Cryptography and Network Security Chapter 4

Fifth Edition

by William Stallings

Lecture slides by Lawrie Brown

(with edits by RHB)

Chapter 4 – Basic Concepts in Number Theory and Finite Fields

The next morning at daybreak, Star flew indoors, seemingly keen for a lesson. I said, "Tap eight." She did a brilliant exhibition, first tapping it in 4, 4, then giving me a hasty glance and doing it in 2, 2, 2, 2, before coming for her nut. It is astonishing that Star learned to count up to 8 with no difficulty, and of her own accord discovered that each number could be given with various different divisions, this leaving no doubt that she was consciously thinking each number. In fact, she did mental arithmetic, although unable, like humans, to name the numbers. But she learned to recognize their spoken names almost immediately and was able to remember the sounds of the names. Star is unique as a wild bird, who of her own free will pursued the science of numbers with keen interest and astonishing intelligence.

— *Living with Birds*, Len Howard

Outline

- will consider:
  - divisibility & GCD
  - modular arithmetic with integers
  - concept of groups, rings, fields
  - Euclid's algorithm for GCD & inverse
  - finite fields GF(p)
  - polynomial arithmetic in general and in GF(2^n)

Introduction

- we build up to introduction of finite fields
- of increasing importance in cryptography
  - AES, Elliptic Curve, IDEA, Public Key
- concern operations on "numbers"
  - where what constitutes a "number" and the type of operations varies considerably
- start with basic number theory concepts

Divisors

- say a non-zero number b **divides** a if for some m have a = m.b (a,b,m all integers)
- that is b divides into a with no remainder
- write this b|a
- and say that b is a **divisor** of a
- eg. all of 1,2,3,4,6,8,12,24 divide 24
- eg. 13|182 ; -5|30 ; 17|289 ; -3|33 ; 17|0

Properties of Divisibility

- If a|1, then a = ±1.
- If a|b and b|a, then a = ±b.
- Any b ≠ 0 divides 0.
- If a|b and b|c, then a|c
  - e.g. 11|66 and 66|198 implies 11|198
- If b|g and b|h, then b|(mg + nh)

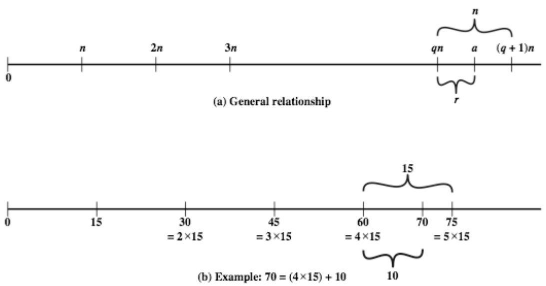
  (for arbitrary integers m and n)

  e.g. b = 7 ; g = 14 ; h = 63 ; m = 3 ; n = 2

  7|14 and 7|63, hence 7|(3.14 + 2.63)

Division Algorithm

- if divide a by n get integer quotient q and integer remainder r such that:
  - a = qn + r where 0 <= r < n; q = floor(a/n)
- remainder r often referred to as a **residue**



Modular Arithmetic

- define **modulo operation** a mod n to yield remainder b when a is divided by n
  - where integer n is called the **modulus**
- b is called a **residue** of a mod n
  with integers can always write: a = qn + b
  - usually choose smallest positive remainder as residue
    - ie. 0 <= b <= n-1
  - known as **modulo reduction**
    - eg. -12 mod 7 = -5 mod 7 = 2 mod 7 = 9 mod 7
- a and b are **congruent** if a mod n = b mod n
  - a and b have same remainder when divided by n
  - eg. 100 = 34 mod 11

Modular Arithmetic Operations

- can perform arithmetic with residues
- use a finite number of values, and loop back from either end

  $Z_n = \{0, 1, \ldots, (n-1)\}$
- modular arithmetic is doing addition and multiplication and modulo reduce answer
- can do reduction at any point, i.e.

  a + b mod n = [a mod n + b mod n] mod n

Modular Arithmetic Operations

1. [(a mod n) + (b mod n)] mod n = (a + b) mod n
2. [(a mod n) – (b mod n)] mod n = (a – b) mod n
3. [(a mod n) x (b mod n)] mod n = (a x b) mod n

e.g.

[(11 mod 8) + (15 mod 8)] mod 8 = 10 mod 8 = 2 (11 + 15) mod 8 = 26 mod 8 = 2

[(11 mod 8) – (15 mod 8)] mod 8 = –4 mod 8 = 4 (11 – 15) mod 8 = –4 mod 8 = 4

[(11 mod 8) x (15 mod 8)] mod 8 = 21 mod 8 = 5 (11 x 15) mod 8 = 165 mod 8 = 5

Modulo 8 Addition

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Modulo 8 Multiplication

+	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Modulo 8 Inverses

w	$-w$	w^{-1}
0	0	—
1	7	1
2	6	—
3	5	3
4	4	—
5	3	5
6	2	—
7	1	7

(c) Additive and multiplicative inverses modulo 8

Modular Arithmetic Properties

Property	Expression
Commutative laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive law	$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 \times w) \bmod n = w \bmod n$
Additive inverse ($-w$)	For each $w \in Z_n$, there exists a z such that $w + z = 0 \bmod n$

Greatest Common Divisor (GCD)

- a common problem in number theory
- `GCD(a,b)` of `a` and `b` is the largest integer that divides exactly into both `a` and `b`
  - eg. `GCD(60,24) = 12`
- define `GCD(0,0) = 0`
- often want **no common factors** (except 1) such numbers **relatively prime / coprime**
  - eg. `GCD(8,15) = 1`
  - hence 8 are 15 are relatively prime or coprime

Euclidean Algorithm

- an efficient way to find the GCD(a,b)
- uses theorem that:
  - `GCD(a,b) = GCD(b, a mod b)`
- Euclidean Algorithm to compute `GCD(a,b)` is:

```
Euclid(a,b)
  if (b=0) then return a;
  else return Euclid(b, a mod b);
```

Example GCD(1970,1066)

```
1970 = 1 x 1066 + 904      gcd(1066, 904)
1066 = 1 x 904 + 162       gcd(904, 162)
904 = 5 x 162 + 94         gcd(162, 94)
162 = 1 x 94 + 68          gcd(94, 68)
94 = 1 x 68 + 26           gcd(68, 26)
68 = 2 x 26 + 16           gcd(26, 16)
26 = 1 x 16 + 10           gcd(16, 10)
16 = 1 x 10 + 6            gcd(10, 6)
10 = 1 x 6 + 4             gcd(6, 4)
6 = 1 x 4 + 2              gcd(4, 2)
4 = 2 x 2 + 0              gcd(2, 0)
```

GCD(1160718174, 316258250)

Dividend	Divisor	Quotient	Remainder
a = 1160718174	b = 316258250	q1 = 3	r1 = 211943424
b = 316258250	r1 = 211943424	q2 = 1	r2 = 104314826
r1 = 211943424	r2 = 104314826	q3 = 2	r3 = 3313772
r2 = 104314826	r3 = 3313772	q4 = 31	r4 = 1587894
r3 = 3313772	r4 = 1587894	q5 = 2	r5 = 137984
r4 = 1587894	r5 = 137984	q6 = 11	r6 = 70070
r5 = 137984	r6 = 70070	q7 = 1	r7 = 67914
r6 = 70070	r7 = 67914	q8 = 1	r8 = 2516
r7 = 67914	r8 = 2516	q9 = 31	r9 = 1078
r8 = 2516	r9 = 1078	q10 = 2	r10 = 0

Extended Euclidean Algorithm

- get not only GCD but x and y such that

  ax + by = d = GCD(a,b)
- useful for later crypto computations
- follow sequence of divisions for GCD but at each step i, keep track of x and y:

  r = ax + by
- at end find GCD value and also x and y
- if GCD(a,b) = 1 = ax + by then

  x is inverse of a mod b (or mod y)

Finding Inverses

```
EXTENDED EUCLID(m, b)
1. (A1, A2, A3)=(1, 0, m);
   (B1, B2, B3)=(0, 1, b)
2. if B3 = 0
   return A3 = GCD(m, b); no inverse
3. if B3 = 1
   return B3 = GCD(m, b); B2 = b^-1 mod m
4. Q = A3 div B3
5. (T1, T2, T3)=(A1 – Q B1, A2 – Q B2, A3 – Q B3)
6. (A1, A2, A3)=(B1, B2, B3)
7. (B1, B2, B3)=(T1, T2, T3)
8. goto 2
```

Inverse of 550 in GF(1759)

Q	A1	A2	A3	B1	B2	B3
—	1	0	1759	0	1	550
3	0	1	550	1	–3	109
5	1	–3	109	–5	16	5
21	–5	16	5	106	–339	4
1	106	–339	4	–111	355	1

Inverse of 550 in GF(1759)

Q	A1	A2	A3	B1	B2	B3
—	1	0	1759	0	1	550
3	0	1	550	1	–3	109
5	1	–3	109	–5	16	5
21	–5	16	5	106	–339	4
1	106	–339	4	–111	355	1

Group

- a set of elements or "numbers"
  - may be finite or infinite
- with some operation whose result is also in the set (closure)
- obeys:
  - associative law: `(a.b).c = a.(b.c)`
  - has identity `e`: `e.a = a.e = a`
  - has inverses a^{-1}: $a.a^{-1} = e$
- if commutative `a.b = b.a`
  - then forms an **abelian group**

Cyclic Group

- define **exponentiation** as repeated application of operator
  - example: $a^3 = a.a.a$
- and write identity as: $e = a^0$
- a group is cyclic if every element `b` is a power of some fixed element `a`
  - i.e. every $b = a^k$ for some `k`
- `a` is said to be a generator of the group

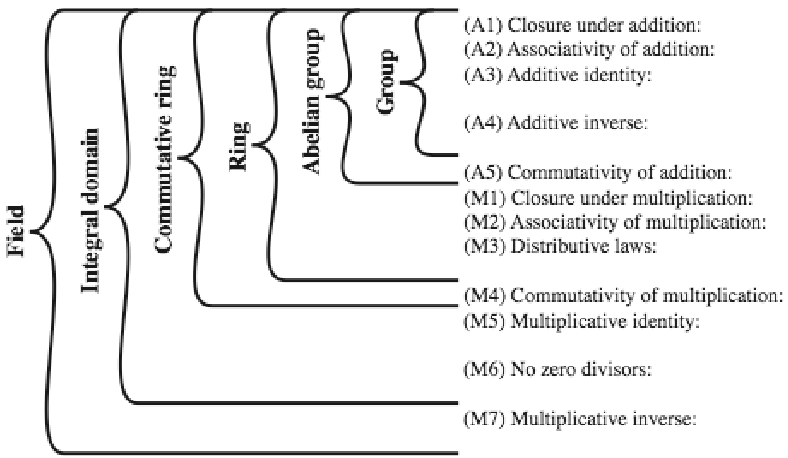
Ring

- a set of elements or “numbers”
- with two operations (addition and multiplication) which form:
- an abelian group with respect to addition
- and multiplication:
  - has closure
  - is associative
  - distributive over addition: $a(b+c) = ab + ac$
- if multiplication operation is *commutative*, we have a **commutative ring**
- if multiplication operation has an *identity* and *no zero divisors*, it forms an **integral domain**

Field

- a set of elements or “numbers”
- with two operations which form:
  - abelian group for addition
  - abelian group for multiplication (ignoring 0)
  - ring
- have hierarchy with more axioms/laws
  - group → ring → field

Group, Ring, Field



Finite (Galois) Fields

- finite fields play a key role in cryptography
- can show number of elements in a finite field **must** be a power of a prime p^n
- known as Galois fields
- denoted $GF(p^n)$
- in particular often use the fields:
  - $GF(p)$
  - $GF(2^n)$

Galois Fields GF(p)

- GF(p) is the set of integers `{0,1,…,p-1}` with arithmetic operations modulo prime `p`
- these form a finite field
  - `1…p-1` coprime to `p`, so have multiplicative inv.
  - find inverse with Extended Euclidean algorithm
- hence arithmetic is "well-behaved" and can do addition, subtraction, multiplication, and division without leaving the field GF(p)
- **everything works as expected**

GF(7) Multiplication

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Arithmetic in GF(7)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) Addition modulo 7

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(b) Multiplication modulo 7

w	$-w$	w^{-1}
0	0	—
1	6	1
2	5	4
3	4	5
4	3	2
5	2	3
6	1	6

(c) Additive and multiplicative inverses modulo 7

Polynomial Arithmetic

- can compute using polynomials

  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum a_i x^i$
  - nb. not interested in any specific value of x
  - x is the indeterminate … like an unspecified base
- several alternatives available
  - ordinary polynomial arithmetic
  - poly arithmetic with coefficients mod p
  - poly arithmetic with coefficients mod p and polynomials mod m(x)

Ordinary Polynomial Arithmetic

- add or subtract corresponding coefficients
- multiply all terms by each other
- e.g.
  let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$

  $f(x) + g(x) = x^3 + 2x^2 - x + 3$
  $f(x) - g(x) = x^3 + x + 1$
  $f(x) \times g(x) = x^5 + 3x^2 - 2x + 2$

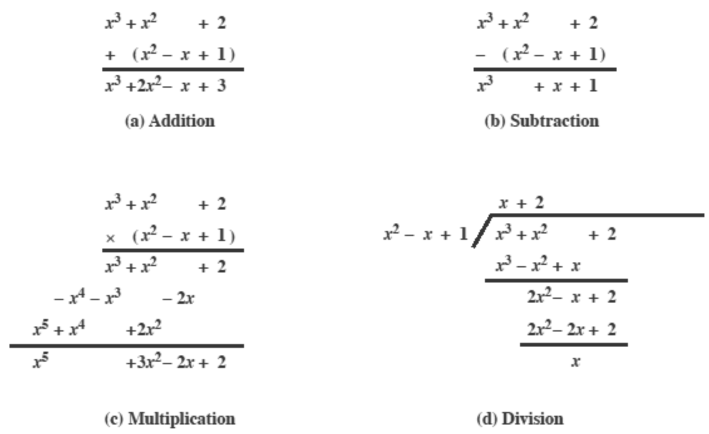


Figure 4.3 Examples of Polynomial Arithmetic

Polynomial Arithmetic with Modulo Coefficients

- when computing value of each coefficient do the calculation modulo some value
  - forms a polynomial ring
- could be modulo any prime
- but we are most interested in mod 2
  - ie all coefficients are 0 or 1
  - eg. let $f(x) = x^3 + x^2$ and $g(x) = x^2 + x + 1$

    $f(x) + g(x) = x^3 + x + 1$
    $f(x) \times g(x) = x^5 + x^2$

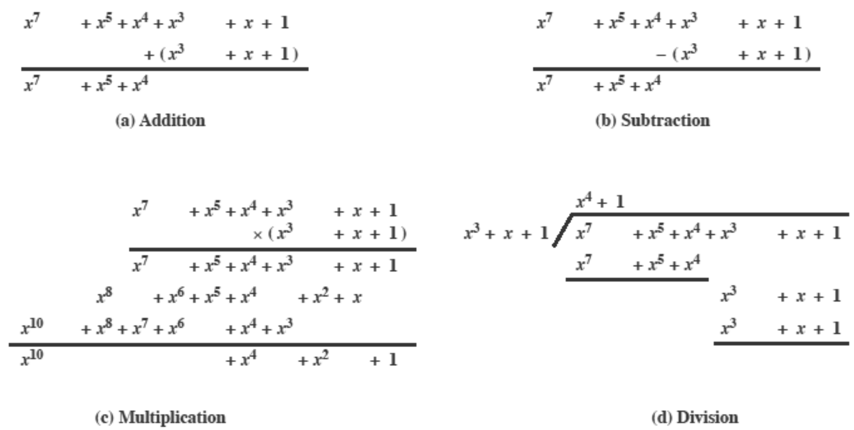


Figure 4.4 Examples of Polynomial Arithmetic over GF(2)

Polynomial Division

We can divide polynomials using 'long division'

- can write any polynomial in the form:
  - $f(x) = q(x)\ g(x) + r(x)$
  - can interpret $r(x)$ as being a remainder
  - $r(x) = f(x) \bmod g(x)$
- if no remainder, say $g(x)$ divides $f(x)$
- if $g(x)$ has no divisors other than itself and 1, say it is **irreducible** (or prime) polynomial
- arithmetic modulo an irreducible polynomial forms a field

Polynomial GCD

- can find greatest common divisor for polys
  - $c(x)$ = GCD($a(x)$, $b(x)$) if $c(x)$ is the poly of greatest degree which divides both $a(x)$, $b(x)$
- can adapt Euclid's Algorithm to find it:

```
Euclid(a(x), b(x))
   if (b(x) = 0) then return a(x);
   else return
            Euclid(b(x), a(x) mod b(x));
```

Modular Polynomial Arithmetic

- can compute in field GF(2^n)
  - elements of GF(2^n) are polynomials with coefficients modulo 2
  - whose degree is less than n
  - hence must reduce modulo an irreducible poly of degree n (when you multiply)
- form a finite field
- can always find an inverse
  - use Extend Euclid Algorithm to find inverse

Computational Considerations

- since coefficients are 0 or 1, can represent any such polynomial as a bit string
- addition becomes XOR of these bit strings
- multiplication is shift and XOR
  - cf. long multiplication
- modulo reduction done by repeatedly substituting highest power with remainder of irreducible poly (also shift and XOR)
- eg. irreducible poly = $x^3 + x + 1$ **means** $x^3 = x + 1$ in the polynomial field

Irreducible polynomial manipulation

- Why is it that if $x^3 + x + 1$ is an irreducible polynomilal in GF(2^n), then $x^3 = x + 1$ in the polynomial field?

  If $x^3 + x + 1$ is irreducible, then $x^3 + x + 1 = 0$ in the field.

  So $x^3 = - x - 1$. But $+1 = -1$ in Z_2 because addition/subtraction is mod 2 in Z_2.

  So $x^3 = x + 1$ after all.

Computational Example

- in GF(2^3) have (x^2+1) is 101_2 & (x^2+x+1) is 111_2
- so addition is
  - (x^2+1) + (x^2+x+1) = x
  - 101 XOR 111 = 010_2
- and multiplication is
  - (x+1).(x^2+1) = x.(x^2+1) + 1.(x^2+1)
    = x^3+x + x^2+1 = x^3+x^2+x+1
  - 011.101 = (101)<<1 XOR (101)<<0 =
    1010 XOR 0101 = 1111_2
- polynomial modulo reduction (to get q(x) & r(x))
  - (x^3+x^2+x+1) mod (x^3+x+1) = 1.(x^3+x+1) + (x^2) = x^2
  - 1111 mod 1011 = 1111 XOR 1011 = 0100_2

Example GF(2^3)

Table 4.7 Polynomial Arithmetic Modulo ($x^3 + x + 1$)

(a) Addition

	+	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
001	1	1	0	$x+1$	x	x^2+1	x^2	x^2+x+1	x^2+x
010	x	x	$x+1$	0	1	x^2+x	x^2+x+1	x^2	x^2+1
011	$x+1$	$x+1$	x	1	0	x^2+x+1	x^2+x	x^2+1	x^2
100	x^2	x^2	x^2+1	x^2+x	x^2+x+1	0	1	x	$x+1$
101	x^2+1	x^2+1	x^2	x^2+x+1	x^2+x	1	0	$x+1$	x
110	x^2+x	x^2+x	x^2+x+1	x^2	x^2+1	x	$x+1$	0	1
111	x^2+x+1	x^2+x+1	x^2+x	x^2+1	x^2	$x+1$	x	1	0

(b) Multiplication

	×	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
010	x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
011	$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
100	x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
101	x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
110	x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
111	x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

Arithmetic in GF(2^3)

	+	000 0	001 1	010 2	011 3	100 4	101 5	110 6	111 7
000	0	0	1	2	3	4	5	6	7
001	1	1	0	3	2	5	4	7	6
010	2	2	3	0	1	6	7	4	5
011	3	3	2	1	0	7	6	5	4
100	4	4	5	6	7	0	1	2	3
101	5	5	4	7	6	1	0	3	2
110	6	6	7	4	5	2	3	0	1
111	7	7	6	5	4	3	2	1	0

(a) Addition

	×	000 0	001 1	010 2	011 3	100 4	101 5	110 6	111 7
000	0	0	0	0	0	0	0	0	0
001	1	0	1	2	3	4	5	6	7
010	2	0	2	4	6	3	1	7	5
011	3	0	3	6	5	7	4	1	2
100	4	0	4	3	7	6	2	5	1
101	5	0	5	1	4	2	7	3	6
110	6	0	6	7	1	5	3	2	4
111	7	0	7	5	2	1	6	4	3

(b) Multiplication

w	$-w$	w^{-1}
0	0	—
1	1	1
2	2	5
3	3	6
4	4	7
5	5	2
6	6	3
7	7	4

(c) Additive and multiplicative inverses

Using a Generator

- equivalent definition of a finite field
- a **generator** g is an element whose powers generate all non-zero elements
  - in F have 0, g^0, g^1, …, g^{q-2}
- can create generator from **root** of the irreducible polynomial
- then implement multiplication by adding exponents of generator
- just a relabelling of the field elements (since only one field of a given size)