CS3222 Exercises 2 (Quantum Theory)

- 1. A physical system can be described by a four dimensional state space, and there is an observable A with eigenvectors $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, $|\delta\rangle$, and with eigenvalues α , β , γ , δ , respectively. Assume that α , β , γ , δ , are all different. Consider the state $|\psi\rangle = 1/2(|\alpha\rangle + |\beta\rangle + |\gamma\rangle + |\delta\rangle$). What outcomes are possible and with what probabilities if A is measured in state $|\psi\rangle$? Now suppose $\beta = \gamma$. Answer the same question. Now do everything again starting with the state $|\psi'\rangle = 1/\sqrt{2}|\alpha\rangle + \sqrt{3/8}|\beta\rangle + 1/4|\gamma\rangle + 1/4|\delta\rangle$.
- 2. In Q1, suppose that the unitary operator $H \oplus H$ (where H is the Hadamard transformation on Q, and the first H acts on $\{|\alpha\rangle, |\beta\rangle\}$, and the second on $\{|\gamma\rangle, |\delta\rangle\}$) is applied to the states $|\psi\rangle$ and $|\psi'\rangle$ prior to measurement. How are the probabilities of the various outcomes affected?
- 3. What would be involved if in Q2 we applied $H \otimes H$ instead of $H \oplus H$?
- 4. Verify that the expectation value of the spin polarisation $\boldsymbol{n}.\underline{\boldsymbol{\sigma}}$ in the state $|\psi\rangle = [\cos(\theta/2), e^{i\phi}\sin(\theta/2)]^T$ is $\langle \psi | \boldsymbol{n}.\underline{\boldsymbol{\sigma}} | \psi \rangle = \boldsymbol{n}.\underline{\psi} = \cos(\angle(\boldsymbol{n},\underline{\psi}))$.
- 5. Check that $\langle +|\boldsymbol{n}.\underline{\boldsymbol{\sigma}}|-\rangle = n_{\mathsf{X}} \mathrm{i}\,n_{\mathsf{Y}}$ and that $\langle -|\boldsymbol{n}.\underline{\boldsymbol{\sigma}}|+\rangle = n_{\mathsf{X}} + \mathrm{i}\,n_{\mathsf{Y}}$, where $|+\rangle$ and $|-\rangle$ are $\boldsymbol{\sigma}_{\mathsf{Z}}$ eigenstates.
- 6. Check that the spin correlation $\langle sing | (n_1.\underline{\sigma}_1) \otimes (n_2.\underline{\sigma}_2) | sing \rangle = -n_1.n_2 = -\cos(\angle(n_1,n_2))$ as advertised.
- 7. In Q, consider the operator $a = (\sigma_X i \sigma_y)/2$. What is its adjoint a^{\dagger} ? What are their matrix representations? Show that $a|+\rangle = |-\rangle$, $a|-\rangle = 0$; and that $a^{\dagger}|-\rangle = |+\rangle$, $a^{\dagger}|+\rangle = 0$. Work out the commutator and anticommutator of a and a^{\dagger} , namely $[a, a^{\dagger}]_- = aa^{\dagger} a^{\dagger}a$, and $[a, a^{\dagger}]_+ = aa^{\dagger} + a^{\dagger}a$. What are aa^{\dagger} and $a^{\dagger}a$?
- 8. (The Fly with apologies to Jeff Goldblum.) We model the state of a human being using Q (this is a little simplistic, maybe). State $|+\rangle$ corresponds to 'human being present', written hereafter as $|hb\rangle$, while $|-\rangle$ corresponds to 'human being absent', $|hb\rangle$. Likewise for the Californian stonefly (allocapnia vivipara), $|av\rangle$, $|av\rangle$. A mad scientist, not having mastered quantum mechanics very thoroughly but intending to build a teleportation mechanism nevertheless, builds by mistake the apparatus $A = (a^{\dagger}{}_{h} \otimes a_{a} + a_{h} \otimes a^{\dagger}{}_{a})$ where the subscripts 'h' and 'a' refer to human and fly subsystems respectively. Check that this is hermitian (i.e. that it corresponds to a valid physical observable). What are the eigenvalues and eigenvectors of A?

He now applies the apparatus A to himself, i.e. to $|hb,av\rangle$. What happens? Determine whether the result is separable or entangled.

(Gruesome isn't it? Pity now subatomic particles, which have to put up with this kind of thing all the time. It's not for nothing that a^{\dagger} and a are known in the trade as creation and annihilation operators.)

9. (Bell's Inequality.) Assume experimental arrangements as for the CHSH inequality but assume that the orientation of B and C are the same. Assume further that the classical measured outcomes for B and C are guaranteed to be the same. (This is true for photons but not for electrons.) Show that $D(C - A) = \pm (1 - CA) = \pm (1 - BA)$ for any classical measurement. Hence show that if quantum theory was a local realistic theory, the corresponding quantum expectation values should satisfy $|\langle DC \rangle - \langle DA \rangle| \le 1 - \langle CA \rangle$ which is Bell's Inequality.

What was the role of *B* in the preceding?

Show that the corresponding result if the measured outcomes for *B* and *C* are guaranteed to be *anticorrelated* (as for electrons) is $|\langle DC \rangle - \langle DA \rangle| \le 1 + \langle CA \rangle$.

10. (Trickier.) Determine the extent to which the various versions of Bell's Inequality can be derived as special cases of the argument used to derive the CHSH inequality.