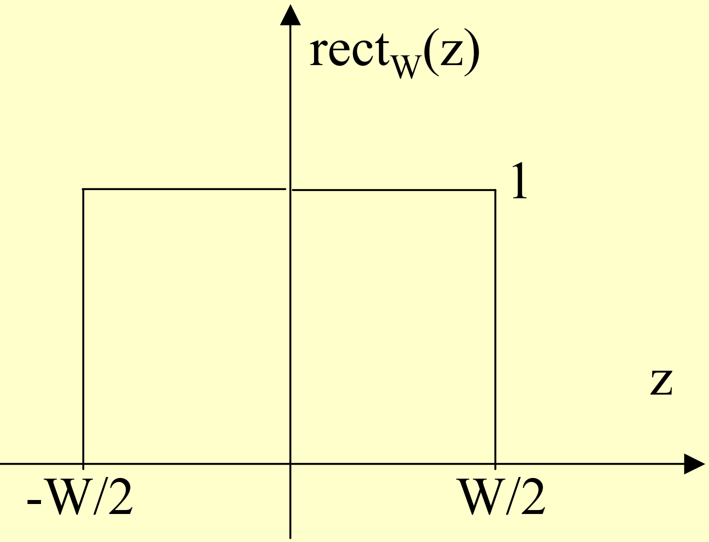
$$X(j\omega) = X((f)) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j f t} dt$$

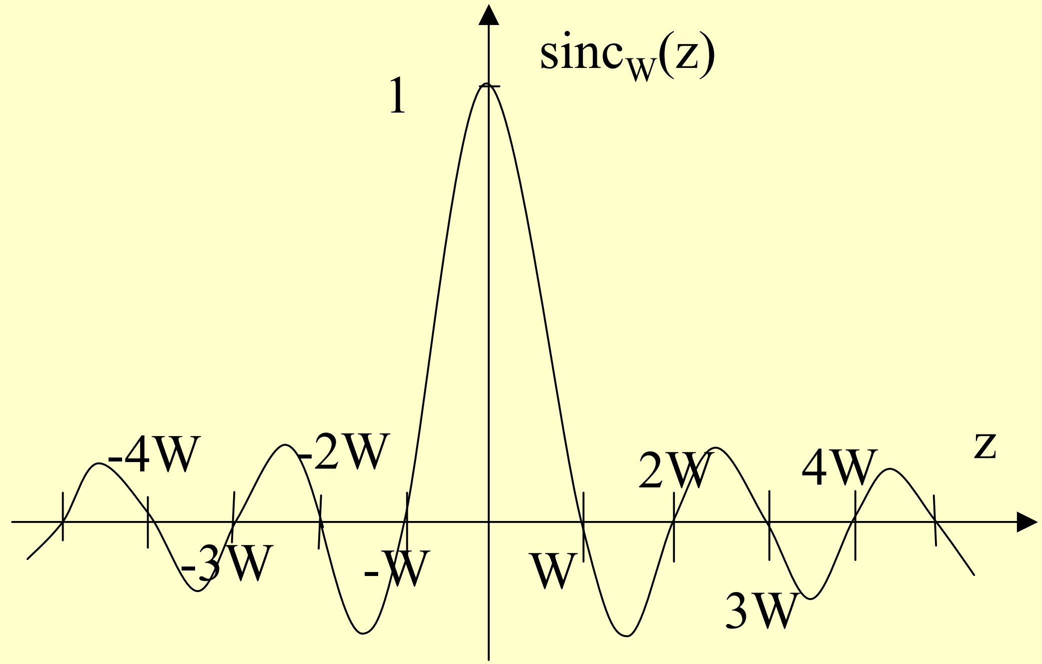
<u>Inverse FT</u>: -

$$x(t) = \int_{-\infty}^{\infty} X((f)) e^{2\pi j f t} df$$

If x(t) real, X((-$f$)) = $\overline{X((f))}$

## **2.2.** Consider $rect_W(z)$ & $sinc_W(z)$:




Refer to $rect_1(z)$ as $rect(z)$ & $sinc_1(z)$ as $sinc(z)$:

$$rect(z) = \begin{cases} 1 & : \ |z| < 0.5 \\ 0.5 & : \ |z| = 0.5 \\ 0 & : \ |z| > 0.5 \end{cases}$$

$$\text{sinc}(z) = \begin{cases} \sin(\pi z) / (\pi z) & : \quad z \neq 0 \\ 1 & : \quad z = 0 \end{cases}$$

Clearly, $\text{rect}_W(z) = \text{rect}(z/W)$ & $\text{sinc}_W(z) = \text{sinc}(z/W)$.

Under the Fourier transform with W=T:

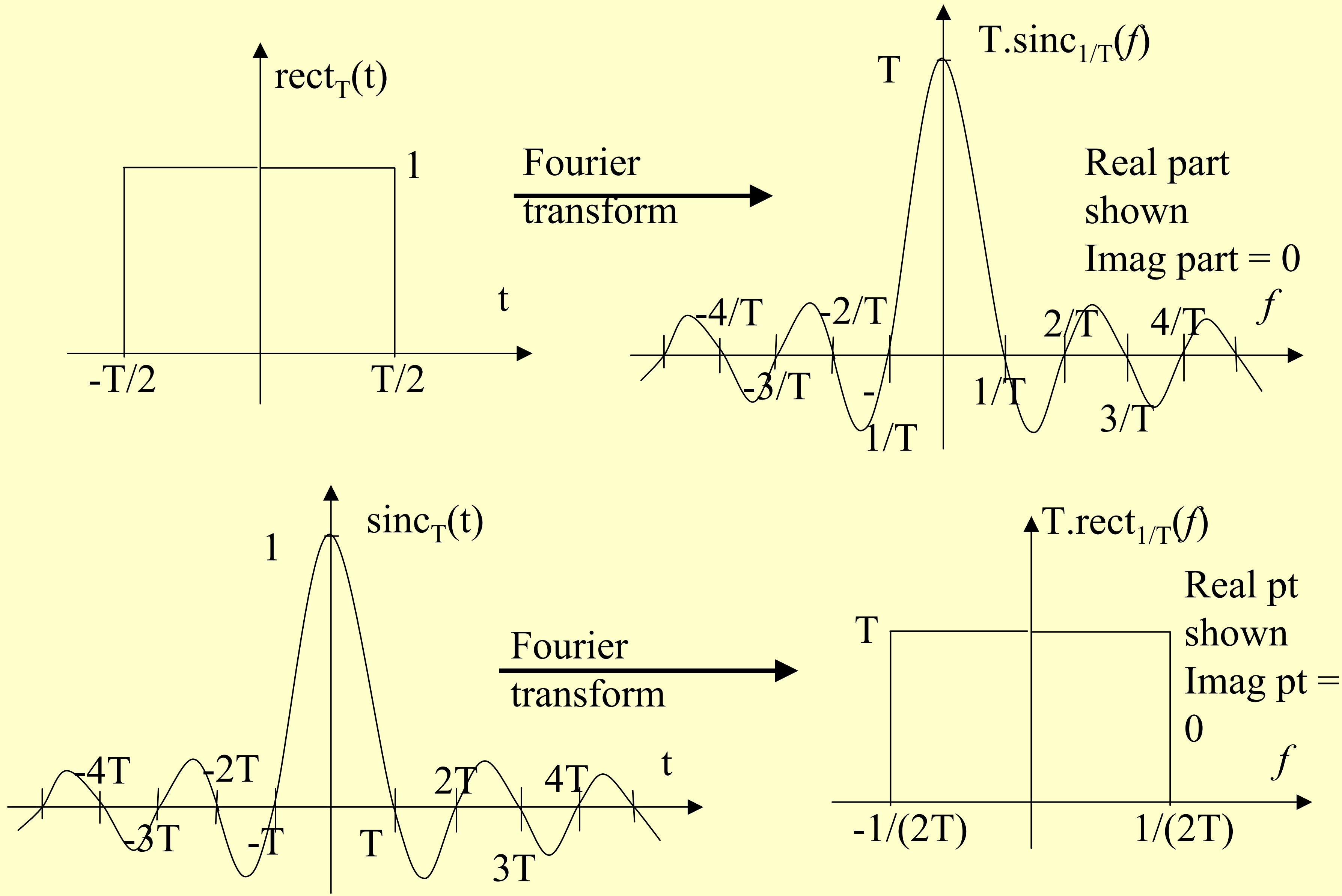
$$\text{rect}_T(t - D)\, e^{2\pi j F t} \longrightarrow T.\text{sinc}_{1/T}(f - F)\, e^{-2\pi j f D}$$

$$\text{sinc}_T(t - D)\, e^{2\pi j F t} \longrightarrow T.\text{rect}_{1/T}(f - F)\, e^{-2\pi j f D}$$

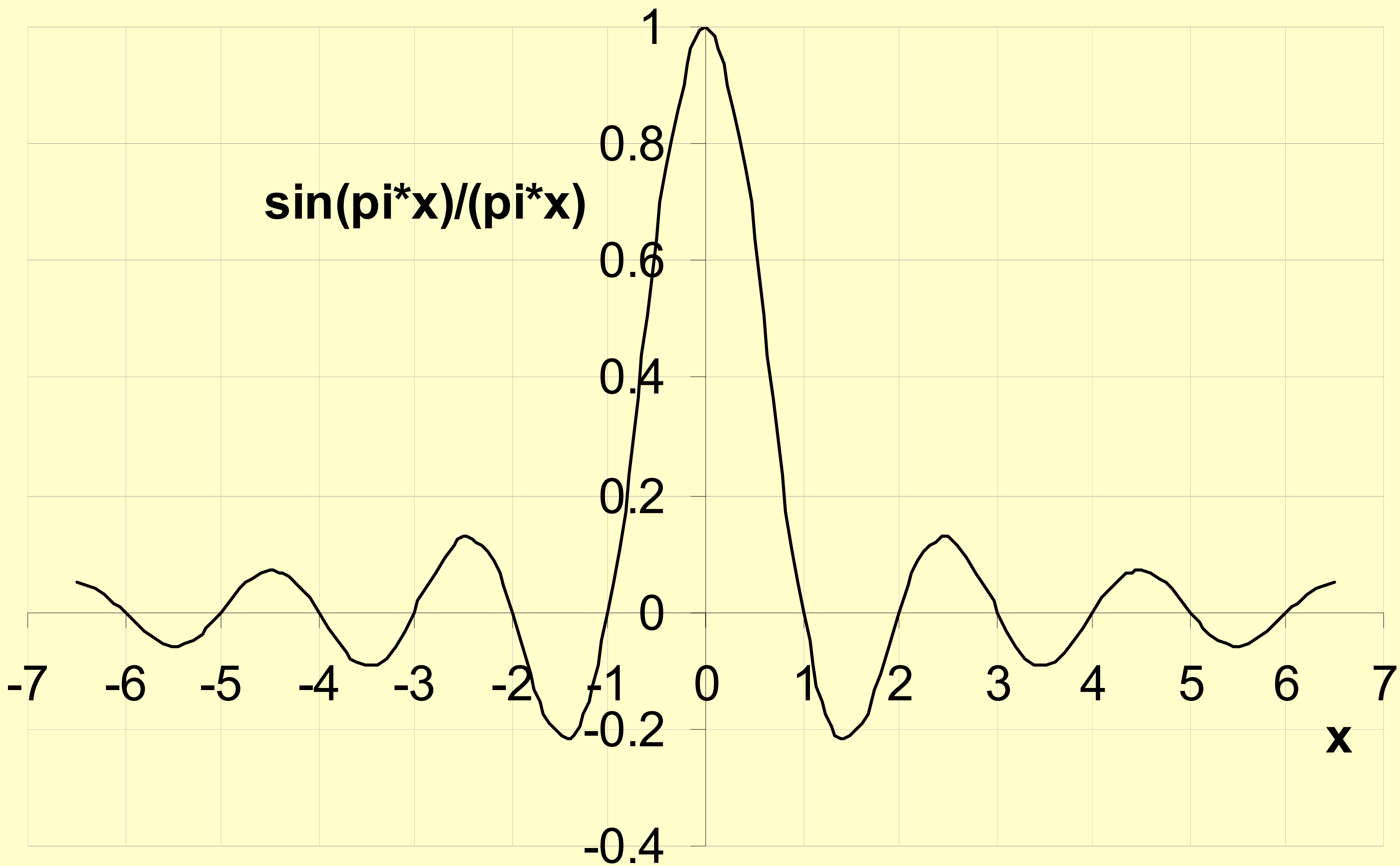
If D=0, F=0, the expressions become:

$$\text{rect}_T(t) \longrightarrow T.\text{sinc}_{1/T}(f)$$

$$\text{sinc}_T(t) \longrightarrow T.\text{rect}_{1/T}(f)$$

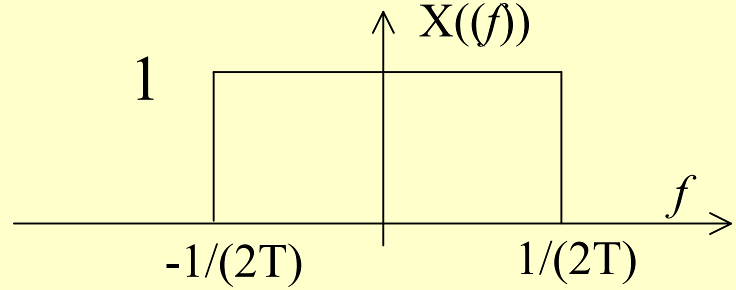


$rect_T(t)$
1
t
-T/2
T/2
Fourier transform
$T.sinc_{1/T}(f)$
T
Real part shown
Imag part = 0
$f$
-4/T
-2/T
2/T
4/T
-3/T
-1/T
1/T
3/T
$sinc_T(t)$
1
-4T
-2T
2T
4T
t
-3T
-T
T
3T
Fourier transform
$T.rect_{1/T}(f)$
Real pt shown
Imag pt = 0
T
$f$
-1/(2T)
1/(2T)

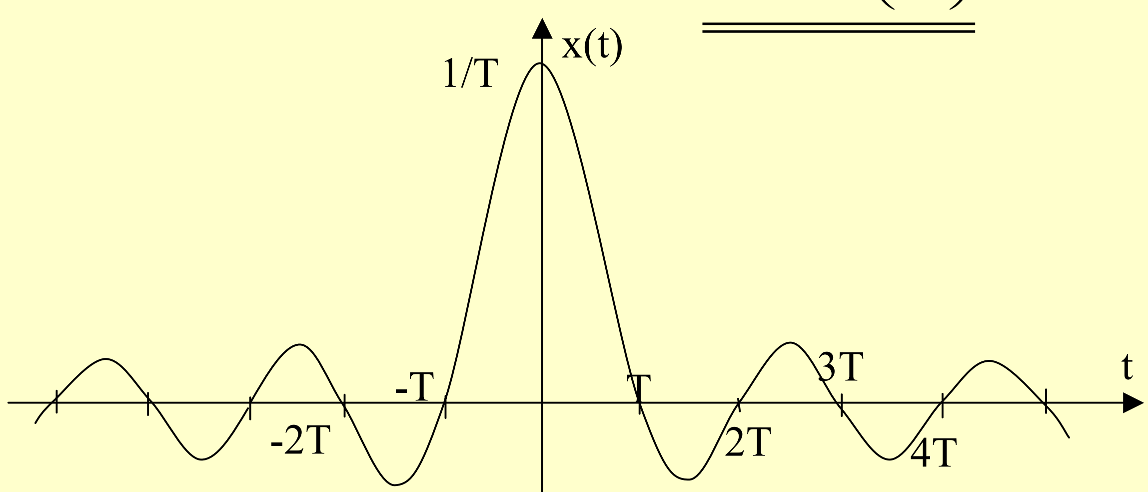


1
0.8
sin(pi*x)/(pi*x)
0.6
0.4
0.2
0
-0.2
-0.4
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
x

Example 2.3: Calculate x(t) when X((*f*)) has real part shown below & zero imaginary part for all *f* :




Solution: $$x(t) = \frac{1}{T}\sin c_T(t) \;= \frac{1}{T}\sin c\left(\frac{t}{T}\right)$$



In Examples 2.1 & 2.2, signal is of finite time duration & infinite bandwidth.

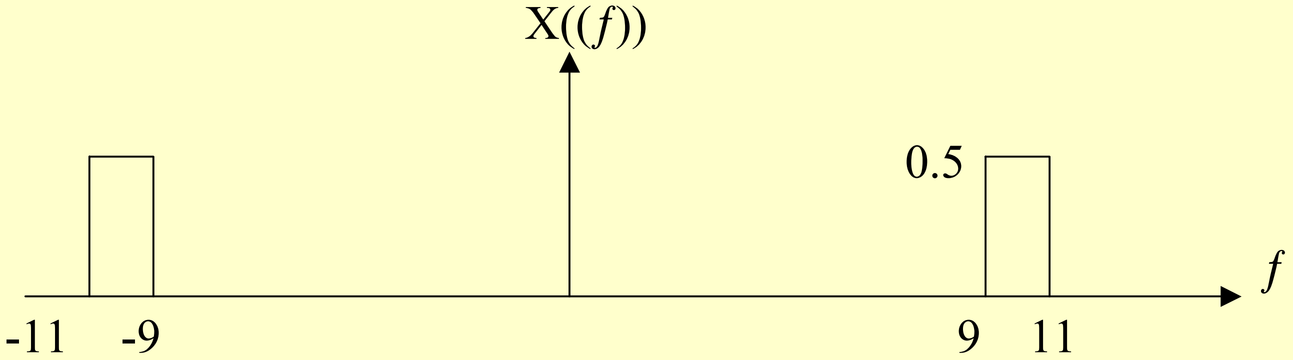
In Example 2.3, signal strictly band-limited & has infinite time duration.

It is worth remembering that under the inverse Fourier transform

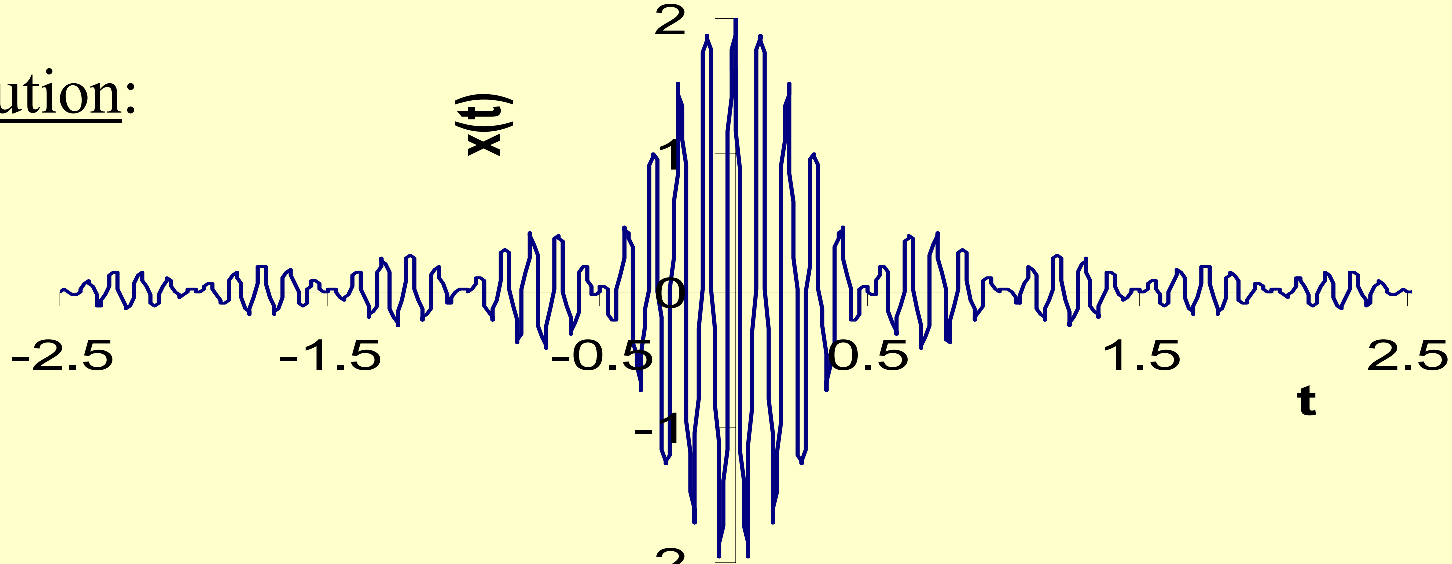
$$(1/T).\mathrm{rect}_T(\ t - D\ )\ e^{+2\pi jFt} \longleftarrow \mathrm{sinc}_{1/T}\ (\ f - F\ )\ e^{-2\pi jfD}$$

$$(1/T).\mathrm{sinc}_T(\ t - D\ )\ e^{+2\pi jFt} \longleftarrow \mathrm{rect}_{1/T}\ (\ f - F)\ e^{-2\pi jfD}$$

Example 2.4: Find inverse FT of:-




Solution:



## Solution to example 2.4:

$$X((f)) = 0.5\text{rect}_2(f + 10) + 0.5\text{rect}_2(f - 10)$$

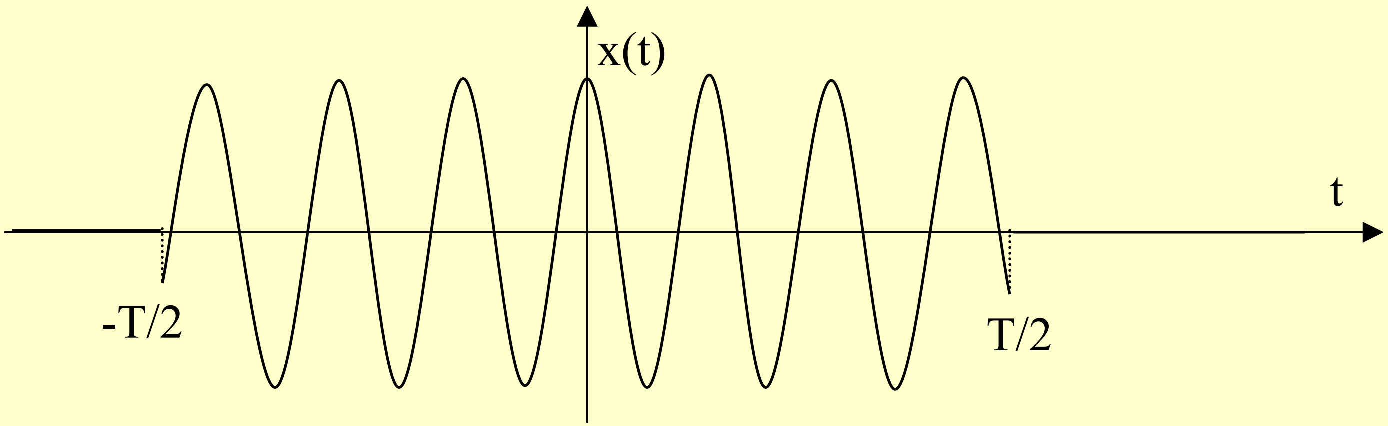
$$\therefore x(t) = \text{sinc}_{0.5}(t)e^{2\pi j10t} + \text{sinc}_{0.5}(t)e^{-2\pi j10t}$$

$$= \text{sinc}(t/0.5)\ [\ e^{2\pi j10t} + e^{-2\pi j10t}]$$

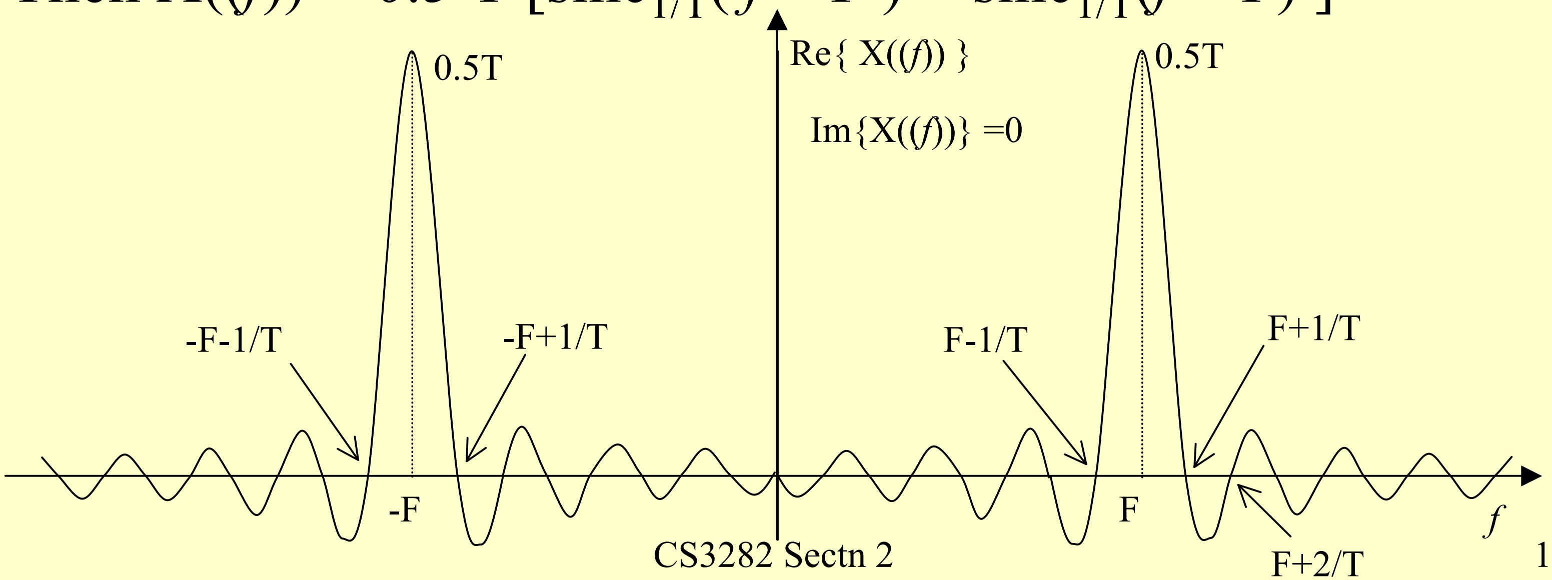
$$= \text{sinc}(2t)\ [\ 2 \cos (20\pi t)\ ] \quad = 2\ \text{sinc}(2t) \cos (20\pi t)$$

## Example 2.5: FT of rect windowed sinusoid:

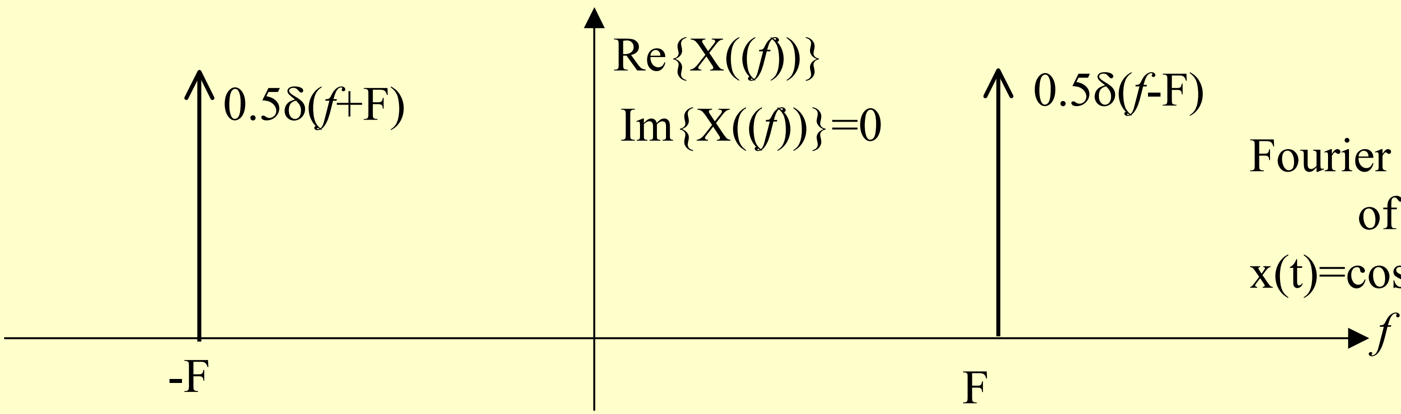
$$x(t) = rect_T(t)\cos(2\pi Ft) \quad = 0.5\, rect_T(t)\,\{e^{2\pi jFt} + e^{-2\pi jFt}\}$$




Then $X((f)) = 0.5\,T\,[\mathrm{sinc}_{1/T}(f - F) + \mathrm{sinc}_{1/T}(f + F)]$



# Example 2.6: FT of un-windowed sinusoid




Fourier transform of $x(t)=\cos(2\pi Ft)$

## 2.3. Multiplication and Convolution

Given $x_1(t) \rightarrow X_1((f))$ & $x_2(t) \rightarrow X_2((f))$ by FT.

FT of time-domain **product** of $x_1(t)$ & $x_2(t)$ is:

$$P((f)) = X_1((f)) \otimes X_2((f))$$

$$= \int_{-\infty}^{\infty} X_1((p)) X_2((f-p)) dp \quad = \quad \int_{-\infty}^{\infty} X_2((p)) X_1((f-p)) dp$$

$(Frequency\ domain\ 'complex'\ convolution)$

FT of time-domain **convolution** between $x_1(t)$ & $x_2(t)$ is:

$$C(f) = X_1((f)).X_2((f))$$

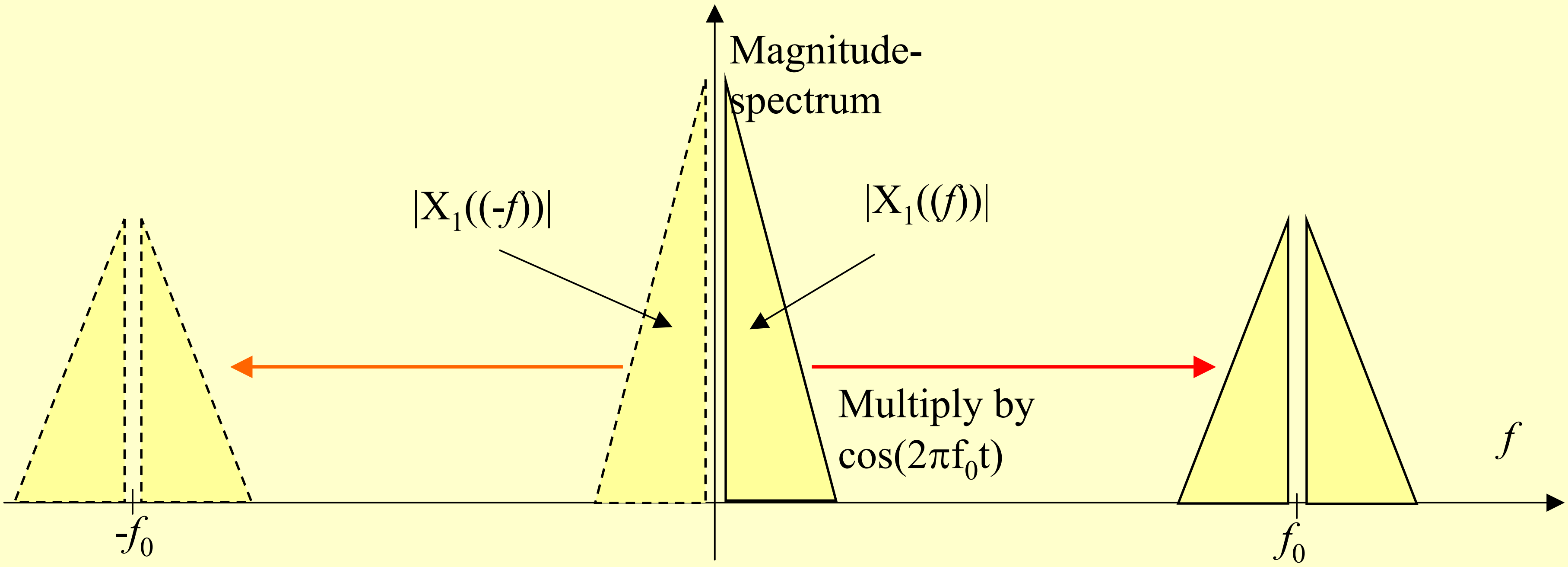
$(Frequency\ domain\ product)$

$$\left( x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1((\tau)) x_2((t-\tau)) d\tau = \int_{-\infty}^{\infty} x_2((\tau)) x_1((t-\tau)) d\tau \right)$$

| Time-domain | Frequency-domain |
| --- | --- |
| Product (modulation or windowing) | Complex convolution |
| Convolution (filtering) | Product of Fourier transforms |

Example 2.7: Multiplication of $x_1(t)$ by $\cos(2\pi f_o\, t)$.

Let $x_2(t) = \cos(2\pi f_o\, t)$ . Then the spectrum of $x_1(t)\, x_2(t)$:



# Proof by FD convolutn: If $x_2(t) = \cos(2\pi f_o\, t)$

Then the spectrum of $x_1(t)\, x_2(t)$:

$$= \int_{-\infty}^{\infty} X_2((p)) X_1((f-p))dp$$

$$= \int_{-\infty}^{\infty} \left(0.5\delta(p-f_0) + 0.5\delta(p+f_0)\right) X_1((f-p))dp$$

$$= \int_{-\infty}^{\infty} 0.5\delta(p-f_0) X_1((f-p))dp + \int_{-\infty}^{\infty} 0.5\delta(p+f_0) X_1((f+p))dp$$

$$= \underline{\underline{0.5X_1((f-f_0)) + 0.5X_1((f+f_0))}}$$

## 2.4.Complex Fourier Series

Periodic signal expressed as Fourier series in 3 ways.

Read this!

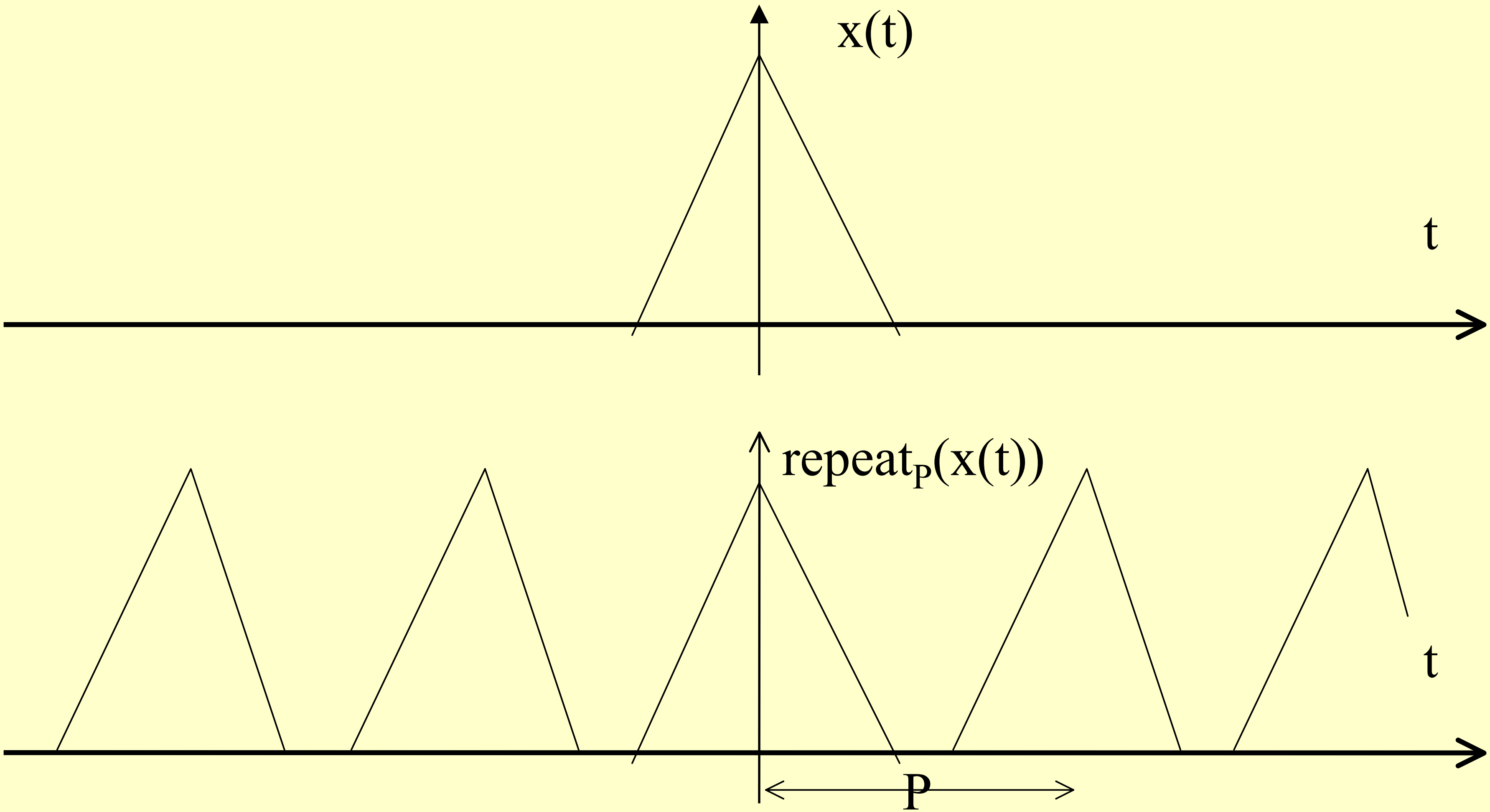
## 2.4.4. 'Repeat' and 'Sample':

Given the signal x(t) with Fourier transform X((*f*)) define:

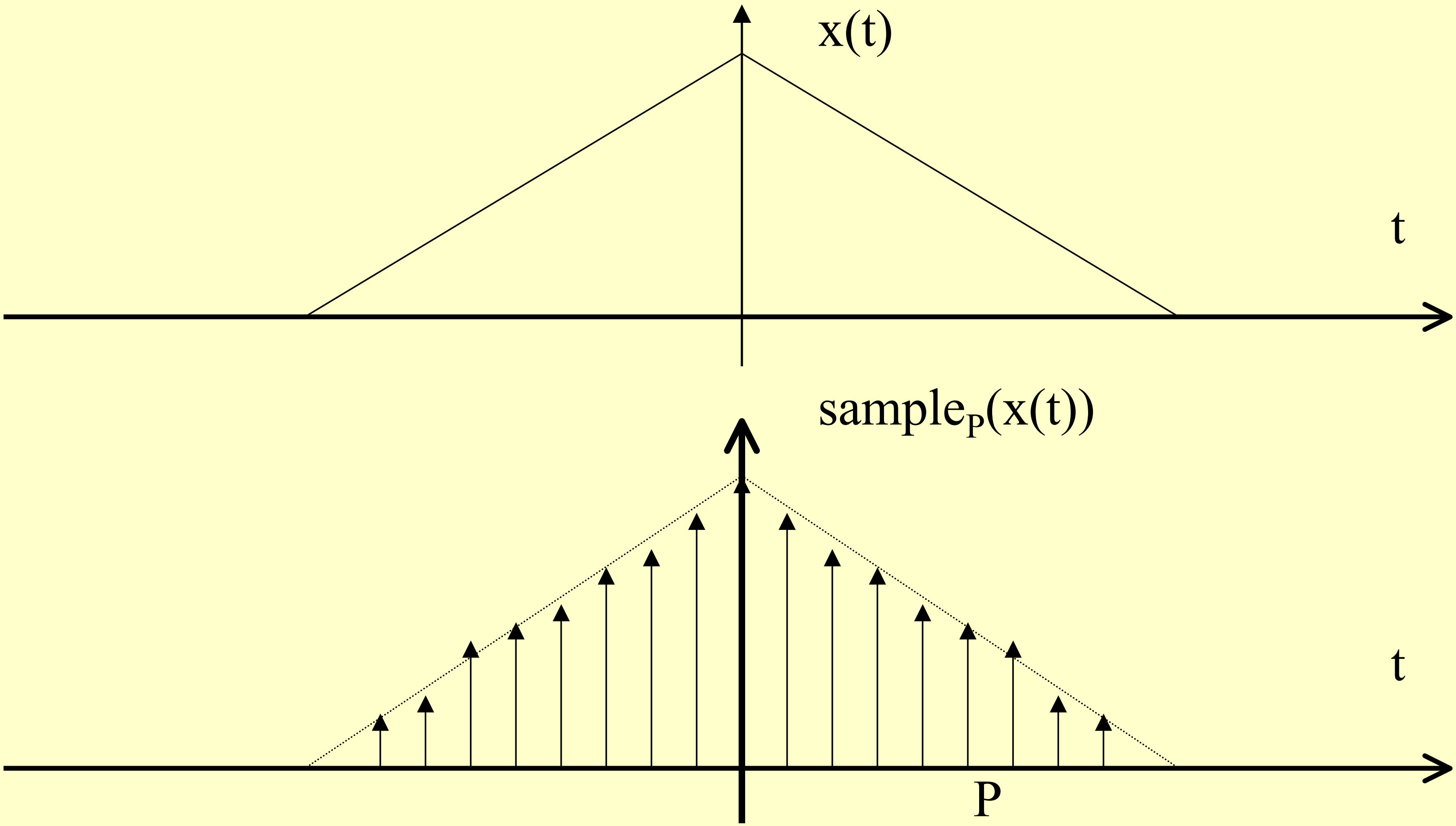
$$\text{repeat}_P(\, x(t)\,) = \sum_{n=-\infty}^{\infty} x(t - nP)$$

$$\text{sample}_P(\, x(t)\,) = x(t)s_P(t) \qquad \text{where} \quad s_P(t) = \sum_{n=-\infty}^{\infty} \delta\,(t - nP)$$

‘ $\mathrm{repeat}_P(x(t)$ ’ repeats $x(t)$ at intervals of P seconds.



'$\text{sample}_P(x(t))$' gives series of impulses at intervals of P, each weighted by a sample of x(t). [mistake in notes not periodic]



These functions may also be applied in freq domain, i.e.

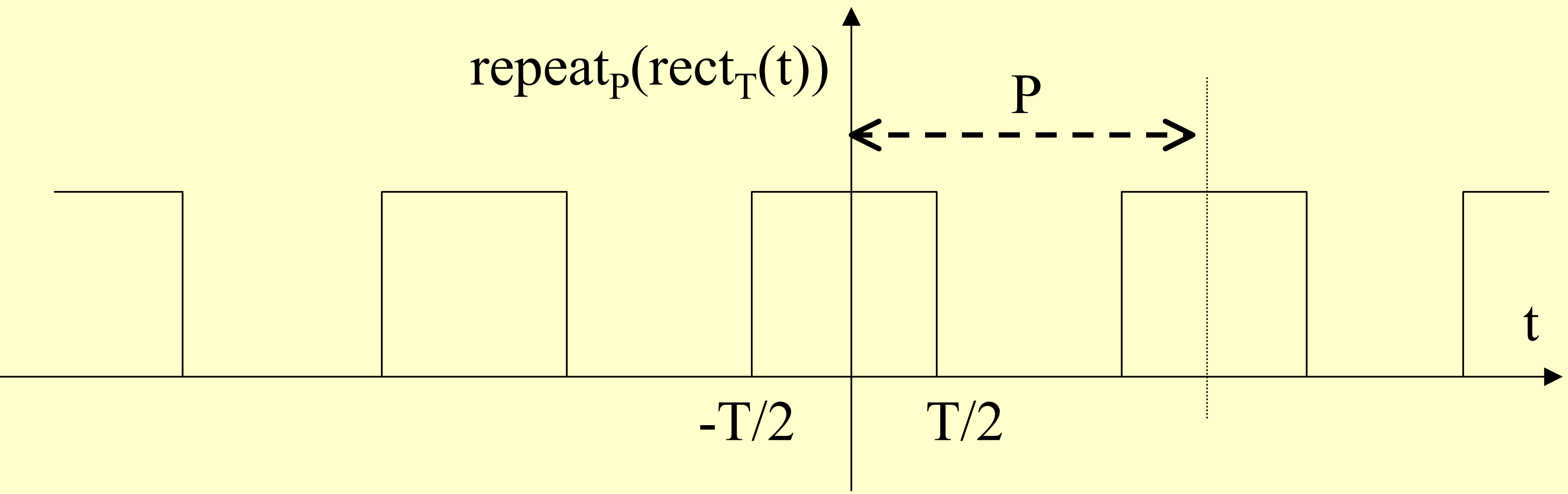
$$\text{repeat}_F(\,X(f)\,) = \sum_{n=-\infty}^{\infty} X(f - nF) \quad \&$$

$$\text{sample}_F(\,X(f)\,) = X(f)S_F(f) \;\text{ where }\; S_F(f) = \sum_{n=-\infty}^{\infty} \delta(f - nF)$$

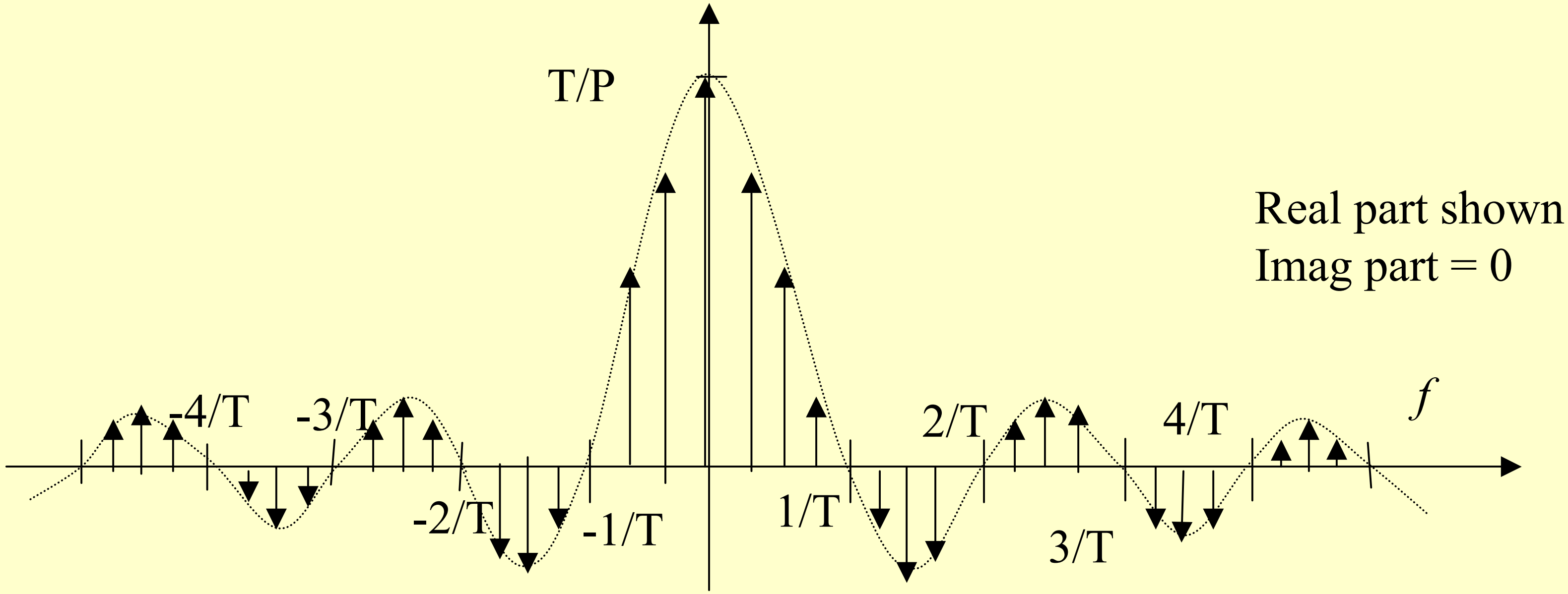
It may be shown that under the FT:

$$\text{repeat}_P(x(t)) \longrightarrow (1/P)\text{sample}_{1/P}(\,X(f)\,)$$
$$\text{sample}_P(x(t)) \longrightarrow (1/P)\text{repeat}_{1/P}(\,X(f)\,)$$

- These FT relationships can be used to derive Fourier series.
- Remember that a periodic signal does not have a ‘normal’ FT because its energy is infinite.
- So it needs delta functions (like sine waves).
- Consider FT for a periodic rectangular wave
- Note it is symmetrical abt t=0.  Otherwise its FT would be complex




Its FT is $(1/P)\ \text{sample}_{1/P}(\ T\text{sinc}_{1/T}(f)\ )$



- Impulses at frequencies 0, ±1/P, ±2/P, …
- Let $f_0$ =1/P denote fundamental frequency
- Impulse at n^th harmonic $nf_0$ ( =n/P) is $(T/P)\mathrm{sinc}_{1/T}(n/P)$
- To get Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi j (n f_0) t} \qquad f_0 = \frac{1}{P} Hz$$

take impulse strengths as values of $C_n$.

(Replacing the impulses δ(t-nP) by unit length lines)

$$C_n = (T/P)\mathrm{sinc}_{1/T}(n/P) = (T/P)\mathrm{sinc}(nT/P)$$

- So if we have any pulse shape, $x(t)$ say, with finite energy it has a nice FT $X((f))$ with no need for impulses.
- If we repeat pulse shape at intervals P we get a periodic signal.
- Its FT now needs impulses, and it has a Fourier series.
- Its impulses are at $0, \quad \pm 1/P, \quad \pm 2/P, \quad \ldots \quad \pm n/P, \ldots.$

with strengths $X((0)), \; X((\pm 1/P)), \; X((\pm 2/P)), \ldots X((n/P)), \ldots$

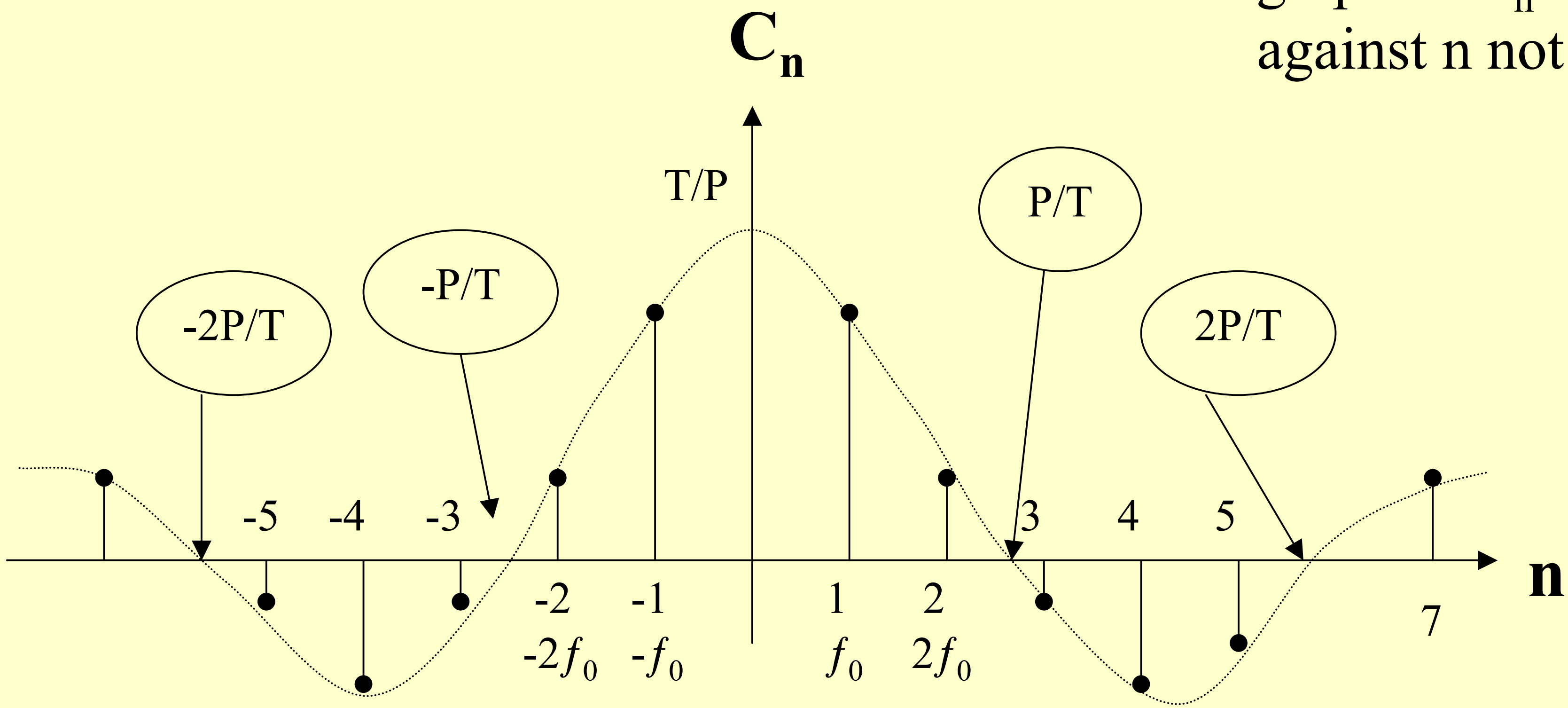
- Its Fourier series coeffs are:

$$X((0)), X((\pm 1/P)), X((\pm 2/P)), \ldots X((n/P)), \ldots$$

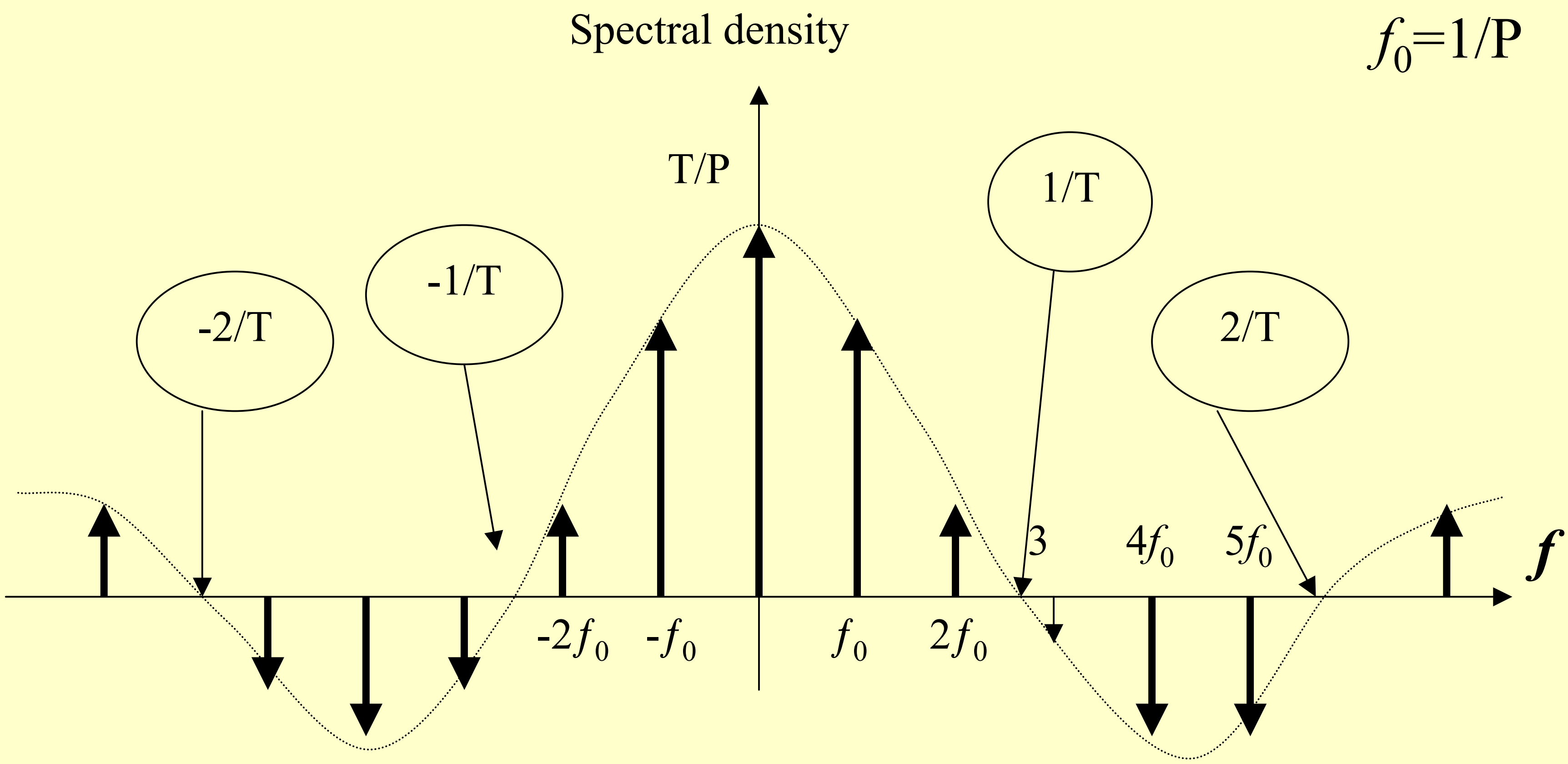
i.e. make the impulses into voltages or 'lines'.

# "Line spectrum" for a Fourier series

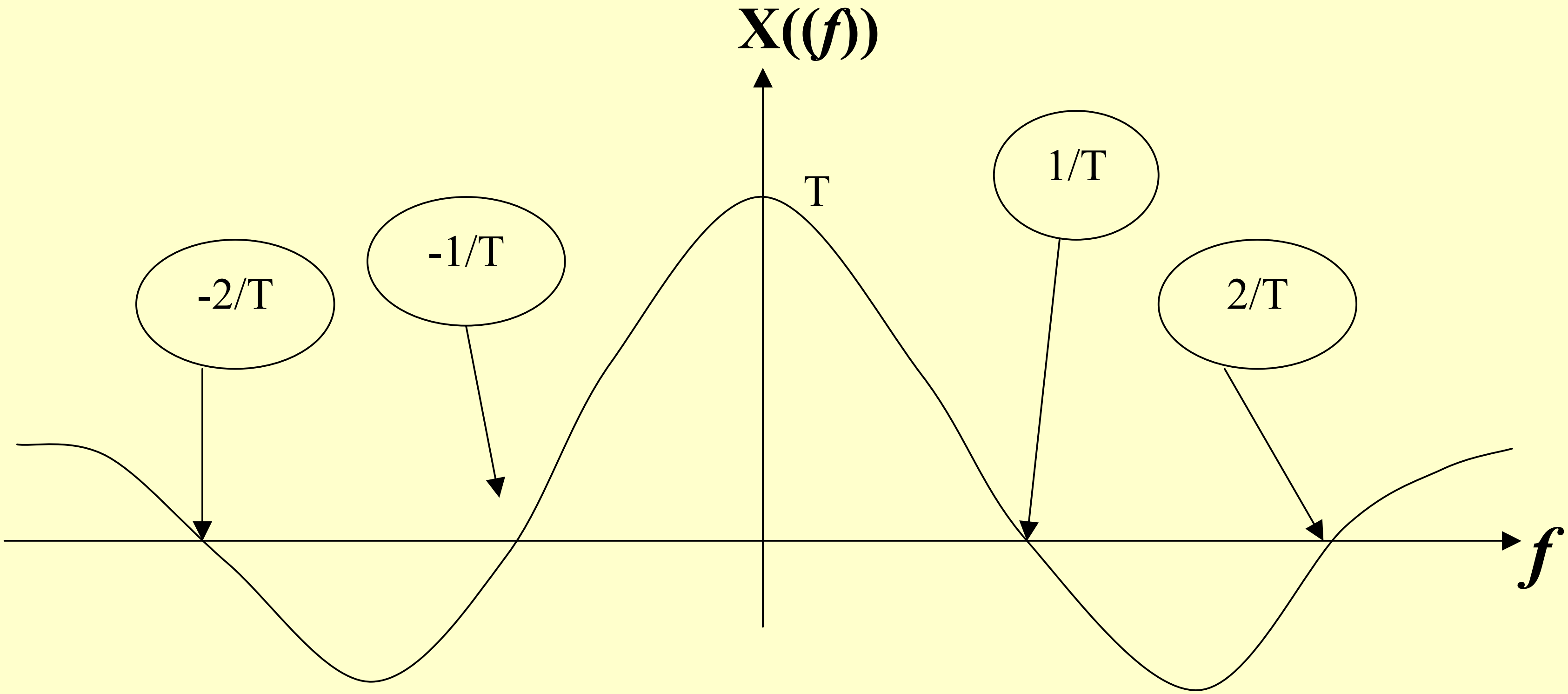
Note this is graph of $C_n$ against n not $f$



# Fourier transform spectrum for a Fourier series



# FT for a single rectangular pulse: $rect_T(t)$



**2.5. Energy and power:** For an analogue signal x(t):

$$\text{Energy} = \int_{-\infty}^{\infty} [\mathrm{x(t)}]^2 \, dt$$

Energy, in Joules, in Joules "relative to 1 Ohm".

Power of x(t) is a bit more tricky:

$$x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ \quad 0 \;\; : \;\; otherwise \end{cases} \qquad \text{Power} = \lim_{\mathrm{D}\to\infty} \frac{1}{\mathrm{D}} \int_{-\infty}^{\infty} [\mathrm{x_D(t)}]^2 \mathrm{dt}$$

Average power in "Watts relative to 1 Ω".
If x(t) periodic with period D, "lim" can be omitted.

## 2.6. ‘Energy-signals’ and ‘power-signals’

It is useful to refer to two types of signal:

*1. ‘Energy-signal’* or ‘*finite energy signal’*: has finite energy and zero average power. An example is a single pulse.

*2. ‘Power-signal’* or ‘*finite power signal’*: has finite average power and therefore infinite energy; e.g. a sine wave or a pulse sequence.

## 2.7. Parseval's Theorem (in two forms)

For a 'finite energy' signal, x(t), it may be shown that its energy is:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X((f))|^2 \, df$$

This means that the energy of a finite energy signal can be calculated either in time-domain, or in frequency-domain from a knowledge of X(($f$)).

For a 'finite power' signal, x(t), define:

$$X_D((f)) = Fourier\ Transform\ \ of\ \ x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ 0 \quad : otherwise \end{cases}$$

which has finite energy for any constant D and therefore has a straightforward FT.
A 2nd version of Parseval's theorem exists proving that under reasonable assumptions:

$$\int_{-\infty}^{\infty} \left( \lim_{D\to\infty} \frac{1}{D} |X_D((f))|^2 \right) df = \lim_{D\to\infty} \frac{1}{D} \int_{-\infty}^{\infty} |X_D((f))|^2 \, df = \lim_{D\to\infty} \frac{1}{D} \int_{-\infty}^{\infty} |x_D(t)|^2 \, dt$$

Referring to the definition earlier, this is the power of x(t).

So now we can calculate the power of x(t) from a knowledge of the frequency-domain function:

$$P(f) = \lim_{D\to\infty} \frac{1}{D} \left| X_D((f)) \right|^2$$

The power of x(t) is the integral of P(*f*) over $-\infty < f < \infty$.

## 2.8. Energy and power spectral density

$|X((f))|^2$ is two-sided "energy spectral density" (ESD).
Units are Joules per Hz.
$2|X((f))|^2$ is the 'one-sided' ESD again in Joules per Hz.
Refer to $10\log_{10}(|X((f))|^2)$ as the "2-sided ESD" in dB.

Under reasonable assumptions, we can define:

$$P((f)) = \lim_{D \to \infty} \frac{1}{D} \left| X_D((f)) \right|^2$$

as "2-sided power spectral density (PSD) of power signal x(t).
Units are Watts per Hz.
The '1-sided' PSD is $2P(f)$ Watt/Hz. Define PSD($f$) =P($f$).
"Density" means "energy per Hz" and "power per Hz".

## Effect of filtering on ESD & PSD

• If 'finite energy' signal x(t) with '2-sided' ESD E($f$) is passed thro' a filter with frequency-response H(($f$)), output is finite energy signal with ESD E($f$)|H(($f$))|$^2$.

• If 'finite power' signal x(t) with 2-sided PSD P($f$) is passed thro' filter with frequency-response H(($f$)), output is finite energy signal with PSD P($f$)|H(($f$))|$^2$.

•These results are needed for Section 5.

•Must write a simple convincing proof of these properties.

## 2.9. Auto-correlation function

•For an energy signal x(t):

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

•It may be shown that $R_x(\tau) = R_x(-\tau)$.

•For a power signal, x(t),

$$R_x(\tau) = \lim_{D \to \infty} \frac{1}{D} \int_{-D/2}^{D/2} x(t)x(t+\tau)dt$$

•In most cases, $R_x(\tau) = R_x(-\tau)$ for power signals, though it is possible to conceive of cases where this is not true.

- $R_x(\tau)$ is function of delay $\tau$ measured in seconds.
- For any $\tau$, gives measure of similarity between x(t) & $x(t+\tau)$ which is x(t) advanced in time by $\tau$ seconds.
- Cross-correlation between energy signals x(t) & y(t) is:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t+\tau) dt$$

- Auto-correlation function of x(t) is cross-correlation between x(t) and itself.
- N.B. $R_{xy}(\tau) \neq R_{yx}(\tau)$. In fact, $R_{xy}(\tau) = R_{yx}(-\tau)$.

- Note the similarity between $R_{xy}(\tau)$ & time-domain convolution:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(\xi) y(\xi + \tau) d\xi \qquad x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\xi) y(t - \xi) d\xi$$

- Therefore $R_{xy}(-\tau) = \int_{-\infty}^{\infty} x(\xi) y(\xi - \tau) d\xi$ & $x(t) \otimes y(-t) = \int_{-\infty}^{\infty} x(\xi) y(\xi - t) d\xi$

- Convolution between x(t) & y(-t) = cross-correlation between y(t) & x(t).
- Convolution between x(t) & y(t) = cross-correlation between y(t) & x(-t). (Careful with signs of $\tau$).
- Time-reversing one of the signals turns convolution into cross-correlation and vice-versa.

## 2.10. Weiner-Khinchine Theorem

- For a "finite energy" signal,
  '2-sided' ESD E(($f$)) is Fourier transform of $R_x(t)$.

- For a "finite power" signal,
  '2-sided' PSD P(($f$)) is Fourier transform of $R_x(t)$

## 2.11. Random Signals

- Given x(t), without exact periodicity or obvious structure it is sometimes useful to assume that at any time t, its value is a sample of a random variable X.
- Statistical properties of X defined by PDF.
- In this case, x(t) is considered to be a "random" signal.

## PDF, $p_x(x)$, of a random variable X

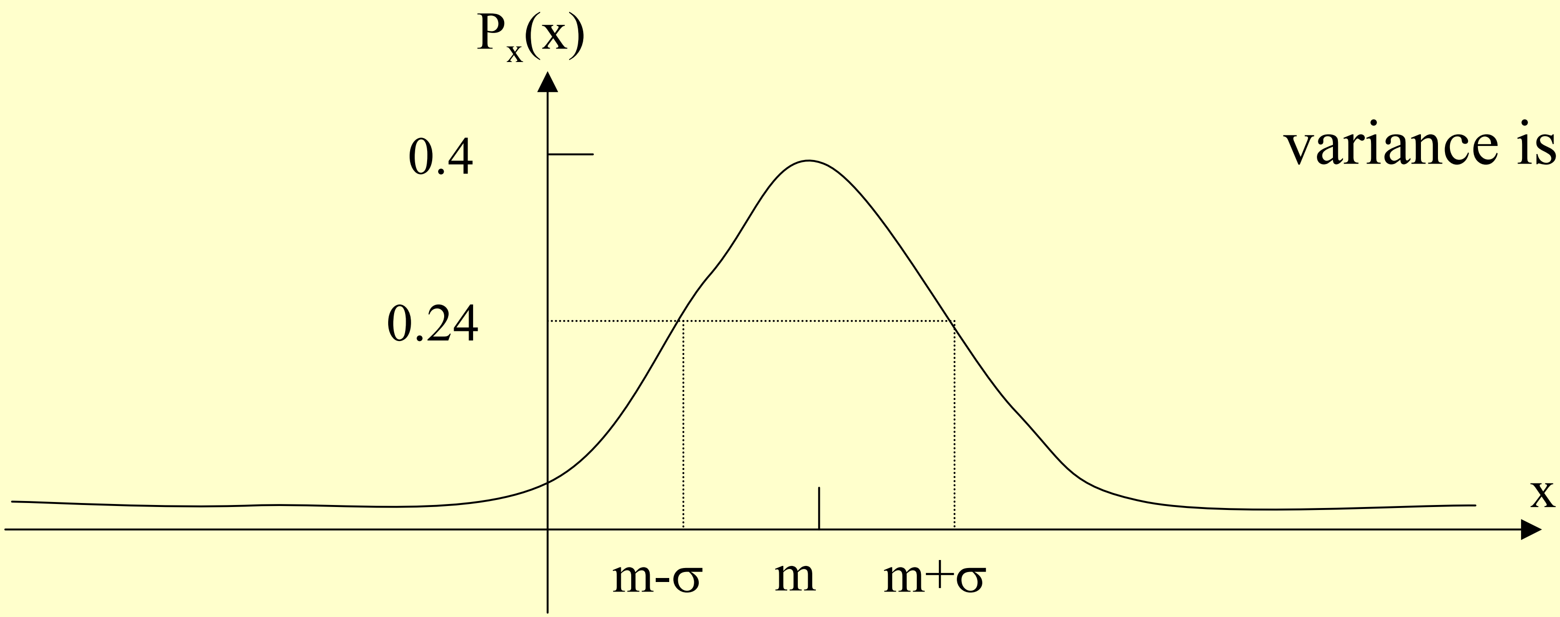
- Given a & b, with a < b,
  probability of an observation of X lying between a and b is :

$$\Pr ob(a < X < b) = \int_a^b p_X(x)dx$$

- Applies to any measured voltage of random signal with this PDF.

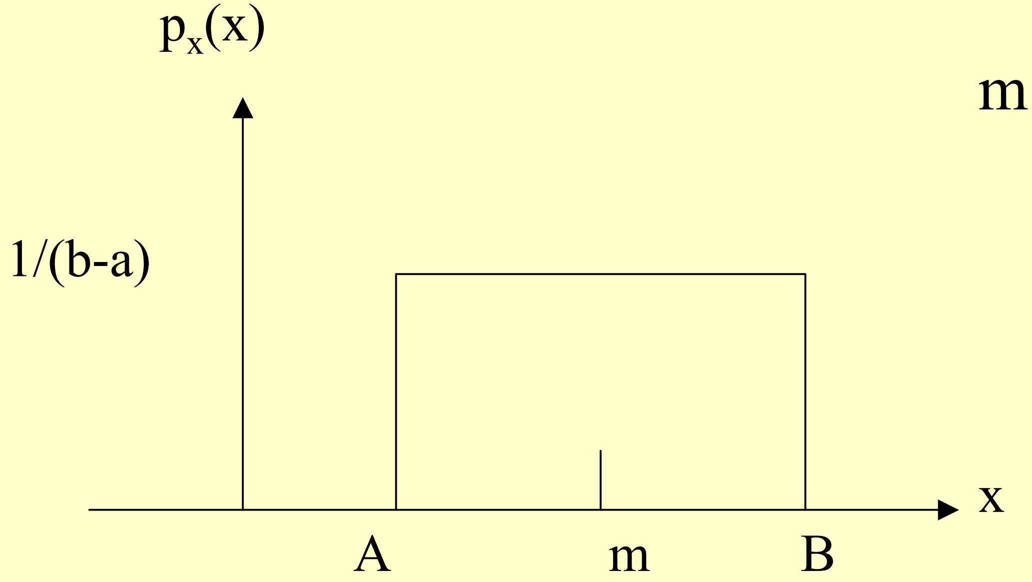
**Two commonly encountered PDFs are**:

1. Gaussian (normal) with mean m & standard deviation $\sigma$ :




variance is $\sigma^2$

# 2. Uniform between x = A & x = B:



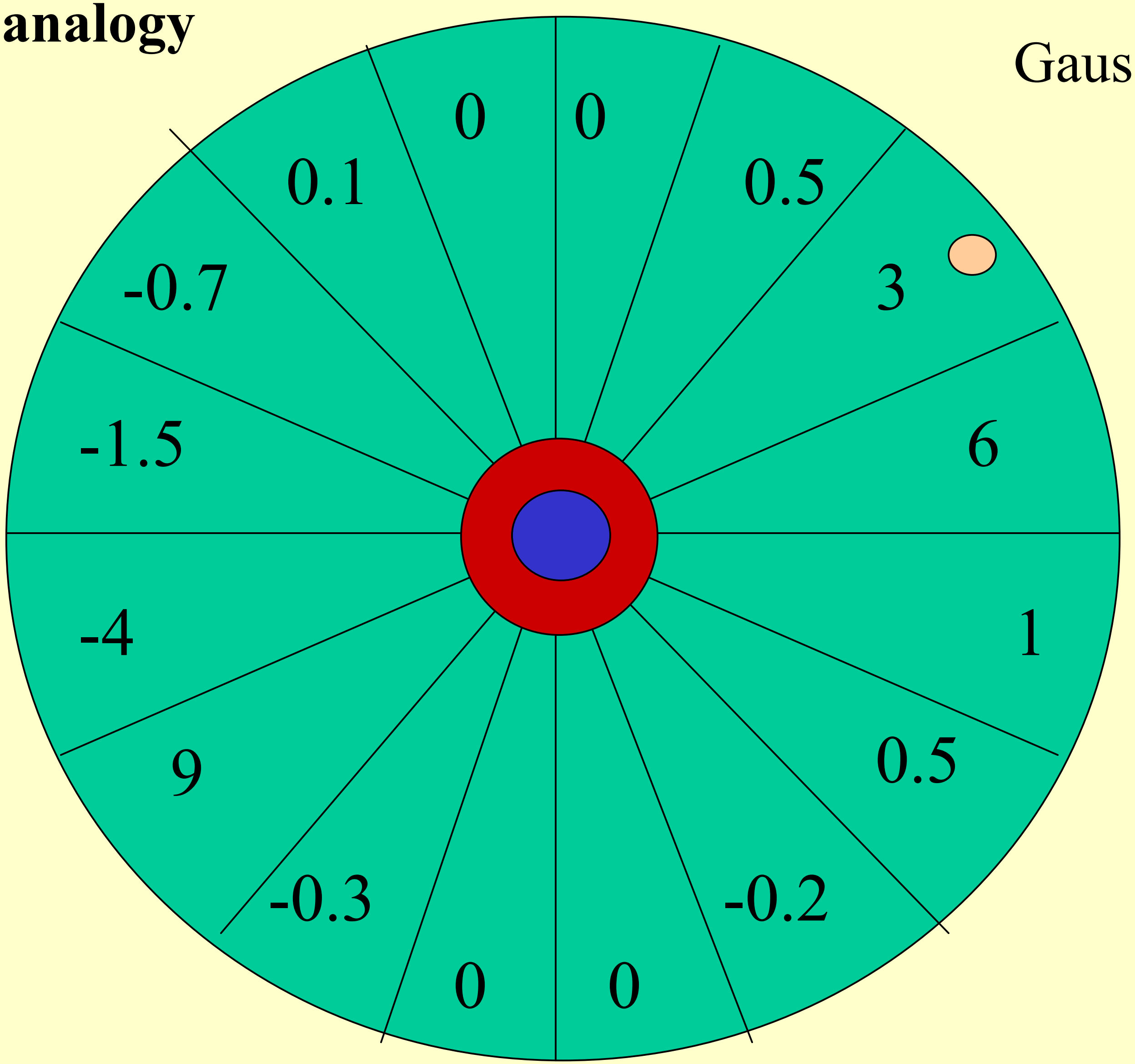

m = (B+A)/2

From $p_X(x)$ we can deduce the most commonly encountered statistical measurements of the random variable X such as its mean, mean-square value and variance.

Exercise 2.12: Give formulae for these measurements.

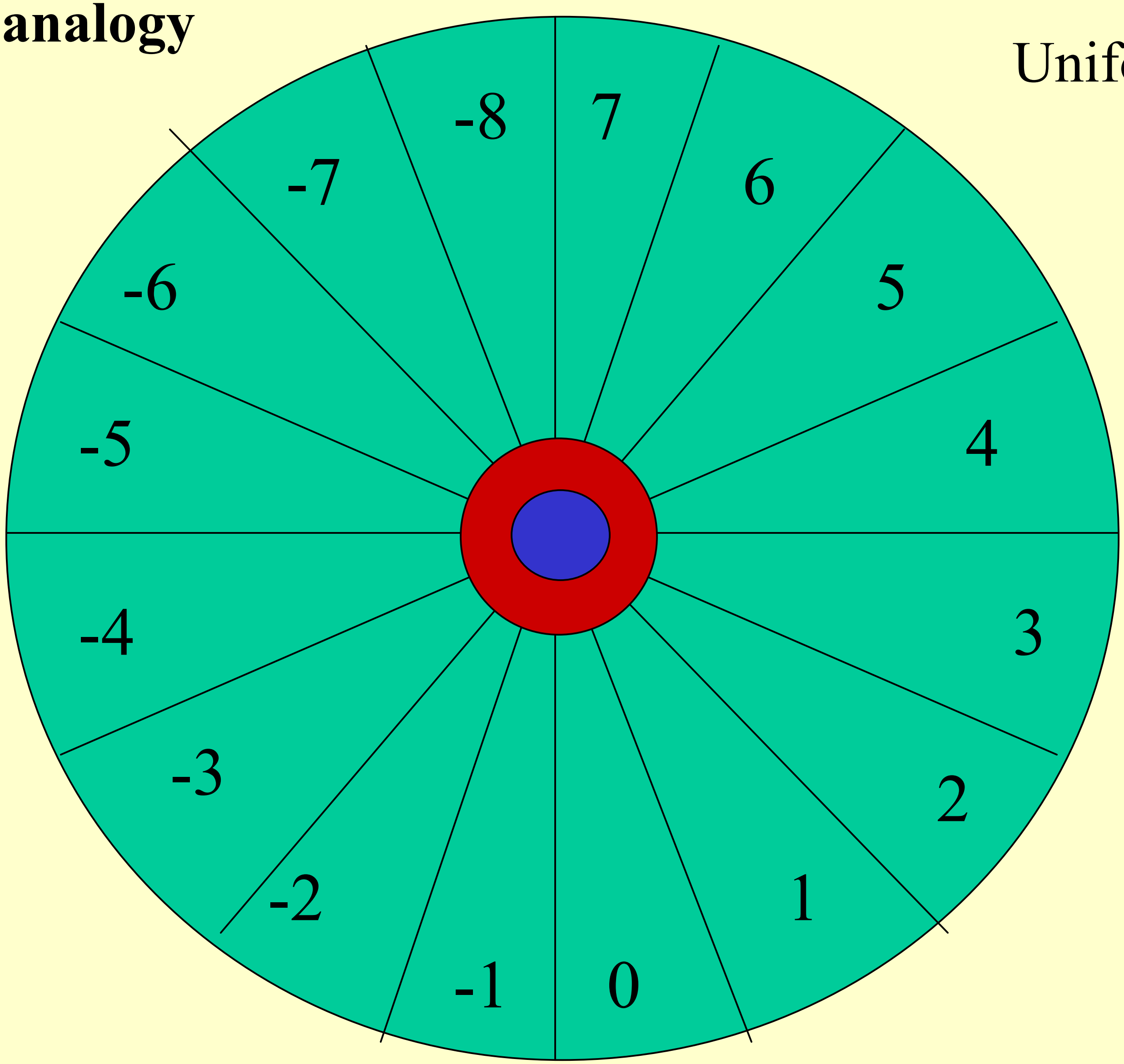
# ‘Casino’ analogy

Gaussian PDF



## 'Casino' analogy

Uniform PDF



Example 2.13:

- How are "dc offset", power & RMS value of x(t) related to mean, mean-square value, standard deviation & variance of X, assuming that x(t) is ergodic?
- How are these relationships affected if the dc offset is zero?

## Spectral properties of a random signal

Although x(t) is random, can often make prediction from previous history. For example, if the values are:

-2 , 10, 21, 33, 41, 55, 62, 69, …

we may predict that the next sample is around 80.

Can attempt prediction because of correlation between samples.

Auto-correlation function $R_x(\tau)$ non-zero for values of $\tau>0$.

This will affect the power spectrum i.e. the Fourier transform of $R_x(\tau)$ making it non-flat.

Considering a second example with no predictability

-2, 44, -4, -17, 9, 61, 2, -19, 3, -16, 1, -7, 30, -1, …

No correlation between signal & signal delayed for any $\tau$.

$R_x(\tau) = 0$ for $\tau \neq 0$.

Power spectrum, i.e. FT of $R_x(\tau)$ will be flat or "white".

Example 2.14: Power signal x(t) of bandwidth -B to B Hz and power P Watts has $R_x(\tau)=0$ for all $\tau \neq 0$. Sketch its 2-sided PSD.

Example 2.15: Considering the sequences of samples of x(t), which one is more likely to be Gaussian and why?

Time properties of x(t) (governing correlation & shape of power spectrum) & statistical properties are independent.

Can have white or spectrally coloured signal with same PDF.

## 2.12. Signal Correlation, Similarity and Matching

Correlation is a signal matching process.
Cross-correlation function between energy signals x(t) & y(t):

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$

Measures "matching" or "similarity" between shapes of these signals when y(t) is advanced by τ seconds.

Auto-correlation of x(t) is cross-correlation between x(t) & itself:

$$R_x(\tau) = R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

## 2.13. Conclusions

Understanding of this section will be useful in the study of digital communications where analogue pulse shapes representing bits or bit-sequences must be visualized in both the time-domain and the frequency-domain.

Ability to adapt easily remembered formulae to given wave-form shapes will often eliminate most of the mathematics and give insight into the nature of communication signals.

The formulae are listed in one of three appendices attached to these notes which will also be attached to the CS3282 examination paper.

## Problems on Section 2

2.1. What are units of: (a) PSD (b) ESD (c) SNR ?

2.2 Classify as energy or power signal, & find energy & power of:

(a) $x(t)=8\cos(6\pi t)$ (b) $x(t)=0$ for $t<0$ & $4e^{-3t}$ for $t \geq 0$

2.3 For finite energy x(t), show that its ESD is FT of its ACF. Hence show that:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X((f))|^2 \, df$$

Give relationship between PSD & ACF and a version of Parseval's Theorem for "finite power" signals.

2.4. Power signal has "2-sided" PSD given below. What is its "1-sided" PSD?. It is passed thro' ideal band-pass filter with cut-off freqs 1Hz & 3Hz & gain 0dB. Calculate power of output.

$$PSD((f)) = \begin{cases} 4-|f| & :|f| < 4\ Hz \\ 0 & :|f| \geq 4 \end{cases}$$

2.5. Gaussian noise n(t) of zero mean & power $W$ has “1-sided” PSD:

$$PSD(f) = \begin{cases} N_0 : |f| \le \text{B Hz} \\ 0 : |f| > \text{B Hz} \end{cases} \quad \text{Watts/Hz}$$

If samples of *n(t)* taken at intervals of $T$ , estimate:

(a) mean, (b) variance (c) pdf of these samples.

(d) Show that $W=N_0B$.

If *n(t)* passed thro’ ideal low-pass filter with $f_c=B/10$ Hz, give

(e) mean, (f) variance (g) pdf of output?

2.6. If X(($f$)) is FT of x(t) show that FT of x(-t) is X((-$f$)),

& show that if x(t) is real then X((-$f$))=X*(($f$)).

Show that FT of x(t-T) is X((f))$e^{-2\pi Tf}$

& hence that FT of x(T-t) is X*(($f$))$e^{2\pi Tf}$.

If $x(t)$ =0 for $t$<0 & $t$>$T$, & A(1-t) for 0 ≤$t$≤$T$, sketch $x(-t)$ & $x(T-t)$.

2.7. Answer TRUE or FALSE:-

(a) All zero mean Gaussian noise signals are spectrally white.

(b) Finite power signals have infinite energy & finite energy signals have infinite power.

(c) Any non-zero periodic signal must have finite power.

(d) ACF symmetric about $\tau=0$ i.e. $ACF(\tau)=ACF(-\tau)$.

(e) Inverse FT of ESD is the ACF of an energy signal.

(f) Signal of finite bandwidth must be of infinite duration.

(g) Cross-correlation between two real signals is same as convolution except that one of the signals is time-reversed.

(h) Cross-correlation between real x(t) & y(t) has FT $X((f))Y^*((f))$

(i) FT of $dx(t)/dt$ is $2\pi j f X((f))$

Some solutions:

2.1. (a) Watts/Hz (b) Joules/Hz (c) Dimensionless power ratio

2.2. (a) Power signal with energy 32 Watts & infinite energy

(b) Energy signal of energy 8/3 Joules & zero average power

2.3 $PSD((f)) = \begin{cases} 8 - 2|f| & :|f| < 4\ Hz \\ 0 & :|f| \geq 4 \end{cases}$

Power = $\int_1^3 (8-2f)df = \left[8f - f^2\right]_1^3 = 16-8 = 8\,watt$

2.5. (a) 0 (b) W (c) Gaussian (d) $N_0$ is Watts/Hz & bandwidth is B Hz hence total power is $N_0B$. (e) 0 (f) W/10 (g) Gaussian.

2.7 (a) False (b) False (c) True(I think) (d) True (e) True (f) True (g) True (h) True (i) True

**Extra question:**

(a) If a 'finite energy' signal x(t) with 2-sided ESD $E(f)$ is passed through a filter with frequency-response $H(f)$, show that the output is a finite energy signal with ESD = $E(f)|H(f)|^2$. joules/Hz. What is the total energy of the filter's output?

(b) If a 'finite power' signal x(t) with 2-sided PSD $P(f)$ is passed through a filter with frequency-response $H(f)$, show that the output is a finite power signal with PSD = $P(f)|H(f)|^2$ watts/Hz. What is the total power of the filter's output?

**Answer:**

(a) $|X(f)|^2$ was defined in the notes as the two-sided ESD of a signal x(t) with Fourier Transfm $X(f)$.

If x(t) passed thro' a filter, the output spectrum is: $X(f)H(f)$ where $H(f)$ is the freq response.

The '2-sided' ESD of the filter output is therefore $|X(f)H(f)|^2$

$= |X(f)|2|H(f)|2 = E(f)\ |H(f)|2$

The total energy of the filter output is:

$$\int_{-\infty}^{\infty} E(f)\,|\,H(f)\,|^2\, df \ldots joules$$

(b) Two-sided PSD of x(t) is:

$$PSD(f) = \lim_{D\to\infty} \frac{1}{D} \left| X_D(f) \right|^2 = \lim_{D\to\infty} \frac{1}{D} E_D(f)$$

where $E_D(f) = ESD \;\; of \;\; x_D(t)$

and $x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ 0 \;\; : \;\; otherwise \end{cases}$

Result of passing $x_D(t)$ thro' H(*f*) is signal whose 2-sided ESD is:

$E_D(f) \mid H(f) \mid^2$

As D → ∞, $x_D(t)$ →x(t) and (1/D){ESD of filter output} → PSD of filter output

∴ as D → ∞, $(1/D\{E_D(f) \mid H(f) \mid^2 \}$ → PSD of filter output

As D → ∞, (1/D{$E_D(f)$ | H($f$) |$^2$ } tends to

$$\lim_{D\to\infty} \frac{1}{D} E_D(f)\left|H(f)\right|^2 = PSD(f)\left|H(f)\right|^2$$

Therefore this is the PSD of the filter output.

The total power of the filter output is:

$$\int_{-\infty}^{\infty} PSD(f)\,|\,H(f)\,|^2\,df$$