Retrenchment: an overview

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1. Origins ... model oriented refinement.
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1. Origins ... model oriented refinement.

Refinement is a word used to mean a large number of different things in different contexts.

In model oriented refinement, we build models of the system, by specifying:

- the state (and I/O) space of a model
- the operations (or events) of a model: via eg. transition systems, programming notations, predicate transformers, etc.
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- the state (and I/O) space of a model
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  transition systems,
  programming notations,
  predicate transformers,
  etc.

Models can then be related pairwise by REFINEMENT. This usually involves a notion of correctness, relying on the substitutivity, of some concrete system behaviours for some abstract system ones, intended to help move closer to an implementation, and leading to sufficient conditions for refinement. Often there are two key conditions:

  Applicability       and       Correctness
If the abstract system can make an $Op_A$ move ...
If the abstract system can make an $Op_A$ move, then the concrete system can make an $Op_C$ move.
Correctness

If the concrete system actually makes an $Op_C$ move ...
If the concrete system actually makes an $Op_C$ move, then the move can be simulated by the abstract system making an $Op_A$ move.
Correctness

If the concrete system actually makes an $Op_C$ move, then the move can be simulated by the abstract system making an $Op_A$ move.

The forward simulation criterion, the focus of our interest from now on.
Refinement Success Stories

There are various detailed theoretical variations on model oriented refinement, and various specific languages and implementations. The implementations of refinement have had some outstanding successes in the construction of dependable industrial scale systems of high criticality.
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Some languages: Z, VDM, B, RAISE, ASM.

Some key projects:

- Mondex Purse, Multos OS (Z)
- MÉTÉOR (and many more French and other railway systems) (B)
- Prolog, C, Java (and many more) language definitions (ASM)
2. What’s up with model oriented refinement?

Refinement can work wonderfully well in certain circumstances. Its paradigm of building models of the system at various levels of abstraction fits very naturally with the desire of designers to manipulate ‘solid’ representations of the system.

However, there are aspects of the real world design activity that can get into tension with refinement.
Refinement 1: Utopia

1. We design the abstract model. It captures the requirements of the system.
2. We refine the abstract model, moving towards a more efficient one.
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____________________

Wonderful when it works.

Crucially, it presupposes that the natural intuitions about the structures of the desired system correspond fairly closely to the ingredients of the abstract model.
Refinement 2: Reality

1. Designers have a fairly reasonable intuitive idea of how the concrete model looks. They are much less clear about how the abstract model looks.

2. They work backwards to reverse engineer the abstract model from the concrete one, removing those elements which experience has taught that refinement can reinstate.
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Where have the system requirements gone?
What is the relationship of the manufactured abstract model to the desired system?
What use is the refinement activity if the abstract model bears little or no relation to original system desiderata?
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Thinking hard about a problem is always beneficial.
Refinement 3: Tragedy

Communities who would make eager and serious use of refinement ... often can’t.
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1. Critical systems developers:
   • Need techniques giving very high assurance.
   • Understand and can benefit from the formal approach.
   • Can finance expensive technology (eg. intellectual) to gain high assurance.
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2. Often physical models are involved:
   - The continuous / discrete transition in modelling (as understood by engineers) is invariably not doable within (strict) refinement, restricting the scope of formal modelling.
   - So at best, the abstract model ends up *already* in the discrete domain, bypassing most of the serious design.
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   • So at best, the abstract model ends up already in the discrete domain, bypassing most of the serious design.

3. Even for purely discrete applications:
   • The real world never starts from a blank sheet, impeding ideal refinement.
   • The complexity of real world applications can prohibit 100% faithful models.
   • Management issues can prevent 100% adherence to refinement ideals.
More on Management Issues:

The management usually *don’t care* about the technical niceties of refinement.
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- The need of the specification to communicate *may override* its need to be refinable to implementation.
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Things like these may compromise the pursuit of refinement in the ideal sense.
Example: Adding an element to a set
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Abstract world: 

\[ Add_{elem}\_A(new) = \]

\[ myset := myset \cup \{new\} \]
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Concrete world: 

\[ Add_{elem_C}(new) = \]
\[ \text{IF} \ new \notin \text{ran}(myseq) \]
\[ \text{THEN} \ myseq := myseq \ ^ [new] \]
Example: Adding an element to a set

Abstract world: \( myset \) is just a set

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Concrete world: \( myset \) is represented as a sequence

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Implementation world: \( myseq \) is represented as an array

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Add_{\text{elem}}_I(new) = \\
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\text{IF} & \text{ new } \notin \text{ran}(\text{myseq}) \\
\text{THEN} & \text{myseq} := \text{myseq} ^ [\text{new}] \\
\end{align*}
\]

Implementation world: \( myseq \) is represented as an array with a finite bound

\[
\text{Add}_{\text{elem}}(\text{new}) =
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\text{IF} & \text{ new } \notin \text{ran}(\text{myarray}) \\
\text{THEN} & \text{myarray}(\text{next}) := \text{new} ; \text{next} := \text{next} + 1 \\
\end{align*}
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Concrete world:

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Implementation world:

\[ Add\_elem_I(new) = \]
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Example: Jet engine fuel supply control

Engineer’s View
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Step 1: Physics of hot fluids
   High temperature materials
   Aerodynamics
   etc.
Example: Jet engine fuel supply control

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Step n: Discrete algorithm
## Example: Jet engine fuel supply control

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| Why? ... |
**Example: Jet engine fuel supply control**

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Why? ... Because refinement is a kind of conservative extension.

It is normally relatively straightforward to impose additional constraints during refinement, but much more difficult to contradict previously adopted properties.
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OK if it makes sense to domain experts, more questionable if not.
Example: Small and large applications in general

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Example: Small and large applications in general

‘Textbook’ world

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& \quad \downarrow \text{refine} \\
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\end{align*}
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‘Real’ world

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This level becomes impenetrable.
Example: Small and large applications in general

'Textbook' world

\( \text{Abs} \)

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\[ \downarrow \text{refine} \]

\[ \downarrow \text{refine} \]

\( \text{Conc} \)

'Real' world

\( \text{True Abs} \)

\[ \downarrow \text{refine} \]

\( \text{Conc Abs} \)

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This level becomes impenetrable.

The bigger the application gets, the worse the problem becomes.

And it is absolutely a question of **SCALE**.
3. The inception of retrenchment.

The ferocity of the refinement POs is what restricts their application in many areas.
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The ferocity of the refinement POs is what restricts their application in many areas. What can we do about the ferocity of the refinement POs? We can attempt to judiciously weaken them.
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What can we do about the ferocity of the refinement POs?

We can attempt to judiciously weaken them.

This of course would have consequences ... Refinement is derived from the prior assumption of substitutivity of concrete for abstract. Any interference with the refinement POs can fatally wound this link with substitutivity properties.

We are prepared to forgo this highly desirable aspect for the sake of:
- Being able to address more (and more of) applications contexts formally.
- Being able to live with real world and management constraints.
The refinement PO

\[ \text{Abs} \quad \vdash \quad \frac{u, i \quad \text{Op}_A(u, i, u', o) \quad u', o}{G(u, v) \land (i = j)} \]

\[ \text{Conc} \quad \vdash \quad \frac{v, j \quad \text{Op}_C(v, j, v', p) \quad v', p}{(G(u', v') \land (o = p))} \]

\[ G(u, v) \land (i = j) \land \text{Op}_C(v, j, v', p) \Rightarrow (\exists u', o \cdot \text{Op}_A(u, i, u', o) \land (G(u', v') \land (o = p))) \]
The retrenchment PO

\[ G(u,v) \land P_{Op}(i,j,u,v) \Rightarrow (\exists u',o \cdot Op_A(u,i,u',o) \land ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \lor C_{Op}(u',v',o,p;u,v,i,j))) \]
The retrenchment PO

\[ G(u,v) \land P_{Op}(i,j,u,v) \Rightarrow ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \lor C_{Op}(u',v',o,p;u,v,i,j)) \]

Abs

\[ Op_A(u,i,u',o) \]

Conc

\[ Op_C(v,j,v',p) \]

\[ u,i \rightarrow u',o \]

\[ v,j \rightarrow v',p \]
Definition of a retrenchment between \textit{Abs} and \textit{Conc}

A retrenchment is defined by the following data:

\begin{itemize}
  \item \textbf{Ops}_A \subseteq \textbf{Ops}_C \hspace{1cm} \text{(the inclusion of operation names can be proper)}
  \item For each abstract operation name a relation \( O_{P_A} : U \times I_{O_{P_A}} \leftrightarrow U \times O_{O_{P_A}} \)
  \item For each concrete operation name a relation \( O_{P_C} : V \times J_{O_{P_C}} \leftrightarrow V \times P_{O_{P_C}} \)
  \item \( G(u,v) \) \hspace{1cm} \text{(the retrieve (or glueing) relation)}
  \item \( P_{Op}(i,j,u,v) \) \hspace{1cm} \text{(the within (or provided) relation)}
  \item \( O_{Op}(o,p;u',v',u,v,i,j) \) \hspace{1cm} \text{(the output relation)}
  \item \( C_{Op}(u',v',o,p;u,v,i,j) \) \hspace{1cm} \text{(the concedes relation)}
\end{itemize}

\text{Initialisation PO:} \hspace{1cm} \textcolor{red}{Init}_C(v') \Rightarrow (\exists u' \cdot \text{Init}_A(u') \land G(u',v')) \hspace{1cm} \text{(as for refinement)}

\text{Operation PO:} \hspace{1cm} \text{per } \text{Op} \in \text{Ops}_A \hspace{1cm}
\begin{align*}
\text{G}(u,v) \land P_{Op}(i,j,u,v) \land O_{P_C}(v,j,v',p) & \Rightarrow \\
(\exists u' \cdot O_{P_A}(u,i,u',o) \land ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \\
\lor C_{Op}(u',v',o,p;u,v,i,j)))
\end{align*}
Remarks

• The output form; cf. $G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op}))$ ‘primitive retrenchment’. As just a container for properties, primitive retrenchment is fine. For theoretical work, the output form is much more convenient.
Remarks

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• The operation $PO$ defaults to the refinement operation $PO$; (set $P_{Op}(i,j,u,v) = (i = j)$; $O_{Op}(o,p;u',v',u,v,i,j) = (o = p)$; $C_{Op}(u',v',o,p;u,v,i,j) = \text{false}$ ) There are other defaults.
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- The operation $PO$ is extremely expressive; it can accommodate *any* property. Nevertheless the $PO$ as a whole still has to be provable.
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Refinement and retrenchment compared
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## Refinement and retrenchment compared

### Refinement

- Guarantees concrete model is faithful to abstract model via $G$.
- Few properties addressable via $G$.
- Forces designer to discharge PO; PO is highly constrained; in particular it is nonlinear in $G$.

### Retrenchment

- Documents model change in a structured way via $G$, $P_{Op}$, $O_{Op}$, $C_{Op}$.
- Many properties addressable via $G$, $P_{Op}$, $O_{Op}$, $C_{Op}$.
- Forces designer to discharge PO; PO is very liberal; in particular, despite being nonlinear in $G$, it is linear in $P_{Op}$, $O_{Op}$, $C_{Op}$.
# Refinement and retrenchment compared

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Design and risk

Refinement
Refinement tends to expel design and risk from the concretisation/accretion process. The process can become monolithic.

Retrenchment
Retrenchment embraces design and risk within the evolution/concretisation process. The process can separate concerns.

'Black Box'

\[ Abs \rightarrow G \rightarrow Conc \]

'Design Risk'

'Glass Box'

\[ Abs \rightarrow G, P_{Op}, \bar{O}_{Op}, C_{Op} \rightarrow Conc \]

'Design Risk'
The real world again

‘Real’ world

‘True’ Abs

‘Conc’ Abs

Conc
The real world again

‘Real’ world

‘True’ Abs

↓ ret ret ret ret

‘Conc’ Abs

↓ refine

Conc

The autopsy gives a rationalisation in domain experts’ terms of the structure of the refinable abstract model.

No need to reverse engineer the true abstract model from the concrete model.
The real world again

‘Real’ world

‘True’ Abs

\[ \Downarrow \text{ret} \]

\[ \Downarrow \text{ret} \]

‘Conc’ Abs

\[ \Downarrow \text{ret} \]

\[ \Downarrow \text{ret} \]

\[ \Downarrow \text{refine} \]

Conc

The autopsy gives a rationalisation in domain experts’ terms of the structure of the refinable abstract model.

No need to reverse engineer the true abstract model from the concrete model.

An idealisation of the process ... many variations possible.

No technical commitment to which \textit{ret} steps are ‘requirements engineering’ and which are ‘development’.
# 4. Retrenchment: key issues and opportunities.

- Default retrenchments 74
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Default retrenchments
Default retrenchments

1. Defaulting to refinement:

\[ P_{Op}(i,j,u,v) := (i = j) \; ; \; O_{Op}(o,p;u',v',u,v,i,j) := (o = p) \; ; \; C_{Op}(u',v',o,p;u,v,i,j) := \text{false} \]
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   Given \( G(u,v), P_{\text{Op}}(i,j,u,v), O_{\text{Op}}(o,p;u',v',u,v,i,j) \) define:
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   Given \( G(u,v), P_{Op}(i,j,u,v), O_{Op}(o,p;u',v',u,v,i,j) \) define:
   \[ P_{Def_{Op}}^{Def}(i,j,u,v) := (G(u,v) \land P_{Op}(i,j,u,v) \land (\exists u',o,v',p \cdot Op_A(u,i,u',o) \land Op_C(v,j,v',p))) \]
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   • ‘Any two systems can be related via a default retrenchment.’
Default retrenchments

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   \]
   \[
   C_{Def}^{Op}(u',v',o,p;u,v,i,j) :=
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- ‘Any two systems can be related via a default retrenchment.’
- ‘Any jumble of words can be assembled into a meaningless phrase.’
- Retrenchment, like speech, should be undertaken with a purpose
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   \[ \neg((G(u', v') \land O_{op}(o, p; u', v', u, v, i, j))) \]

- ‘Any two systems can be related via a default retrenchment.’
- ‘Any jumble of words can be assembled into a meaningless phrase.’
- Retrenchment, like speech, should be undertaken with a purpose ... validation.
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- ‘Any two systems can be related via a default retrenchment.’
- ‘Any jumble of words can be assembled into a meaningless phrase.’
- Retrenchment, like speech, should be undertaken with a purpose ... validation.

Default and trivial retrenchments are top and bottom of a ‘lattice’.
Composition/decomposition

Vertical composition

Suppose given:

\(\text{Abs/Conc retrenchment: } OPA \preceq_{G_1, P_1, O_1, C_1} OPC\) from \((u, i, u', o)\) to \((v, j, v', p)\)

\(\text{Conc/Imp retrenchment: } OPC \preceq_{G_2, P_2, O_2, C_2} OPI\) from \((v, j, v', p)\) to \((w, k, w', q)\)
Composition/decomposition

Vertical composition

Suppose given:

*Abs/Conc* retrenchment: \( O_{A} \leq_{G_{1},P_{1},O_{1},C_{1}} O_{C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

*Conc/Imp* retrenchment: \( O_{C} \leq_{G_{2},P_{2},O_{2},C_{2}} O_{I} \) from \((v,j,v',p)\) to \((w,k,w',q)\)

Then there is an

*Abs/Imp* retrenchment: \( O_{A} \leq_{G_{(1,2)},P_{(1,2)},O_{(1,2)},C_{(1,2)}} O_{I} \) from \((u,i,u',o)\) to \((w,k,w',q)\)

given by:
Composition/decomposition

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Suppose given:

*Abs*/Conc retrenchment: \( O_{PA} \preceq_{G_1,P_1,O_1,C_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

*Conc*/Imp retrenchment: \( O_{PC} \preceq_{G_2,P_2,O_2,C_2} O_{PI} \) from \((v,j,v',p)\) to \((w,k,w',q)\)

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given by:

\[ G_{(1,2)}(u,w) = (\exists v \cdot G_1(u,v) \land G_2(v,w)) \]
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\]

\[
P_{Op,(1,2)}(i,k,u,w) = (\exists \, v,j \cdot G_1(u,v) \land G_2(v,w) \land P_{Op,1}(i,j,u,v) \land P_{Op,2}(j,k,v,w))
\]
Composition/decomposition

Vertical composition

Suppose given:
- **Abs/Conc** retrenchment: \( O_{PA} \preceq_{G_1,P_1,O_1,C_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)
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Then there is an
- **Abs/Imp** retrenchment: \( O_{PA} \preceq_{G_{(1,2)},P_{(1,2)},O_{(1,2)},C_{(1,2)}} O_{PI} \) from \((u,i,u',o)\) to \((w,k,w',q)\)
given by:

\[
\begin{align*}
G_{(1,2)}(u,w) &= (\exists v \cdot G_1(u,v) \land G_2(v,w)) \\
P_{Op,(1,2)}(i,k,u,w) &= (\exists v,j \cdot G_1(u,v) \land G_2(v,w) \land P_{Op,1}(i,j,u,v) \land P_{Op,2}(j,k,v,w)) \\
O_{Op,(1,2)}(o,q;u',w' \ldots) &= (\exists v',p,v,j \cdot O_{Op,1}(o,p;u',v' \ldots) \land O_{Op,2}(p,q;v',w' \ldots))
\end{align*}
\]
Composition/decomposition

Vertical composition

Suppose given:

- **Abs/Conc** retrenchment: \( O_{PA} \preceq_{G_1,P_1,O_1,C_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)
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given by:

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G_{(1,2)}(u,w) = (\exists \; v \cdot G_1(u,v) \land G_2(v,w))
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\[
P_{Op,(1,2)}(i,k,u,w) = (\exists \; v,j \cdot G_1(u,v) \land G_2(v,w) \land P_{Op,1}(i,j,u,v) \land P_{Op,2}(j,k,v,w))
\]

\[
O_{Op,(1,2)}(o,q;u',w'...) = (\exists \; v',p,v,j \cdot O_{Op,1}(o,p;u',v'...) \land O_{Op,2}(p,q,v',w'...))
\]

\[
C_{Op,(1,2)}(u',w',o,q;...) = (\exists \; v',p,v,j \cdot (G_1(u,v) \land O_{Op,1}(o,p;u',v'...) \land C_{Op,2}(v',w',p,q;...)) \lor
\]

\[
(C_{Op,1}(u',v',o,p;...) \land G_2(v,w) \land O_{Op,2}(p,q,v',w'...)) \lor
\]

\[
(C_{Op,1}(u',v',o,p;...) \land C_{Op,2}(v',w',p,q;...)))
\]
Composition/decomposition

Vertical composition

Suppose given:

\(\text{Abs/Conc retrenchment: } \bigcirc_{P,1}^{O_1,A_1,C_1} \rightarrow \bigcirc_{P,2}^{O_2,C_2} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

\(\text{Conc/Imp retrenchment: } \bigcirc_{P,1}^{O_1,C_2} \rightarrow \bigcirc_{P,2}^{O_2,C_2} \) from \((v,j,v',p)\) to \((w,k,w',q)\)

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given by:

\[
\begin{align*}
G_{(1,2)}(u,w) &= (\exists v \cdot G_1(u,v) \land G_2(v,w)) \\
\bigcirc_{P,(1,2)}^{O,(1,2)}(i,k,u,w) &= (\exists v,j \cdot G_1(u,v) \land G_2(v,w) \land \bigcirc_{P,(1,2)}^{O,(1,2)}(i,j,u,v) \land \bigcirc_{P,(1,2)}^{O,(1,2)}(j,k,v,w)) \\
\bigcirc_{O,(1,2)}^{P,(1,2)}(o,q,u',w') &= (\exists v',p,v,j \cdot \bigcirc_{O,(1,2)}^{P,(1,2)}(o,p,u',v') \land \bigcirc_{O,(1,2)}^{P,(1,2)}(p,q,v',w')) \\
\bigcirc_{C,(1,2)}^{P,(1,2)}(u',w',o,q;) &= (\exists v',p,v,j \cdot \\
&\quad (G_1(u,v) \land \bigcirc_{O,(1,2)}^{P,(1,2)}(o,p,u',v') \land \bigcirc_{C,(1,2)}^{P,(1,2)}(v',w',p,q;)) \lor \\
&\quad (C_{O,(1,2)}^{P,(1,2)}(u',v',o,p;) \land G_2(v,w) \land \bigcirc_{O,(1,2)}^{P,(1,2)}(p,q,v',w') \land \\
&\quad (C_{O,(1,2)}^{P,(1,2)}(u',v',o,p;) \land C_{O,(1,2)}^{P,(1,2)}(v',w',p,q;))
\end{align*}
\]

‘Plain vanilla’ vertical composition. Associative. There are many stronger forms.
Horizontal composition

Unrestricted horizontal composition is in general problematic. What about when the final configuration of one step cannot satisfy the $G \land P_{op}$ of the next one?
Horizontal composition

Unrestricted horizontal composition is in general problematic. What about when the final configuration of one step cannot satisfy the $G \land P_{Op}$ of the next one?

The unrestricted case is best studied directly via the simulation relation:

\[
G(u, v) \land P_{Op}(i, j, u, v) \land P_{Op}(i, j, u', v') \land \forall \Sigma^1 \land (G(u', v') \land O_{Op}(o, p; u', v', u, v, i, j)) \lor C_{Op}(u', v', o, p; u, v, i, j))
\]
Unrestricted horizontal composition is in general problematic. What about when the final configuration of one step cannot satisfy the $G \land P_{Op}$ of the next one?

The unrestricted case is best studied directly via the simulation relation:

$$G(u,v) \land P_{Op}(i,j,u,v) \land Op_{A}(u,i,u',o) \land Op_{C}(v,j,v',p) \land ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \lor C_{Op}(u',v',o,p;u,v,i,j))$$

This can capture ‘retrenchment is like refinement except round the edges’.
Horizontal composition

Suppose given:

*Abs/Conc* retrenchment: $Op_{1,A} \preceq_{G,P_{1},O_{1},C_{1}} Op_{1,C}$ from $(u, i_{1}, v, o_{1})$ to $(v_{1}, j_{1}, v, p_{1})$

*Abs/Conc* retrenchment: $Op_{2,A} \preceq_{G,P_{2},O_{2},C_{2}} Op_{2,C}$ from $(u, i_{2}, v', o_{2})$ to $(v_{2}, j_{2}, v', p_{2})$
Horizontal composition

Suppose given:

Abs/Conc retrenchment: \( Op_{1,A} \preceq_{G,P_1,o_1,c_1} Op_{1,C} \) from \((u, i_1, u, o_1)\) to \((v, j_1, v, p_1)\)

Abs/Conc retrenchment: \( Op_{2,A} \preceq_{G,P_2,o_2,c_2} Op_{2,C} \) from \((u, i_2, u', o_2)\) to \((v, j_2, v', p_2)\)

Then there is an Abs/Conc retrenchment:

\( Op_{(1;2),A} \preceq_{G,P_{(1;2)},o_{(1;2)},c_{(1;2)}} Op_{(1;2),C} \) from

\((u, [i_1, i_2], u', [o_1, o_2])\) to \((v, [j_1, j_2], v', [p_1, p_2])\)
Horizontal composition

Suppose given:

Abs/Conc retrenchment: $Op_{1,A} \leq_{G,P_{1,0_{1},C_{1}}} Op_{1,C}$ from $(u,i_{1},u,o_{1})$ to $(v,j_{1},v,p_{1})$

Abs/Conc retrenchment: $Op_{2,A} \leq_{G,P_{2,0_{2},C_{2}}} Op_{2,C}$ from $(u,i_{2},u',o_{2})$ to $(v,j_{2},v',p_{2})$

Then there is an Abs/Conc retrenchment:

$Op_{(1;2),A} \leq_{G,P_{(1;2),0_{(1;2)},C_{(1;2)}}} Op_{(1;2),C}$ from $(u, [i_{1},i_{2}], u', [o_{1},o_{2}])$ to $(v, [j_{1},j_{2}], v', [p_{1},p_{2}])$

given by:

$Op_{(1;2),A} = Op_{1,A} ; Op_{2,A}$  \hspace{1cm} $Op_{(1;2),C} = Op_{1,C} ; Op_{2,C}$
Horizontal composition

Suppose given:

Abs/Conc retrenchment: \( \text{Op}_1, A \leq_{G, P, O, C} \text{Op}_1, C \) from \((u, i_1, u, o_1)\) to \((v, j_1, v, p_1)\)

Abs/Conc retrenchment: \( \text{Op}_2, A \leq_{G, P, O, C} \text{Op}_2, C \) from \((u, i_2, u', o_2)\) to \((v, j_2, v', p_2)\)

Then there is an Abs/Conc retrenchment:

\( \text{Op}_{(1;2), A} \leq_{G, P_{(1;2), O}, C_{(1;2)}} \text{Op}_{(1;2), C} \) from

\( (u, [i_1, i_2], u', [o_1, o_2]) \) to \((v, [j_1, j_2], v', [p_1, p_2])\)

given by:

\[
\begin{align*}
\text{Op}_{(1;2), A} &= \text{Op}_{1, A}; \text{Op}_{2, A} \\
\text{Op}_{(1;2), C} &= \text{Op}_{1, C}; \text{Op}_{2, C} \\
P_{\text{Op}, (1;2)}([i_1, i_2], [j_1, j_2], u, v) &= G(u, v) \land P_{\text{Op}, 1}(i_1, j_1, u, v) \land \\
&\quad wp(\Sigma^1(\text{Op}_1), (G(u, v) \land P_{\text{Op}, 2}(i_2, j_2, u, v)))
\end{align*}
\]
Horizontal composition

Suppose given:

Abs/Conc retrenchment: $O_{p_1,A} \leq_{G.P_i,o_1,c_1} O_{p_1,C}$ from $(u,i_1,u,o_1)$ to $(v,j_1,v,p_1)$

Abs/Conc retrenchment: $O_{p_2,A} \leq_{G.P_i,o_2,c_2} O_{p_2,C}$ from $(u,i_2,u',o_2)$ to $(v,j_2,v',p_2)$

Then there is an Abs/Conc retrenchment:

$O_{p_1,2,A} \leq_{G.P_{1,2},o_{1,2},c_{1,2}} O_{p_1,2,C}$ from

$(u,[i_1,i_2],u',[o_1,o_2])$ to $(v,[j_1,j_2],v',[p_1,p_2])$

given by:

$O_{p_1,2,A} = O_{p_1,A}; O_{p_2,A}$

$O_{p_1,2,C} = O_{p_1,C}; O_{p_2,C}$

$P_{O_{p_1,2}}([i_1,i_2],[j_1,j_2],u,v) = G(u,v) \land P_{O_{p_1}}(i_1,j_1,u,v) \land$

$wp(\Sigma^1 (O_{p_1}), (G(u,v) \land P_{O_{p_2}}(i_2,j_2,u,v)))$

$O_{O_{p_1,2}}([o_1,o_2],[p_1,p_2];...) = (\exists \ u,v \cdot O_{O_{p_1}}(o_1,p_1,u',v';...) \land O_{O_{p_2}}(o_2,p_2,u,v;...))$
Horizontal composition

Suppose given:

Abs/Conc retrenchment: $\text{Op}_1,A \preceq_{G,p_1,o_1,c_1} \text{Op}_1,C$ from $(u,i_1,u,o_1)$ to $(v,j_1,v,p_1)$

Abs/Conc retrenchment: $\text{Op}_2,A \preceq_{G,p_2,o_2,c_2} \text{Op}_2,C$ from $(u,i_2,u',o_2)$ to $(v,j_2,v',p_2)$

Then there is an Abs/Conc retrenchment:

$\text{Op}_{(1;2),A} \preceq_{G,p_{(1;2)},o_{(1;2)},c_{(1;2)}} \text{Op}_{(1;2),C}$ from $(u,[i_1,i_2],u',[o_1,o_2])$ to $(v,[j_1,j_2],v',[p_1,p_2])$

given by:

$\text{Op}_{(1;2),A} = \text{Op}_1,A \; ; \; \text{Op}_2,A$

$\text{Op}_{(1;2),C} = \text{Op}_1,C \; ; \; \text{Op}_2,C$

$P_{\text{Op}_{(1;2)}}([i_1,i_2],[j_1,j_2],u,v) = G(u,v) \land P_{\text{Op}_1}(i_1,j_1,u,v) \land wp(\Sigma^1(\text{Op}_1),(G(u,v) \land P_{\text{Op}_2}(i_2,j_2,u,v)))$

$O_{\text{Op}_{(1;2)}}([o_1,o_2],[p_1,p_2];\ldots) = (\exists u,v \ldots O_{\text{Op}_1}(o_1,p_1;u,v;\ldots) \land O_{\text{Op}_2}(o_2,p_2;u,v;\ldots))$

$C_{\text{Op}_{(1;2)}}(u',v',[o_1,o_2],[p_1,p_2];\ldots) = (\exists u,v \ldots$

$(G(u,v) \land O_{\text{Op}_1}(o_1,p_1;u,v;\ldots) \land C_{\text{Op}_2}(u',v',o_2,p_2;\ldots)) \lor$

$(C_{\text{Op}_1}(u,v,o_1,p_1;\ldots) \land G(u',v') \land O_{\text{Op}_2}(o_2,p_2;u',v';\ldots)) \lor$

$(C_{\text{Op}_1}(u,v,o_1,p_1;\ldots) \land C_{\text{Op}_2}(u',v',o_2,p_2;\ldots))$
**Horizontal composition**

Suppose given:

Abs/Conc retrenchment: $\text{Op}_{1,A} \preceq_{G,P_{1,o_{1},c_{1}}} \text{Op}_{1,C}$ from $(u,i_{1},u,o_{1})$ to $(v,j_{1},v,p_{1})$

Abs/Conc retrenchment: $\text{Op}_{2,A} \preceq_{G,P_{2,o_{2},c_{2}}} \text{Op}_{2,C}$ from $(u,i_{2},u',o_{2})$ to $(v,j_{2},v',p_{2})$

Then there is an Abs/Conc retrenchment:

$\text{Op}_{(1;2),A} \preceq_{G,P_{(1;2),o_{(1;2)},c_{(1;2)}}} \text{Op}_{(1;2),C}$ from $(u,[i_{1},i_{2}],u',[o_{1},o_{2}])$ to $(v,[j_{1},j_{2}],v',[p_{1},p_{2}])$

given by:

$\text{Op}_{(1;2),A} = \text{Op}_{1,A} \cdot \text{Op}_{2,A}$ \hspace{1cm} $\text{Op}_{(1;2),C} = \text{Op}_{1,C} \cdot \text{Op}_{2,C}$

$P_{\text{Op}_{(1;2)}}([i_{1},i_{2}],[j_{1},j_{2}],u,v) = G(u,v) \land P_{\text{Op}_{1}}(i_{1},j_{1},u,v) \land wp(\Sigma^{1}(\text{Op}_{1}),(G(u,v) \land P_{\text{Op}_{2}}(i_{2},j_{2},u,v)))$

$O_{\text{Op}_{(1;2)}}([o_{1},o_{2}],[p_{1},p_{2}];\ldots) = (\exists \ u,v \cdot O_{\text{Op}_{1}}(o_{1},p_{1};u',v'\ldots) \land O_{\text{Op}_{2}}(o_{2},p_{2};u,v\ldots))$

$C_{\text{Op}_{(1;2)}}(u',v',[o_{1},o_{2}],[p_{1},p_{2}];\ldots) = (\exists \ u,v \cdot$

$(G(u,v) \land O_{\text{Op}_{1}}(o_{1},p_{1};u,v\ldots) \land C_{\text{Op}_{2}}(u',v',o_{2},p_{2};\ldots)) \lor$

$(C_{\text{Op}_{1}}(u,v,o_{1},p_{1};\ldots) \land G(u',v') \land O_{\text{Op}_{2}}(o_{2},p_{2};u',v'\ldots)) \lor$

$(C_{\text{Op}_{1}}(u,v,o_{1},p_{1};\ldots) \land C_{\text{Op}_{2}}(u',v',o_{2},p_{2};\ldots))$

N.B. $wp(\Sigma^{1}(\text{Op}_{1}),(G \land P_{\text{Op}_{2}})) = (\forall \ u,v,o_{1},p_{1} \cdot (u,i_{1},u,o_{1}) \Sigma^{1}(\text{Op}_{1}) (v,j_{1},v,p_{1}) \Rightarrow (G \land P_{\text{Op}_{2}}))$
Horizontal composition

Suppose given:

<table>
<thead>
<tr>
<th>Abs/Conc retrenchment:</th>
<th>Op1,A ≤_G.P1,O1,C1 Op1,C from (u,i,u,o) to (v,j,v,p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abs/Conc retrenchment:</td>
<td>Op2,A ≤_G.P2,O2,C2 Op2,C from (u,i,u',o) to (v,j,v',p)</td>
</tr>
</tbody>
</table>

Then there is an Abs/Conc retrenchment:

| Op(1,2),A ≤_G.P(1,2),O(1,2),C(1,2) Op(1,2),C from |
|------------------------|--------------------------------------------------|
| (u,[i_1,i_2],u',[o_1,o_2]) to (v,[j_1,j_2],v',[p_1,p_2]) |

given by:

\[
\begin{align*}
    Op_{(1,2),A} &= Op_{1,A} ; Op_{2,A} \\
    P_{Op,(1,2)}([i_1,i_2],[j_1,j_2],u,v) &= G(u,v) \land P_{Op,1}(i_1,j_1,u,v) \land \\
    & \quad \wp(\Sigma^1(Op_1),(G(u,v) \land P_{Op,2}(i_2,j_2,u,v))) \\
    O_{Op,(1,2)}([o_1,o_2],[p_1,p_2];\ldots) &= (\exists u,v \cdot Op_{1,1}(o_1,p_1;u,v;\ldots) \land Op_{2,1}(o_2,p_2;u,v;\ldots)) \\
    C_{Op,(1,2)}(u',v',[o_1,o_2],[p_1,p_2];\ldots) &= (\exists u,v \cdot \\
    & \quad (G(u,v) \land Op_{1,1}(o_1,p_1;u,v;\ldots) \land C_{Op,2}(u',v',o_2,p_2;\ldots)) \lor \\
    & \quad (C_{Op,1}(u,v,o_1,p_1;\ldots) \land G(u',v') \land Op_{2,1}(o_2,p_2;u',v';\ldots)) \lor \\
    & \quad (C_{Op,1}(u,v,o_1,p_1;\ldots) \land C_{Op,2}(u',v',o_2,p_2;\ldots)) \\
\end{align*}
\]

N.B. \( \wp(\Sigma^1(Op_1),(G \land P_{Op,2})) = (\forall u,v,o_1,p_1 \cdot (u,i_1,u,o) \Sigma^1(Op_1) (v,j_1,v,p) \Rightarrow (G \land P_{Op,2})) \)

This can capture 'bulk sequential composition' of operations, but it's rather stronger than the simulation relation.
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

*Abs/Conc* retrenchment: \( O_{p1,A} \leq_{s,G,p1,o1,c1} O_{p1,C} \) from \((i_0,o_1)\) to \((j_0,p_1)\)

*Abs/Conc* retrenchment: \( O_{p2,A} \leq_{s,G,p2,o2,c2} O_{p2,C} \) from \((i_1,o_2)\) to \((j_1,p_2)\)

and \( (o_1 = i_1) \) and \( (p_1 = j_1) \)
**Dataflow composition**

Trivial state \(*\); outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( O_{1,A} \preceq_{G, P_1, o_1, c_1} O_{1,C} \) from \((i_0, o_1)\) to \((j_0, p_1)\)

Abs/Conc retrenchment: \( O_{2,A} \preceq_{G, P_2, o_2, c_2} O_{2,C} \) from \((i_1, o_2)\) to \((j_1, p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment:

\( O_{(1\,\Rightarrow \,2), A} \preceq_{G, P_{(1\,\Rightarrow \,2)}, o_{(1\,\Rightarrow \,2)}, c_{(1\,\Rightarrow \,2)}} O_{(1\,\Rightarrow \,2), C} \) from \((i_0, o_2)\) (via \((o_1 = i_1)\)) to \((j_0, p_2)\) (via \((p_1 = j_1)\))

given by:
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( \text{Op}_{1,A} \leq_{G,p_1,o_1,c_1} \text{Op}_{1,C} \) from \((i_0,o_1)\) to \((j_0,p_1)\)

Abs/Conc retrenchment: \( \text{Op}_{2,A} \leq_{G,p_2,o_2,c_2} \text{Op}_{2,C} \) from \((i_1,o_2)\) to \((j_1,p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment:

\( \text{Op}_{(1\rightarrow2),A} \leq_{G,p_{(1\rightarrow2)},o_{(1\rightarrow2)},c_{(1\rightarrow2)}} \text{Op}_{(1\rightarrow2),C} \) from \((i_0,o_2)\) (via \((o_1 = i_1)\)) to \((j_0,p_2)\) (via \((p_1 = j_1)\))

given by:

\( \text{Op}_{(1\rightarrow2),A} = \text{Op}_{1,A} \land \text{Op}_{2,A} \) \hspace{1cm} \( \text{Op}_{(1\rightarrow2),C} = \text{Op}_{1,C} \land \text{Op}_{2,C} \)
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( Op_{1,A} \trianglelefteq_{G,P_1,o_1,c_1} Op_{1,C} \) from \((i_0, o_1)\) to \((j_0, p_1)\)

Abs/Conc retrenchment: \( Op_{2,A} \trianglelefteq_{G,P_2,o_2,c_2} Op_{2,C} \) from \((i_1, o_2)\) to \((j_1, p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment: \( Op_{(1\to2),A} \trianglelefteq_{G,P_{(1\to2)},o_{(1\to2)},c_{(1\to2)}} Op_{(1\to2),C} \) from \((i_0, o_2)\) (via \((o_1 = i_1)\)) to \((j_0, p_2)\) (via \((p_1 = j_1)\))

given by:

\[
\begin{align*}
Op_{(1\to2),A} &= Op_{1,A} \cdot Op_{2,A} \\
Op_{(1\to2),C} &= Op_{1,C} \cdot Op_{2,C} \\
P(Op_{1\cdot Op_{2}}(i_0,j_0)) &= P_{Op_{1}}(i_0,j_0)
\end{align*}
\]
Dataflow composition

Trivial state ∗; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( O_{P_1,A} \preceq_{G,P_1,o_1,c_1} O_{P_1,C} \) from \((i_0,o_1)\) to \((j_0,p_1)\)

Abs/Conc retrenchment: \( O_{P_2,A} \preceq_{G,P_2,o_2,c_2} O_{P_2,C} \) from \((i_1,o_2)\) to \((j_1,p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment:

\( O_{(1\triangleright 2),A} \preceq_{G,P_{(1\triangleright 2)},o_{(1\triangleright 2)},c_{(1\triangleright 2)}} O_{(1\triangleright 2),C} \) from \((i_0,o_2)\) (via \((o_1 = i_1)\)) to \((j_0,p_2)\) (via \((p_1 = j_1)\))

given by:

\[
O_{(1\triangleright 2),A} = O_{P_1,A} \circ O_{P_2,A} \quad O_{(1\triangleright 2),C} = O_{P_1,C} \circ O_{P_2,C}
\]

\[
P_{(O_{p_1},O_{p_2})(i_0,j_0)} = P_{O_{p_1},1(i_0,j_0)}
\]

\[
O_{(O_{p_1},O_{p_2})(o_2,p_2;i_0,j_0)} = (\exists \ a,c \cdot O_{O_{p_2},2(o_2,p_2;a,c)} \land O_{O_{p_1},1(a,c;i_0,j_0)}))
\]
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step. Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( Op_{1,A} \preceq_{G_1,P_1,o_1,c_1} Op_{1,C} \) from \((i_0,o_1)\) to \((j_0,p_1)\)

Abs/Conc retrenchment: \( Op_{2,A} \preceq_{G_2,P_2,o_2,c_2} Op_{2,C} \) from \((i_1,o_2)\) to \((j_1,p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment:

\( Op_{(1\cdot2),A} \preceq_{G_{(1\cdot2)},P_{(1\cdot2)},o_{(1\cdot2)},c_{(1\cdot2)}} Op_{(1\cdot2),C} \) from \((i_0,o_2)\) (via \((o_1 = i_1)\)) to \((j_0,p_2)\) (via \((p_1 = j_1)\))

given by:

\[
\begin{align*}
Op_{(1\cdot2),A} &= Op_{1,A} \cdot Op_{2,A} \\
Op_{(1\cdot2),C} &= Op_{1,C} \cdot Op_{2,C} \\
P_{(Op_{1,Op_{2}}(i_0,j_0))} &= P_{Op_{1}}(i_0,j_0) \\
o_{(Op_{1},Op_{2})(o_2,p_2;i_0,j_0)} &= (\exists a, c \cdot Op_{2}(o_2,p_2;a,c) \land Op_{1}(a,c;i_0,j_0)) \\
C_{(Op_{1},Op_{2})(o_2,p_2;i_0,j_0)} &= (\exists a, c \cdot (Op_{2}(o_2,p_2;a,c) \land C_{Op_{1}}(a,c;i_0,j_0)) \lor (C_{Op_{2}}(o_2,p_2;a,c) \land Op_{1}(a,c;i_0,j_0)) \lor (C_{Op_{2}}(o_2,p_2;a,c) \land C_{Op_{1}}(a,c;i_0,j_0)))
\end{align*}
\]
Dataflow composition

Trivial state *; outputs of one step plug into inputs of next step.
Much better behaved than unrestricted horizontal composition.

Suppose given:

Abs/Conc retrenchment: \( O_{p1,A} \leq_{g,p1,o1,c1} O_{p1,C} \) from \((i_0,o_1)\) to \((j_0,p_1)\)

Abs/Conc retrenchment: \( O_{p2,A} \leq_{g,p2,o2,c2} O_{p2,C} \) from \((i_1,o_2)\) to \((j_1,p_2)\)

and \((o_1 = i_1)\) and \((p_1 = j_1)\)

Then there is an Abs/Conc retrenchment:

\( O_{p(1\triangleright 2),A} \leq_{g,p_{(1\triangleright 2)},o_{(1\triangleright 2)},c_{(1\triangleright 2)}} O_{p(1\triangleright 2),C} \) from \((i_0,o_2)\) (via \((o_1 = i_1)\)) to \((j_0,p_2)\) (via \((p_1 = j_1)\))

given by:

\[
\begin{align*}
O_{p(1\triangleright 2),A} &= O_{p1,A} ; O_{p2,A} \\
O_{p(1\triangleright 2),C} &= O_{p1,C} ; O_{p2,C} \\
P_{(op_1,op_2)(i_0,j_0)} &= P_{op_1}(i_0,j_0) \\
O_{(op_1,op_2)(o_2,p_2;i_0,j_0)} &= (\exists a,c \cdot O_{op_2}(o_2,p_2;a,c) \land O_{op_1}(a,c;i_0,j_0)) \\
C_{(op_1,op_2)(o_2,p_2;i_0,j_0)} &= (\exists a,c \cdot (O_{op_2}(o_2,p_2;a,c) \land C_{op_1}(a,c;i_0,j_0)) \lor (C_{op_2}(o_2,p_2;a,c) \land O_{op_1}(a,c;i_0,j_0)) \lor (C_{op_2}(o_2,p_2;a,c) \land C_{op_1}(a,c;i_0,j_0)))
\end{align*}
\]

provided \( O_{op_1} \lor C_{op_1} \Rightarrow P_{op_2} \)
Synchronous parallel composition

Partial states.
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \( O_{p_1,A} \leq_{g_1,p_1,o_1,c_1} O_{p_1,C} \) from \((u_1,i_1,u'_1,o_1)\) to \((v_1,j_1,v'_1,p_1)\)

Abs/Conc retrenchment: \( O_{p_2,A} \leq_{g_2,p_2,o_2,c_2} O_{p_2,C} \) from \((u_2,i_2,u'_2,o_2)\) to \((v_2,j_2,v'_2,p_2)\)
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \( Op_{1,A} \preceq_{G_{1},p_{1},o_{1},c_{1}} Op_{1,C} \) from \((u_{1},i_{1},u'_{1},o_{1})\) to \((v_{1},j_{1},v'_{1},p_{1})\)

Abs/Conc retrenchment: \( Op_{2,A} \preceq_{G_{2},p_{2},o_{2},c_{2}} Op_{2,C} \) from \((u_{2},i_{2},u'_{2},o_{2})\) to \((v_{2},j_{2},v'_{2},p_{2})\)

Then there is an Abs/Conc retrenchment:

\( Op_{(1|2),A} \preceq_{G_{(1|2)},p_{(1|2)},o_{(1|2)},c_{(1|2)}} Op_{(1|2),C} \) from

\( ((u_{1},u_{2}),(i_{1},i_{2}),(u'_{1},u'_{2}),(o_{1},o_{2})) \) to \( ((v_{1},v_{2}),(j_{1},j_{2}),(v'_{1},v'_{2}),(p_{1},p_{2})) \)

given by:
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \(Op_{1,A} \leq_{G_1,p_1,o_1,c_1} Op_{1,C}\) from \((u_1,i_1,u'_1,o_1)\) to \((v_1,j_1,v'_1,p_1)\)

Abs/Conc retrenchment: \(Op_{2,A} \leq_{G_2,p_2,o_2,c_2} Op_{2,C}\) from \((u_2,i_2,u'_2,o_2)\) to \((v_2,j_2,v'_2,p_2)\)

Then there is an Abs/Conc retrenchment:

\[Op_{(1|2),A} \leq_{G_{(1|2)},p_{(1|2)},o_{(1|2)},c_{(1|2)}} Op_{(1|2),C}\]

from:

\[((u_1,u_2),(i_1,i_2),(u'_1,u'_2),(o_1,o_2))\]

to:

\[((v_1,v_2),(j_1,j_2),(v'_1,v'_2),(p_1,p_2))\]

given by:

\[Op_{(1|2),A} = Op_{1,A} \land Op_{2,A} \quad Op_{(1|2),C} = Op_{1,C} \land Op_{2,C}\]
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \( Op_{1,A} \preceq_{G_{1,p_1,o_1,c_1}} Op_{1,C} \) from \((u_1, i_1, u'_1, o_1)\) to \((v_1, j_1, v'_1, p_1)\)

Abs/Conc retrenchment: \( Op_{2,A} \preceq_{G_{2,p_2,o_2,c_2}} Op_{2,C} \) from \((u_2, i_2, u'_2, o_2)\) to \((v_2, j_2, v'_2, p_2)\)

Then there is an Abs/Conc retrenchment:

\[
Op_{(1|2),A} \preceq_{G_{(1|2),p_{(1|2)},o_{(1|2)},c_{(1|2)}}} Op_{(1|2),C} \text{ from }
((u_1, u_2), (i_1, i_2), (u'_1, u'_2), (o_1, o_2)) \text{ to } ((v_1, v_2), (j_1, j_2), (v'_1, v'_2), (p_1, p_2))
\]

given by:

\[
\begin{align*}
Op_{(1|2),A} &= Op_{1,A} \land Op_{2,A} \\
Op_{(1|2),C} &= Op_{1,C} \land Op_{2,C} \\
G_{(1|2)}((u_1, u_2), (v_1, v_2)) &= G_1(u_1, v_1) \land G_2(u_2, v_2)
\end{align*}
\]
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \( \text{Op}_{1,A} \preceq_{G_1,p_1,o_1,c_1} \text{Op}_{1,C} \) from \((u_1,i_1,u'_1,o_1)\) to \((v_1,j_1,v'_1,p_1)\)

Abs/Conc retrenchment: \( \text{Op}_{2,A} \preceq_{G_2,p_2,o_2,c_2} \text{Op}_{2,C} \) from \((u_2,i_2,u'_2,o_2)\) to \((v_2,j_2,v'_2,p_2)\)

Then there is an Abs/Conc retrenchment:

\( \text{Op}_{(1|2),A} \preceq_{G_{(1|2)},p_{(1|2)},o_{(1|2)},c_{(1|2)}} \text{Op}_{(1|2),C} \) from 
\((u_1,u_2,(i_1,i_2),(u'_1,u'_2),(o_1,o_2))\) to 
\((v_1,v_2,(j_1,j_2),(v'_1,v'_2),(p_1,p_2))\)

given by:

\[ \text{Op}_{(1|2),A} = \text{Op}_{1,A} \land \text{Op}_{2,A} \]

\[ \text{Op}_{(1|2),C} = \text{Op}_{1,C} \land \text{Op}_{2,C} \]

\[ G_{(1|2)}((u_1,u_2),(v_1,v_2)) = G_1(u_1,v_1) \land G_2(u_2,v_2) \]

\[ P_{\text{Op}_{(1|2)}}((i_1,i_2),(j_1,j_2),(u_1,u_2),(v_1,v_2)) = P_{\text{Op}_{1}}(i_1,j_1,u_1,v_1) \land P_{\text{Op}_{2}}(i_2,j_2,u_2,v_2) \]
Synchronous parallel composition

Partial states.

Suppose given:

Abs/Conc retrenchment: \( O_{p1,A} \leq_{G_1,p_1,o_1,c_1} O_{p1,C} \) from \((u_1,i_1,u'_1,o_1)\) to \((v_1,j_1,v'_1,p_1)\)

Abs/Conc retrenchment: \( O_{p2,A} \leq_{G_2,p_2,o_2,c_2} O_{p2,C} \) from \((u_2,i_2,u'_2,o_2)\) to \((v_2,j_2,v'_2,p_2)\)

Then there is an Abs/Conc retrenchment:

\( O_{p(1|2),A} \leq_{G_{(1|2)},p_{(1|2)},o_{(1|2)},c_{(1|2)}} O_{p(1|2),C} \) from

\(((u_1,u_2),(i_1,i_2),(u'_1,u'_2),(o_1,o_2))\) to \(((v_1,v_2),(j_1,j_2),(v'_1,v'_2),(p_1,p_2))\)

given by:

\( O_{p(1|2),A} = O_{p1,A} \land O_{p2,A} \)
\( O_{p(1|2),C} = O_{p1,C} \land O_{p2,C} \)
\( G_{(1|2)}((u_1,u_2),(v_1,v_2)) = G_1(u_1,v_1) \land G_2(u_2,v_2) \)
\( P_{O_{p(1|2)}}((i_1,i_2),(j_1,j_2),(u_1,u_2),(v_1,v_2)) = P_{O_{p1}}(i_1,j_1,u_1,v_1) \land P_{O_{p2}}(i_2,j_2,u_2,v_2) \)
\( O_{O_{p(1|2)}}((o_1,o_2),(p_1,p_2);...) = O_{O_{p1}}(o_1,p_1;u'_1,v'_1...) \land O_{O_{p2}}(o_2,p_2;u'_2,v'_2...) \)
Synchronous parallel composition

Partial states.

Suppose given:

*Abs/Conc* retrenchment: \( Op_{1,A} \leq_{G_1,p_1,o_1,c_1} Op_{1,C} \) from \((u_1,i_1,u'_1,o_1)\) to \((v_1,j_1,v'_1,p_1)\)

*Abs/Conc* retrenchment: \( Op_{2,A} \leq_{G_2,p_2,o_2,c_2} Op_{2,C} \) from \((u_2,i_2,u'_2,o_2)\) to \((v_2,j_2,v'_2,p_2)\)

Then there is an *Abs/Conc* retrenchment:

\[
Op_{\langle 1\mid 2 \rangle,A} \leq_{G_{\langle 1\mid 2 \rangle},p_{\langle 1\mid 2 \rangle},o_{\langle 1\mid 2 \rangle},c_{\langle 1\mid 2 \rangle}} Op_{\langle 1\mid 2 \rangle,C}
\]
from
\[
(((u_1,u_2),(i_1,i_2),(u'_1,u'_2),(o_1,o_2)) \to ((v_1,v_2),(j_1,j_2),(v'_1,v'_2),(p_1,p_2))
\]
given by:

\[
Op_{\langle 1\mid 2 \rangle,A} = Op_{1,A} \land Op_{2,A} \quad Op_{\langle 1\mid 2 \rangle,C} = Op_{1,C} \land Op_{2,C}
\]

\[
G_{\langle 1\mid 2 \rangle}((u_1,u_2),(v_1,v_2)) = G_1(u_1,v_1) \land G_2(u_2,v_2)
\]

\[
P_{\langle 1\mid 2 \rangle}(i_1,i_2),(j_1,j_2),(u_1,u_2),(v_1,v_2)) = P_{Op,1}(i_1,j_1,u_1,v_1) \land P_{Op,2}(i_2,j_2,u_2,v_2)
\]

\[
O_{Op,\langle 1\mid 2 \rangle}(o_1,o_2),(p_1,p_2);\ldots) = O_{Op,1}(o_1,p_1,u'_1,v'_1;\ldots) \land O_{Op,2}(o_2,p_2,u'_2,v'_2;\ldots)
\]

\[
C_{Op,\langle 1\mid 2 \rangle}(u'_1,u'_2),(v'_1,v'_2),(o_1,o_2),(p_1,p_2);\ldots) = \\
((G_1(u_1,v_1) \land O_{Op,1}(o_1,p_1,u'_1,v'_1;\ldots) \land C_{Op,2}(u'_2,v'_2,o_1,p_1;\ldots)) \lor \\
(C_{Op,1}(u'_1,v'_1,o_1,p_1;\ldots) \land G_2(u_2,v_2) \land O_{Op,2}(o_2,p_2,u'_2,v'_2;\ldots)) \lor \\
(C_{Op,1}(u'_1,v'_1,o_1,p_1;\ldots) \land C_{Op,2}(u'_2,v'_2,o_1,p_1;\ldots)))
\]
Fusion composition

Simultaneous retrenchments. Conjunctive composition.
Equiextensional simulation relation.
Fusion composition

Simultaneous retrenchments. Conjunctive composition.
Equiextensional simulation relation. Add guards if necessary.
Fusion composition


Suppose given:

Abs/Conc retrenchment: $O_{PA} \leq G, P_1, O_1, C_1 \quad O_{PC}$ from $(u, i, u', o)$ to $(v, j, v', p)$

Abs/Conc retrenchment: $O_{PA} \leq G, P_2, O_2, C_2 \quad O_{PC}$ from $(u, i, u', o)$ to $(v, j, v', p)$
Fusion composition

Simultaneous retrenchments. Conjunctive composition.
Equiextensional simulation relation. Add guards if necessary.

Suppose given:

Abs/Conc retrenchment: \( O_{PA} \preceq_{G,P_1,O_1,C_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( O_{PA} \preceq_{G,P_2,O_2,C_2} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\( O_{PA} \preceq_{G,P_{(1\lor2)},O_{(1\lor2)},C_{(1\lor2)}} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:
Fusion composition

Simultaneous retrenchments. Conjunctive composition.
Equiextensional simulation relation. Add guards if necessary.

Suppose given:

Abs/Conc retrenchment: \( O_{PA} \preceq_{G,P_1,O_1,C_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( O_{PA} \preceq_{G,P_2,O_2,C_2} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\( O_{PA} \preceq_{G,P_{(1\land 2)},O_{(1\land 2)},C_{(1\land 2)}} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\[ P_{O_{p,(1\land 2)}(i,j,u,v)} = P_{O_{p,1}(i,j,u,v)} \land P_{O_{p,2}(i,j,u,v)} \]
Fusion composition

Simultaneous retrenchments. Conjunctive composition.  
Equiextensional simulation relation. Add guards if necessary.

Suppose given:

Abs/Conc retrenchment: \( O_{P_A} \leq_{G,P_1,O_1,C_1} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( O_{P_A} \leq_{G,P_2,O_2,C_2} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\[ O_{P_A} \leq_{G,P_{(1\land2)},O_{(1\land2)},C_{(1\land2)}} O_{P_C} \] from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\[ P_{Op,(1\land2)}(i,j,u,v) = P_{Op,1}(i,j,u,v) \land P_{Op,2}(i,j,u,v) \]

\[ O_{Op,(1\land2)}(o,p;u',v' \ldots) = O_{Op,1}(o,p;u',v' \ldots) \land O_{Op,2}(o,p;u',v' \ldots) \]
Fusion composition


Suppose given:

*Abs/Conc* retrenchment: \( O_{P_A} \lesssim_{G,P_1,O_1,C_1} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

*Abs/Conc* retrenchment: \( O_{P_A} \lesssim_{G,P_2,O_2,C_2} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an *Abs/Conc* retrenchment:

\( O_{P_A} \lesssim_{G,P_{(1\wedge 2)},O_{(1\wedge 2)},C_{(1\wedge 2)}} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\[
\begin{align*}
P_{Op,(1\wedge 2)}(i,j,u,v) &= P_{Op,1}(i,j,u,v) \land P_{Op,2}(i,j,u,v) \\
O_{Op,(1\wedge 2)}(o,p;u',v'\ldots) &= O_{Op,1}(o,p;u',v'\ldots) \land O_{Op,2}(o,p;u',v'\ldots) \\
C_{Op,(1\wedge 2)}(u',v',o,p;\ldots) &= (G(u,v) \land O_{Op,1}(o,p;u',v'\ldots) \land C_{Op,2}(u',v',o,p;\ldots)) \lor \\
& \quad (C_{Op,1}(u',v',o,p;\ldots) \land G(u,v) \land O_{Op,2}(o,p;u',v'\ldots)) \lor \\
& \quad (C_{Op,1}(u',v',o,p;\ldots) \land C_{Op,2}(u',v',o,p;\ldots))
\end{align*}
\]
Fusion composition

Simultaneous retrenchments. Disjunctive composition.
Fusion composition

Simultaneous retrenchments. Disjunctive composition.

Suppose given:

Abs/Conc retrenchment: $O_{PA} \preceq_{G,P_1,o_1,c_1} O_{PC}$ from $(u,i,u',o)$ to $(v,j,v',p)$

Abs/Conc retrenchment: $O_{PA} \preceq_{G,P_2,o_2,c_2} O_{PC}$ from $(u,i,u',o)$ to $(v,j,v',p)$
Fusion composition

Simultaneous retrenchments. Disjunctive composition.

Suppose given:

Abs/Conc retrenchment: \( \text{Op}_A \leq_{G,P_1,0_1,C_1} \text{Op}_C \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( \text{Op}_A \leq_{G,P_2,0_2,C_2} \text{Op}_C \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\( \text{Op}_A \leq_{G,P_{(1\lor 2)},0_{(1\lor 2)},C_{(1\lor 2)}} \text{Op}_C \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:
Fusion composition

Simultaneous retrenchments. Disjunctive composition.

Suppose given:

Abs/Conc retrenchment: \( O_{PA} \preceq_{G.P_1.o_1.c_1} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( O_{PA} \preceq_{G.P_2.o_2.c_2} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\( O_{PA} \preceq_{G.P_{(1\lor 2)}.o_{(1\lor 2)}.c_{(1\lor 2)}}} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\( P_{Op,(1\lor 2)}(i,j,u,v) = P_{Op,1}(i,j,u,v) \lor P_{Op,2}(i,j,u,v) \)
Fusion composition

Simultaneous retrenchments. Disjunctive composition.

Suppose given:

Abs/Conc retrenchment: \( O_{P_A} \preceq_{G,P_1,o_1,c_1} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( O_{P_A} \preceq_{G,P_2,o_2,c_2} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\( O_{P_A} \preceq_{G,P_{(1\lor 2)},o_{(1\lor 2)},c_{(1\lor 2)}} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\[
P_{O_{P,(1\lor 2)}(i,j,u,v)} = P_{O_{P,1}(i,j,u,v)} \lor P_{O_{P,2}(i,j,u,v)}
\]

\[
O_{O_{P,(1\lor 2)}(o,p;u',v'...)} = O_{O_{P,1}(o,p;u',v'...)} \lor O_{O_{P,2}(o,p;u',v'...)}
\]
Fusion composition

Simultaneous retrentions. Disjunctive composition.

Suppose given:

Abs/Conc retrenchment: \( \mathcal{O}_A \leq_{G,P_1,o_1,c_1} \mathcal{O}_C \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Abs/Conc retrenchment: \( \mathcal{O}_A \leq_{G,P_2,o_2,c_2} \mathcal{O}_C \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Then there is an Abs/Conc retrenchment:

\[ \mathcal{O}_A \leq_{G,(P_1 \lor P_2),(o_1 \lor o_2),(c_1 \lor c_2)} \mathcal{O}_C \] from \((u,i,u',o)\) to \((v,j,v',p)\)

given by:

\[ P_{\mathcal{O}_A,(1 \lor 2)}(i,j,u,v) = P_{\mathcal{O}_A,1}(i,j,u,v) \lor P_{\mathcal{O}_A,2}(i,j,u,v) \]

\[ O_{\mathcal{O}_A,(1 \lor 2)}(o,p;u',v' \ldots) = O_{\mathcal{O}_A,1}(o,p;u',v' \ldots) \lor O_{\mathcal{O}_A,2}(o,p;u',v' \ldots) \]

\[ C_{\mathcal{O}_A,(1 \lor 2)}(u',v',o,p;\ldots) = C_{\mathcal{O}_A,1}(u',v',o,p;\ldots) \lor C_{\mathcal{O}_A,2}(u',v',o,p;\ldots) \]
For all proposed notions of composition, the key issues are:

- Does the notion compose to give a retrenchment?
- Does the notion compose to give a retrenchment satisfying its own definition?
- Does the notion yield a composition which is **associative**?
- Whenever the notion is defined by propositional manipulations of retrenchment data, then there will be stronger variants.
Decomposition (Mike Poppleton)

Suppose given:

\[
\text{Abs/Conc retrenchment: } O_{P_A} \preceq_{G,P,O,C} O_{P_C} \text{ from } (u, i, u', o) \text{ to } (v, j, v', p)
\]
Decomposition (Mike Poppleton)

Suppose given:

Abs/Conc retrenchment: \( O_{P_A} \preceq_{G,P,O,C} O_{P_C} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Suppose that the domain of \( O_{P_A} \) decomposes as \( \text{dom}(O_{P_A}) = a_{O_{P_A},1} \uplus a_{O_{P_A},2} \uplus \cdots a_{O_{P_A},k} \)

Suppose that the domain of \( O_{P_C} \) decomposes as \( \text{dom}(O_{P_C}) = c_{O_{P_C},1} \uplus c_{O_{P_C},2} \uplus \cdots c_{O_{P_C},l} \)

Let \( O_{P_A,i} = a_{O_{P_A},i} \mathrel{\triangleleft} O_{P_A} \) and \( O_{P_C,j} = c_{O_{P_C},j} \mathrel{\triangleleft} O_{P_C} \) and \( P_{O,p,i} = a_{O_{P},i} \mathrel{\triangleleft} P_{O} \mathrel{\triangleright} c_{O_{P},j} \)
Decomposition (Mike Poppleton)

Suppose given:

Abs/Conc retrenchment: \( O_{PA} \preceq_{G,P,O,C} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Suppose that the domain of \( O_{PA} \) decomposes as \( \text{dom}(O_{PA}) = a_{Op,1} \cup a_{Op,2} \cup \ldots a_{Op,k} \)

Suppose that the domain of \( O_{PC} \) decomposes as \( \text{dom}(O_{PC}) = c_{Op,1} \cup c_{Op,2} \cup \ldots c_{Op,l} \)

Let \( O_{PA,i} = a_{Op,i} \ll O_{PA} \) and \( O_{PC,j} = c_{Op,j} \ll O_{PC} \) and \( P_{Op,ij} = a_{Op,i} \ll P_{Op} \gg c_{Op,j} \)

Then for all pairs \( i,j: O_{PA} \preceq_{G,P_{ij},O,C} O_{PC} \) and \( O_{PA,i} \preceq_{G,P_{ij},O,C} O_{PC,j} \)
Decomposition (Mike Poppleton)

Suppose given:

*Abs/Conc* retrenchment: \( O_{PA} \preceq_{G,P,O,C} O_{PC} \) from \((u,i,u',o)\) to \((v,j,v',p)\)

Suppose that the domain of \( O_{PA} \) decomposes as 
\[ \text{dom}(\text{dom}(O_{PA})) = a_{Op,1} \uplus a_{Op,2} \uplus \ldots \uplus a_{Op,k} \]

Suppose that the domain of \( O_{PC} \) decomposes as 
\[ \text{dom}(\text{dom}(O_{PC})) = c_{Op,1} \uplus c_{Op,2} \uplus \ldots \uplus c_{Op,l} \]

Let 
\[ O_{PA,i} = a_{Op,i} \prec O_{PA} \] and 
\[ O_{PC,j} = c_{Op,j} \prec O_{PC} \] and 
\[ P_{Op,ij} = a_{Op,i} \prec P_{Op} \uplus c_{Op,j} \]

Then for all pairs \( i,j \):
\[ O_{PA} \preceq_{G,P_{ij},O,C} O_{PC} \] and 
\[ O_{PA,i} \preceq_{G,P_{ij},O,C} O_{PC,j} \]

The original retrenchment \( O_{PA} \preceq_{G,P,O,C} O_{PC} \) can now be recovered by fusion composition.
Decomposition (Mike Poppleton)

Suppose given:

Abs/Conc retrenchment: $O_p_A \preceq_{G,P,O,C} O_p_C$ from $(u,i,u',o)$ to $(v,j,v',p)$

Suppose that the domain of $O_p_A$ decomposes as $\text{dom}(O_p_A) = a_{O_p,1} \uplus a_{O_p,2} \uplus \ldots a_{O_p,k}$

Suppose that the domain of $O_p_C$ decomposes as $\text{dom}(O_p_C) = c_{O_p,1} \uplus c_{O_p,2} \uplus \ldots c_{O_p,l}$

Let $O_{p_A,i} = a_{O_p,i} \triangleleft O_p_A$ and $O_{p_C,j} = c_{O_p,j} \triangleleft O_p_C$ and $P_{op,ij} = a_{op,i} \triangleleft P_{op} \triangleright c_{op,j}$

Then for all pairs $i,j$: $O_{p_A} \preceq_{G,P_{ij},O,C} O_{p_C}$ and $O_{p_A,i} \preceq_{G,P_{ij},O,C} O_{p_C,j}$

The original retrenchment $O_{p_A} \preceq_{G,P,O,C} O_{p_C}$ can now be recovered by fusion composition. ■ ■ ■
Varieties of retrenchment; general retrenchment
Varieties of retrenchment; general retrenchment

Primitive retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]
Varieties of retrenchment; general retrenchment

Primitive retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]

Output retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \land O_{Op}) \lor C_{Op})) \]
Varieties of retrenchment; general retrenchment

Primitive retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]

Output retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \land O_{Op}) \lor C_{Op})) \]

Sharp retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \lor C_{Op}) \land V_{Op})) \]
Varieties of retrenchment; general retrenchment

Primitive retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]

Output retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \land O_{Op}) \lor C_{Op})) \]

Sharp retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \lor C_{Op}) \land V_{Op})) \]

Sharp output retrenchment

\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (((G \land O_{Op}) \lor C_{Op}) \land V_{Op})) \]
Varieties of retrenchment; general retrenchment

Primitive retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]

Output retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \land O_{Op}) \lor C_{Op})) \]

Sharp retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \lor C_{Op}) \land V_{Op})) \]

Sharp output retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (((G \land O_{Op}) \lor C_{Op}) \land V_{Op})) \]

General retrenchment
\[ \Phi_{Op} \ldots \Rightarrow (\exists \ldots \Theta_{Op}) \ldots \text{where } \Phi_{Op} \text{ and } \Theta_{Op} \text{ are as general as possible.} \]
Varieties of retrenchment; general retrenchment

Primitive retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (G \lor C_{Op})) \]

Output retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \land O_{Op}) \lor C_{Op})) \]

Sharp retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots ((G \lor C_{Op}) \land V_{Op})) \]

Sharp output retrenchment
\[ G \land P_{Op} \ldots \Rightarrow (\exists \ldots (((G \land O_{Op}) \lor C_{Op}) \land V_{Op})) \]

General retrenchment
\[ \Phi_{Op} \ldots \Rightarrow (\exists \ldots \Theta_{Op}) \ldots \text{where } \Phi_{Op} \text{ and } \Theta_{Op} \text{ are as general as possible.} \]

Associativity!
Algebraic theory of retrenchment and refinement
Algebraic theory of retrenchment and refinement

Factorisations of a retrenchment (Czeslaw Jeske’s M.Sc.)
Algebraic theory of retrenchment and refinement

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Why? Why?
Algebraic theory of retrenchment and refinement

Factorisations of a retrenchment (Czeslaw Jeske’s M.Sc.)

Why?
Lift new requirements to Abs level automatically.

Why?
Relate Abs to ‘Conc’ Abs automatically.
Completion constructions I (Czeslaw Jeske's Ph.D.)

\[ \text{Abs} \xrightarrow{\text{RET}} \text{Ret} \]

\[ \text{Ref} \xrightarrow{\text{REF}} \text{Ret} \xrightarrow{\text{RET}} \text{Conc} \]
Completion constructions I (Czeslaw Jeske’s Ph.D.)

\[
\begin{align*}
Abs & \xrightarrow{\text{RET}} Ret \\
Ref & \xrightarrow{\text{RET}} Univ \\
Univ & \xrightarrow{\text{RET}} Ref \\
Ret & \xrightarrow{\text{RET}} Conc
\end{align*}
\]
Completion constructions I (Czeslaw Jeske's Ph.D.)

Abs \rightarrow Ret

Ref \rightarrow Univ

Xtra

Univ \rightarrow Ref

Ret \rightarrow Conc
Completion constructions I (Czeslaw Jeske’s Ph.D.)
Completion constructions I (Czeslaw Jeske's Ph.D.)

Why?
Generate refinement of new requirements automatically.

Why?
Generate abstraction of low level unpatch automatically.
Completion constructions II (Czeslaw Jeske's Ph.D.)

Abs  
\[\text{RET}\]  
Ref  
\[\text{RET}\]  
Conc

Abs  
\[\text{RET}\]  
Ret  
\[\text{REF}\]  
Conc
Completion constructions II (Czeslaw Jeske’s Ph.D.)

\[
\begin{align*}
\text{Abs} &\xrightarrow{\text{RET}} \text{Univ} \\
\text{Ref} &\xrightarrow{\text{RET}} \text{Conc}
\end{align*}
\]
Completion constructions II (Czeslaw Jeske’s Ph.D.)

Abs \rightarrow \text{Univ}
\text{Ref} \rightarrow \text{Conc}

Abs \rightarrow \text{Ret}
\text{Univ} \rightarrow \text{Conc}

Abs \rightarrow Xtra

Abs \rightarrow RET
Univ \rightarrow CONC
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Patterns of Retrenchment

Just as software has identified commonly occurring ‘patterns’, we can see something similar for retrenchment, *independent of the issue dealt with in the retrenchment.*
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Most evident is the ‘**Tower Pattern**’, built top down or bottom up ...:
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![Diagram]

‘Ideal development’

‘Realistic development’
Patterns of Retrenchment

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![Diagram](image_url)

- 'Ideal development'
- 'Realistic development'

Generate
(Semi-)
Automatically
Applicability/correctness

Various refinement theories augment the operation PO with other criteria that a pair of operations must satisfy to be in the corresponding refinement relation.

These criteria depend strongly on the programming/systems theory used.

Examples:
- Applicability ... eg. Z methodology, ASM methodology
- Termination ... eg. Refinement calculus, B-Method
- Feasibility ... eg. Refinement calculus, B-Method

These give rise to POs that are implications ‘θ_A ⇒ θ_C’ from abstract to concrete, or ‘θ_C ⇒ θ_A’ from concrete to abstract, depending on the criterion and the theory.
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These give rise to POs that are implications \( \theta_A \Rightarrow \theta_C \) from abstract to concrete, or \( \theta_C \Rightarrow \theta_A \) from concrete to abstract, depending on the criterion and the theory.

Moreover, all the criteria known, depend on just the before states and inputs, either abstract or concrete.

Retrenchment advocates a generally applicable way of embracing all such theories.
Broad Principles:

1. Useful systems consist of successful operation steps; failure of applicability/termination/feasibility etc. is connected with unsuccessful steps. Retrenchment should be primarily concerned with useful/successful steps. These are steps for which **ALL** the antecedents are valid (and thus all the consequents too).

2. Retrenchment wants to *favour* ‘stepwise simulation’ theorems.
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The simplest solution:

Given a specific theory of correctness, and a prospective retrenchment between systems Abs/Conc given by $G, P_{Op}, O_{Op}, C_{Op}$:

Insist that for every relevant $\theta$ of the theory in question:

$$\text{dom}(G \land P_{Op}) \subseteq \theta_{A,Op}$$
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\end{align*}
\]

This ensures that the transitions about which the retrenchment operation PO speaks, are indeed well behaved, useful/successful steps. Also the PO itself is unaffected.
Simulation and behavioural properties

The simulation relation holds when all the facts spoken of in the PO are verified:

\[ G(u,v) \land P_{Op}(i,j,u,v) \land \Sigma^1 \land ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \lor C_{Op}(u',v',o,p;u,v,i,j)) \]

\[ G(u,v) \land P_{Op}(i,j,u,v) \land O_{Op}(i,j,u,v) \land O_{Op}(u,i,u,o) \land ((G(u',v') \land O_{Op}(o,p;u',v',u,v,i,j)) \lor C_{Op}(u',v',o,p;u,v,i,j)) \]
Simulation and behavioural properties

The simulation relation holds when all the facts spoken of in the PO are verified:

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G(u, v) \land P_{Op}(i, j, u, v) \land \forall u' \exists o \exists v' \exists p \exists \Sigma^1 (\Sigma^1) \land O_{Op}(o, p; u', v', u, v, i, j) \lor C_{Op}(u', v', o, p; u, v, i, j)
\]

A **fragment** is a sequence of steps \( S = [u_0 \rightarrow (i_0, Op_{A,0}, o_1) \rightarrow u_1 \rightarrow (i_1, Op_{A,1}, o_2) \rightarrow u_2 \ldots ] \)

A **multifragment** is a sequence of fragments \( S = [S_0, S_1, \ldots ] \)
The ideal for retrenchment is the stepwise simulation relationship for fragments, i.e. an induction-like result thus:
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![Diagram of stepwise simulation relationship](image)
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Retrenchment can *encourage* if not guarantee this, as the final configuration of one step need not satisfy the $G \land P_{Op}$ of the next one (cf. horizontal composition).

With additional (strong, effectively refinement-like) assumptions, it can be achieved.
The ideal for retrenchment is the stepwise simulation relationship for fragments, i.e. an induction-like result thus:

Retrenchment can *encourage* if not guarantee this, as the final configuration of one step need not satisfy the $G \land P_{Op}$ of the next one (cf. horizontal composition).

With additional (strong, effectively refinement-like) assumptions, it can be achieved. But ... that misses the point.

Retrenchment is primarily for cases when the induction fails. So it’s best to deal with simulation directly.
Multifragment simulation permits situations $S \Sigma \mathcal{T}$ like the following:
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Vertical composition of multifragment simulations is *encouraged* by biretrenchment.

Biretrenchment:

\[
G \land P_{op} \land P_{op_c} \Rightarrow (\exists \ldots O_{pa} \land ((G' \land O_{op}) \lor C_{op}))
\]

\[
G \land P_{op} \land P_{op_a} \Rightarrow (\exists \ldots O_{pc} \land ((G' \land O_{op}) \lor C_{op}))
\]
Suitably constrained to avoid excessive fragmentation, sets of multifragments define properties.

The multifragment simulation relation yields simulation transformers for properties:

\[
\begin{align*}
[\Sigma] SS &= \{ T | (\exists S \cdot S \in SS \land S \Sigma T) \land (\forall S \cdot S \Sigma T \Rightarrow S \in SS) \} \\
\langle\Sigma\rangle SS &= \{ T | (\exists S \cdot S \in SS \land S \Sigma T) \} \\
[\Sigma] TT &= \{ S | (\exists T \cdot T \in TT \land S \Sigma T) \land (\forall T \cdot S \Sigma T \Rightarrow T \in TT) \} \\
\langle\Sigma\rangle TT &= \{ T | (\exists T \cdot T \in TT \land S \Sigma T) \}
\end{align*}
\]

These map abstract properties to concrete properties and vice versa, and act like box and diamond operators in a modal algebra.
Suitably constrained to avoid excessive fragmentation, sets of multifragments define **properties**.

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\langle \Sigma \rangle SS = \{ T | (\exists S \cdot S \in SS \land S \Sigma T) \}
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[\Sigma] TT = \{ S | (\exists T \cdot T \in TT \land S \Sigma T) \land (\forall T \cdot S \Sigma T \Rightarrow T \in TT) \}
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\[
\langle \Sigma \rangle TT = \{ T | (\exists T \cdot T \in TT \land S \Sigma T) \}
\]

These map abstract properties to concrete properties and vice versa, and act like box and diamond operators in a modal algebra.

Unfortunately, the \([\Sigma] SS, \langle \Sigma \rangle SS, [\Sigma] TT, \langle \Sigma \rangle TT\) transformers leave the mapping of the non-simulable parts of multifragments completely unconstrained. This can be addressed by imposing additional constraints:

- via meta-linguistic means ...
- explicitly.
Coarse grained retrenchment

When a collection of individual steps is to be regarded as a whole, in effect the only issue is \textit{atomicity}. 
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When a collection of individual steps is to be regarded as a whole, in effect the only issue is atomicity. (Without atomicity, what’s the point?)

In coarse grained retrenchment atomicity is assumed.

In coarse grained retrenchment a partial states picture is adopted, the syntactic role is taken by relevant collections of individual steps which are encapsulated in prime event structures (ESs) $\mathcal{E} = (E, \leq, \#)$.

\[ \rightarrow \downarrow \alpha \quad \rightarrow \downarrow \beta \quad \downarrow \# \]
\[ \downarrow \beta \]
\[ \leq = (\bigcup \alpha < \alpha)^* \]
\[ e_1 \# e_2 \leq e_3 \Rightarrow e_1 \# e_3 \]

Variable values at root and leaf events of $E$ extend to the before- and after- values of the global state.
An execution structure is the corresponding (conflict free) runtime notion. It yields an execution fragment via serialisation. Fragments thus yield **deployments** of ESs.
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Variable values at root and leaf events of $E$ extend to before- and after- values of the global state.

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These global state values appear in the retrenchment PO for corresponding abstract and concrete ESs.
Interaction of coarse grained retrenchment with composition/decomposition mechanisms yields a fertile infrastructure for sophisticated analyses of system behaviour.
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Example: Aspect oriented programming

• Start with base language and functional aspect of top level operations.

• Identify ‘weave points’ for other aspects. Decompose top level operations at weave points using sequential/parallel decomposition, into finer grained actions.

• Capture the ‘weaving in’ of additional aspects as suitable coarse grained retrenchments of the original decomposed top level operations.

• Analyse the resulting coarse grained retrenchments. Do they yield the properties expected of the included additional aspects?
Example: Complex Systems

- Identify basic components of the complex system.
- Are they already too complicated to understand properly? If so simplify them.
- Determine the assembly of basic components to make up the global system, describe it using sequential/parallel/dataflow composition to get top level operations of the global system.
- Capture the incorporation of more of the properties of the basic components into the global model using successive coarse grained retrenchments. Can emerging properties be identified in the retrenchment analysis?
- Compare calculated retrenchment for the composed system with any ad hoc retrenchment written for it. Compare calculated retrenchments for the decomposed components with ad hoc retrenchments describing increments of basic component properties.
Probabilistic retrenchment

Fundamental Abs transition probabilities: $p_{Op_A}(u', o | u, i)$ so $\sum_{(u', o)} p_{Op_A}(u', o | u, i) = 1$
Probabilistic retrenchment

Fundamental $Abs$ transition probabilities: $p_{OpA}(u',o | u,i)$ so $\sum_{(u',o)} p_{OpA}(u',o | u,i) = 1$

Given a before- distribution $p(u,i)$, the probability of achieving $\Phi(u',o)$ in one step is:

$$\sum_{(u',o,u,i)} p_{OpA}(u',o | u,i) \cdot p(u,i) \cdot \Phi(u',o)$$  ({$\Phi(u',o)$ the characteristic function of $\Phi$})
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Similarly, calculate the probability of achieving a free stepwise simulation of length $n$, starting from $(u, v)$ and with I/O $(is, js, os, ps)$:
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$$n = 1: \quad \Pi_{GO,1} = \sum_{(u_1, v_1)} \Pi_{GO,1}(u_1, v_1)$$

where \(\Pi_{GO,1}(u_1, v_1) = p_{OpA,1}(u_1, os_1 \mid u, is_0) \cdot p_{OpC,1}(v_1, ps_1 \mid v, js_0) \cdot G(u_1, v_1) \cdot O_{Op1}(os_1, ps_1; u_1, v_1, u, v, is_0, js_0) \cdot G(u, v) \cdot P_{Op1}(is_0, js_0, u, v)$$
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Similarly, calculate the probability of achieving a free stepwise simulation of length \( n \), starting from \( (u,v) \) and with I/O \((is,js,os,ps)\):

\( n = 1: \quad \Pi_{GO,1} = \sum_{(u_1,v_1)} \Pi_{GO,1}(u_1,v_1) \) where \( \Pi_{GO,1}(u_1,v_1) = \)

\[
p_{\text{Op}_A,1}(u_1,os_1 \mid u,is_0) \cdot p_{\text{Op}_C,1}(v_1,ps_1 \mid v,js_0). G(u_1,v_1). O_{\text{Op}_1}(os_1,ps_1;u_1,v_1,u,v,is_0,js_0). G(u,v). P_{\text{Op}_1}(is_0,js_0,u,v)
\]

\( \Pi_{C,1} = \sum_{(u_1,v_1)} \Pi_{C,1}(u_1,v_1) \) where \( \Pi_{C,1}(u_1,v_1) = \)

\[
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\( n = 1: \quad \Pi_{\text{GO}, 1} = \sum_{(u_1, v_1)} \Pi_{\text{GO}, 1}(u_1, v_1) \) where \( \Pi_{\text{GO}, 1}(u_1, v_1) = \)

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\( \Pi_{\text{C}, 1} = \sum_{(u_1, v_1)} \Pi_{\text{C}, 1}(u_1, v_1) \) where \( \Pi_{\text{C}, 1}(u_1, v_1) = \)

\[
p_{\text{Op}_A, 1}(u_1, os_1 \mid u, is_1) \cdot p_{\text{Op}_C, 1}(v_1, ps_1 \mid v, js_1) \cdot C_{\text{Op}_1}(u_1, v_1, os_1, ps_1; u, v, is_0, js_0) \cdot G(u, v) \cdot P_{\text{Op}_1}(is_0, js_0, u, v)
\]

\[
\Pi_{\text{GO}, 1} + \Pi_{\text{C}, 1} - \Pi_{\text{GO} \text{C}, 1} = 1
\]
1 < l < n: \[ \Pi_{GO,l} = \sum_{(u_l, v_l)} \Pi_{GO,l}(u_l, v_l) \] where \[ \Pi_{GO,l}(u_l, v_l) = \sum_{(u_{l-1}, v_{l-1})} P_{OP_{A,l}}(u_{l-1}, os_l | u_{l-1}, is_{l-1}) \cdot P_{OP_{C,l}}(v_{l-1}, ps_l | v_{l-1}, js_{l-1}) \cdot \Pi_{GO,l-1}(u_{l-1}, v_{l-1}) \cdot G(u_l, v_l) \cdot O_{Op_l}(os_l, ps_l, u_l, v_l, u_{l-1}, v_{l-1}, is_{l-1}, js_{l-1}) \cdot P_{Op_l}(is_{l-1}, js_{l-1}, u_{l-1}, v_{l-1}) \]
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n: \[ \Pi_{GO,n} = \sum_{(u_n, v_n)} \Pi_{GO,n}(u_l, v_l) \quad \text{where} \quad \Pi_{GO,n}(u_n, v_n) = \sum_{(u_{n-1}, v_{n-1})} P_{OpA,n}(u_n, os_n | u_{n-1}, is_{n-1}) \cdot P_{OpC,n}(v_n, ps_n | v_{n-1}, js_{n-1}) \cdot \Pi_{GO,n-1}(u_{n-1}, v_{n-1}) \cdot G(u_n, v_n) \cdot O_{Op}(os_n, ps_n; u_n, v_n, u_{n-1}, v_{n-1}, is_{n-1}, js_{n-1}) \cdot P_{Op}(is_{n-1}, js_{n-1}, u_{n-1}, v_{n-1}) \]
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\[
\Pi_{GO,l}(u_l,v_l) = \\
\sum_{(u_{l-1},v_{l-1})} \cdot \cdot \cdot 
\]
\[
= \sum_{(u_{l-1},v_{l-1})} \cdot \cdot \cdot 
\]
\[
G(u_l,v_l) \cdot Op_l (os_l,ps_l|u_l,v_l,u_{l-1},v_{l-1},is_{l-1},js_{l-1}) \cdot Op_l (is_{l-1},js_{l-1},u_{l-1},v_{l-1})
\]
\[
\Pi_{GO,n} = \sum_{(u_n,v_n)} \Pi_{GO,n}(u_n,v_n) \] where
\[
\Pi_{GO,n}(u_n,v_n) = \\
\sum_{(u_{n-1},v_{n-1})} \cdot \cdot \cdot 
\]
\[
= \sum_{(u_{n-1},v_{n-1})} \cdot \cdot \cdot 
\]
\[
G(u_n,v_n) \cdot Op_n (os_n,ps_n|u_n,v_n,u_{n-1},v_{n-1},is_{n-1},js_{n-1}) \cdot Op_n (is_{n-1},js_{n-1},u_{n-1},v_{n-1})
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\Pi_{C,n} = \sum_{(u_n,v_n)} \Pi_{C,n}(u_n,v_n) \] where
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\]
\[
= \sum_{(u_{n-1},v_{n-1})} \cdot \cdot \cdot 
\]
\[
C_{Op_n}(u_1,v_1,os_1,ps_1|u,v,is_0,js_0) \cdot Op_n (is_0,js_0,u,v)
\]
\[
\Pi_{GO,1}
\]
\[ 1 < l < n: \quad \Pi_{GO,l} = \sum_{(u_l, v_l)} \Pi_{GO,l}(u_l, v_l) \quad \text{where} \quad \Pi_{GO,l}(u_l, v_l) = \sum_{(u_{l-1}, v_{l-1})} P_{OpA,l}(u_{l-1}, os_{l-1}, v_{l-1}, is_{l-1}) \cdot P_{OpC,l}(v_{l-1}, ps_{l-1}, v_l, js_{l-1}) \cdot \Pi_{GO,l-1}(u_{l-1}, v_{l-1}) \cdot G(u_l, v_l) \cdot O_{Op}(os_{l}, ps_{l}, u_l, v_l, u_{l-1}, v_{l-1}, is_{l-1}, js_{l-1}) \cdot P_{Op}(is_{l-1}, js_{l-1}, u_{l-1}, v_{l-1}) \]

\[ n: \quad \Pi_{GO,n} = \sum_{(u_n, v_n)} \Pi_{GO,n}(u_n, v_n) \quad \text{where} \quad \Pi_{GO,n}(u_n, v_n) = \sum_{(u_{n-1}, v_{n-1})} P_{OpA,n}(u_n, os_{n-1}, u_{n-1}, is_{n-1}) \cdot P_{OpC,n}(v_n, ps_{n}, v_{n-1}, js_{n-1}) \cdot \Pi_{GO,n-1}(u_{n-1}, v_{n-1}) \cdot G(u_n, v_n) \cdot O_{Op}(os_n, ps_n, v_n, u_{n-1}, v_{n-1}, u_n, v_n, u_{n-1}, v_{n-1}, is_{n-1}, js_{n-1}) \cdot P_{Op}(is_{n-1}, js_{n-1}, u_{n-1}, v_{n-1}) \]

\[ \Pi_{C,n} = \sum_{(u_n, v_n)} \Pi_{C,n}(u_n, v_n) \quad \text{where} \quad \Pi_{C,n}(u_n, v_n) = \sum_{(u_{n-1}, v_{n-1})} P_{OpA,n}(u_n, os_{n}, u_{n-1}, is_{n}) \cdot P_{OpC,n}(v_n, ps_{n}, v_{n-1}, js_{n}) \cdot \Pi_{GO,n-1}(u_{n-1}, v_{n-1}) \cdot C_{Op}(u_1, v_1, os_1, ps_1, u, v, is_0, js_0) \cdot P_{Op}(is_0, js_0, u, v) \]
\[
\begin{align*}
1 < l < n: \quad & \Pi_{GO,l} = \sum_{(u_l,v_l)} \Pi_{GO,l}(u_l,v_l) \quad \text{where} \quad \Pi_{GO,l}(u_l,v_l) = \\
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& G(u_l,v_l) \cdot O_{Op_l}(os_l,ps_l;u_l,v_l,u_{l-1},v_{l-1},is_{l-1},js_{l-1}) \cdot P_{Op_l}(is_{l-1},js_{l-1},u_{l-1},v_{l-1}) \\
\end{align*}
\]

\[
\begin{align*}
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& G(u_n,v_n) \cdot O_{Op_n}(os_n,ps_n;u_n,v_n,u_{n-1},v_{n-1},is_{n-1},js_{n-1}) \cdot P_{Op_n}(is_{n-1},js_{n-1},u_{n-1},v_{n-1}) \\
\end{align*}
\]

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& C_{Op_n}(u_1,v_1,os_1,ps_1;u,v,is_0,js_0) \cdot P_{Op_n}(is_0,js_0,u,v)
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\]
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\[ \Pi_{GO,4} \]
\[ 1 < l < n: \quad \Pi_{GO,l} = \sum_{(u_l, v_l)} \Pi_{GO, l}(u_l, v_l) \quad \text{where} \quad \Pi_{GO, l}(u_l, v_l) = \]
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Easy variations produce:

- Probabilities for stepwise simulation of *given* concrete fragment.
- Probabilities for stepwise simulation with unrestricted I/O.
- Probabilities for coarse grained retrenchments.
- etc.
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Probabilistic analysis can be mapped over the various composition/decomposition mechanisms.
Applications: special purpose retrenchments

Toy case studies (couple of pages ... discrete ones *always* doable by refinement)

- Finite precision: sets/multisets $\rightarrow$ sequences, bounded arithmetic ...
- Simple resource control
- Gas turbine control
- Control theory

Larger case studies (several pages $\rightarrow$ entire paper $\rightarrow$ several papers)

- Feature interaction
- Mondex purse
- Comparative evaluation of graphics algorithms
- More control theory
- Radiotherapy dose calculations ... Other scientific apps. ...
Special purpose retrenchments (designed for further processing to some objective)

• Fault tree generation (with Marco Bozzano):
  retrenchment for any given component changes according to its place in the complete system

• Quantum retrenchment; obvious connections with probabilistic retrenchment

There may be several different retrenchments worth contemplating for the same pair of systems for different purposes. ... Fusion composition.
Tool support

Simon Fraser’s M.Phil. project incorporated retrenchment PO generation into B-Core’s B-Toolkit™ environment.

- It’s not public domain.
- Many aspects of the B-Toolkit are optimised for a linear B-Method development.
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Simon Fraser’s Ph.D. project is to design and build an open source, flexible, extensible, toolkit for experimenting with retrenchment.

- ISO-Z (extended) used as the mathematical language
- A ‘wrapper’ syntax for building systems out of pieces of maths
- PO generation for a variety of theorem provers:
  - PVS, B-Tool, Rodin, Isabelle, Vampire, KIV, ...
  - Engineering of environment information ...
- Ultimately, a meta-language for expressing myriad composition techniques etc.
Methodology

The big question is:

‘How does all the formal stuff fit with engineering practice?’
Methodology

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Two aspects:

1. How should individual ingredients of retrenchment impact practice?
2. How should practice impact the details of retrenchment?
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Re. 1., as a start, practice can systematise checklists of things to oversee during periodic reviews of a development utilising retrenchment steps (the ‘glass box’), eg.:
   • Adequacy/coverage of the domain/range of the within relation,
   • Adequacy/coverage of the domain/range of the output and concedes relations,
   • Adequacy/coverage of the abstract and concrete operations ...
   termination/feasibility ... etc.
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Re. 2., as a start, retrenchment ought to consider how its remit could be widened to take on board more of the development’s non-functional requirements.
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Retrenchment strives to make quantitative, the system engineering process as it is.
5. The Retrenchment Homepage

See the Retrenchment Homepage: Google knows where it is.

http://www.cs.man.ac.uk/retrenchment

• Bibliographical pointers.
• This tutorial.
• Etc.