University of Manchester

CS3282: Digital Communications '06 sols
Section 2: Notes on the Fourier Transform

2.1. Introduction:

In Communications, the analogue Fourier transform of a time-domain signal, x(t) say, is often expressed in terms of f (frequency in Hz) rather than ω (radians/second). The formulas and the notation are slightly different, though their mathematical significance is identical. To ease the transition from ω to f, and to avoid possible confusion with previous definitions of X(jω) or X(ω), we will, for a while, refer to the Fourier transform of x(t) as X((f)). The formulae are given below. Note the absence of the 1/2π factor at the beginning of the inverse formula.

Fourier Transform: - Given signal x(t) for $-\infty < t < \infty$,

$$X(j\omega) = X((f)) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\,dt$$

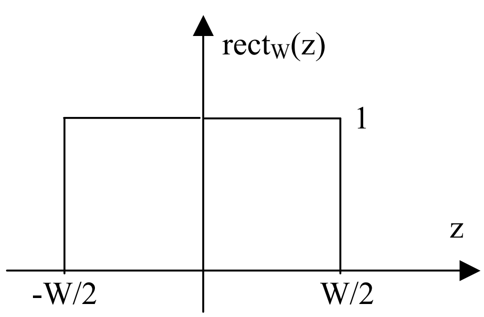
Inverse Fourier Transform: - Given X((f)) for $-\infty < f < \infty$,

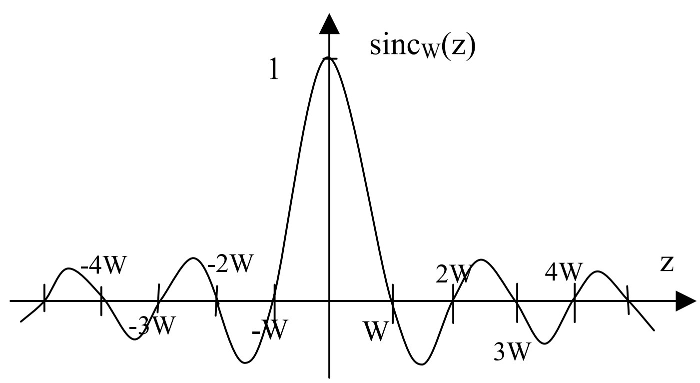
$$x(t) = \int_{-\infty}^{\infty} X((f))e^{2\pi jft}\,df$$

If x(t) is real, X((-f)) = $\overline{X((f))}$ where f is frequency in Hz and $f = \omega/2\pi$

2.2. Functions 'rect' and 'sinc'

Consider the following two functions, $rect_W(z)$ and $sinc_W(z)$, where W is a constant and z is any continuous variable.





We refer to $rect_1(z)$ as rect(z) and $sinc_1(z)$ as sinc(z) where:

$$rect(z) = \begin{cases} 1 & : \ |z| < 0.5 \\ 0.5 & : \ |z| = 0.5 \\ 0 & : \ |z| > 0.5 \end{cases} \qquad \text{sinc}(z) = \begin{cases} \sin(\pi z) \big/ (\pi z) & : \ z \neq 0 \\ 1 & : \ z = 0 \end{cases}$$

Clearly, $rect_W(z) = rect(z/W)$ and $sinc_W(z) = sinc(z/W)$.

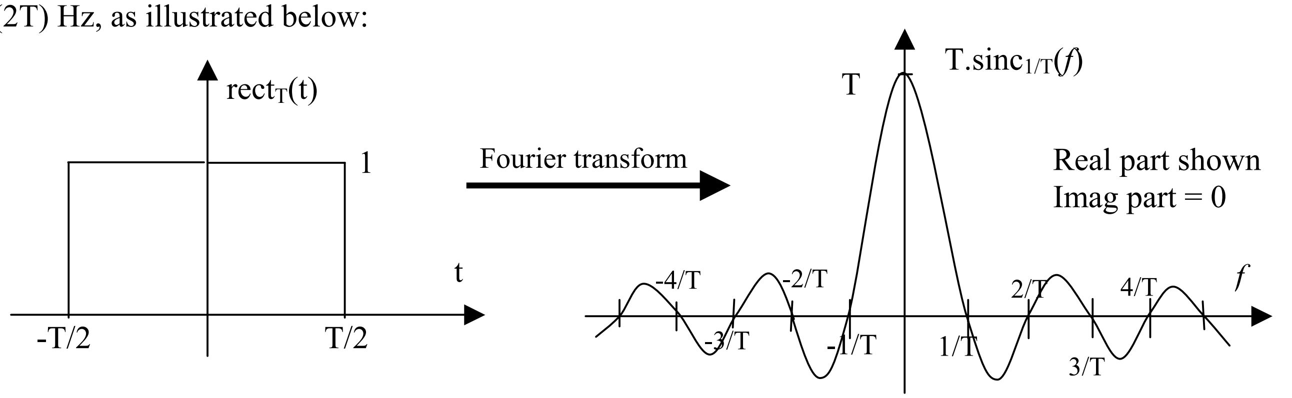
It can be shown that under the Fourier transform:

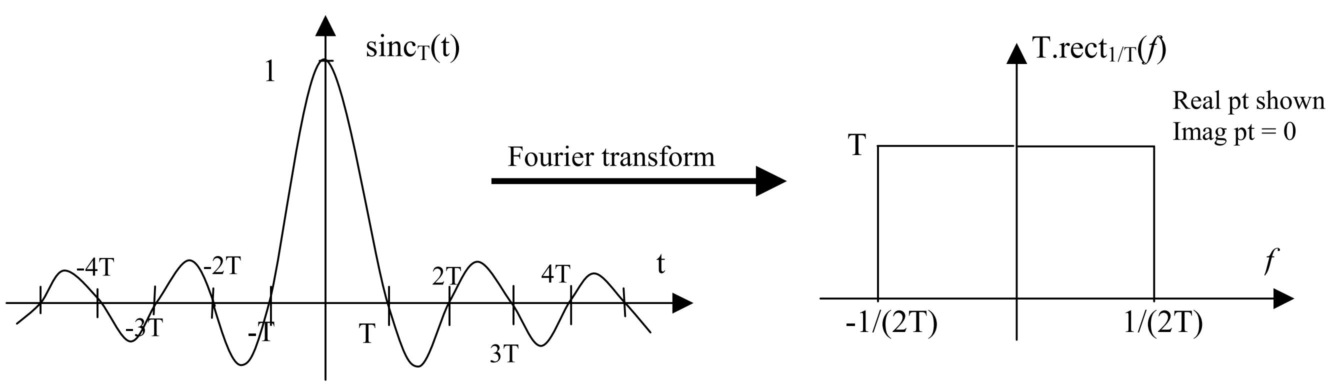
$$rect_T(\, t - D\,)\, e^{2\pi jFt} \longrightarrow T.sinc_{1/T}\,(\, f - F\,)\, e^{-2\pi jfD}$$
$$sinc_T(\, t - D\,)\, e^{2\pi jFt} \longrightarrow T.rect_{1/T}\,(\, f - F)\, e^{-2\pi jfD}$$

The constants D and F are any delay (in seconds) and frequency shift (in Hz) respectively. Taking the simplest case, where D=0, F=0 and W = T, the expressions become:

$$\text{rect}_T(t) \longrightarrow T.\text{sinc}_{1/T}(f)$$
$$\text{sinc}_T(t) \longrightarrow T.\text{rect}_{1/T}(f)$$

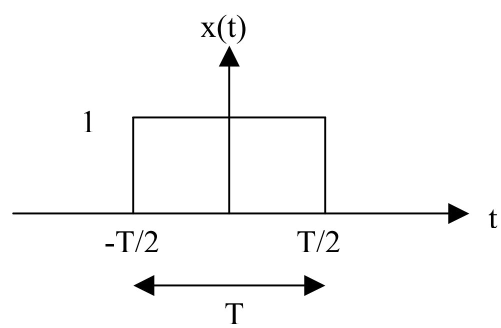
Hence a rectangular time-domain pulse of height one and width T seconds (±T/2) centred on t=0 has a Fourier transform whose real part is a 'sinc' frequency-domain function of height T, 'main lobe' width 2/T Hz centred on f=0 and with zero-crossings at f = ±1/T, ±2/T, ±3/T Hz, etc. Its imaginary part is zero for all f.
A '$\text{sinc}_T(t)$' unit height time-domain pulse centred on t=0 with zero-crossings at t = ±T, ±2T, ±3T, seconds etc., transforms to a '$T.\text{rect}_{1/T}(f)$' frequency-domain function whose value is T between ±1/(2T) Hz, as illustrated below:



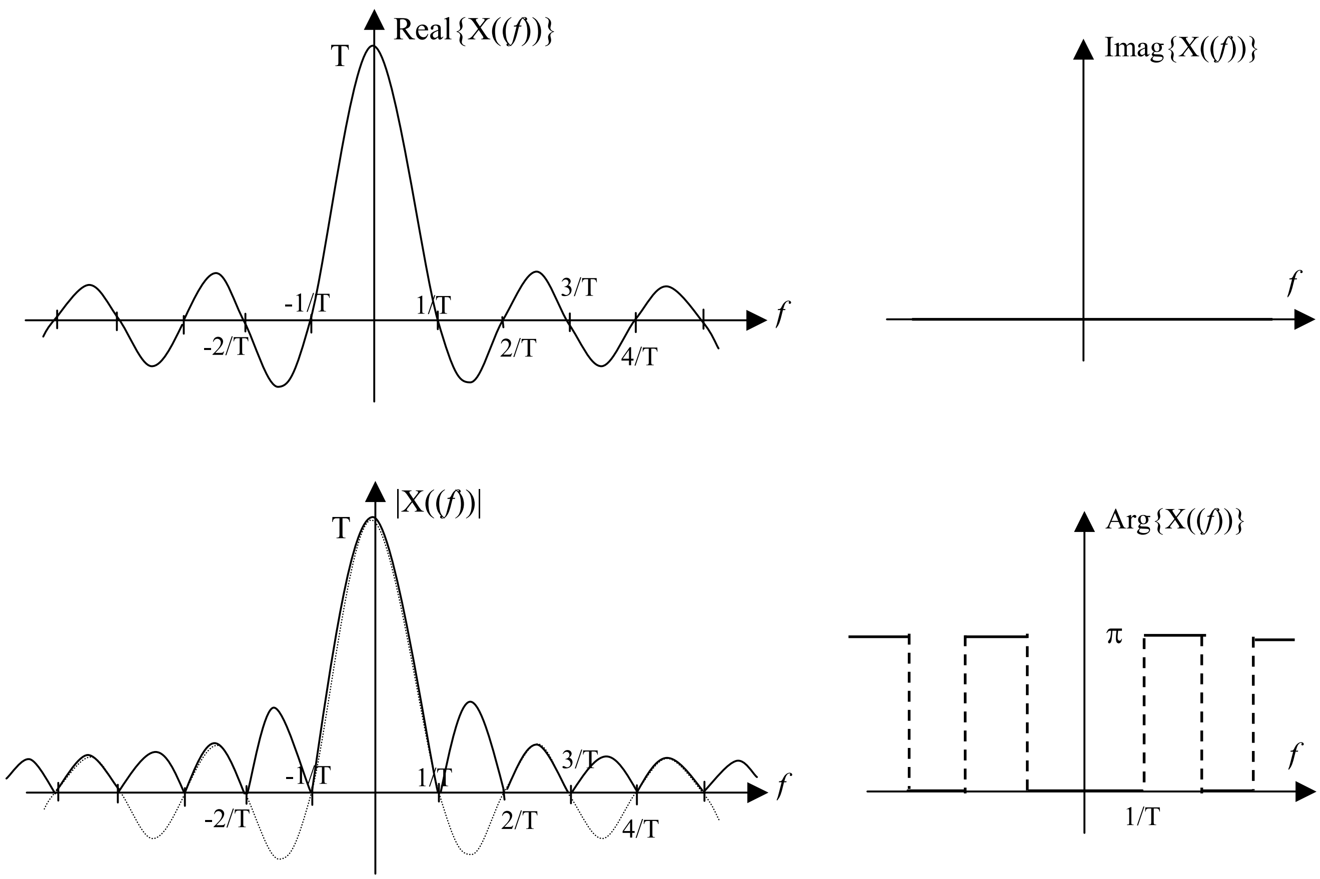


Example 2.1: Find the Fourier transform, in magnitude/phase form, of the 'rectangular pulse' x(t) where:-

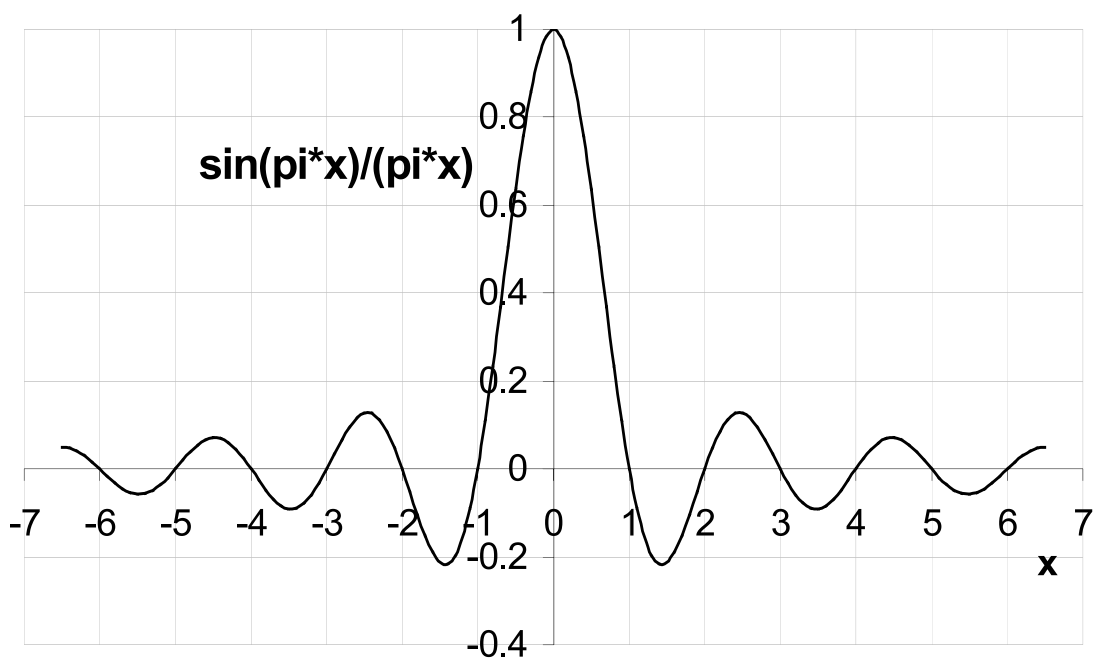
$$x(t) = \begin{cases} 1 & : -T/2 \le t \le T/2 \\ 0 & : elsewhere \end{cases}$$



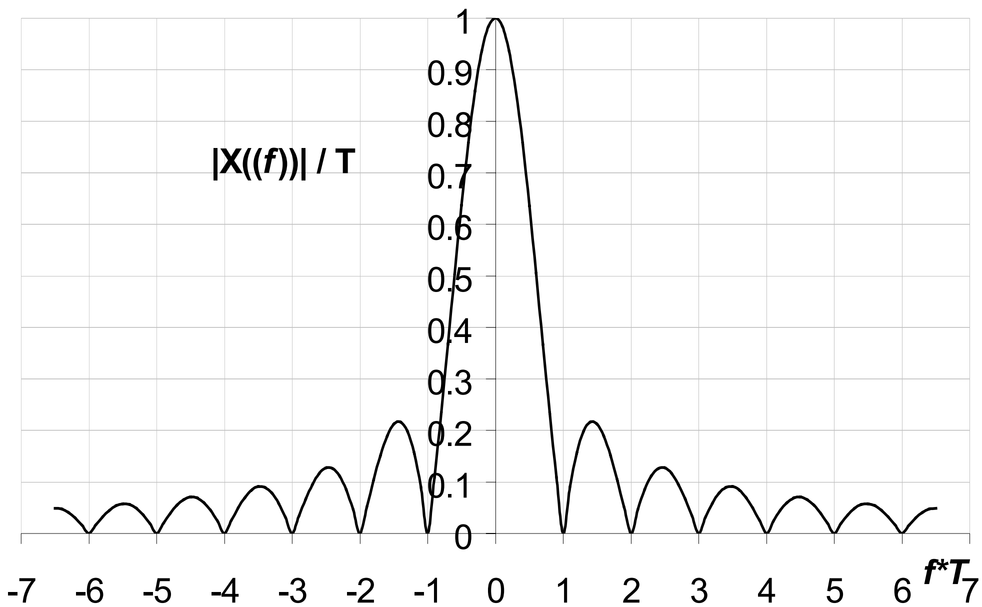
Solution: $x(t) = \text{rect}_T(t)$
Therefore, $X((f)) = T \sin c_{1/T}(f) + j0$ for all f.



Note that $T\text{sinc}_{1/T}(f) = T\ \text{sinc}(fT)$ where $\text{sinc}(x) = \sin(\pi x)/(\pi x)$. Some DSP people define sinc(x) as sin(x)/x when $x \neq 0$, but Communications textbooks tend to use the definition given here. A more accurate graph of sinc(x) against x is shown below. Note the 'zero-crossings at $x = \pm 1, \pm 2, \pm 3, \ldots$



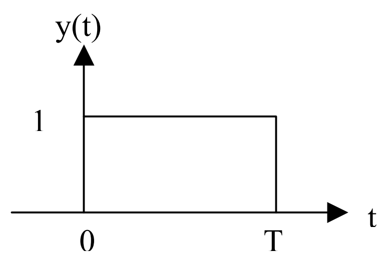
In general, the Fourier transform of a real signal is complex-valued. For the rectangular pulse in Example 2.1, the imaginary part of X((*f*)) is zero for all *f*. This is because the rectangular pulse is symmetric about t=0. The modulus of X((*f*)) is the modulus of a "sinc" function as drawn more accurately below. The phase or argument of X((*f*)) is zero when X((*f*)) > 0 and π radians (180^o) when X((*f*)) < 0. The bandwidth of x(t) is infinite since X((*f*)) has non-zero values outside any given frequency range.



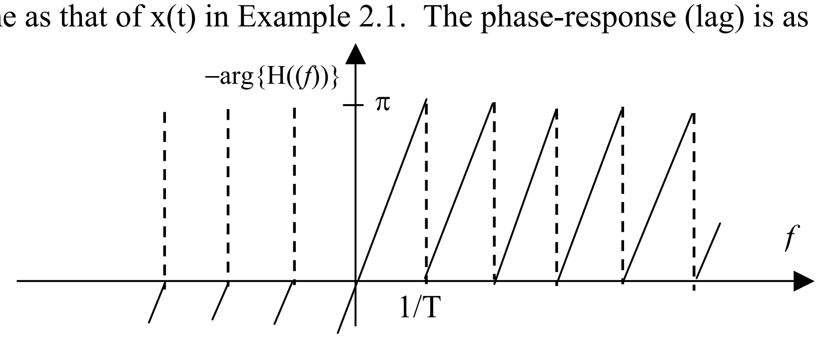
Example 2.2**:**

Calculate in real/imaginary form and in modulus/phase form the spectrum of the following rectangular signal which is not symmetric about t=0.

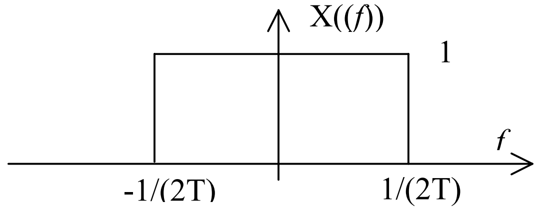
$$y(t) = \begin{cases} 1 & : 0 \le t \le T \\ 0 & : elsewhere \end{cases}$$



Solution: y(t) = $rect_T$(t-T/2) therefore Y((*f*)) = T.sinc(T*f*).$e^{-2\pi j f T/2}$.
The pulse y(t) is the same as x(t) in Example 2.1 except that it is delayed by T/2 seconds.
Y((*f*)) = T.sinc(T*f*)cos(2π*f*T/2) - j T.sinc(T*f*) sin(2π*f*T/2) so this spectrum is not purely real as before.
However |Y((*f*))| = |Tsinc(T*f*)| $|e^{-2\pi j f T/2}|$ = |Tsinc(T*f*)| so the magnitude spectrum of y(t) is exactly the same as that of x(t) in Example 2.1. The phase-response (lag) is as sketched below.

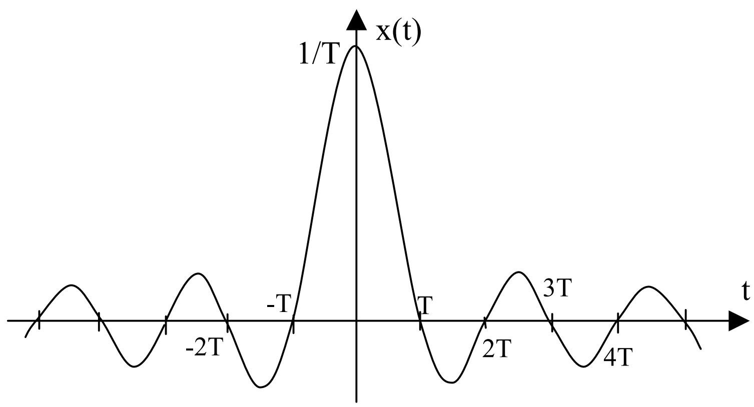


Example 2.3:
Calculate x(t) when X((*f*)) has the real part shown below and zero imaginary part for all *f* :



Solution:

$$x(t) = \frac{1}{T}\sin c_T(t) \;\; = \frac{1}{T}\sin c\left(\frac{t}{T}\right)$$

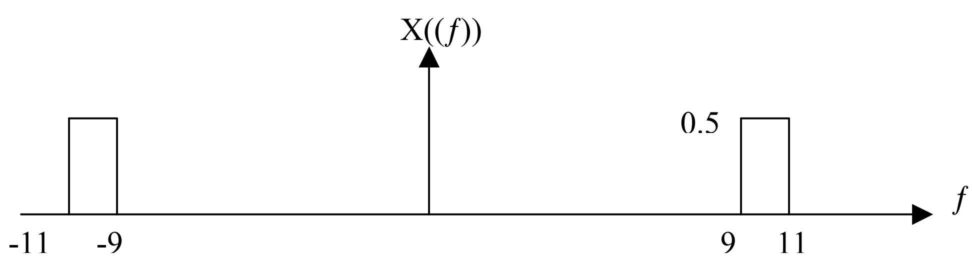


In Examples 2.1 & 2.2, the signal is of finite time duration and has infinite bandwidth. In Example 2.3, the signal is strictly band-limited and has infinite time duration. It is worth remembering that under the inverse Fourier transform

$$(1/T).\text{rect}_T(\, t - D\,)\, e^{+2\pi jFt} \longleftarrow \text{sinc}_{1/T}\,(\, f - F\,)\, e^{-2\pi jfD}$$

$$(1/T).\text{sinc}_T(\, t - D\,)\, e^{+2\pi jFt} \longleftarrow \text{rect}_{1/T}\,(\, f - F)\, e^{-2\pi jfD}$$

Example 2.4: Calculate the time-domain waveform whose Fourier transform has the real part sketched below with imaginary part zero:



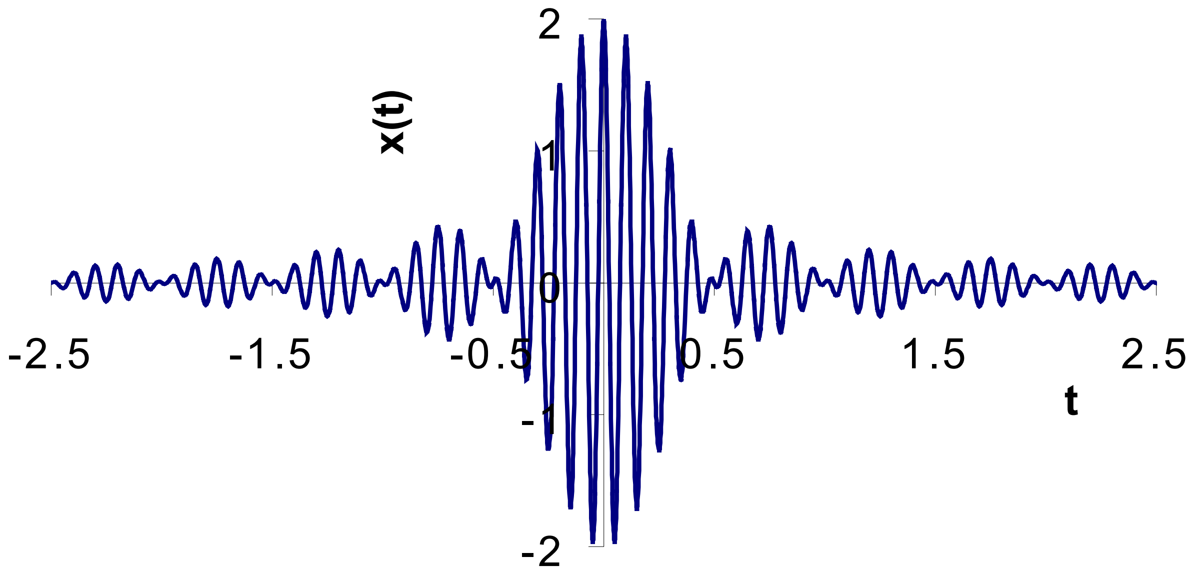
Solution:

$X((f)) = 0.5\text{rect}_2(f+10) + 0.5\text{rect}_2(f-10)$ $\;\therefore x(t) = \text{sinc}_{0.5}(t)e^{2\pi j10t} + \text{sinc}_{0.5}(t)e^{-2\pi j10t}$

$$= \text{sinc}(t/0.5)\,[\, e^{2\pi j10t} + e^{-2\pi j10t}]$$

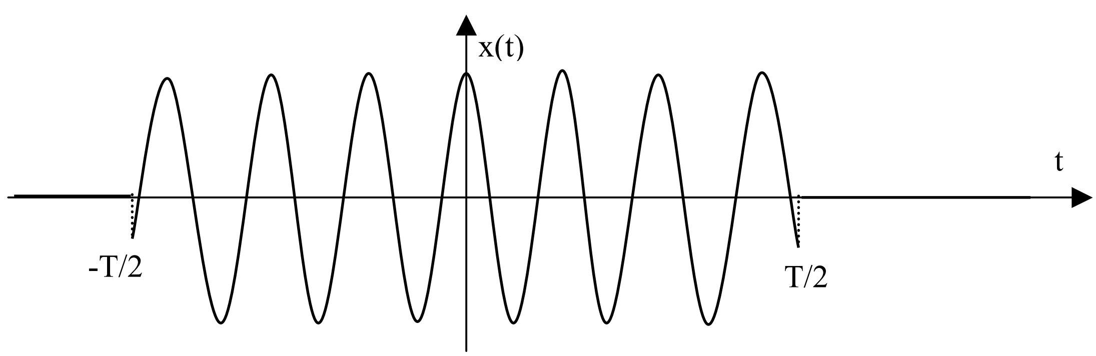
$$= \text{sinc}(2t)\,[\, 2\cos(20\pi t)\,] \;\;= 2\,\text{sinc}(2t)\cos(20\pi t)$$

This is a carrier sinusoid of frequency 10Hz whose amplitude is modulated by 2 sinc(2t). Zero crossings of the sinc shaped envelope occur at t = ± 0.5, ±1, ±1.5, …. Note again that |X((*f*))| is strictly bandlimited and x(t) is of infinite time duration.

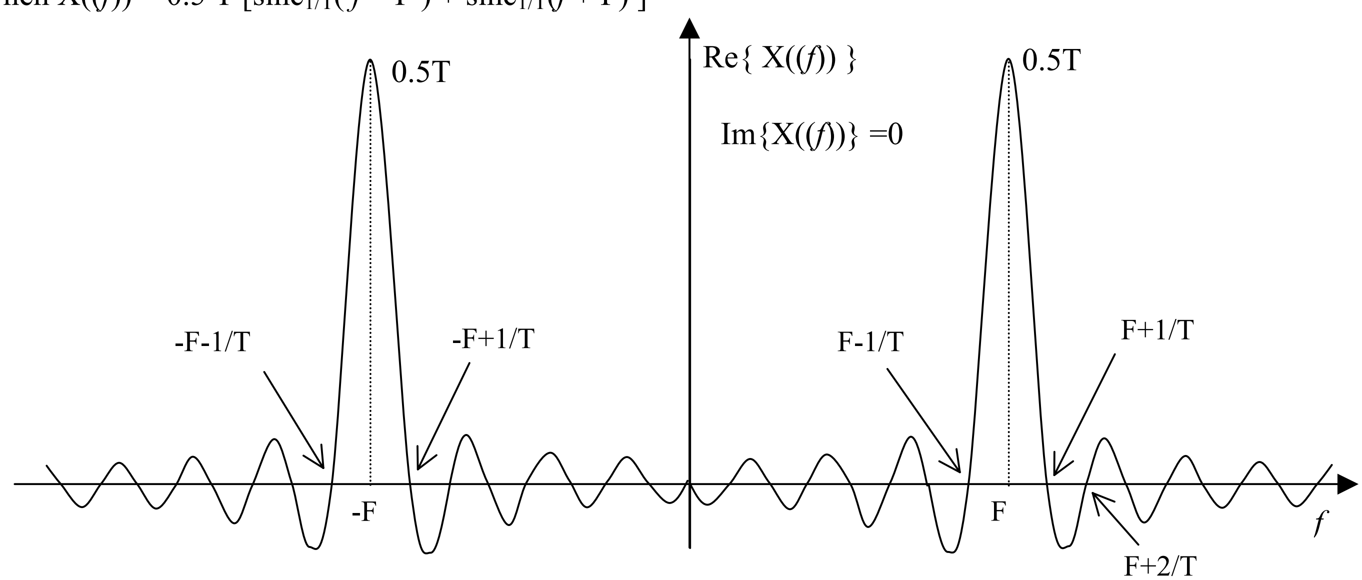


Example 2.5: Fourier Transform of rectangularly windowed sinusoid of frequency F Hz

Given $x(t) = \text{rect}_T(t)\cos(2\pi Ft) = 0.5\,\text{rect}_T(t)\,\{e^{2\pi jFt} + e^{-2\pi jFt}\}$

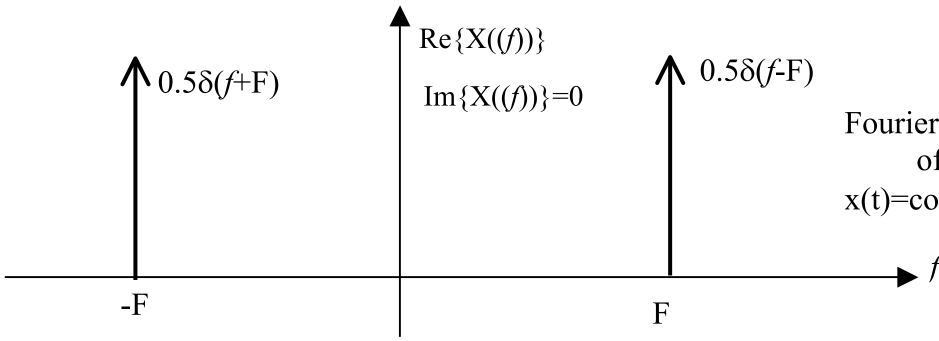


Then $X((f)) = 0.5\,T\,[\text{sinc}_{1/T}(f - F) + \text{sinc}_{1/T}(f + F)]$



Example 2.6: Fourier Transform of un-windowed sinusoid of frequency F Hz

If T is made very large, the 0.5 $T.\text{sinc}_{1/T}(f)$ peak becomes very large, and (F±1/T), (F±2/T), etc. get closer and closer to F. As $T \longrightarrow \infty$, $T.\text{sinc}_{1/T}(f) \longrightarrow \delta(f)$ where $\delta(f)$ is a Dirac delta function of frequency (sometimes called an impulse of frequency). As $T \longrightarrow \infty$, we are getting closer and closer to an un-windowed cosine wave. Therefore we define the Fourier transform of an un-windowed cosine wave: $x(t) = \cos(2\pi Ft)$ as $X((f)) = 0.5\delta(f\text{-}F) + 0.5\,\delta(f+F)$.



The Dirac delta function δ(t) of time or δ(f) of frequency
Two properties: -

(i) $\delta(t) = 0$ for $t \neq 0$ and (ii) $\int_{-\infty}^{\infty} \delta(t)dt = 1$ Can replace t by f.

2.3. Multiplication and Convolution

Given $x_1(t) \rightarrow X_1((f))$ & $x_2(t) \rightarrow X_2((f))$ by the Fourier transform.

What is the Fourier transform, $P((f))$ say, of the time-domain **product** of $x_1(t)$ and $x_2(t)$?

$$P((f)) = X_1((f)) \otimes X_2((f))$$
$$= \int_{-\infty}^{\infty} X_1((p))X_2((f-p))dp = \int_{-\infty}^{\infty} X_2((p))X_1((f-p))dp$$
$$(\textit{Frequency domain 'complex' convolution})$$

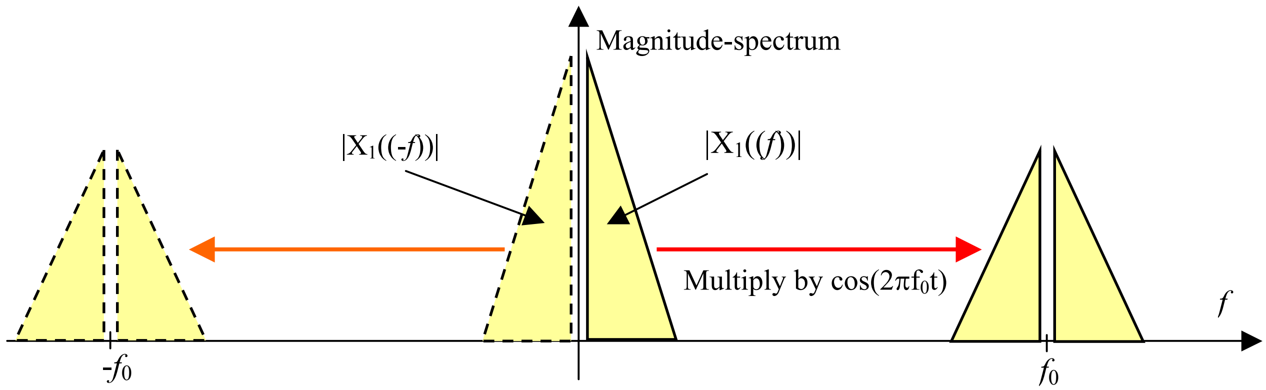
What is the Fourier transform, $C((f))$ say, of the time-domain **convolution** between $x_1(t)$ and $x_2(t)$:

$$C(f) = X_1((f)).X_2((f))$$
$$(\textit{Frequency domain product})$$
$$\left(x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1((\tau))x_2((t-\tau))d\tau = \int_{-\infty}^{\infty} x_2((\tau))x_1((t-\tau))d\tau \right)$$

Time-domain	Frequency-domain
Product (modulation or windowing)	Complex convolution
Convolution (filtering)	Product of Fourier transforms

Example 2.7: Multiplication of any signal $x_1(t)$, with Fourier Transform $X_1((f))$, by $\cos(2\pi f_o t)$.
By method in Example 2.5, spectrum is $0.5X_1((f-f_0)) + 0.5X_1((f+f_0))$. So easy.
To get the same answer by convolution, let $x_2(t) = \cos(2\pi f_o t)$. Then

$$X_1(f) \otimes X_2(f) = \int_{-\infty}^{\infty} X_2((p)) X_1((f-p)) dp$$
$$= \int_{-\infty}^{\infty} \left(0.5\delta(p-f_0) + 0.5\delta(p+f_0)\right) X_1((f-p)) dp$$
$$= \int_{-\infty}^{\infty} 0.5\delta(p-f_0) X_1((f-p)) dp + \int_{-\infty}^{\infty} 0.5\delta(p+f_0) X_1((f+p)) dp$$
$$= \underline{\underline{0.5X_1((f-f_0)) + 0.5X_1((f+f_0))}}$$



Remember: If $H((f))$ is the frequency response of a filter, the inverse FT of $H((f))$, call it $h(t)$, is the impulse-response of the filter. If the input to the filter is a signal $x(t)$ with FT $X((f))$ then the output is the signal: $y(t) = x(t) \otimes h(t)$ with Fourier Transform $Y((f)) = X((f))H((f))$

A filter performs time-domain convolution (with its impulse-response) which is equivalent to frequency-domain multiplication (by its frequency-response).

2.4. Complex Fourier Series:

A periodic signal of period P seconds may be expressed as a Fourier series in three ways:

2.4.1. Complex form of Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi j (n f_0) t} \qquad f_0 = \frac{1}{P} Hz$$

$$\text{where} \qquad C_n = \frac{1}{P} \int_{-P/2}^{P/2} x(t) e^{-2\pi j (n f_0) t} dt \qquad \text{(complex valued)}$$

If $x(t)$ is real, then C_0 must be real and $C_n = C_{-n}^*$ for all n where * denote complex conjugate.

2.4.2. Sine/cosine form of Fourier series:

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t) \right) \text{ where } f_0 = 1/P \text{ Hz}$$

Relationship to complex form: $A_0 = C_0$, $A_n = 2.Real(C_n)$, $B_n = 2.Imag(C_n)$ for n = 1, 2,…

2.4.3. Modulus& phase form of Fourier series:

$$x(t) = \sum_{n=0}^{\infty} M_n \cos(2\pi f_0 t + \phi_n) \qquad \text{where } f_0 = 1/P \text{ Hz}$$

Relationship to complex form: $M_0 = |C_0|$ and $\phi_0 = \arg(C_0)$ normally 0 or π

$$M_n = 2|C_n| \text{ and } \phi_n = \arg(C_n) \text{ for } n = 1, 2, \ldots$$

Example 2.8: Prove the relationship between the different forms of Fourier series coefficients:
Solution: Let $C_n = R_n + jI_n$, then:

$$x(t) = \sum_{n=-\infty}^{\infty}(R_n + jI_n)\left(\cos(2\pi nf_0t) + j\sin(2\pi nf_0t)\right)$$

$$= R_0 + jI_0 + \sum_{n=1}^{\infty}\left((R_n + R_{-n} + jI_n + jI_{-n})\cos(2\pi nf_0t) + (jR_n - jR_{-n} - I_n + I_{-n})\sin(2\pi nf_0t)\right)$$

$$= R_0 + jI_0 + \sum_{n=1}^{\infty}\left(2R_n\cos(2\pi nf_0t) - 2I_n\sin(2\pi nf_0t)\right)$$

Since if $C_n = \overline{C}_{-n}$ for all n, $I_{-n} = -I_n$, $R_n = R_{-n}$ and $I_0 = 0$. It follows that

$$x(t) = R_0 + \sum_{n=1}^{\infty}\left(M_n\cos(2\pi nf_0t + \phi_n)\right)$$

where $M_n = \sqrt{(2R_n)^2 + (2I_n)^2}$ and $\tan(\phi_n) = -I_n / R_n$.

2.4.4. 'Repeat' and 'Sample':

Given the signal x(t) with Fourier transform X((*f*)) define:

$$\text{repeat}_P(\,x(t)\,) = \sum_{n=-\infty}^{\infty} x(t - nP)$$

$$\text{sample}_P(\,x(t)\,) = x(t)s_P(t) \quad \text{where} \quad s_P(t) = \sum_{n=-\infty}^{\infty}\delta(t - nP)$$

$$\text{repeat}_F(\,X(f)\,) = \sum_{n=-\infty}^{\infty} X(f - nF) \quad \&$$

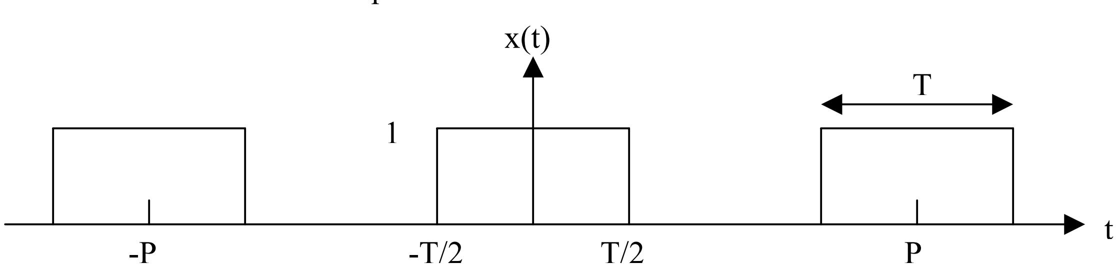
$$\text{sample}_F(\,X(f)\,) = X(f)S_F(f) \text{ where } S_F(f) = \sum_{n=-\infty}^{\infty}\delta(f - nF)$$

The '$\text{repeat}_P(x(t))$' function produces a periodic signal by repeating x(t) at intervals of P seconds.
The '$\text{sample}_P(x(t))$' function produces a periodic series of impulses (Dirac deltas) at intervals of P seconds, each impulse being weighted by the sample of x(t) occurring at the point of the impulse.
These functions may be applied in the time-domain or the frequency-domain, and it may be shown that under the Fourier transform:

$$\text{repeat}_P(x(t)) \longrightarrow (1/P)\text{sample}_{1/P}(\,X(f)\,)$$
$$\text{sample}_P(x(t)) \longrightarrow (1/P)\text{repeat}_{1/P}(\,X(f)\,)$$

These Fourier transform relationships can be used to derive the Fourier series for $\text{repeat}_P(x(t))$ by taking the impulse strengths as the values of C_n. This may be thought of as replacing the impulses $\delta(t-nP)$ by unit length lines.

Example 2.9: Calculate the Fourier series for the following periodic signal. This is periodic with period P seconds. It is not a square-wave unless P=2T.



Solution:

$$C_n = \frac{1}{P}\int_{-T/2}^{T/2} 1 \bullet e^{-2\pi j(nf_0)t} dt \quad with\ f_0 = 1/P$$

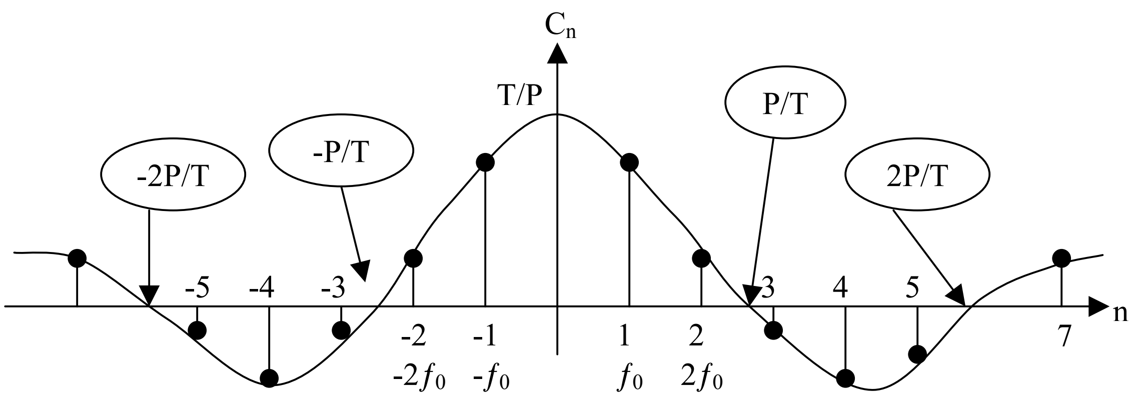
$$= \frac{1}{-2\pi jnf_0 P}\left[e^{-2\pi jnf_0 t}\right]_{-T/2}^{T/2} \quad when\ n \neq 0$$

$$= \frac{1}{\pi nf_0 P}\sin(\pi nf_0 T) = \frac{T}{P}\sin c(nf_0 T) \quad n \neq 0$$

When n = 0 it is easily confirmed that C_n = T/P. Therefore,

$$C_n = \frac{T}{P}\sin c\left(\frac{nT}{P}\right) \quad for -\infty < n < \infty$$

It is common to plot the real and imaginary parts of C_n against n as a "line spectral graph". Such a graph must not be confused with a Fourier spectrum though there are clear similarities.



In this case, C_n is real for all n but in general there may be non-zero real and imaginary parts.

Compare the line spectral graph above for C_n against n with the spectral graph of $T\text{sinc}_{1/T}(f)$ (= Tsinc(Tf)) against f obtained for the Fourier transform of the single pulse specified by Example 2.1. Repeating this single pulse at intervals of P seconds gives a "line spectral graph" for C_n where the length of each line is a sample of $T\text{sinc}_{1/T}(f)$ (=Tsinc(Tf)) scaled by 1/P.

Relationship between Fourier series of periodic pulse-train and Fourier transform of original pulse:

Given an analogue signal x(t) with Fourier transform X((f)), if x(t) is repeated periodically at intervals P(=1/f_0) seconds in the time-domain, the resulting periodic signal has a Fourier series whose complex coefficients C_n for n = 0, ±1, ±2, ±3, … are samples of (1/P) X((f)) at f = 0, ±f_o, ±2f_0, ±3f_0, … in the frequency-domain.

The Fourier series coefficients C_n are often plotted against n as a "line-spectrum", though this must not be confused with a Fourier transform spectrum. To convert the line spectrum for a Fourier series to a Fourier transform spectrum, each line for C_n becomes an impulse of strength equal to C_n (normally complex) at frequency nf_0 Hz. This is because the Fourier transform of $C_n\ e^{2\pi jnf0t}$ is $C_n\ \delta(f - nf_0)$ as observed earlier.

Therefore a reasonable strategy for finding the Fourier transform of a periodic signal is first to find its complex Fourier series, draw its line-spectrum and then replace each line of strength C_n by an impulse of strength C_n.

Exercise 2.10**:** Draw the line-spectrum and the Fourier transform spectrum for the pulse sequence in Exercise 2.4 when P = 2T. The pulse sequence becomes a square-wave. What happens to the even and odd harmonics?

Solution:

$C_n = 0.5\text{sinc}(n/2)$ which is zero for n = ±2, ±4, ±6, ±8,… All even harmonics are zero.

Since $C_n = \dfrac{0.5\sin(\pi n/2)}{(\pi n/2)}$ then $C_n = 0.5, \frac{1}{\pi}, -\frac{1}{3\pi}, \frac{1}{5\pi}, -\frac{1}{7\pi}, ...$ when n = 0, ±1, ±3, ±5, ±7, ...

Hence we can draw a line spectrum for the real parts of the Fourier series coefficients, noting that the imaginary part of C_n is zero for all n.

The real part of the Fourier transform spectrum X((*f*)) is similar except that the spectral lines at n = 0, ±1, ±3, etc. are replaced by impulses of strength equal to 0.5 at *f* = 0, 1/π at *f* = ±1/P, $-1/(3\pi)$, $1/(5\pi)$, $-1(7\pi)$,… at *f* = ±3/P, etc. and the value of X((*f*)) between these frequencies is zero. The imaginary part of X((*f*)) is zero for all *f* in this example.

2.5. Definitions of energy and power

For an analogue signal x(t):

$$\text{Energy} = \int_{-\infty}^{\infty} [x(t)]^2\, dt$$

This is the total energy, in Joules, that would be delivered to a 1 Ohm resistor if a voltage of value x(t) were applied to it for all time. Therefore the units are "Joules relative to 1 Ohm".

To define the power of x(t) is a bit more tricky. Let $x_D(t)$ be defined as follows for any constant D:

$$x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ \quad 0 \;:\; otherwise \end{cases}$$

$$\text{Power} = \lim_{D\to\infty} \frac{1}{D}\int_{-\infty}^{\infty} [x_D(t)]^2 dt$$

This is the average power in Watts that would be dissipated (probably as heat) by a 1 Ω resistor when a voltage x(t) is applied for all time. The units are "Watts relative to 1 Ω".

Some textbooks refer to the units simply as squared-volts.

If x(t) is periodic with period D, the "lim" in the power definition above can be omitted.

2.6. 'Energy-signals' and 'power-signals'

It is useful to refer to two types of signal:

1. *'Energy-signal'* or *'finite energy signal'*: has finite energy and zero average power. An example is a single pulse.

2. *'Power-signal'* or *'finite power signal'*: has finite average power and therefore infinite energy; e.g. a sine wave or a pulse sequence.

2.7. Parseval's Theorem (in two forms)

For a 'finite energy' signal, x(t), it may be shown that its energy is:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X((f))|^2 \, df$$

This means that the energy of a finite energy signal can be calculated either directly in the time-domain, or in the frequency-domain from a knowledge of X((*f*)). The theorem is valid even if x(t) is a complex signal.

For a 'finite power' signal, x(t), define:

$$X_D((f)) = Fourier\ Transform\ \ of\ \ x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ 0 \quad : otherwise \end{cases}$$

which has finite energy for any constant D and therefore has a straightforward Fourier transform.
A second version of Parseval's theorem exists proving that under reasonable assumptions:

$$\int_{-\infty}^{\infty} \left(\lim_{D\to\infty} \frac{1}{D} |X_D((f))|^2 \ \right) df = \lim_{D\to\infty} \frac{1}{D} \int_{-\infty}^{\infty} |X_D((f))|^2 \, df = \lim_{D\to\infty} \frac{1}{D} \int_{-\infty}^{\infty} |x_D(t)|^2 \, dt$$

Referring to the definition earlier, this is the power of x(t).
So now we can calculate the power of x(t) from a knowledge of the frequency-domain function:

$$P(f) = \lim_{D\to\infty} \frac{1}{D} \left| X_D((f)) \right|^2$$

The power of x(t) is the integral of P(*f*) over $-\infty < f < \infty$.

<u>Significance of Parseval's theorem:</u>

Let's say we now wanted to calculate the energy of a filtered version of x(t) where the filter is a band-pass filter centred on the frequency f_0 Hz. Let the filter be an ideal band-pass filter with cut-off frequencies f_L and f_U Hz. The Fourier transform of the filtered signal is:

$$X_F((f)) = \begin{cases} X((f)) \ : \ f_L \le f \le f_U \\ X((f)) \ : \ -f_U \le f \le -f_L \\ 0 \quad : \quad otherwise \end{cases}$$

and by Parseval's theorem, the energy of the filtered signal is:

$$\int_{-\infty}^{\infty} |X_F((f))|^2 \, df$$

Now this is clearly equal to:

$$\int_{-f_U}^{-f_L} |X((f))|^2 \, df + \int_{f_L}^{f_U} |X((f))|^2 \, df = \int_{f_L}^{f_U} 2\,|X((f))|^2 \, df$$

since it is easily shown that

$$|X((f))|^2 = |X((-f))|^2$$

So to calculate the energy of x(t) in the band from f_L to f_U, we simply have to integrate $|X((f))|^2$ between $-f_U$ and $-f_L$ and between f_L and f_U. Equivalently, we can just integrate $2|X((f))|^2$ between f_L and f_U.

Energy per Hz: If we make the ideal band-pass filter considered above a 1 Hz bandwidth filter centred on f_0 Hz, with $f_L = f_0 - 0.5$ and $f_U = f_0 + 0.5$ and assume that $X((f))$ does not change significantly over this 1Hz band, we obtain an energy of $2|X((f_0))|^2$.
Hence we can consider $2|X((f))|^2$ to be the "1-sided energy per Hz " at frequencies close to any frequency f. Its units are Joules.
The "2-sided energy per Hz" is $|X((f))|^2$ Joules.
If we want to observe how the energy of x(t) is distributed in the frequency-domain, we can sweep f_0 from zero to infinity and observe how the energy in the 1Hz bands centred on f_0 varies.
(A bandwidth of 1Hz around f_0 has been chosen to simplify the argument but we could have chosen a narrower bandwidth, if 1Hz were too wide for $X((f))$ to be considered constant.)

Power per Hz: The argument above can be applied to power signals. The "1-sided power per Hz" is $2P(f)$ as defined earlier for a power signal. The "2-sided power per Hz" is $P(f)$. Units are Watts.
To calculate the power of x(t) in the band from f_L to f_U, we simply have to integrate $P(f)$ between $-f_U$ and $-f_L$ and between f_L and f_U. Equivalently, we can just integrate $2P(f)$ between f_L and f_U.

2.8. Energy and power spectral density

We define $|X((f))|^2$ as the two-sided "energy spectral density" (ESD). Its units are energy (in Joules relative to 1Ω) per Hz. We define $2|X((f))|^2$ as the one-sided energy spectral density (ESD) again in Joules per Hz. It is common to refer to $10\log_{10}(|X((f))|^2)$ as the "2-sided ESD" in dB.

Under reasonable assumptions, we can define:

$$P((f)) = \lim_{D \to \infty} \frac{1}{D} \left| X_D((f)) \right|^2$$

as the "2-sided power spectral density (PSD) of the power signal x(t). The units of $P((f))$ are Watts (relative to 1 Ohm) per Hz. The '1-sided' PSD is $2P(f)$ Watt/Hz. Define $PSD(f) = P(f)$.
Note the meaning of "density" and the concepts of "energy per Hz" and "power per Hz".

Effect of filtering on ESD & PSD:

If a 'finite energy' signal x(t) with 2-sided ESD $E(f)$ is passed through a filter with frequency-response $H((f))$, the output is a finite energy signal with ESD = $E(f)|H((f))|^2$ Joules/Hz

If a 'finite power' signal x(t) with 2-sided PSD $P(f)$ is passed through a filter with frequency-response $H((f))$, the output is a finite power signal with PSD = $P(f)|H((f))|^2$ Watts/Hz

2.9. Auto-correlation function

For an energy signal x(t):

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

Clearly,

$$R_x(\tau) = R_x(-\tau) = \int_{-\infty}^{\infty} x(\xi)x(\xi-\tau)d\xi \qquad \text{(Replacing t by } \xi \text{ has no effect).}$$

For a power signal x(t):

$$R_x(\tau) = \lim_{D\to\infty} \frac{1}{D}\int_{-D/2}^{D/2} x(t)x(t+\tau)dt$$

In most cases we can also say that $R_x(\tau) = R_x(-\tau)$ for power signals, though it is possible to conceive of cases where this is not true. It is true for periodic signals and many others which are "wide-sense stationary" (WSS). If x(t) is WSS, its mean and auto-correlation function do not change if the time-origin changes; for such a signal, $R_x(\tau) = R_x(-\tau)$ and the maximum value of $R_x(\tau)$ occurs at $\tau=0$ and equals the energy or power.

$R_x(\tau)$ is always a function of delay τ measured in seconds. For any given value of "delay" τ, this gives a measure of the similarity between x(t) and $x(t+\tau)$ which is x(t) advanced in time by τ seconds. The correlation or cross-correlation function between two energy signals x(t) and y(t) is:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$

Therefore the auto-correlation function of x(t) is the cross-correlation function between x(t) and itself.
N.B. $R_{xy}(\tau) \neq R_{yx}(\tau)$. In fact, $R_{xy}(\tau) = R_{yx}(-\tau)$.
Note the similarity between $R_{xy}(\tau)$ and the formula for time-domain convolution:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(\xi)y(\xi+\tau)d\xi \quad \text{and} \quad x(t)\otimes y(t) = \int_{-\infty}^{\infty} x(\xi)y(t-\xi)d\xi$$

Therefore,

$$R_{xy}(-\tau) = \int_{-\infty}^{\infty} x(\xi)y(\xi-\tau)d\xi \qquad \text{and} \qquad x(t)\otimes y(-t) = \int_{-\infty}^{\infty} x(\xi)y(\xi-t)d\xi$$

Therefore replacing time t by delay τ the convolution between x(t) and y(-t), i.e. between x(t) and a time-reversed version of y(t), is equal to the cross-correlation between y(t) and x(t). Equivalently, the convolution between x(t) and y(t) is equal to the cross-correlation between y(t) and x(-t). Need to be careful with signs of τ. Time-reversing one of the signals turns convolution into cross-correlation and vice-versa.

2.10. Weiner-Khinchine Theorem

For a "finite energy" signal, the '2-sided' ESD E((f)) is the Fourier transform of $R_x(t)$.

For a "finite power" signal, the '2-sided' PSD P((f)) is the Fourier transform of $R_x(t)$

2.11. Random Signals

When examining a signal, x(t) say, without exact periodicity or an obvious structure, it is sometimes useful to assume that at any time t, the value of x(t) is a sample of a random variable, X, with statistical properties defined by a "probability density function"(PDF). In this case, x(t) is considered to be a "random" signal.
The PDF, $p_x(x)$, of a random variable X is defined as follows:

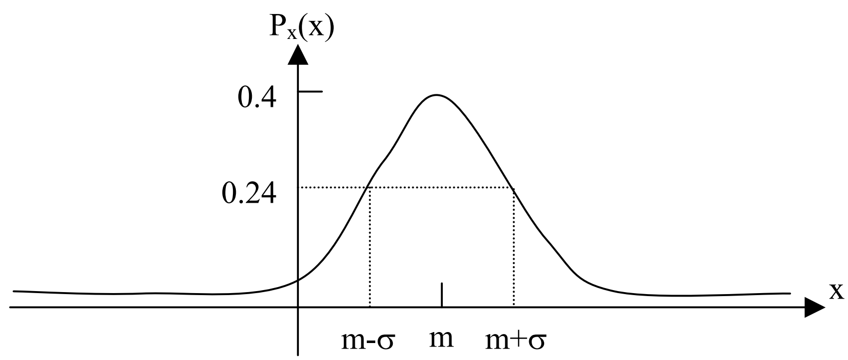
$$p_X(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr ob(x < X < x + \Delta)$$
$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left(\Pr ob(X < x + \Delta) - \Pr ob(X < x) \right) = \frac{d}{dx} \Pr ob(X < x)$$

Given any real number, it gives you a "probability density" telling you how likely you are to see samples of the random variable, X, close to that real number. More precisely, given two real numbers, a and b say, with a < b, the probability of any observation of the random variable lying between a and b is:

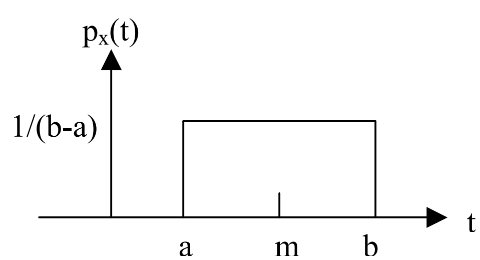
$$\Pr ob(a < X < b) = \int_a^b p_X(x)dx$$

Two commonly encountered PDFs are as follows:

1. Gaussian (normal) with mean = m and "standard deviation" $\left(\sqrt{\text{var}\, iance}\right) = \sigma$: -
   (Look up formula in a textbook)



2. Uniform between x = a and x = b: -
   m =(b+a)/2



From $p_X(x)$ we can deduce the most commonly encountered statistical measurements of the random variable X such as its mean, mean-square value and variance.

<u>Exercise 2.12</u>: Give the formulae for these measurements in terms of $p_X(x)$.

<u>Solution</u>: Exercise

If each value of x(t) is considered to be a different observation of the random variable X, we can estimate the statistical properties of X from these different observations. For example, the average value of x(t) over a period of time would be an estimate of the mean of the random variable X. Similarly, the average of $(x(t))^2$ would be an estimate of the mean-square value of X, and the average value of $(x(t)^2$ minus the average value of x(t) would be an estimate of the variance of X. Signals for which time averages produce reasonable and sensible estimates of statistical averages (for useful statistical descriptions of the signals in question) are said to be "ergodic" signals.

Example 2.13: How are the "dc offset", power and RMS value of x(t) related to the mean, mean-square value, standard deviation and variance of X, assuming that x(t) is ergodic? How are these relationships affected if the dc offset is zero?
Solution**:** Exercise

Although a random signal x(t) will be to a degree unpredictable, it will often be possible to make some prediction to the value of x(t) from a knowledge of previous history. For example, if the values of x(0), x(0.5), x(1), x(1.5), x(2), x(2,5), … are as follows:

-2 , 10, 21, 33, 41, 55, 62, 69, …

we may predict that the next sample is around 80, though this prediction will be to some extent in error because of randomness in x(t). The fact that we can even attempt a prediction is due to correlation between one sample and the next. The auto-correlation function $R_x(\tau)$ will be non-zero for values of $\tau>0$ and this will affect the power spectrum i.e. the Fourier transform of $R_x(\tau)$ making it non-flat and, in fact, biased towards the lower frequencies because of the slowly changing nature of x(t). Considering a second example,

-2, 44, -4, -17, 9, 61, 2, -19, 3, -16, 1, -7, 30, -1, …

there may be no predictability at all, meaning that there is no correlation between the signal and the signal delayed and therefore that $R_x(\tau) = 0$ for $\tau \neq 0$. This means that the power spectrum, i.e. the Fourier transform of $R_x(\tau)$ will be flat or "white".

Example 2.14: A "power" signal x(t) of bandwidth -B to B Hz and power P Watts (relative to 1 Ohm) has $R_x(\tau)=0$ for all $\tau\neq0$. Sketch its 2-sided PSD.

Example 2.15: Considering the two examples above of sequences of samples of x(t), which one is more likely to be Gaussian and why?

Note that the time properties of x(t) (governing correlation and hence the shape of the power spectrum) and the assumed statistical properties of x(t) are largely independent. We can have a white signal with Gaussian PDF or a spectrally coloured signal with the same Gaussian PDF. With random signals having higher spectral density at lower frequencies, giving strong correlation from one sample to the next, the signal will change more slowly than for a spectrally white (uncorrelated) signal with the same PDF; but over infinite time the same distribution of signal values will be observed.

2.12. Signal Correlation, Similarity and Matching

Correlation is introduced in the recommended text as a signal matching process. The cross-correlation function between two real finite energy signals x(t) and y(t):

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$

is said to measure the "matching" or "similarity" between "the shapes of" these signals when y(t) is advanced by τ seconds. The auto-correlation function of x(t) is the cross- correlation between x(t) and itself:

$$R_x(\tau) = R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

and measures the matching or similarity between the shape of x(t) and the shape of x(t) delayed by -τ seconds.

2.13. Conclusions

An understanding of the concepts in this section will be useful in the study of digital communications where analogue pulse shapes representing bits or bit-sequences must be visualized in both the time-domain and the frequency-domain. The ability to adapt easily remembered formulae to given wave-form shapes will often eliminate most of the mathematics and give insight into the nature of communication signals. The formulae are listed in one of three appendices attached to these notes which will also be attached to the CS3282 examination paper.

Problems on Section 2

2.1 What are the units of the following: (a) PSD (b) ESD (c) SNR ?

2.2 Classify the each of the following as an energy or a power signal, & find its energy & power normalised to 1 Ohm.: (a) x(t)=8cos (6πt) (b) x(t)=0 for t<0 & $4e^{-3t}$ for t ≥ 0.

2.3 For a "finite energy" signal x(t), show that the ESD is the Fourier Transform of the auto-correlation function (ACF). Hence show that:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X((f))|^2\, df \quad \text{(Parseval's or Rayleigh's Thm for energy signals)}$$

Give a relationship between PSD and ACF and a version of Parseval's Theorem for "finite power" signals.

2.4. A power signal has "2-sided" PSD as given below. What is its "1-sided" PSD?. This signal is passed through an ideal band-pass filter with cut-off frequencies 1Hz and 3Hz and pass-band gain 0dB. Calculate the power of the output signal.

$$PSD((f)) = \begin{cases} 4-|f| & :|f| < 4\ Hz \\ 0 & :|f| \geq 4 \end{cases}$$

2.5. A Gaussian noise signal n(t) of zero mean and power equal to W $Volt^2$/sec (or Watts normalised to 1 Ohm) has "1-sided" PSD:

$$PSD(f) = \begin{cases} N_0 & : |f| \leq \text{B Hz} \\ 0 & : |f| > \text{B Hz} \end{cases} \quad \text{Watts/Hz}$$

If samples of n(t) are taken at intervals of T>0 seconds, what would be:

(e) the mean, (b) the variance (c) the pdf of these samples.

(d) Show that $W=N_0B$.

If n(t) is passed through an ideal low-pass filter with cut-off frequency f_c=B/10 Hz, what would be

(e) the mean, (f) the variance (g) the pdf of the output signal?

2.6. If X((*f*)) is the FT of x(t) show that the FT of x(-t) is X((-*f*)), and show further that if x(t) is real then X((-*f*))=X*((*f*)) (complex conjugate). Show that the FT of x(t-T) is $X((f))e^{-2\pi Tf}$ and hence that the FT of x(T-t) is $X^*((f))e^{2\pi Tf}$. If x(t) is zero for t<0 and for t>T, and equals A(1-t) volts for

0 ≤t≤T, sketch x(-t) and x(T-t).

2.7. Answer TRUE or FALSE:-

(a) All zero mean Gaussian noise signals are spectrally white.

(b) Finite power signals have infinite energy & finite energy signals have infinite power.

(c) Any non-zero periodic signal must have finite power.

(d) The ACF of any signal is symmetric about delay $\tau=0$ i.e. $ACF(\tau)=ACF(-\tau)$.

(e) The inverse Fourier transform of the ESD is the ACF of an energy signal.

(f) A signal of finite bandwidth must be of infinite duration.

(g) The cross-correlation between two real signals is the same as convolution except that one of the two signals is time-reversed.

(h) The cross-correlation between real x(t) & y(t) has FT $X((f))Y^*((f))$

(i) The FT of dx(t)/dt is $2\pi j f\ X((f))$

Some solutions:

2.1. (a) Watts/Hz (b) Joules/Hz (c) Dimensionless power ratio

2.2. (a) Power signal with energy 32 Watts & infinite energy

(b) Energy signal of energy 8/3 Joules & zero average power

2.4.
$$PSD((f)) = \begin{cases} 8-2|f| & :|f|<4\ Hz \\ 0 & :|f|\ge 4 \end{cases}$$

$$Power = \int_1^3 (8-2f)df = \left[8f - f^2\right]_1^3 = 16-8 = 8 watts$$

2.5. (a) 0 (b) W (c) Gaussian (d) N_0 is Watts/Hz & bandwidth is B Hz hence total power is N_0B.

(e) 0 (f) W/10 (g) Gaussian.

2.7 (a) False (b) False (c) True(I think) (d) True (e) True (f) True (g) True (h) True (i) True

Extra question:

(a) If a 'finite energy' signal x(t) with 2-sided ESD $E(f)$ is passed through a filter with frequency-response $H(f)$, show that the output is a finite energy signal with ESD = $E(f)|H(f)|^2$. joules/Hz. What is the total energy of the filter's output?

(b) If a 'finite power' signal x(t) with 2-sided PSD $P(f)$ is passed through a filter with frequency-response $H(f)$, show that the output is a finite power signal with PSD = $P(f)|H(f)|^2$ watts/Hz. What is the total power of the filter's output?

Answer:

(a) $|X(f)|^2$ was defined in the notes as the two-sided ESD of a signal x(t) with Fourier Transfm $X(f)$.

If x(t) passed thro' a filter, the output spectrum is: $X(f)H(f)$ where $H(f)$ is the freq response.

The '2-sided' ESD of the filter output is therefore $|X(f)H(f)|^2$

$$= |X(f)|^2|H(f)|^2 = E(f)\,|H(f)|^2$$

The total energy of the filter output is:

$$\int_{-\infty}^{\infty} E(f)\,|H(f)|^2\, df \ldots joules$$

(b) Two-sided PSD of x(t) is:

$$PSD(f) = \lim_{D\to\infty} \frac{1}{D} |X_D(f)|^2 = \lim_{D\to\infty} \frac{1}{D} E_D(f)$$

where $E_D(f) = ESD\ \ of\ \ x_D(t)$

and $x_D(t) = \begin{cases} x(t) : -D/2 \le t \le D/2 \\ 0 \ : \ \ otherwise \end{cases}$

Result of passing the finite energy signal $x_D(t)$ thro' H(*f*) is a signal whose 2-sided ESD is:
$E_D(f) \mid H(f) \mid^2$

As D $\rightarrow$ infinity, $x_D(t) \rightarrow x(t)$ and (1/D){ESD of filter output} $\rightarrow$ PSD of filter output
Therefore as D $\rightarrow \infty$, (1/D{$E_D(f) \mid H(f) \mid^2$ } $\rightarrow$ PSD of filter output
As D $\rightarrow \infty$, (1/D{$E_D(f) \mid H(f) \mid^2$ } tends to

$$\lim_{D\to\infty} \frac{1}{D} E_D(f) |H(f)|^2 = PSD(f) |H(f)|^2$$

Therefore this is the PSD of the filter output.
The total power of the filter output is:

$$\int_{-\infty}^{\infty} PSD(f) \mid H(f) \mid^2 df$$

CS3282 Digital Communications

Appendix 1: Graph of complementary error function Q(z) against z.
Appendix 2: Fourier transform properties
Appendix 3: Trigonometric formulae
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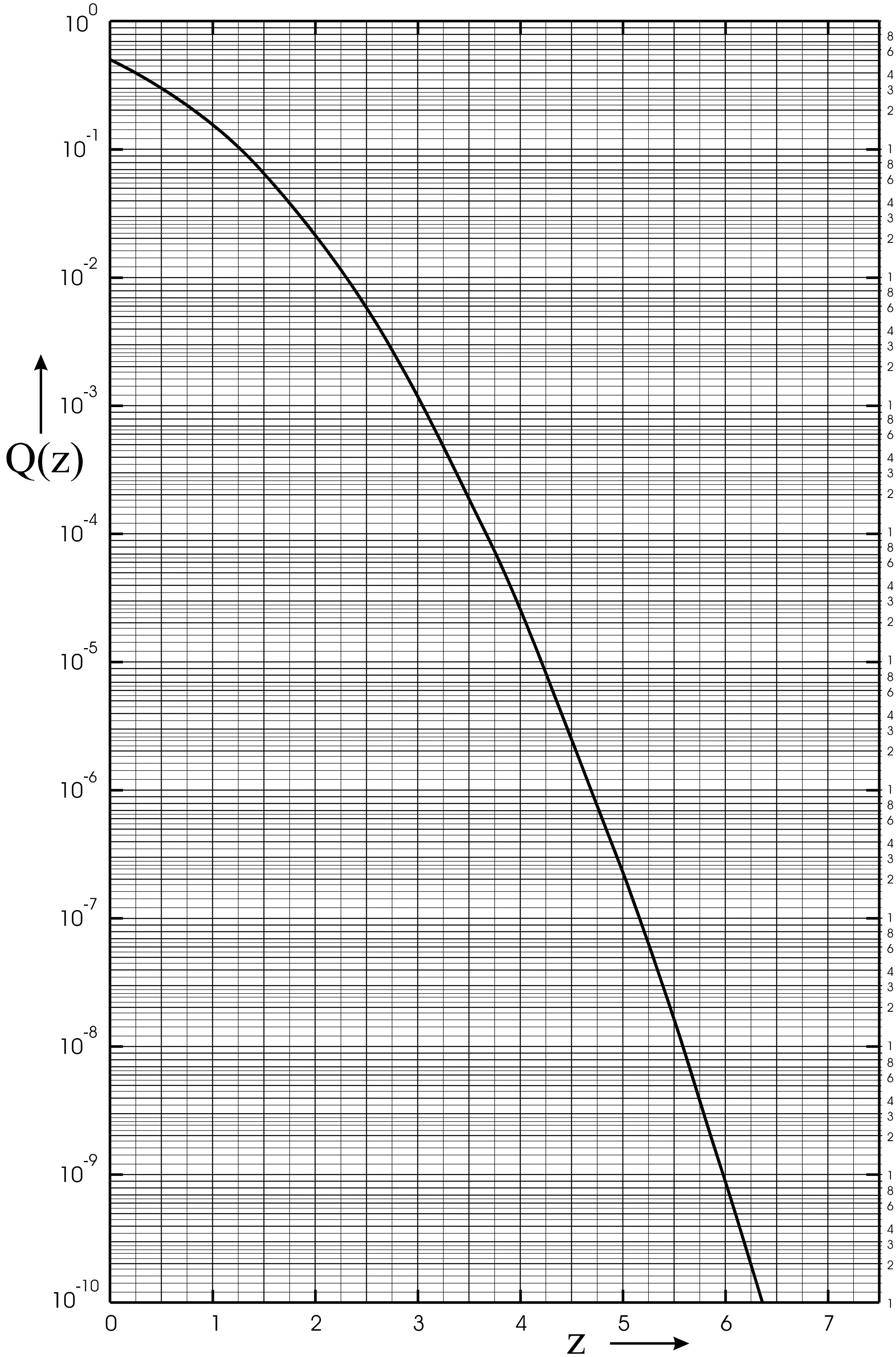

<u>Graph of Complementary Error function</u>, $Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$

Appendix 2: Properties of the Fourier Transform

Property	Signal x(t)	Fourier Transform X(*f*)
Transform & inverse:	$x(t) = \int_{-\infty}^{\infty} X(f)e^{2\pi jft}\,df$	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\,dt$

Similarly let the Fourier transforms of y(t), $y_1(t)$ & $y_2(t)$ be Y(*f*), $Y_1(f)$ & $Y_2(f)$ respectively.

Rect & sinc	$\text{rect}_A(t) = \text{rect}(t/A)$	$A\,\text{sinc}_{1/A}(f) = A\,\text{sinc}(Af)$
	$\text{sinc}_A(t) = \text{sinc}(t/A)$	$A\,\text{rect}_{1/A}(f) = A\,\text{rect}(Af)$
Delay	$y(t-\tau)$	$e^{-2\pi j\tau}Y(f)$
Frequency shift	$e^{2\pi jFt}\,y(t)$	$Y(f-F)$
Amplitude scaling:	$Ay(t)$	$AY(f)$
Time-reversal:	$y(-t)$	$Y(-f)$
Superposition:	$Ay_1(t)+By_2(t)$	$AY_1(f)+BY_2(f)$
Constant:	A	$A\delta(f)$
Impulse:	$A\delta(t)$	A
Gaussian	$(\sqrt{\pi}/\alpha)\exp(-\pi^2 t^2/\alpha^2)$	$\exp(-\alpha^2 f^2)$
Time-scaling:	$y(At)$	$(1/\lvert A\rvert)\,Y(f/A)$
Differentiation:	$d^m\{y(t)\}/dt^m$	$(2\pi jf)^m\,Y(f)$
Product:	$y_1(t)y_2(t)$	$Y_1(f)\otimes Y_2(f) = \int_{-\infty}^{\infty} Y_1(\theta)Y_2(f-\theta)\,d\theta$
Convolution:	$\int_{-\infty}^{\infty} y_1(\tau)y_2(t-\tau)\,d\tau$	$Y_1(f)\,Y_2(f)$
Cross-correlation:	$\int_{-\infty}^{\infty} y_1(\tau)y_2(t+\tau)\,d\tau$	$Y_1(f)\,Y_2^*(f)$
Auto-correlation:	$\int_{-\infty}^{\infty} y(\tau)y(t+\tau)\,d\tau$	$\lvert Y(f)\rvert^2$
Repeat:	$\text{repeat}_P\{y(t)\}$	$(1/P)\text{sample}_{1/P}\{Y(f)\}$
Sample:	$\text{sample}_T\{y(t)\}$	$(1/T)\text{repeat}_{1/T}\{Y(f)\}$

Properties for real signals:-

For all real signals:	$x^*(t)=x(t)$	$X(-f) = X^*(f)$ i.e. $\lvert X(-f)\rvert = \lvert X(f)\rvert$ & $\phi(-f) = -\phi(f)$
Real and even:	$x(t) = x(-t)$	X(*f*) is purely real & $X(-f) = X(f)$
Real and odd:	$x(t) = -x(-t)$	X(*f*) is purely imaginary & $X(-f) = -X(f)$

Formulae:-

$$\text{sinc}(x) = \begin{cases} \sin(\pi x)/(\pi x) & : x \neq 0 \\ 1 & : x = 0 \end{cases} \qquad \text{rect}(x) = \begin{cases} 1 & : |x| < 0.5 \\ 0.5 & : |x| = 0.5 \\ 0 & : |x| > 0.5 \end{cases} \qquad \text{sinc}_A(x) = \text{sinc}(x/A) \qquad \text{rect}_A(x) = \text{rect}(x/A)$$

$$\text{repeat}_P\{x(t)\} = \sum_{n=-\infty}^{\infty} x(t-nP) \qquad \text{sample}_T\{x(t)\} = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \quad = \quad x(t)\,\text{repeat}_T\{\delta(t)\}$$

Fourier series for $\text{repeat}_P\{x(t)\}$:-

$$\sum_{k=-\infty}^{\infty} C_k e^{2\pi jkt/P} = \sum_{k=0}^{\infty} M_k \cos((2\pi k/P)t + \theta_k) = A_0 + \sum_{k=1}^{\infty}\left(A_k \cos((2\pi k/P)t) + B_k \sin((2\pi k/P)t)\right)$$

$$C_k = (1/P)X(k/P) \;;\quad M_0 = |C_0|;\quad M_k = 2|C_k|;\quad \theta_k = \arg(C_k);\; A_0 = C_0;\; A_k = 2\,\text{Re}\{C_k\};\; B_k = 2\,\text{Im}(C_k\}$$

Appendix 3: Trigonometric formulae

$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
$\tan(A \pm B) = (\tan(A) \pm \tan(B)) / (1 \mp \tan(A)\tan(B))$

$\sin(2A) = 2\sin(A)\cos(A)$
$\cos(2A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$
$\tan(2A) = 2\tan(A) / (1 - \tan^2(A))$

$2\cos(A)\cos(B) = \cos(A + B) + \cos(A - B)$
$2\sin(A)\cos(B) = \sin(A + B) + \sin(A - B)$
$2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B)$

$\cos(A) + \cos(B) = 2\cos((A + B)/2)\cos((A - B)/2)$
$\sin(A) + \cos(B) = \sin(A) + \sin(B+\pi/2)$
$\sin(A) + \sin(B) = 2\sin((A + B)/2)\cos((A - B)/2)$

$\cos(\theta) = (e^{j\theta} + e^{-j\theta}) / 2$
$\sin(\theta) = (e^{j\theta} - e^{-j\theta}) / (2j)$

$\lambda\cos(\theta) + \mu\sin(\theta) = R\cos(\theta + \phi)$ where $R^2 = \lambda^2 + \mu^2$ and $\phi = \tan^{-1}(\mu/\lambda) + \{\pi \text{ if } \lambda<0\}$

$$\sum_{n=0}^{N-1}\cos(n\theta) = \frac{\sin(N\theta/2)\cos([n-1]\theta/2)}{\sin\left(\theta/2\right)} \qquad \sum_{n=0}^{N-1}\sin(n\theta) = \frac{\sin(N\theta/2)\sin([n-1]\theta/2)}{\sin\left(\theta/2\right)}$$