University of Manchester

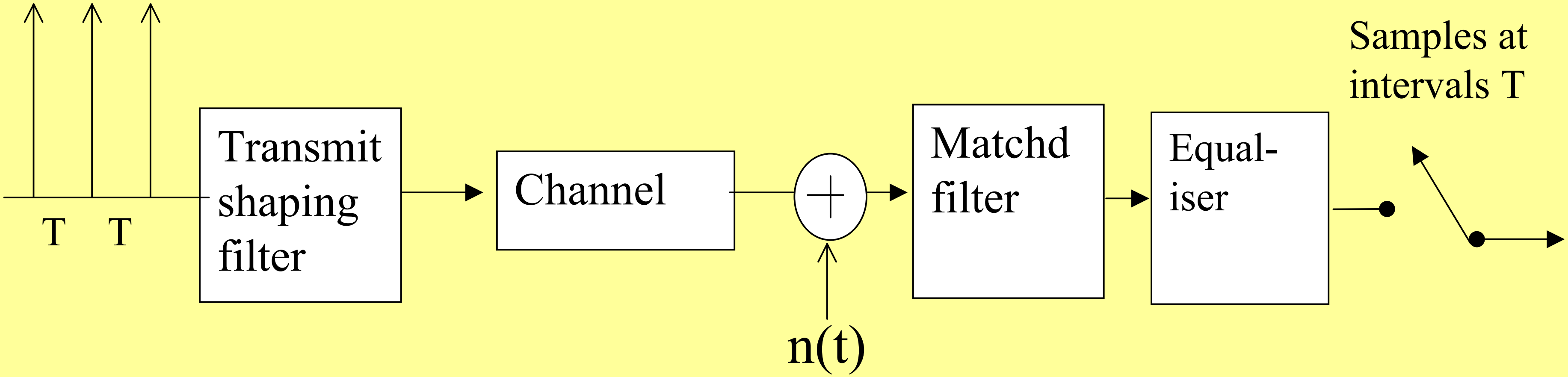
CS3282: Digital Communications 2006

Section 6

Inter-symbol interference and pulse shaping

- Rectangular symbols not suitable for transmitting data at highest possible bit-rates over band-limited channels.
- Time-limited pulse requires infinite bandwidth.
- Symbol with finite bandwidth must have infinite time duration.
- Although using symbols which exist from t = -∞ to +∞ may seem impossible, this is the ideal & we must produce approximations.
- Pulses used in practice may not go on & back in time for ever.
- Must be non-zero for more than T s when signaling rate is 1/T .
- One symbol will run into previous & next symbols.
- Result could be "inter-symbol interference" (ISI).
- Cannot avoid overlap of symbols in time-domain,
- Find ways of making sure that data carried by symbols is not affected by this overlap.
- Solution to this challenge lies in "pulse shaping"

- Generate an impulse of the appropriate strength for each symbol & pass it thro' a "transmit pulse shaping" filter.
- Impulse-response of filter is symbol shape we wish to launch.
- FIR digital filter followed by a DAC will do this job nicely.
- Channel will affect shape of symbol & noise will be added.
- At receiver, to optimize detection, filtering tasks required:.



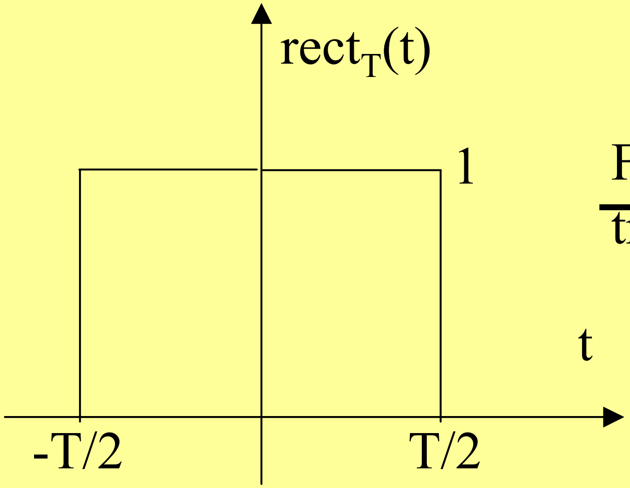
- ISI can occur due to ringing of one symbol into next.
- ISI avoided if transmitter shapes symbols so that zero-crossings at detector occur T, $2T$, ... after (& before) centre of symbol.
- If we sample at $t=0$, T, $2T$, etc, only see centre of one symbol.
- All other symbols are zero at those instants.
- Nice in theory & possible to a fair degree in practice.

- Let a pulse shape be produced by applying a unit impulse to a filter with the ‘brick-wall’ frequency-response:

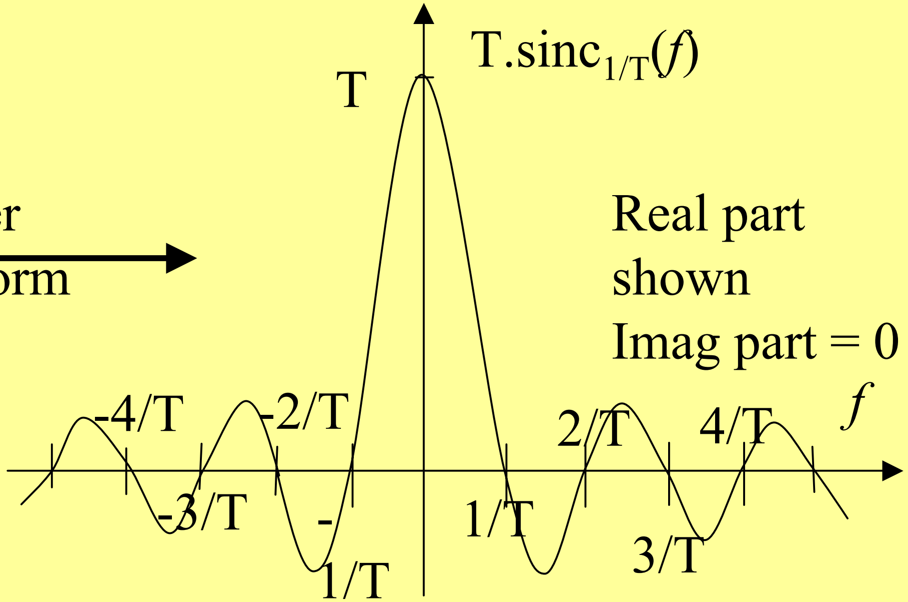
$$H(f) = T\, rect_{1/T}(f)$$

- Pulse-shape is impulse-response; i.e. $sinc_T(t)$.
- This is a ‘pure’ sinc function as illustrated on next slide.
- Has zero crossings at $t = \pm T, \ \pm 2T$, etc

Reminder (see Section 2)

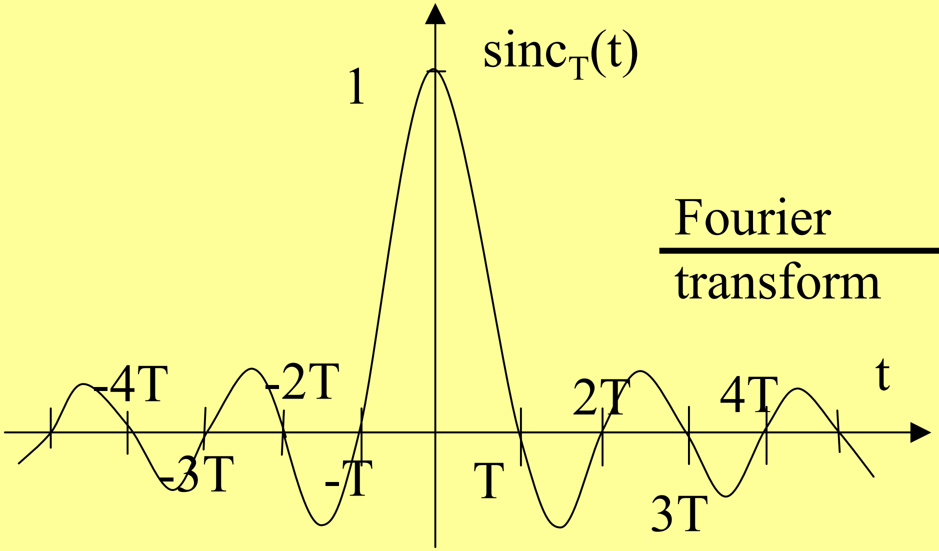


Fourier transform

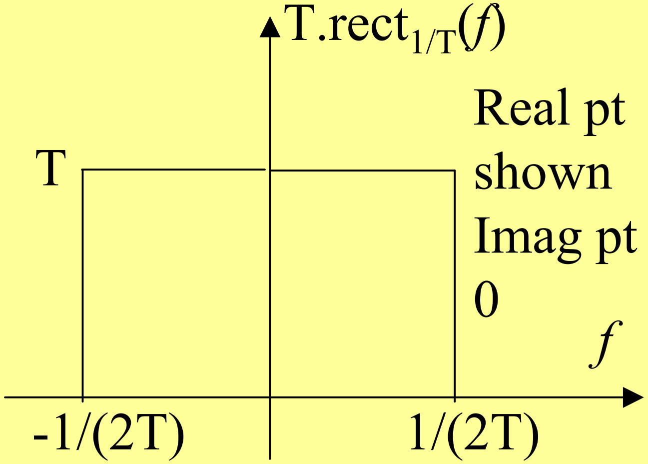


Real part shown

Imag part = 0



Fourier transform



Real pt shown

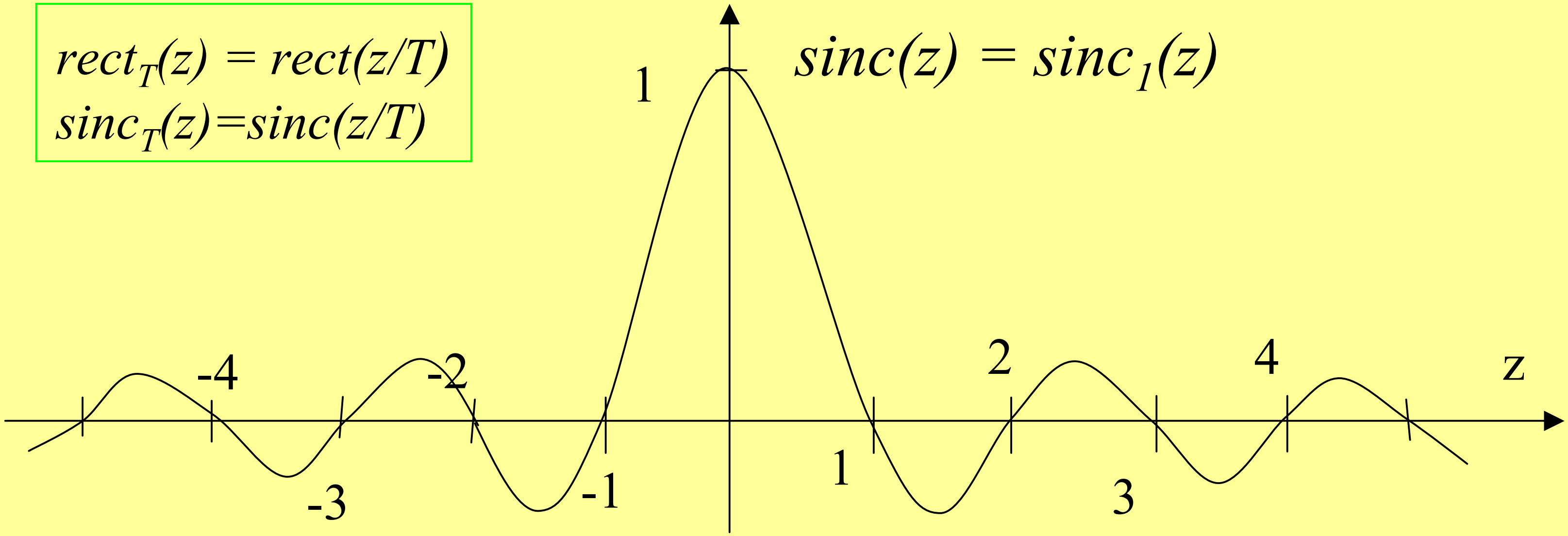
Imag pt = 0

Reminder (continued)

$$rect(z) = \begin{cases} 1 & : \ |z| < 0.5 \\ 0.5 & : \ |z| = 0.5 \\ 0 & : \ |z| > 0.5 \end{cases}$$

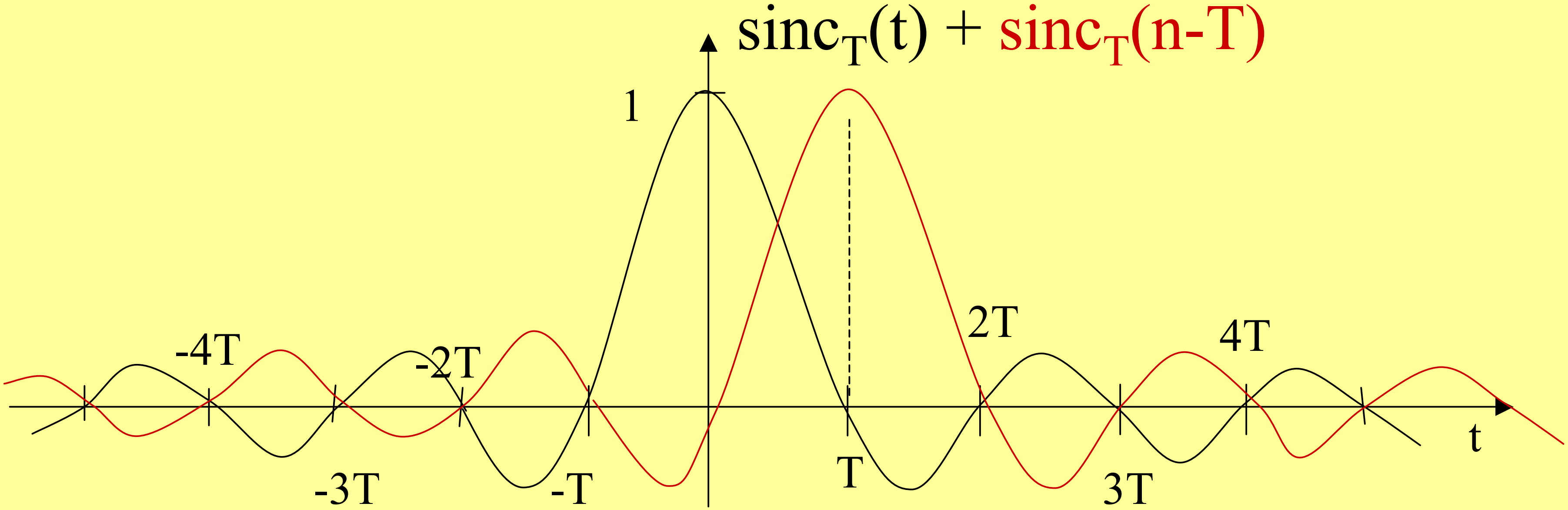
$$\mathrm{sinc}(z) = \begin{cases} \sin(\pi z) \big/ (\pi z) & : \quad z \neq 0 \\ 1 & : \quad z = 0 \end{cases}$$

$rect_T(z) = rect(z/T)$
$sinc_T(z) = sinc(z/T)$



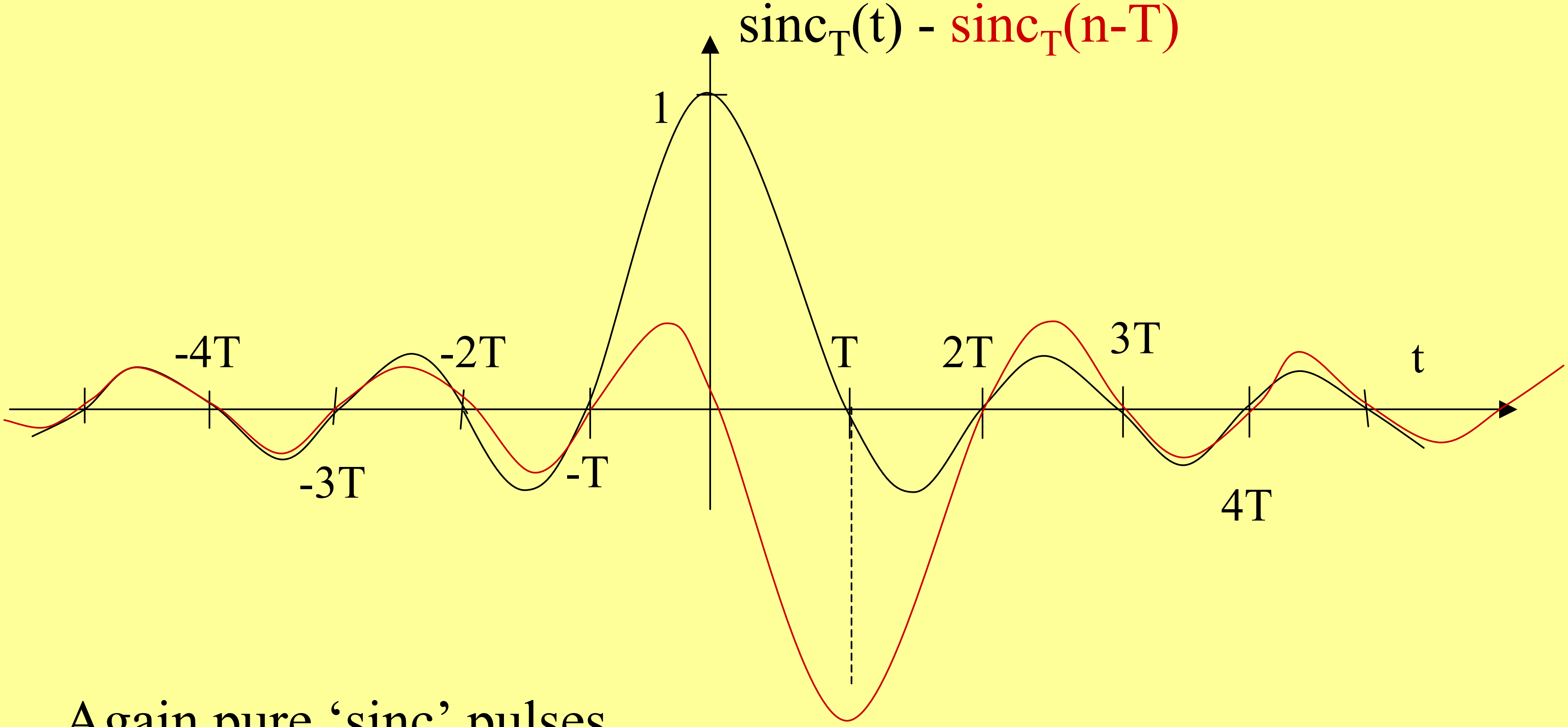
$sinc(z) = sinc_1(z)$

Two pulses with effects of ISI eliminated at t=0 & t=T



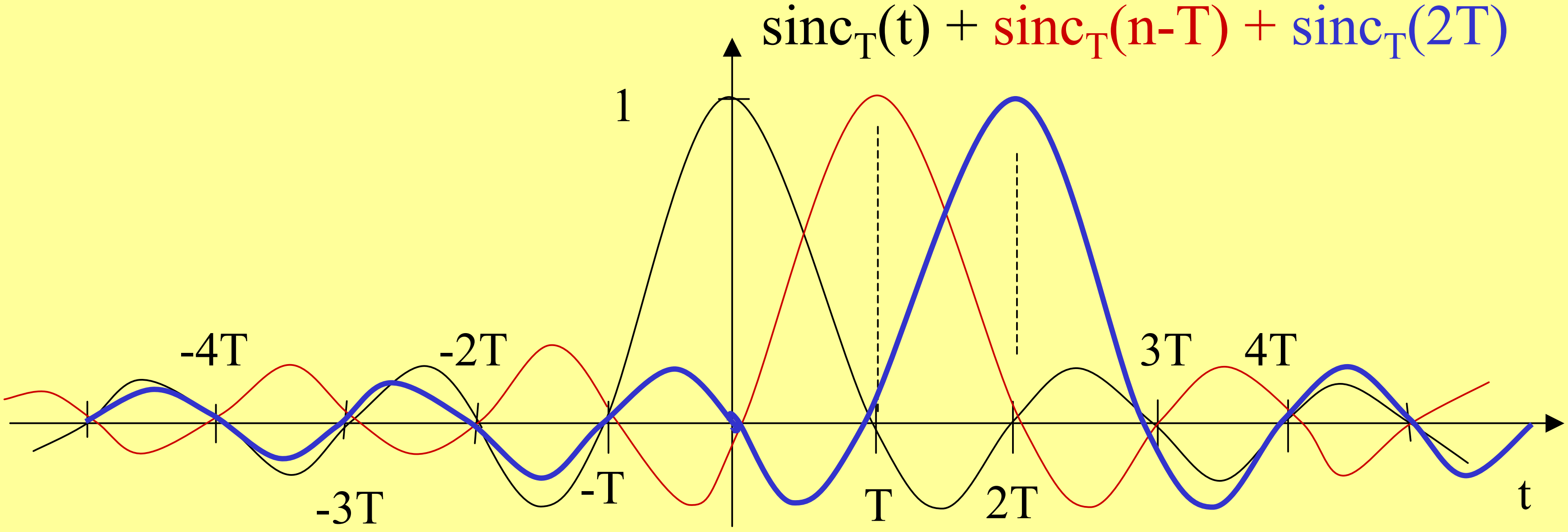
These are pure ‘sinc’ pulses.

It still works when 2nd pulse has opposite polarity



Again pure ‘sinc’ pulses

Three pure sinc pulses with ISI eliminated at t=0, T, 2T



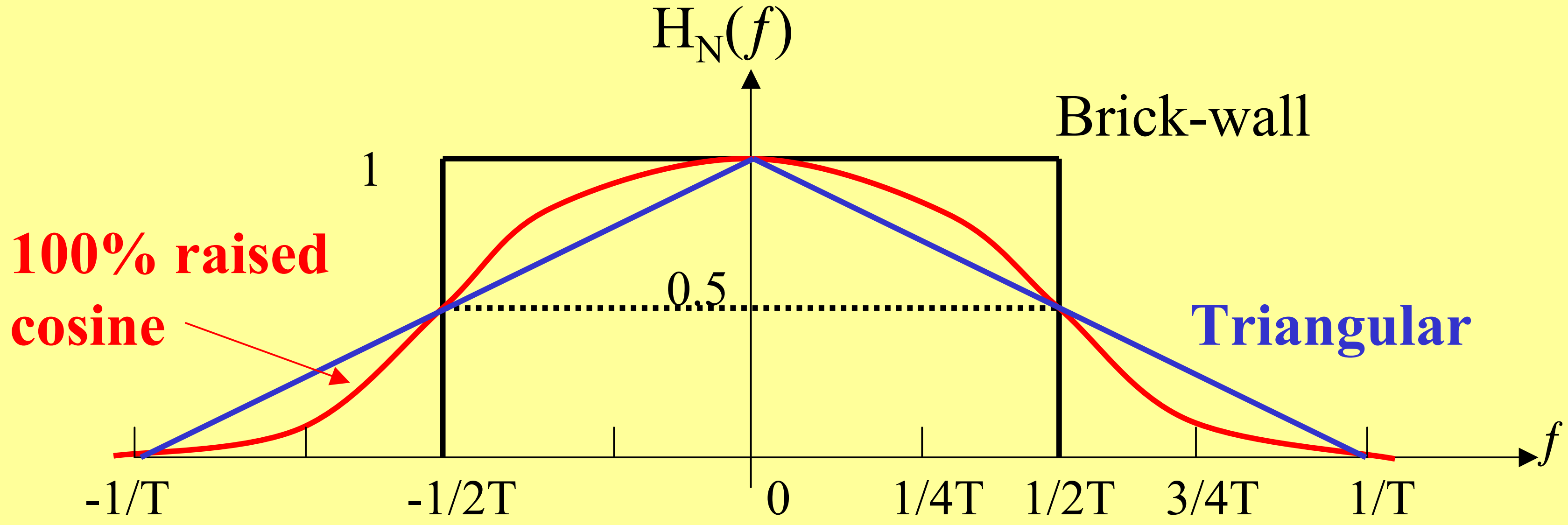
‘Nyquist’ frequency-response

- Pure sinc pulses are produced by exciting the brick-wall filter $\mathrm{H}(f) = T\,\mathrm{rect}_{1/\mathrm{T}}(f)$ with impulses.
- Zero ISI is achievable with pure sinc functions.

  & also with other ‘sinc-like’ pulses.
- Such pulses are produced by exciting, by impulses, any filter whose freq-response is ‘Nyquist’.
- A Nyquist freq-response, $H_N(f)$ say, band-limits the input to $\pm 1/T$ Hz & has a form of odd-symmetry about $1/(2T)$ in that:

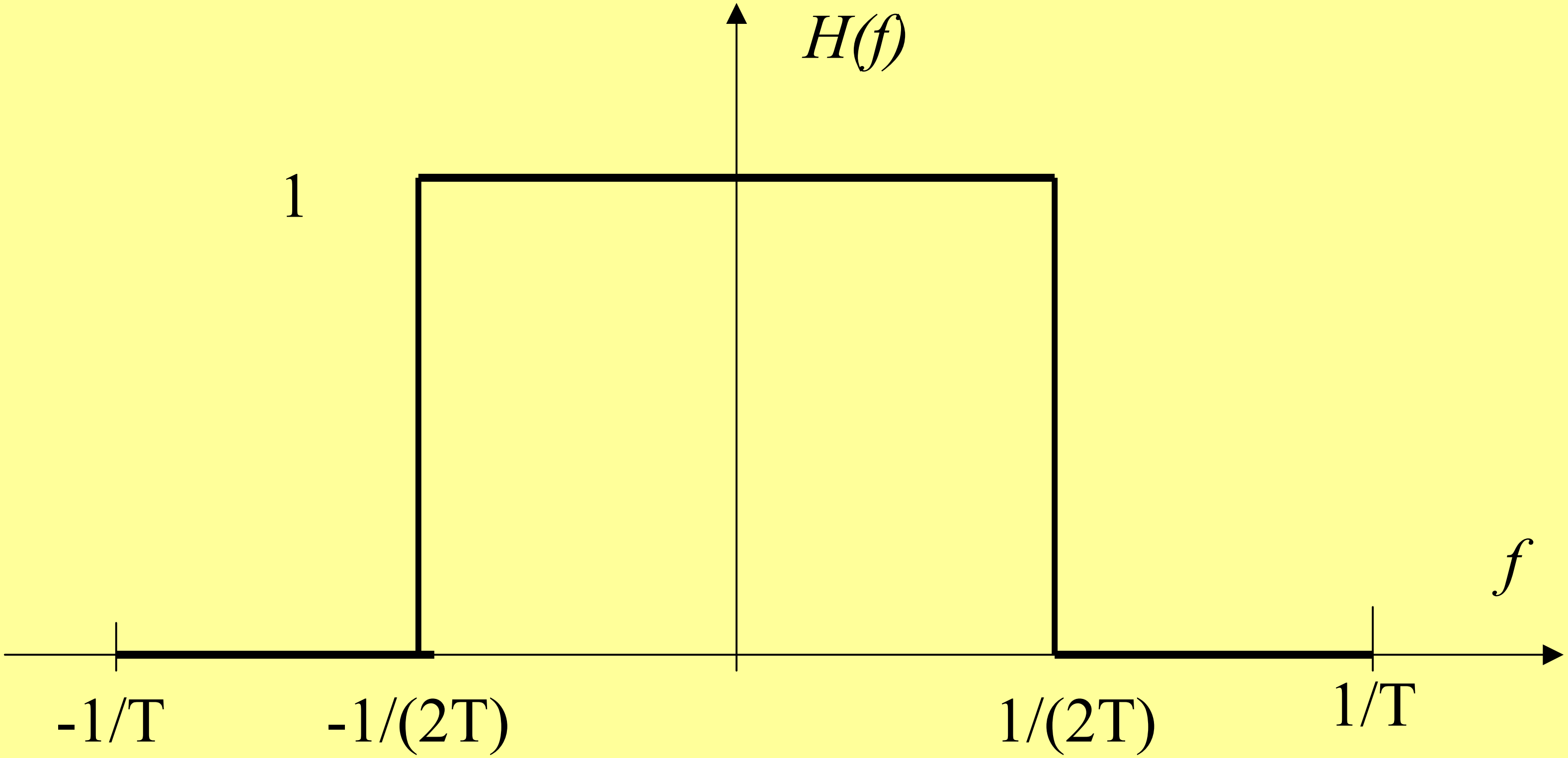
$$H_N(f) + H_N^{*}(1/T - f) = constant \;\; for \; 0 \le |\,f\,| \le 1/T$$

- Three real freq-responses satisfying this property shown below.
- They guarantee that zero crossings occur at t = ±T, ±2T, ±3T, ….

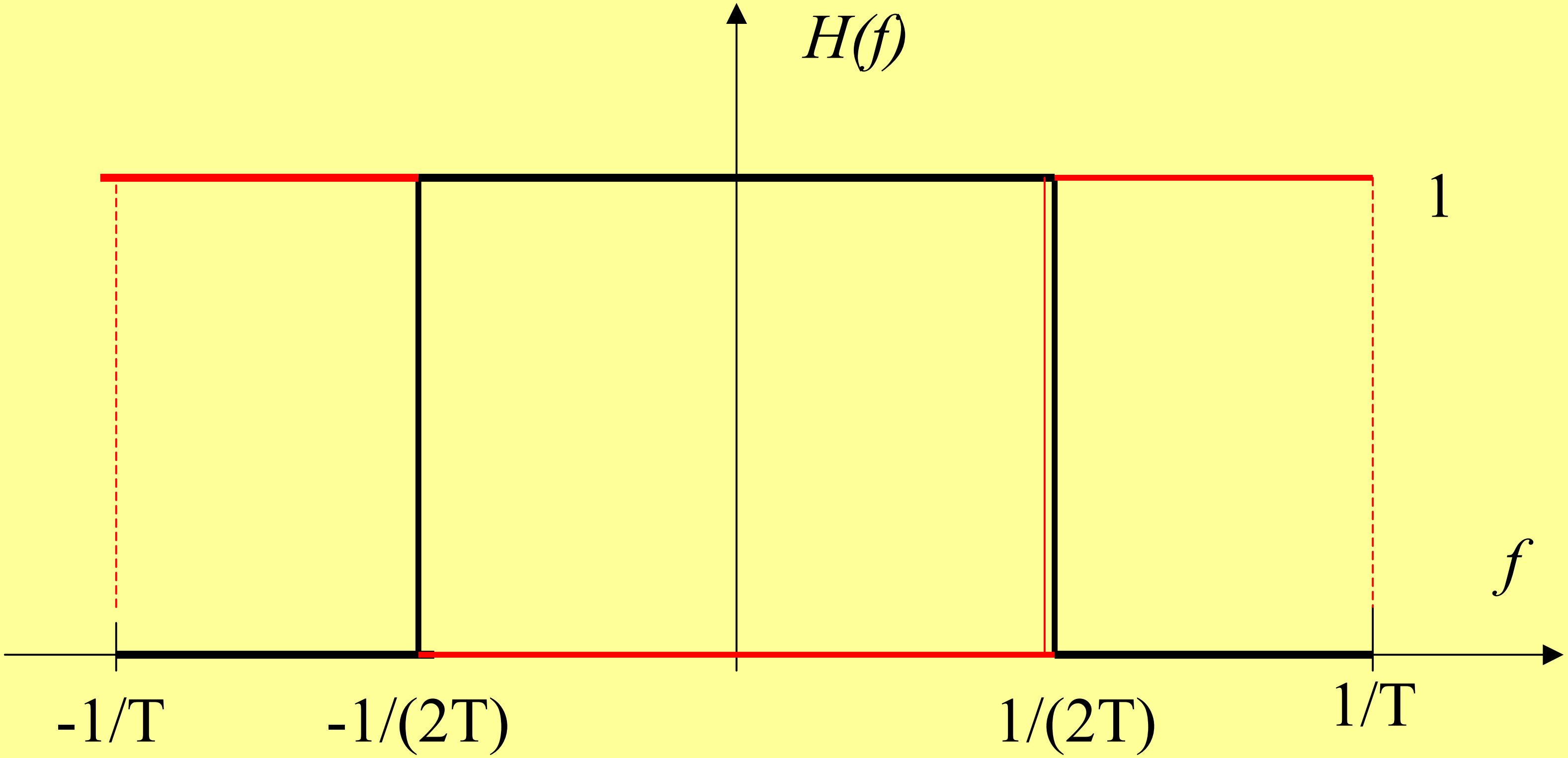


In general $H_N(f)$ may be complex.

Example 6.1: Show that the freq-response below is Nyquist

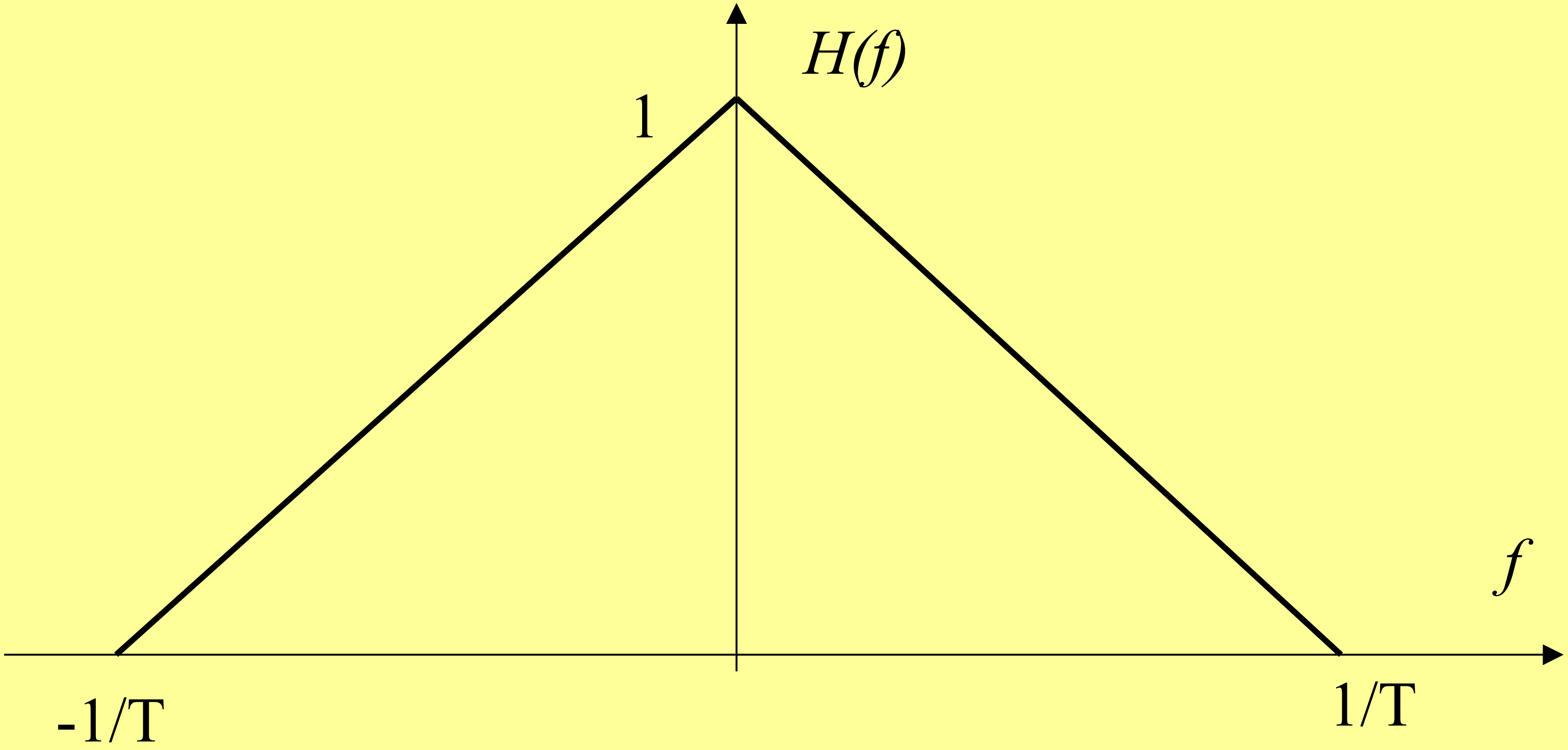


Solution: Draw $H(1/T - f)$ in red for $f > 0$

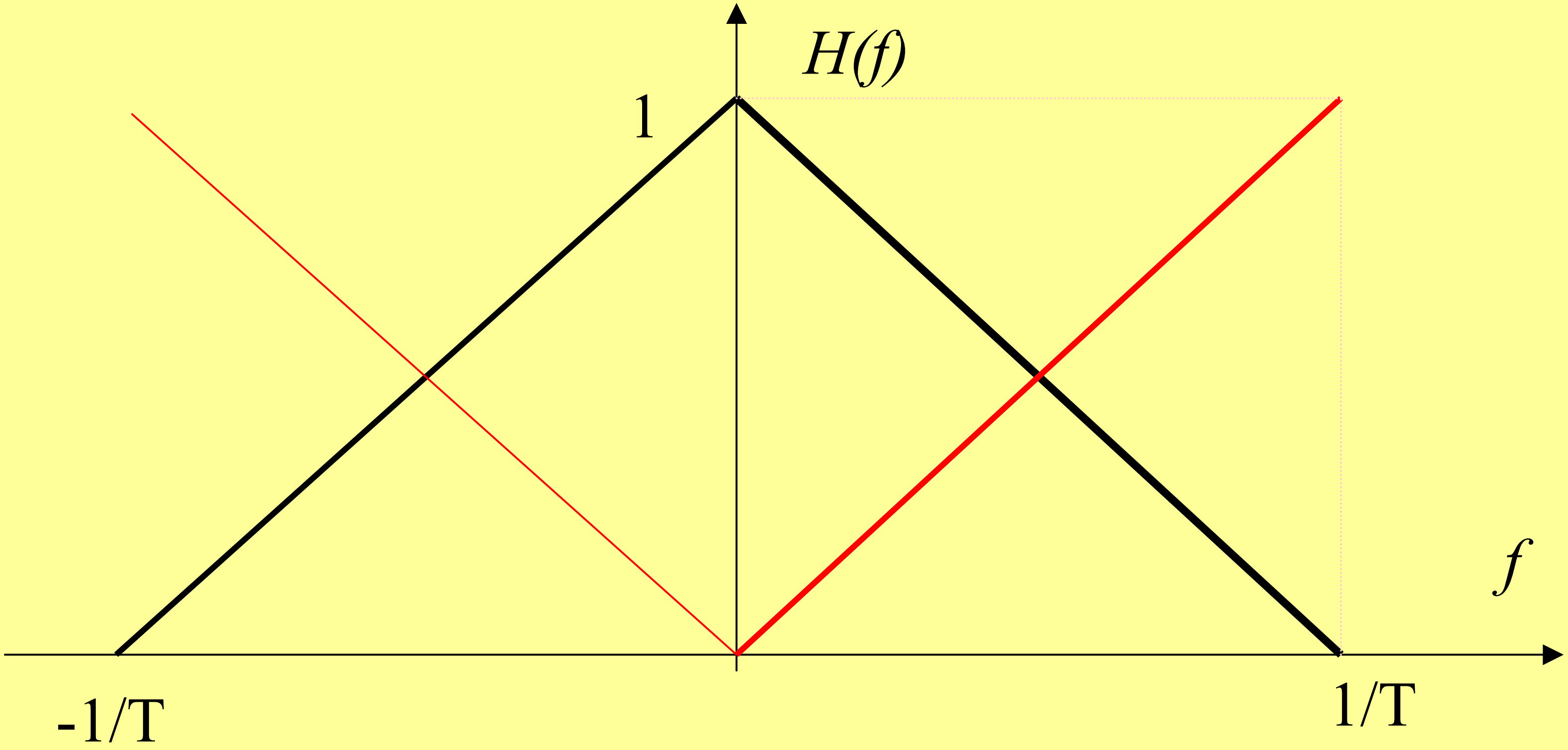


Clearly $H(f) + H^*(1/T - f) = 1$ for $-1/T < f < 1/T$

Example 6.2: Show that the freq-response below is Nyquist

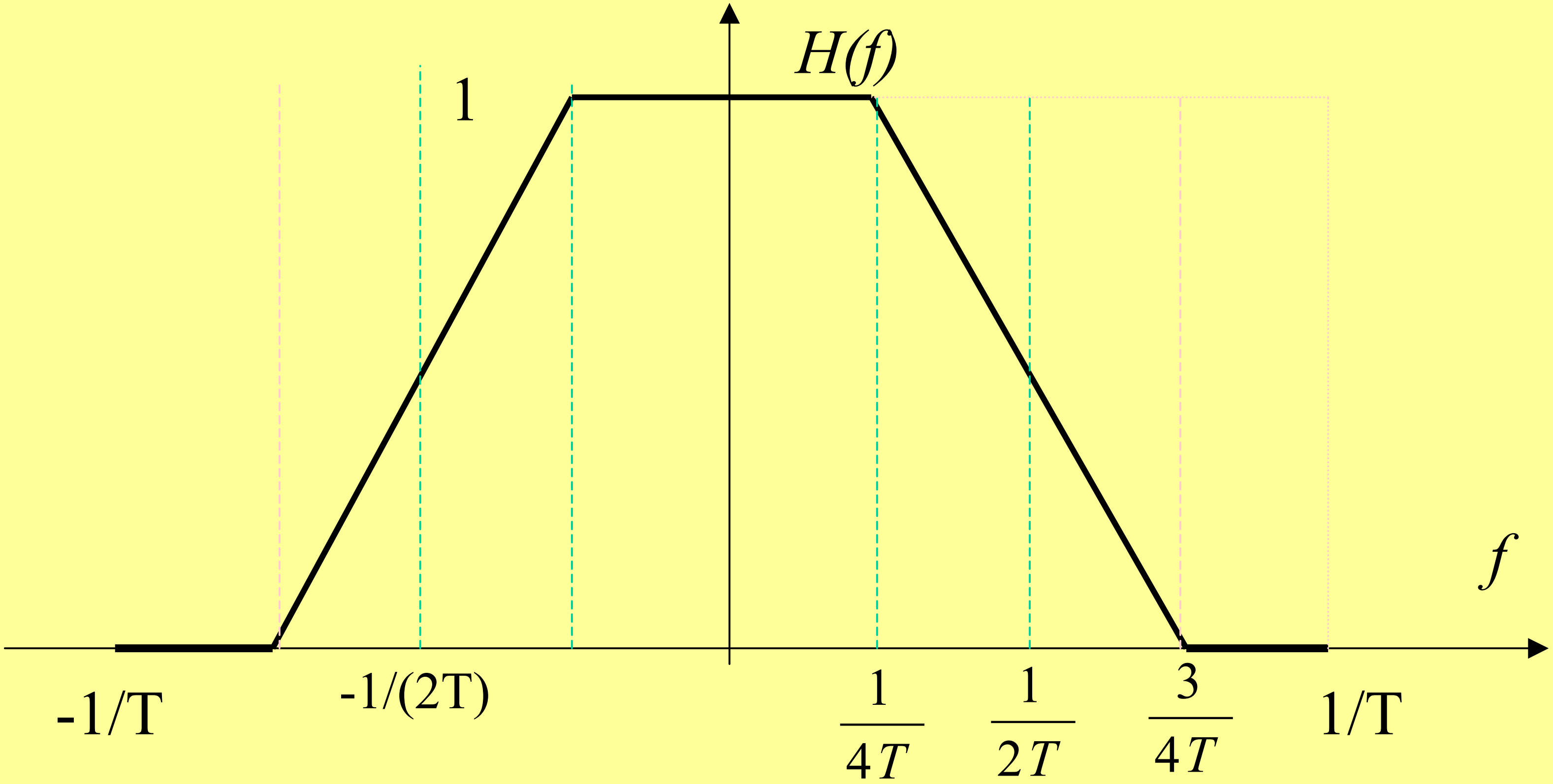


Example: Draw $H^*(1/T - f)$ in red for $0 < f < 1/T$



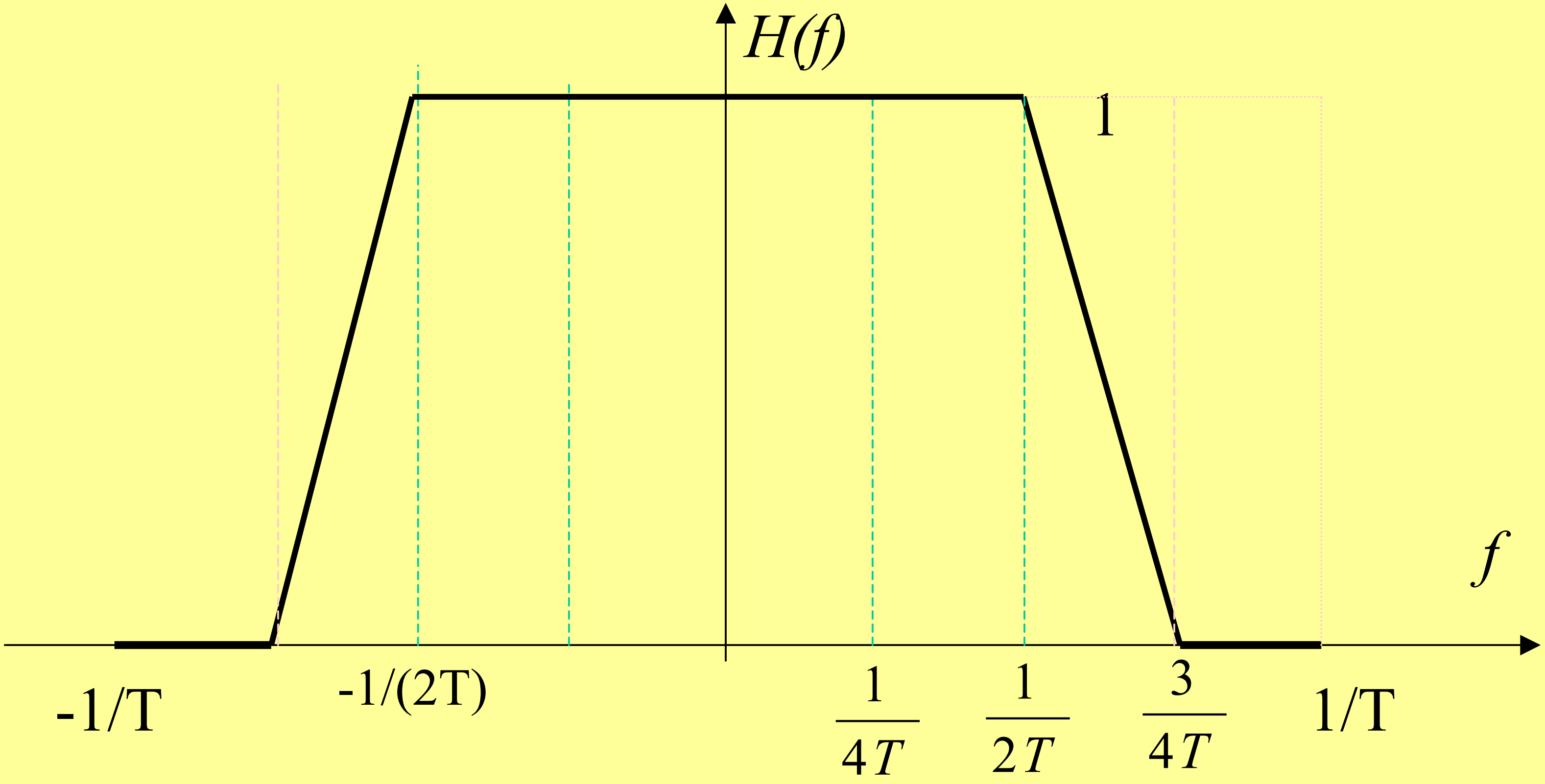
Clearly $H(f) + \boldsymbol{H^*(1/T - f)} = (1 - fT) + \boldsymbol{fT} = 1$

Example 6.3: Is the freq-response below Nyquist ?



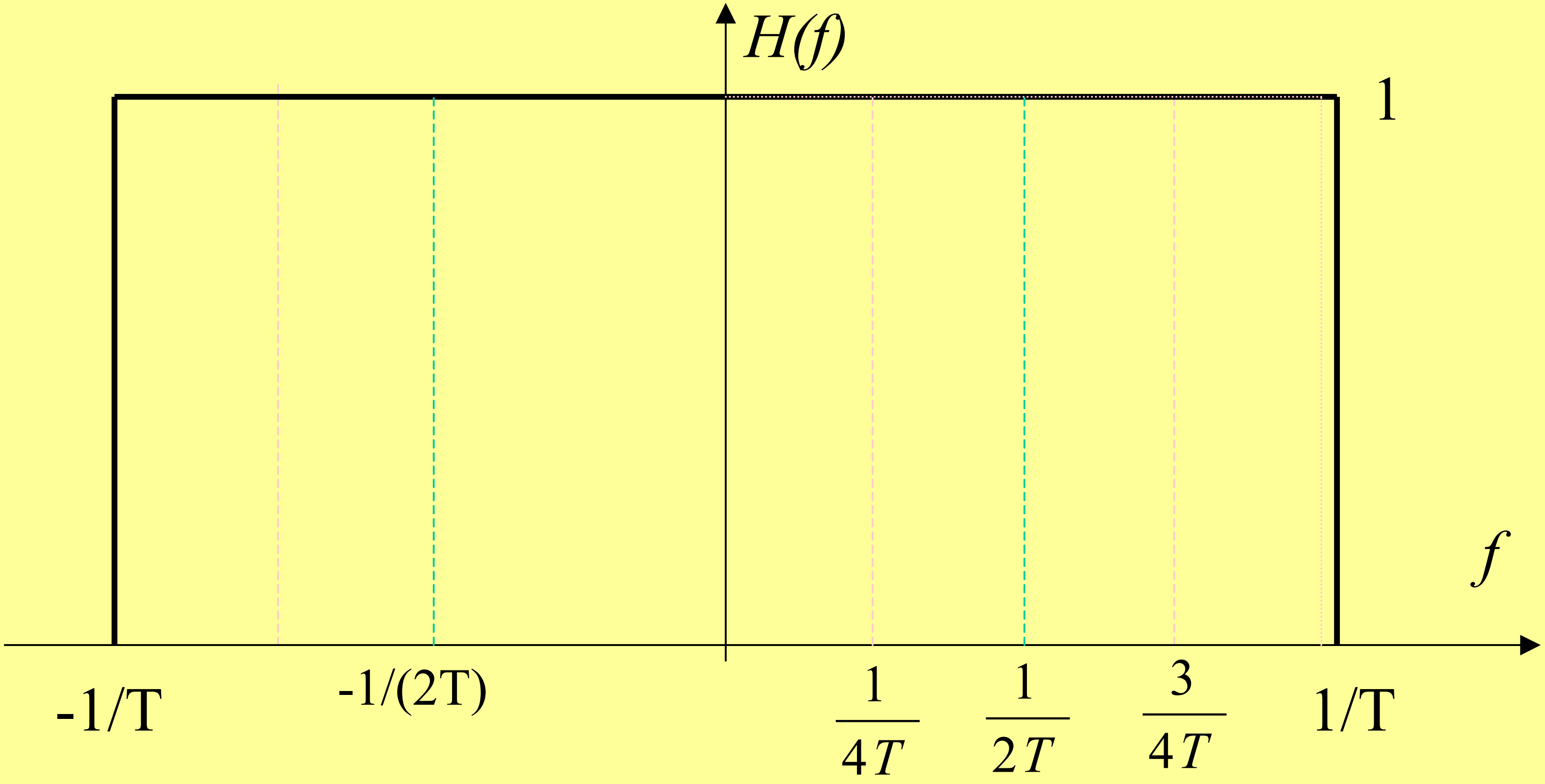
Clearly $H(f) + \textcolor{red}{\boldsymbol{H^*(1/T - f)}} = 1$ so YES

Example 6.4: Is the freq-response below Nyquist ?



NO!

Example 6.5: What about this one ?



YES! Zero crossings at 0, **±T/2**, ±T, **±3T/2**, ±2T, ...

(We don’t actually need the blue ones)

The ‘brick-wall’ filter

$$H(f) = \begin{cases} 1 : |f| < 1/(2T) \\ 0 : |f| \geq 1/(2T) \end{cases}$$

- •Has Nyquist freq-resp & we know that its impulse-response is a sinc function with zero crossings at t=±T, ±2T, etc.
- •It is band-limiting to minimum possible bandwidth,
- •Difficult to deal with because side-lobes of its "sinc" impulse response do not die away too quickly.
- •To get reasonable approximation to this filter, need very long impulse response & hence high order FIR filter.
- •Also, with such a pulse shape, if there is slightest error in timing of sampling point at detector, ISI will occur.

Raised cosine frequency response

- Commonly used family of Nyquist freq-responses is “raised cosine” family parameterised by r (or α) :

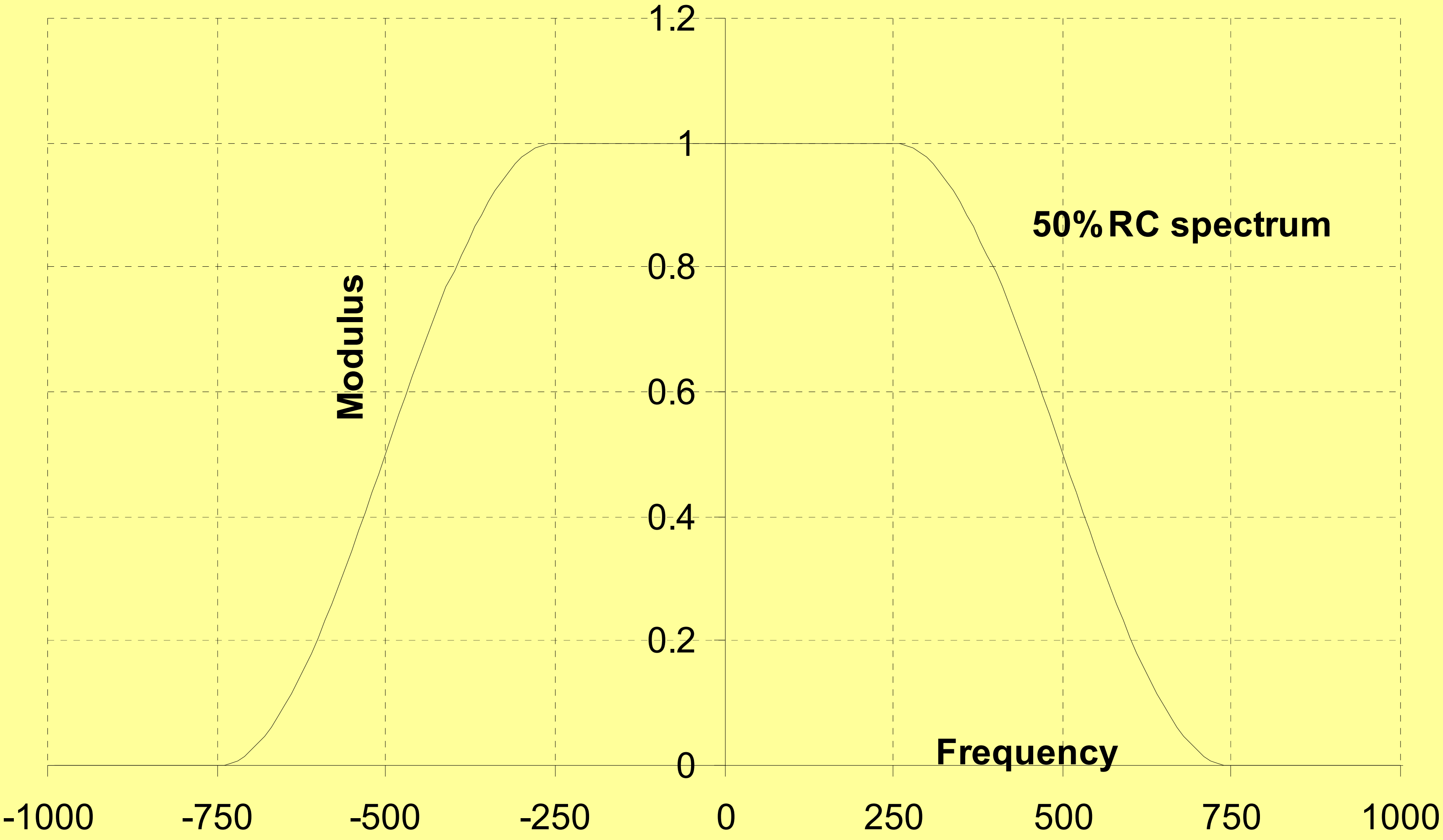
$$H((f)) = \begin{cases} 1 & : \quad |f| \le (1-r)/(2T) \\ 0.5[1+\cos(\pi T\{|f|-(1-r)/(2T)\}/r)] & : \quad (1-r)/(2T) \le |f| \le (1+r)/(2T) \\ 0 & : \quad (1+r)/(2T) < |f| \end{cases}$$

- Impulse-response of such a filter may be shown to be:

$$h(t) = \frac{\sin c(t/T)\cos(\pi r t/T)}{T - 4r^2t^2/T} \quad \text{where} \quad \mathrm{sinc(x)} = \lim \frac{\sin(\pi x)}{\pi x}$$

- When r=0, this becomes “brick-wall” from -1/(2T) to 1/(2T) Hz
- When r = 0.5, $H_{rc}(f)$ is as shown below when T=0.001 s.
- From general formula above, its spectrum is:

$$H((f)) = \begin{cases} 1 & : \quad |f| \le (1-r)/(2T) \\ 0.5[1+\cos(\pi T\{|f|-(1-r)/(2T)\}/r)] & : \quad (1-r)/(2T) \le |f| \le (1+r)/(2T) \\ 0 & : \quad (1+r)/(2T) < |f| \end{cases}$$


1.2
1
0.8
0.6
0.4
0.2
0
-1000
-750
-500
-250
0
250
500
750
1000
Modulus
Frequency
50% RC spectrum

- Note "odd-symmetry" about f=500 Hz for $f>0$

  & about f = -500 Hz = - 1/(2T) for f<0.

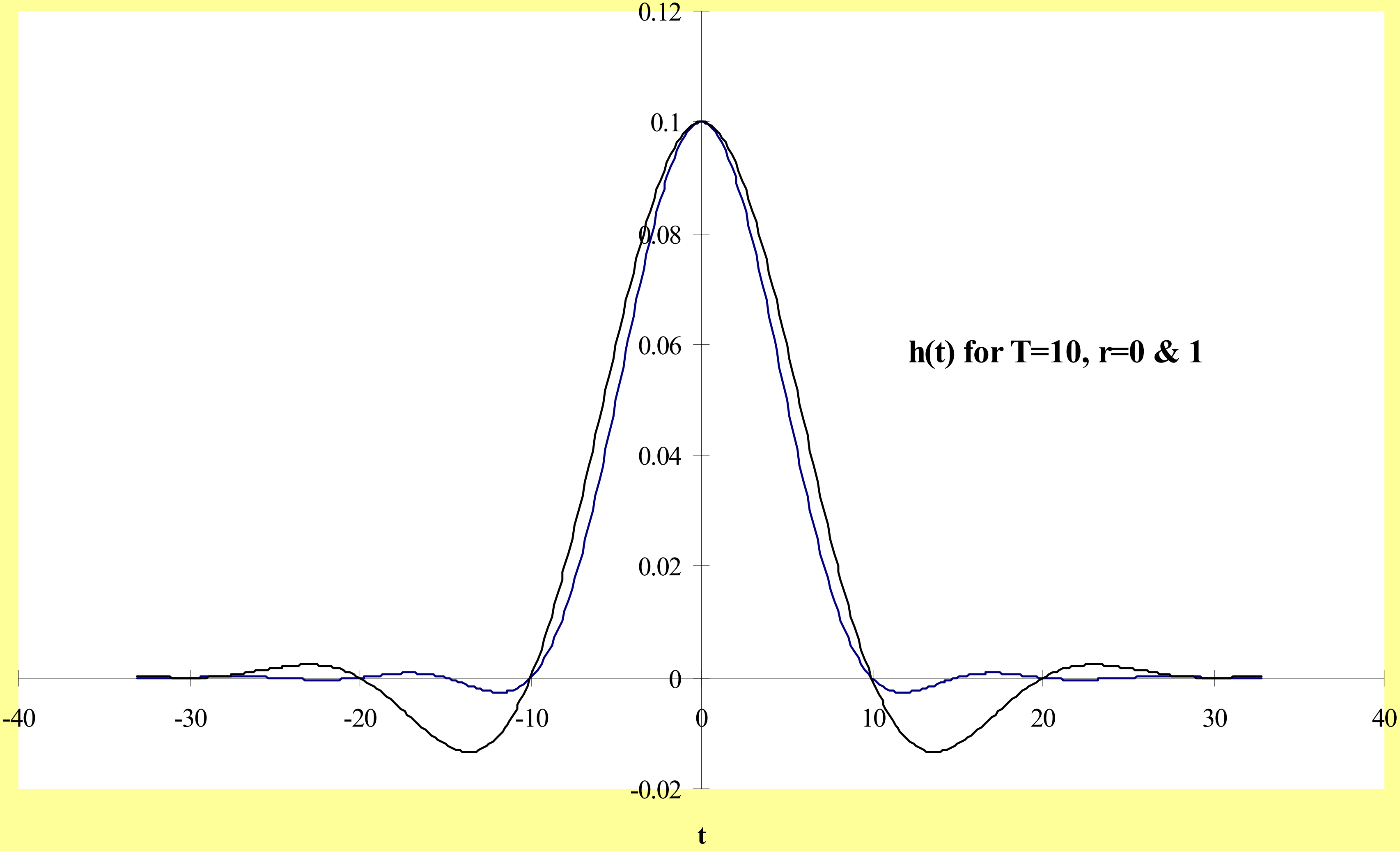
- It may be shown that $H_{rc}((f))+H^*_{rc}((1000 - f)) = 1$.
- When r=1, $H_{rc}((f))$ is pure raised cosine shaped response.
- No “flat-top” & widest bandwidth of family: ±1/T Hz.
- With 1/T=1 kHz, bandwidth would be -1000 to 1000 Hz.

- Impulse-responses $h_{rc}(t)$ corresponding to $H_{rc}((f))$ with r=0 &1 shown below taking T = 10.
- Note reduction of side-lobe ripples when r=1, at expense of doubling the bandwidth.
- Raised cosine spectrally shaped pulse for given value of r is "100r % RC symbol" & has bandwidth ±(1+r)/T Hz.
- When r=0.5, we have a 50% RC spectrally shaped symbol with bandwidth from -750 Hz to 750 Hz if 1/T = 1 kHz..
- This is 50% more than the absolute minimum band-width needed to avoid ISI with the techniques discussed up to now.
- Minimum bandwidth would be achieved with a 0% RC symbol which has a "brick-wall" spectrum from -1/(2*T*) to +1/(2*T*) Hz..

Bandwidth efficiency at baseband

0% RC (brick-wall)	$1/T$ b/s in $1/2T$ Hz	2b/s per Hz
50% RC	$1/T$ b/s in $3/(4T)$ Hz	1.333 b/s per Hz
100% RC	$1/T$ b/s in $1/T$ Hz	1 b/s/Hz

- Apart from the practical difficulties of generating a pulse with such a spectrum, the disadvantage of the brick-wall spectrum is that its time-domain shape is a "sinc" pulse which does not die away quickly enough for our liking.
- The 100% RC spectrum (r=1) produces a time-domain pulse which dies away much faster.
- By increasing r from 0 to 1 we improve the rate of dying away at the expense of extra bandwidth


h(t) for T=10, r=0 & 1
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-40
-30
-20
-10
0
10
20
30
40
t

- Shaping filter is FIR digital filter with coeffs equal to samples of h(t).
- Succession of shaped pulses viewed on oscilloscope as "eye-diagram".
- Scope triggered at beginning of each pulse.
- Open eye as clean new pulse superimposed on previous pulses.
- With noise, eye closes, making threshold detection difficult.

- $H_N((f))$ is required overall response of

  transmitting filter, channel & receiving filters.
- Receiving filters are:
  - Matched filter to minimise effect of AWGN
  - Equalising filter to cancel out filtering effect of channel.

- Pulse shape as seen at detector must have RC spectrum.
- Receiving filters include a matched filter.
- Not correct for transmitter to send symbols with RC spectra.
- If we did, matched filter would have freq-response equal to complx conj of the RC spectrum.
- Resulting output would have squared magnitude of this spectrum.
- No longer be raised cosine but raised cosine squared.
- Not a problem as far as minimising effect of noise.
- But no good for eliminating ISI .
- Detector must see, not a 'raised cosine squared' spectrally shaped pulse but a raised cosine spectrally shaped pulse.

- Distribute desired RC freq-response equally between transmitter & receiving matched filter.
- Make each “100r % root raised cosine” (RRC) freq-response.
- Transmitter sends RRC spectrally shaped symbols.
- In time-domain, RRC symbols not too dissimilar from RC ones.
- They exist for all time & are symmetric about t=0.
- But they do not have zero crossings at t = ±T, ±2T, ±3T, …
- To produce such symbols, we define:

$$H((f)) = \begin{cases} 1 & : \quad |f| \le (1-r)/(2T) \\ \sqrt{0.5[1+\cos(\pi T\{|f|-(1-r)/(2T)\}/r)]} & : (1-r)/(2T) \le |f| \le (1+r)/(2T) \\ 0 & : \quad (1+r)/(2T) < |f| \end{cases}$$

•Chose value of r in range 0 to 1.

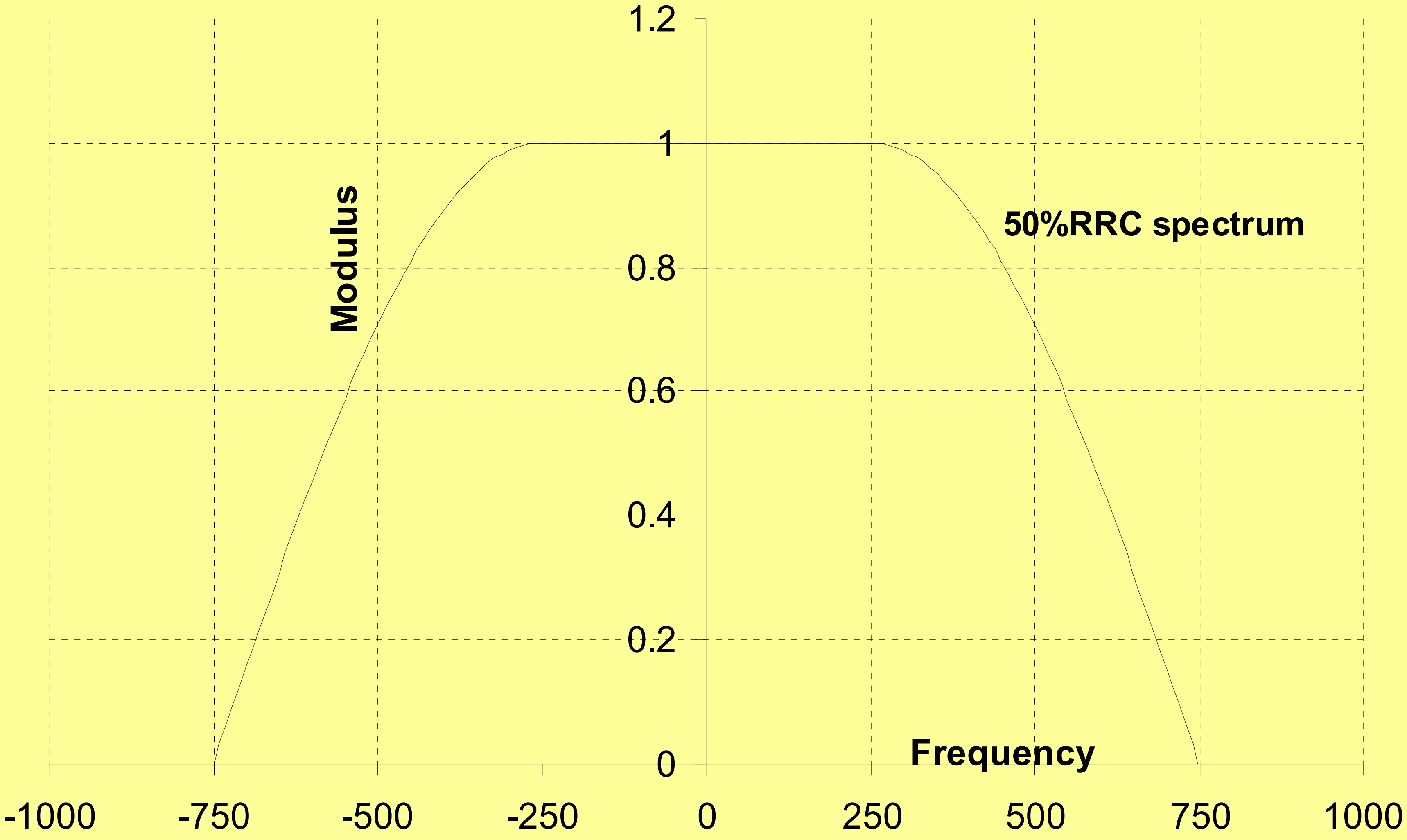
•Apply inverse FT to calculate corresponding impulse resp $h_{rrc}(t)$.

•Normally done in sampled data form.

• $h_{rrc}(t)$ sampled, windowed & delayed (involves approximation)

•Gives impulse-response (& hence coeffs) of an FIR digital filter of manageable order.

•Graph of $H_{rrc}(f)$ against f when 1/T=1000 Hz & r=0.5 next.


1.2
1
0.8
0.6
0.4
0.2
0
Modulus
50%RRC spectrum
Frequency
-1000
-750
-500
-250
0
250
500
750
1000

- Note lack of "odd symmetry" about $f = \pm 500$.
- Matched filter at receiver performs two roles at same time.
- First role is to minimise error probability P_B as usual.
- Its impulse-response must equal s(t) time-reversed & suitably delayed.
- As s(t) = s(-t) for RRC symbol, it also squares spectrum of s(t) to make it RC rather than RRC.
- As well as minimising P_B, matched filter also completes the job of generating RC Nyquist freq-resp as required to eliminate ISI.
- Matched filter has impulse-response s(T-t) i.e. time-reversed symbol delayed by T.
- Have to delay this by several more intervals of T to allow some of s(-t) to be included without making filter non-causal.
- Must also delay decision until several intervals beyond t=0 .
- Each symbol now extends beyond single interval of T s.

Exercise 6.6:

Does a symbol whose spectrum is '100r % RC squared' have zero-crossings at t= ±T, ±2T, ±3T, … as required for zero ISI?

Answer:

Nope (except when r=0). This is the problem.

Exercise 6.7:

A Nyquist freq-response $H_N(f)$ with band-width from -1/T Hz to 1/T Hz has symmetry about 1/(2T) in that:

$$H_N(f) + H_N^*(1/T - f) = constant \; for \; 0 \le |f| \le 1/T$$

Show that this condition guarantees that the zero-crossings of its impulse-response occur in the time-domain at $t = \pm T, \ \pm 2T, \ \pm 3T, \ldots$ as required to eliminate ISI.

Solution: Let impulse-response be denoted by *h(t)*.

- If $h(t)$ has zero crossings at $t = \pm T, \pm 2T, \pm 3T, \ldots$, then

$$\text{sample}_T[h(t)] = h(0)\delta(t)$$

- FT of $\text{sample}_T[h(t)]$ is $(1/T)\ \text{repeat}_{1/T}(\ H_N(f)\ )$
- FT of $\delta(t)$ is 1

$$\therefore \quad \frac{1}{T}\sum_{k=-\infty}^{\infty} H_N(f - k/T) \quad = \quad h(0)$$

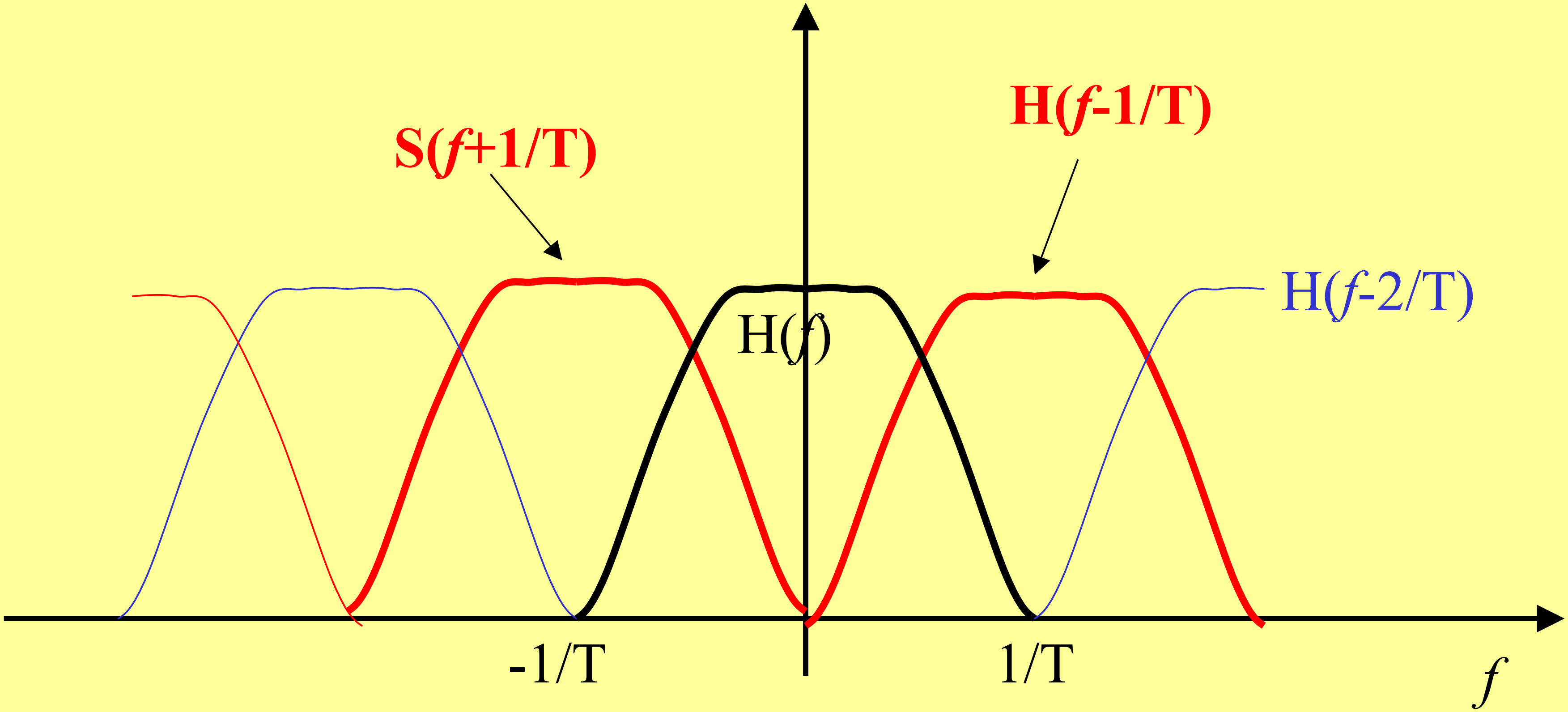
- If $H_N(f)$ band-limited between $f = \pm 1/T$ Hz, only terms with

  $k = 0$ & 1 contribute for f in range $0 \le f \le 1/T$.

$$\therefore\ H_N(f) + H_N(f - 1/T) = T\,h(0) \quad \text{for} \quad 0 \le f \le 1/T$$

- As $h(t)$ is real $H_N(-f) = H_N^*(f)$ for any f.
- Same argument for $-1/T \le f \le 0$.

repeat$_{1/T}$(H(f))



Exercise 6.8: Without calculating $h_{rrc}(t)$, i.e. symbol generated by exciting 100r % RRC filter with an impulse, would you expect its zero-crossings to occur at $t = \pm T,\ \pm 2T,\ \pm 3T, \ldots$?

Answer: Nope.

- Strange that this RRC symbol does not have zero-crossings required for zero ISI.
- Transmission along channel has ISI.
- When symbols received & passed thro' matched filter thus squaring RRC spectrum, condition for zero ISI is satisfied.

Exercise 6.9: If RRC pulses are used, show that if AWGN is added to the received signal, noise component $n_0(t)$ of $z(t)$, i.e. output of the RRC matched filter at receiver, has an auto-correlation function which is zero at delays T, 2T, … Why is this a good thing?

Answer:

ACF is inverse FT of power spectrum. Instead of t we have delay τ.
At delay $\tau = T$, ACF tells us whether the noise we observe at time t is expected to be correlated with what we observed T seconds ago.
If noise sample was high T seconds ago does this mean that we must expect a high noise level again?
If so, a high P_B before will lead to a high P_B now.
Hence errors will tend to occur in "bursts".
A lot of errors at once, then few errors. Not such a good thing.

If ACF of $n_0(t)$ is zero at $\tau = \pm T, \pm 2T$, etc. there will be no correlation between error prob. at $t=nT$, & at $t=mT$ for $m \neq n$. Errors more evenly distributed in time.

Channel noise n(t) only passes through matched filter. So its magnitude spectrum will be RRC shaped.

Its power spectrum will be RRC squared, i.e. RC. Gives required time or "delay" domain zero crossings

Exercise 6.10: Check from the formula given that an 100r % RC filter satisfies the conditions for being a Nyquist filter. If T= 125 μs, what bandwidth is required for binary signalling with 60% RRC pulses?

Equalisation

•Channel distorts symbol-shape because of its non-ideal frequency-response.

•Noise added to the received version of symbol.

•Matched filter at receiver minimises effect of noise.

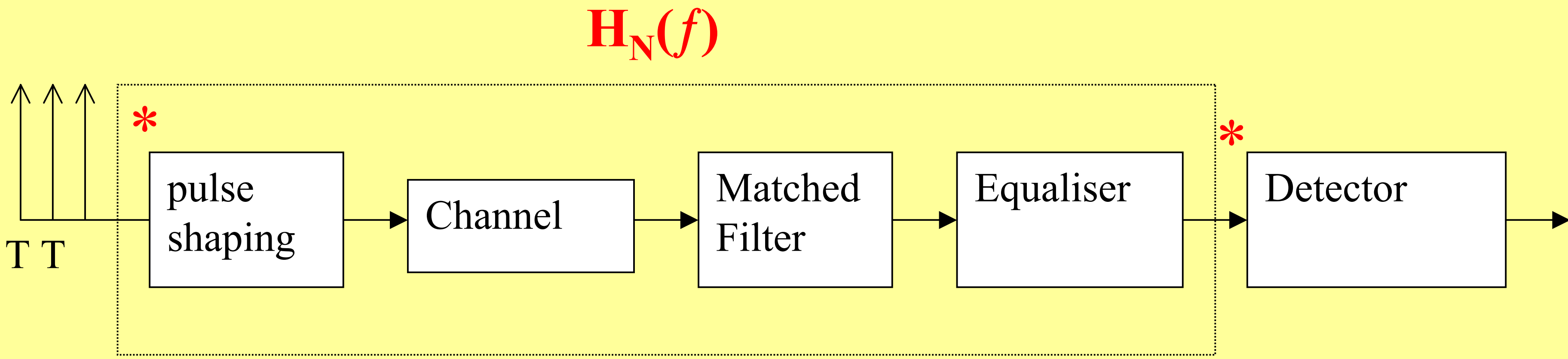
•'Equaliser' filter at receiver cancels out symbol-shape distortion due to frequency-response of channel.

•Frequency-response of a radio channel not ideal because of multi-path propagation.

•With wired channel problem is capacitance & inductance of line.

- To minimise ISI, product of freq-responses of
  pulse shaping filter at transmitter
  channel
  matched filter
  equaliser

must be Nyquist.



$H_N(f)$

*

T T

pulse shaping

Channel

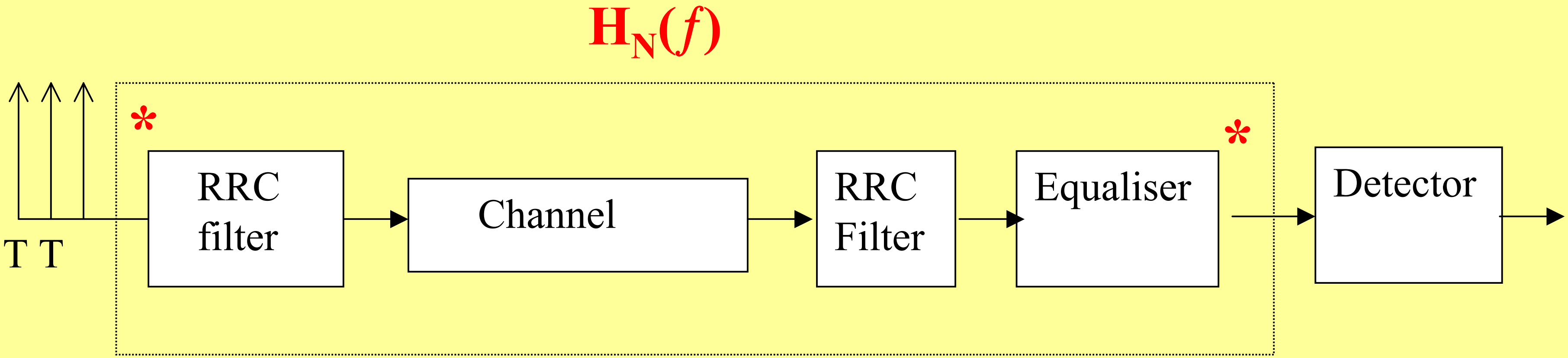
Matched Filter

Equaliser

*

Detector

- To make product of freq-responses of all filters & channel equal to $H_N((f))$, generally RC, take its square root & construct two RRC filters.
- First RRC filter acts as pulse-shaper band-limiting signal launched into channel.
- Receiver's RRC filter completes RC shaping as well as being MF.
- Equaliser cancels frequency response of channel.



$H_N(f)$

T T

*

RRC filter

Channel

RRC Filter

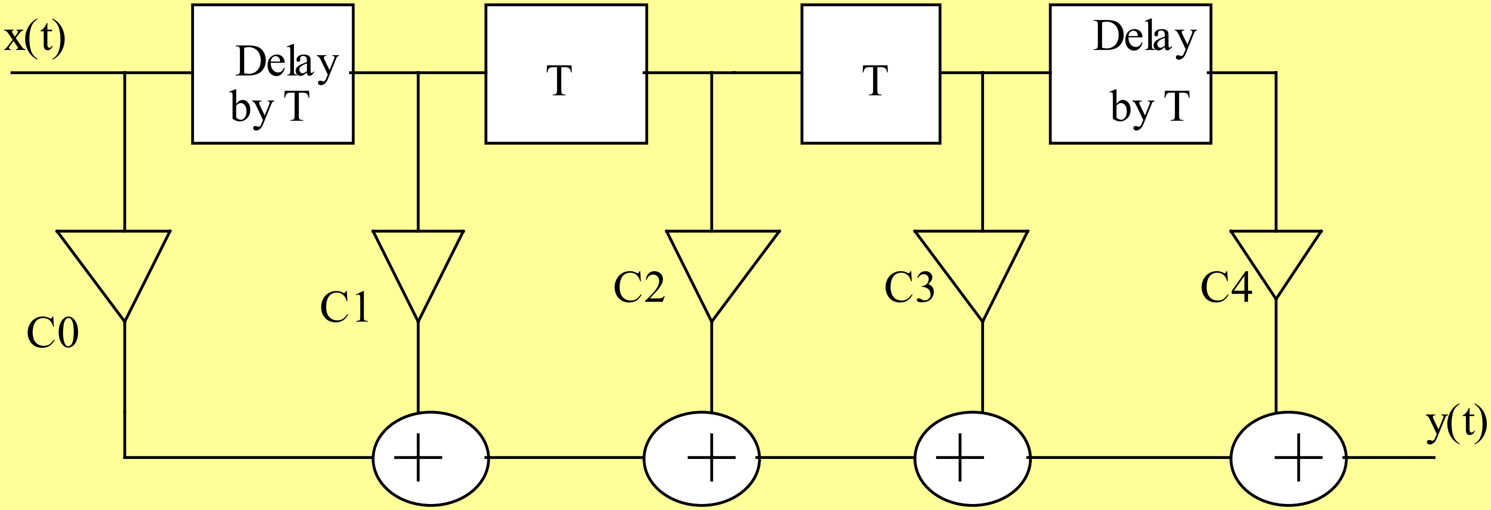
Equaliser

*

Detector

Equalisation of channel may be achieved using adaptive FIR "transversal" filter, (or "tapped delay line")

4th order example shown below:



- Such a filter can be made “zero forcing” transversal equaliser.
- It is FIR filter but note that delay boxes are not z^{-1} but z^{-T}
- Looks at symbols received from RRC/channel/RRC combination.
- Adapts its coefficients C_0, C_1, C_2, etc. such that, in response to an input waveform x(t) centred on t=0, output y(t) has exact zero-crossings at t=0, ±T, ±2T, ...
- An Nth order transversal filter must delay centre of symbol by (N/2)T to do its job.
- Transversal filter shown above is of order N = 4, with five “taps” labelled C_0, C_1, C_2, C_3, C_4.
- Output is:

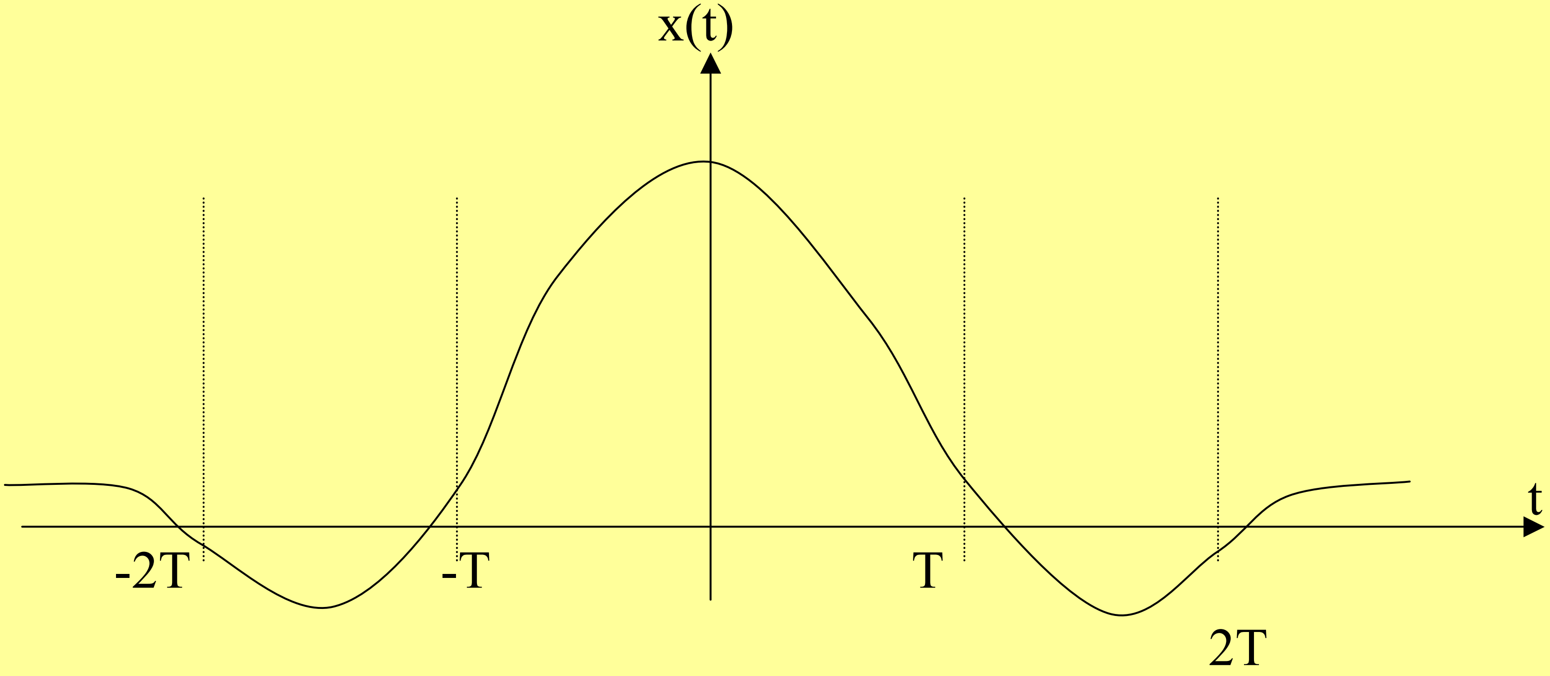
$$y(t) = \sum_{n=0}^{N} C_n x(t - nT)$$

Given symbol x(t) from RRC/channel/RRC filtering with centre at t = 0.

Assume effect of channel makes x(±T) & x(±2T) non-zero.

To take an example assume

$x(-2T) = -0.04$, $x(-T) = 0.2$, $x(0) = 1.0$, $x(T)=0.2$ and $x(2T)=-0.04$.



- To simplify calculation, use 3-tap (2nd order) transversal filter.
- Force zero-crossings at t = +T & -T.
- Centre of symbol is delayed & occurs at t=T.
- Output of 3-tap transversal filter is:

$$y(t) = C_0 x(t) + C_1 x(t-T) + C_2 x(t-2T)$$

- Output at t=0, T, and 2T is as follows:

$$y(0) = C_0 x(0) + C_1 x(-T) + C_2 x(-2T)$$

$$y(T) = C_0 x(T) + C_1 x(0) + C_2 x(-T)$$

$$y(2T) = C_0 x(2T) + C_1 x(T) + C_2 x(0)$$

- We wish to make y(0)=0, y(T)=1, y(2T)=0. Therefore:

$$y(0) = C_0 + 0.2C_1 - 0.04C_2$$

$$y(T) = 0.2C_0 + C_1 + 0.2C_2$$

$$y(2T) = -0.04C_0 + 0.2C_1 + C_2$$

- Set of 3 linear simultaneous eqns in 3 unknowns. In matrix form:

$$\begin{bmatrix} 1 & 0.2 & -0.04 \\ 0.2 & 1 & 0.2 \\ -0.04 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

i.e. $A\underline{c} = \underline{b}$ with solution $\underline{c} = A^{-1}\underline{b}$

Use MATLAB to find A^{-1} as follows:

```
A= [1 .2 -0.04; 0.2 1 0.2; -0.04 0.2  1]
inv(A)
   1.0490   -0.2273    0.0874
   -0.2273    1.0909   -0.2273
    0.0874   -0.2273    1.0490
```

$$\begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1.049 & -0.227 & 0.087 \\ -0.227 & 1.091 & -0.227 \\ 0.087 & -0.227 & 1.049 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore: $C_0 = -0.227$, $C_1 = 1.091$. $C_2 = -0.227$