University of Manchester
CS3282: Digital Communications 2005
Section 7: The Shannon-Hartley Theorem.

•Famous theorem of information theory.
•Gives theoretical maximum bit-rate that can be transmitted with arbitrarily small bit-error rate (BER), with given average signal power, over channel with bandwidth B Hz affected by AWGN.

•By "arbitrarily small BER" this means that for any given BER, we can find a coding technique that achieves it.
•The smaller the BER, the more complicated the technique.
•Maximum achievable bit-rate (with arbitrary BER) is 'channel capacity' C.

•The Shannon-Hartley Theorem (or Law) states that:

$$C = B \log_2 \left(1 + \frac{S}{N} \right) \quad \text{bits/secon d}$$

S/N is mean-square signal to noise power ratio (not in dB).

•Proof beyond syllabus.

•Doubling bandwidth doubles capacity if S/R remains the same.

Exercise 7.1: Show that if the signal power is equal to the noise power, C in b/s is equal to the bandwidth B Hz.

Solution: If S/N=1, $B\log_2(1+S/N) = B\log_2(2) = B$

- To avoid calculating logs to the base 2,

$$C = \frac{1}{\log_{10} 2} B \log_{10}\left(1 + \frac{S}{N}\right) \approx 3.32\, B \log_{10}\left(1 + \frac{S}{N}\right)$$

- This means that if S/N >>1,

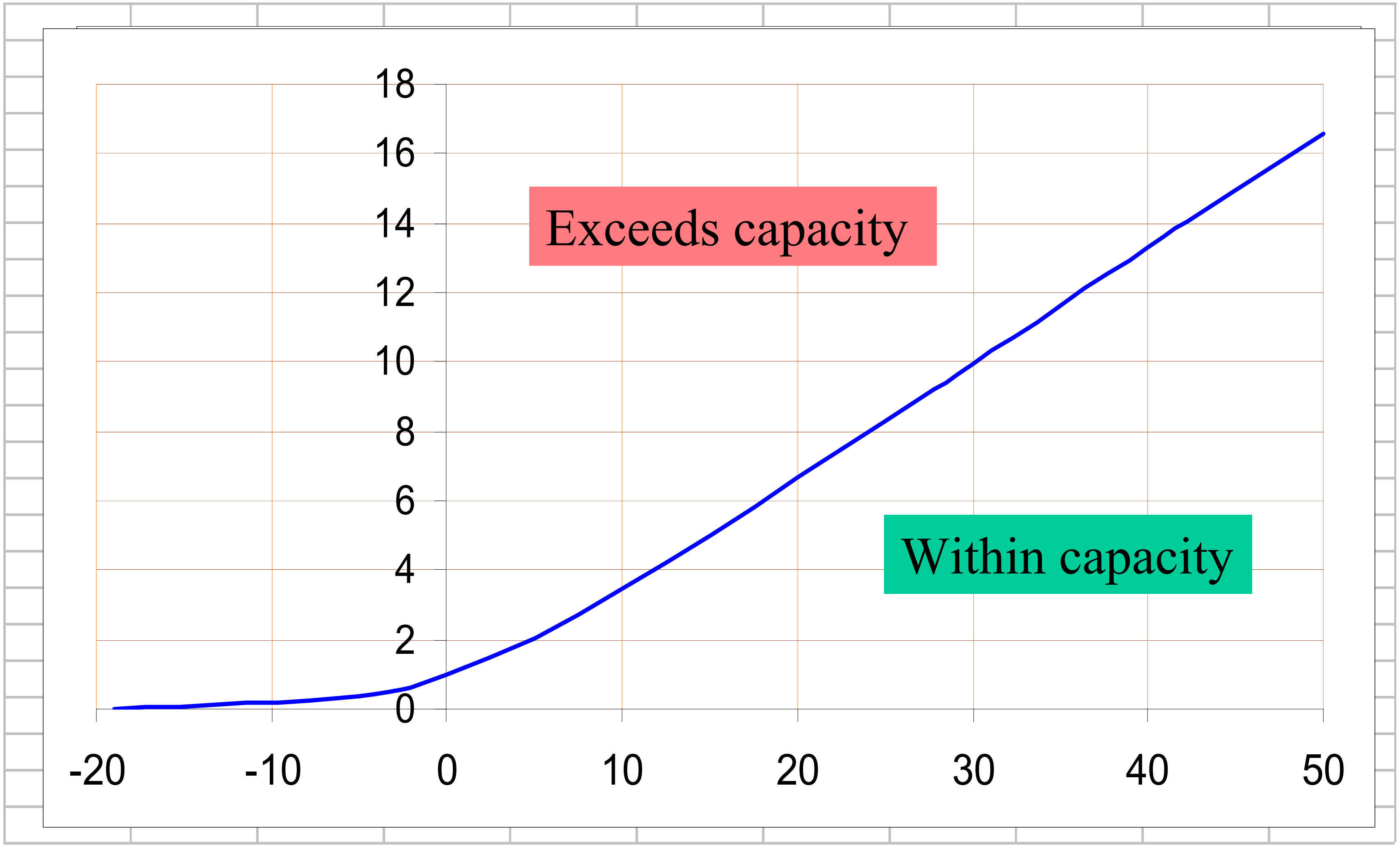
$C \approx 0.332\, B\, 10 \log_{10} (S/N)$

$\therefore$ C ≈ 0.332 times B times the SNR in dB

Exercise 7.2:
What is max bit-rate achievable with arbitrarily low bit-errors by computer modems operating over 3 kHz telephone channels that guarantees only a 30dB SNR?

Solution: 30kb/s.

Graph of capacity C (in b/s/Hz) against SNR (in dB).



Exercise 7.3:

Assuming a usable bandwidth of 0 to 3 kHz with AWGN and a 2 sided noise PSD of $N_0/2$, design a simple modem (using M-ary signalling with $M \approx 5$) for transmitting 30kb/s with a SNR of 30dB.

What is the bit-error rate?

If this BER is too high for your application, how could you reduce it?

$$\text{Capacity limit, } C = B\log_2\left(1+\frac{S}{N}\right)$$

•For a given bandwidth B & S/N, we can find a way of transmitting data at a bit-rate R bits/second, with an bit-error rate (BER) as low as we like, as long as R ≤ C.

•Given B and S/N, assume we transmit R bits/sec & we wish to ensure that R < Shannon-Hartley limit C. Then:

$$R \le B\log_2\left(1+\frac{S}{N}\right)$$

•Assume average energy/bit is E_b (Joules per bit) & AWGN has 2-sided PSD N_0 /2 Watts per Hz.

•Signal power S = E_bR & noise power N = N_0B Watts. Therefore:

$$R/B \le \log_2\left(1+\frac{E_b R}{N_0 B}\right)$$

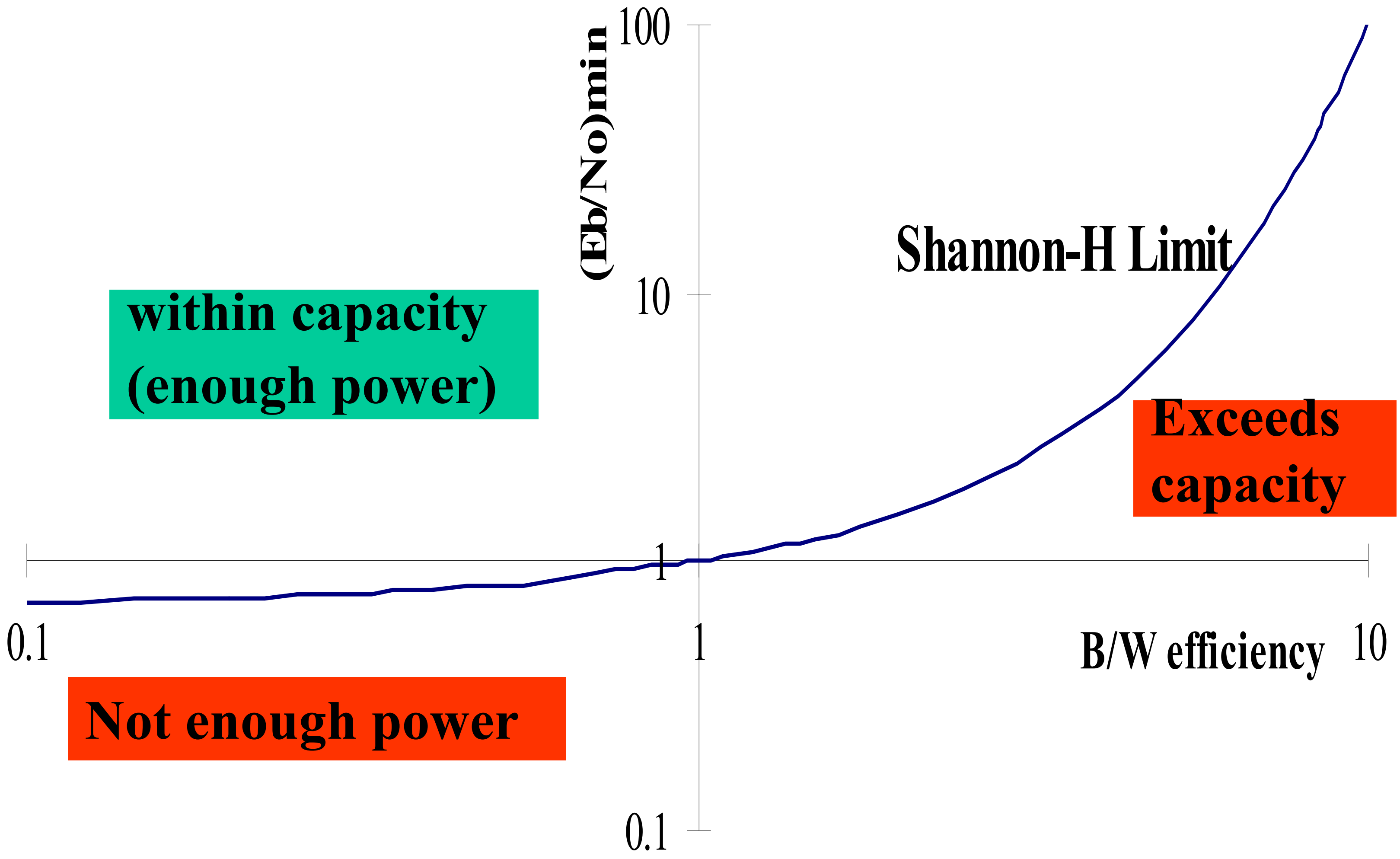
- R/B is called the bandwidth efficiency in bit/second/Hz.
- How many bit/second do I get for each Hz of bandwidth.
- We want this to be a high as possible.
- E_b/N_0 is "normalised average energy/bit" where normalisation is with respect to 1-sided PSD of AWGN.
- It is a sort of signal to noise energy ratio.
- Often converted to dBs as $10\log_{10}(E_b/N_0)$. We can now write:

$$2^{R/B} \le 1 + (E_b / N_0)(R / B)$$

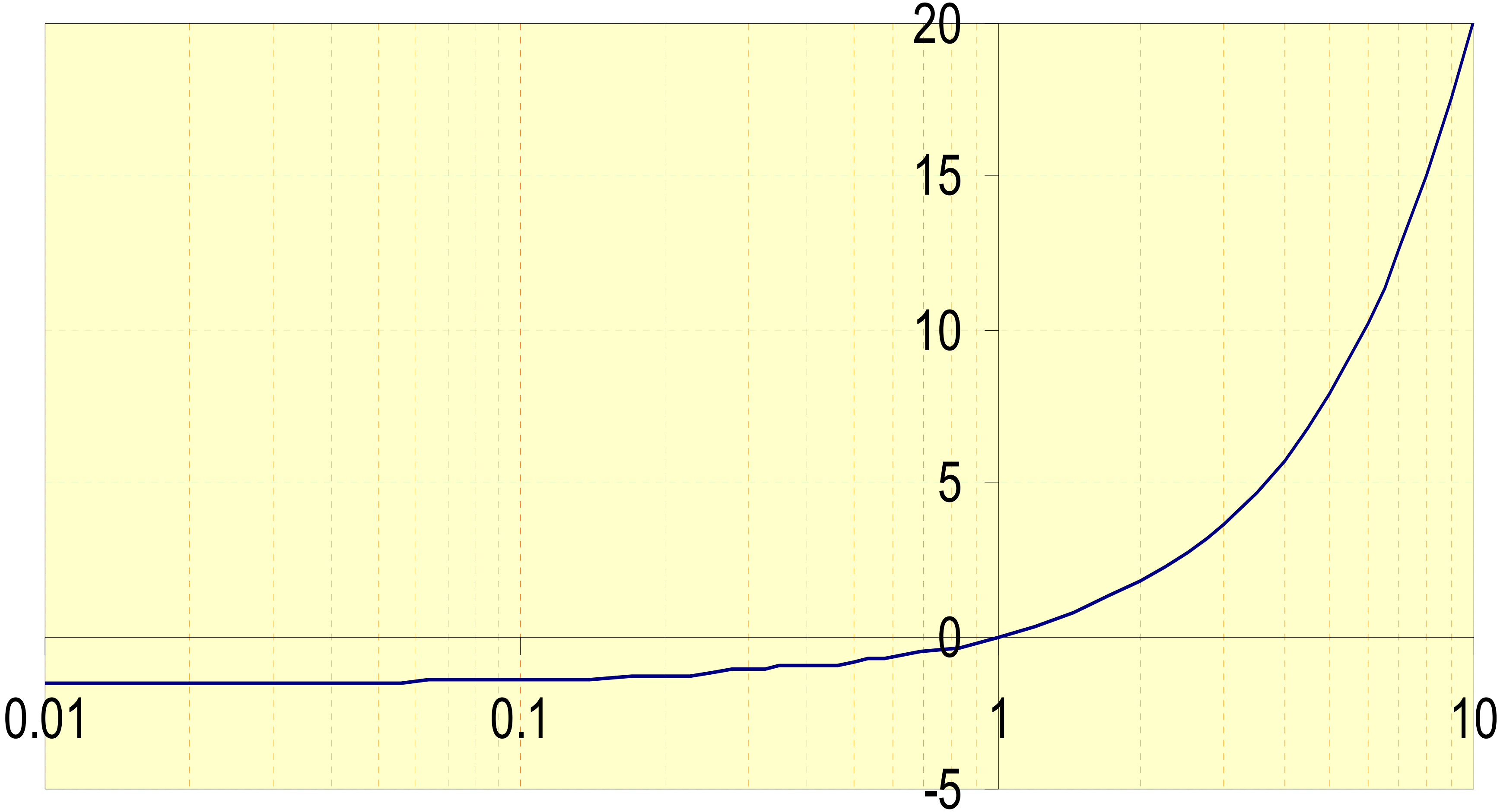
which means that $$E_b / N_0 \ge \frac{2^{R/B} - 1}{R/B}$$

$$(E_b/N_0)_{\min} = \frac{2^{R/B} - 1}{R/B}$$

- This formula gives min possible E_b/N_0 which allows transmission at R/B b/s/Hz with arbitrarily low bit-errors.
- Graph of $(E_b/N_0)_{min}$ against bandwidth efficiency (R/B) shows that $(E_b/N_0)_{min}$ never goes less than about 0.69 i.e. about -1.6 dB.
- If $E_b/N_0 < -1.6$dB, we can never satisfy Shannon-Hartley law however inefficient (in terms of bit/rate/Hz) we are prepared to be.
- Above curve gives values of (E_b/N_0) which satisfy the law for given bandwidth efficiency.
- Here, any BER, however low, can be achieved in theory.
- Below curve, E_b/N_0 is too low for a given bandwidth efficiency & certain bit-error rates become unachievable.


100
(Eb/No)min
Shannon-H Limit
10
within capacity
(enough power)
Exceeds
capacity
1
0.1
1
B/W efficiency
10
Not enough power
0.1

Graph of $(E_b/N_0)_{min}$ (dB) for a required b/s perHz



•It is possible to view the formula in a different way.

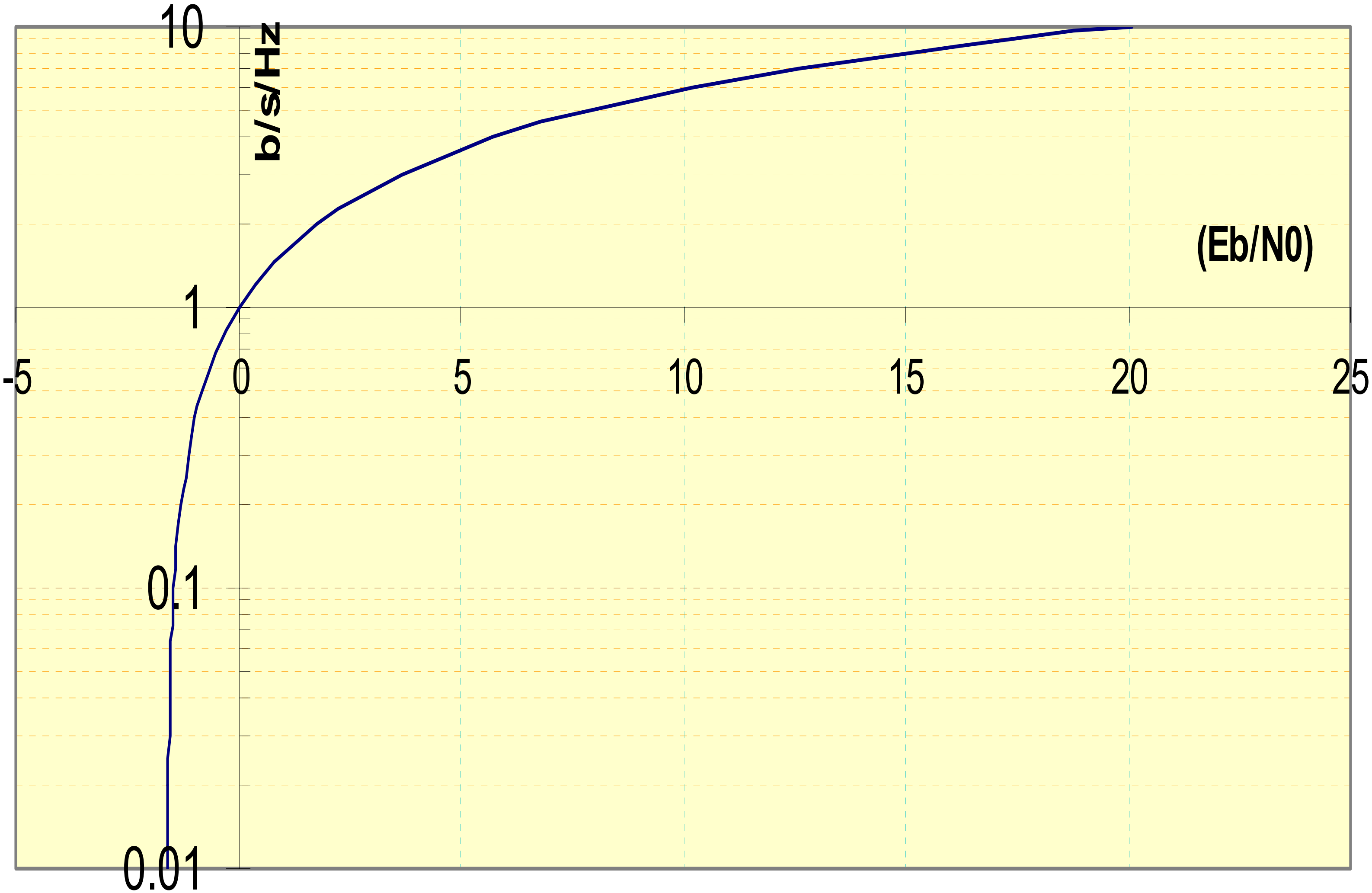
•The following graph shows the max achievable b/s/Hz for a given Eb/$N_{0.}$

•This shows clearly that when Eb/N_0 becomes less than -1.6dB, the max achievable b/s per Hz becomes very low, essentially zero.

•So there is no bit-rate, however low, that will achieve arbitrarily low bit-rate when $E_b/N_0 < -1.6$ dB..

•Not even 1 b/s per year!!!

Graph of max achievable b/s per Hz against (E_b/N_0) (dB)



To see this mathematically, note that

$$2^{R/B} = e^{(\log_e 2)R/B} = e^{0.693R/B} \approx 1 + 0.693\,R/B$$

when R/B is small. Therefore when R/B is small,

$$\left(E_b/N_0\right)_{\min} \approx \frac{1 + 0.69\,R/B - 1}{R/B} = 0.69 = -1.6dB$$