University of Manchester

CS3282: Digital Communications '05-'06
Section 7: The Shannon-Hartley Theorem.

This is a famous theorem of information theory that gives us a theoretical maximum bit-rate that can be transmitted with an arbitrarily small bit-error rate (BER), with a given average signal power, over a channel with bandwidth B Hz which is affected by AWGN. By "arbitrarily small BER" this means that provided the conditions of the theorem are met, for any given BER, however small, we can find a coding technique that achieves this BER; the smaller the given BER, the more complicated will be the technique. The maximum achievable bit-rate (with arbitrary BER) is referred to as the channel capacity C. The Shannon-Hartley Theorem (or Law) states that:

$$C = B\log_2\left(1+\frac{S}{N}\right) bits/\sec ond$$

where S/N is the mean-square signal to noise ratio (not in dB), and the logarithm is to the base 2. A proof of this theorem is beyond our syllabus, but we can argue that it is reasonable. Assume we are managing to transmit at C bits/sec, given a bandwidth B Hz. We can in theory transmit 2B symbols/sec, and doubling B (with no other changes) doubles the achievable Baud rate and hence doubles the bit-rate. To avoid calculating logs to the base 2,

$$C = \frac{1}{\log_{10} 2} B \ \log_{10}\left(1+\frac{S}{N}\right) \quad \approx \quad 3.32\,B\ \log_{10}\left(1+\frac{S}{N}\right)$$

This means that if S/N >>1, the channel capacity C ≈ 0.332 times B times the SNR in dB.

Exercise 7.1: Show that, according to the Shannon-Hartley law, if the signal power is equal to the noise power, the channel capacity in b/s is equal to the bandwidth B Hz.

Exercise 7.2: According to the Shannon-Hartley law, what is the maximum achievable bit-rate for a computer modem operating over a telephone channel with 3 kHz bandwidth and a maximum allowed signal power that guarantees only a 30dB SNR?

Solution: 30kb/s.

Exercise 7.3: Assuming a usable bandwidth of 0 to 3 kHz with AWGN and a 2 sided noise PSD of $N_0/2$, design a simple modem (using M-ary signalling with M≈5) for transmitting 30kb/s with a SNR of 30dB. What is the bit-error rate? If this BER is too high for your application, how could you reduce it?

Further note about the Shannon-Hartley Theorem

Some people call it the Hartley-Shannon Theorem or Law. It tells us that:

$$Capacity\ \lim it, C = B \log_2\left(1 + \frac{S}{N}\right)$$

where C in bits/second and B is 1-sided bandwidth in Hz and S/N is signal/noise ratio (not in dB).

Stated another way, for a given bandwidth B and a given S/N, we can find a way of transmitting data at a bit-rate R bits/second, with an bit-error rate (BER) as low as we like, as long as R ≤ C.

If R is not too close to C and/or the BER is not too low, the transmitter and receiver may be fairly simple. Otherwise, we have to consider more and more complex transmitters and receivers to achieve the required performance: knowing that if R is less than C for the given S/N, such a technique exists in theory. If R is greater than C, we may be able to find a technique for a relatively high BER, but reducing the required BER will soon bring us to a situation that cannot be achieved.

Therefore given B and S/N, assume we are transmitting R bits/sec and we wish to ensure that R is less than the Shannon-Hartley limit C. Then:

$$R \le B \log_2\left(1 + \frac{S}{N}\right)$$

Now assume we wish to transmit average energy/bit of E_b (Joules per bit) and the AWG noise has 2-sided power spectral density N_0 /2 Watts (normalised to 1 Ohm) per Hz. It follows that the signal power $S = E_bR$ and the noise power $N = N_0B$ Watts. Therefore:

$$R/B \le \log_2\left(1 + \frac{E_b R}{N_0 B}\right)$$

R/B is called the bandwidth efficiency in units of bit/second/Hz. How many bit/second do I get for each Hz of bandwidth. This is fairly self-explanatory, and obviously we want this to be a high as possible. We have seen E_b/N_0 before. We can call it "normalised average energy/bit" where the normalisation is with respect to the one sided PSD of the noise. It is a sort of signal to noise energy ratio and is often converted to dBs as $10\log_{10}(E_b/N_0)$. We can now write:

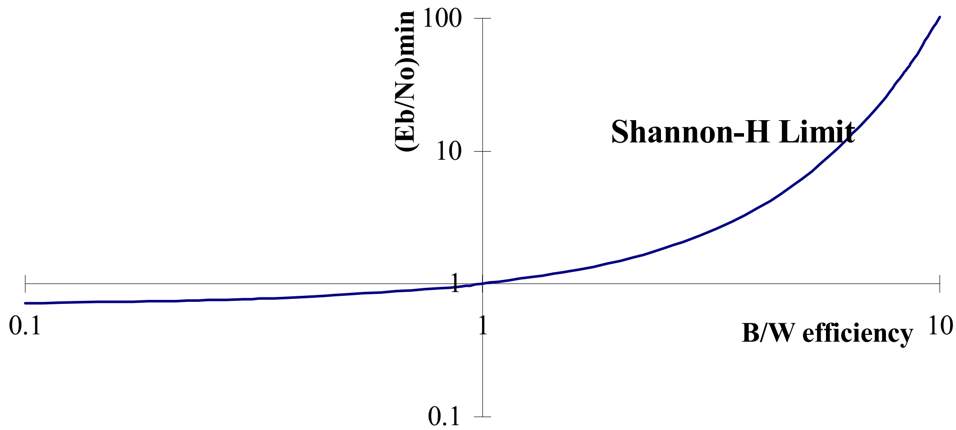
$$2^{R/B} \leq 1 + (E_b / N_0)(R / B)$$

which means that

$$E_b / N_0 \geq \frac{2^{R/B} - 1}{R/B}$$

i.e. $$(E_b / N_0)_{\min} \geq \frac{2^{R/B} - 1}{R/B}$$

This gives the minimum possible normalised energy satisfying the Shannon-H law.
If we draw a graph of $(E_b/N_0)_{min}$ against bandwidth efficiency (R/B) we observe the remarkable result that $(E_b/N_0)_{min}$ never goes less than about 0.69 which is about -1.6 dB. Therefore if our normalised energy per bit is less than -1.6dB, we can never satisfy the Shannon-Hartley law however inefficient (in terms of bit/rate/Hz) we are prepared to be. Above the curve (to the left) gives values of (E_b/N_0) which satisfy the law for given bandwidth efficiency. Here, any BER, however low, can be achieved in theory. Below the curve, to the right, the energy/bit is too low (in relation to N_0) for a given bandwidth efficiency and certain bit-error rates will become unachievable.

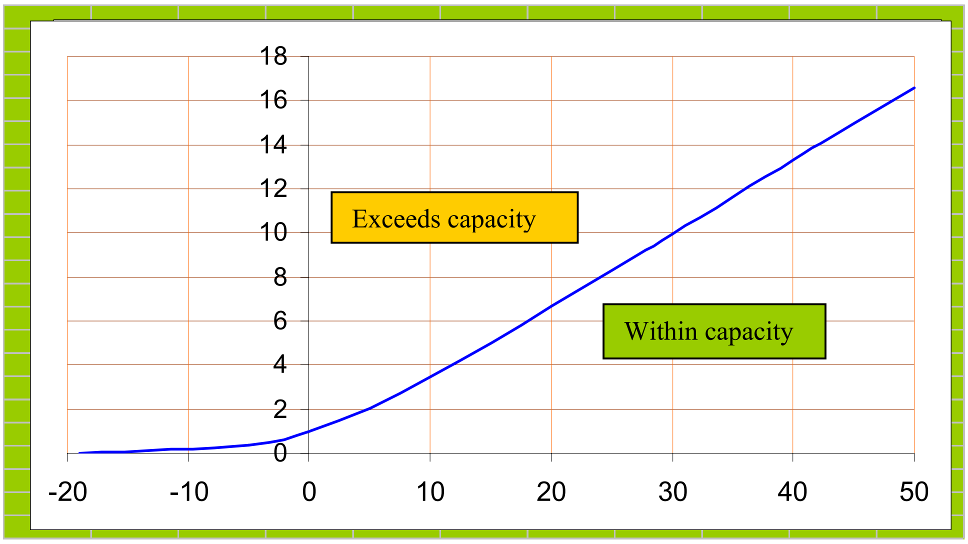


To see this mathematically, note that

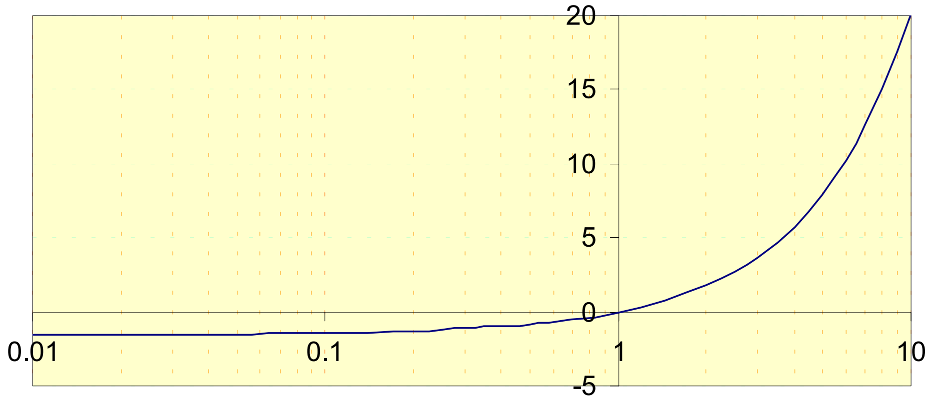
$$2^{R/B} = e^{(\log_e 2) R/B} = e^{0.693 R/B} \approx 1 + 0.693\, R/B$$

when R/B is small. Therefore when R/B is small,

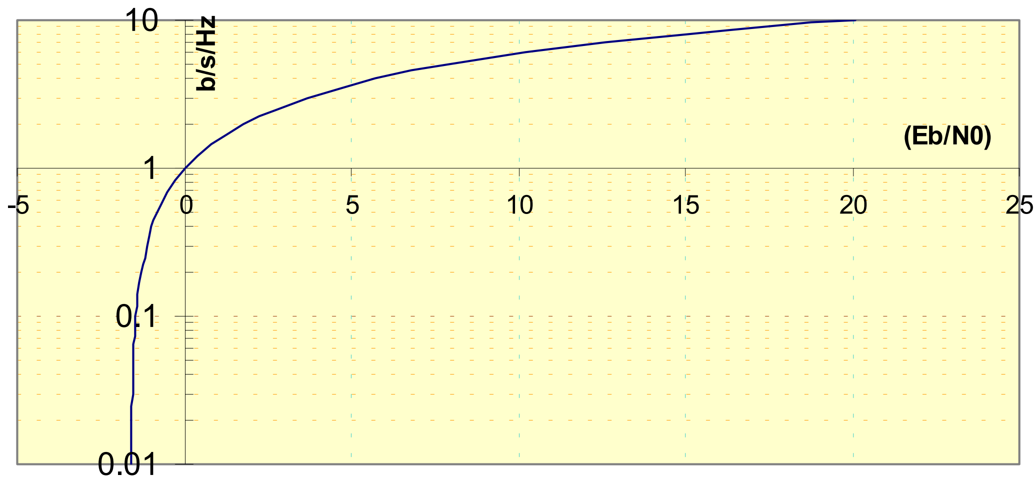
$$(E_b / N_0)_{\min} \approx \frac{1 + 0.69\, R/B - 1}{R/B} = 0.69 = -1.6dB$$



Graph of capacity C (in b/s/Hz) against SNR (in dB):



Graph of $(E_b/N_0)_{min}$ (dB) for a required b/s per Hz



Graph of max achievable b/s per Hz against (E_b/N_0) (dB)

The third graph on the previous page shows the maximum achievable b/s/Hz for a given Eb/N_0. This shows clearly that when E_b/N_0 becomes less than -1.6dB, the max achievable b/s per Hz becomes very low, essentially zero.
So there is no bit-rate, however low, that will achieve arbitrarily low bit-rate when $E_b/N_0 < -1.6$ dB.. Not even 1 b/s per year!!!