CS3222 Linear Algebra: Supplementary Notes

Matrices

Matrix Multiplication

In general, matrix multiplication is defined as follows. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Matrix multiplication is not commutative, as shown.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

Determinants

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ For example: $\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -(-2)$

Column Vectors

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$ = a typical vector in $\mathbb{C}^3$. Addition of vectors works as follows.

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ and multiplication by a scalar: $\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$.

Linear Independence

Are $\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$ independent? Try: $\lambda \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \lambda - \mu \\ 2\lambda - 5\mu \\ -5\lambda + 6\mu \end{bmatrix} = 0$.

The first element in the vector means that $\lambda = \mu$. This, with the second element shows that $2\lambda + 5\lambda = 0 \Rightarrow \lambda = 0$ and $\mu = 0$. The third agrees with this result and so they ARE independent.

$\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$ span a 2-dimensional subspace of $\mathbb{C}^3$. This consists of all vectors that can be expressed as $\lambda \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$. Is $\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$ one of these vectors? Yes, as shown below.

$\lambda - \mu = 0$ and $2\lambda - 5\mu = 7$ and $-5\lambda + 6\mu = 1 \Rightarrow \lambda = 1$ and $\mu = 1$.

$\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$ is not a vector of this type since the equations $\lambda - \mu = 0$, $2\lambda - 5\mu = 5$ and $-5\lambda + 6\mu = 1$ have no solution. There is a solution for any two but the third will not agree.

Coordinates

The elements of an independent set SPAN a linear subspace (of the space of interest).

In the case of $\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$ this is a 2-dimensional subspace of $\mathbb{C}^3$. The MULTIPLES of $\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$ needed to represent any vector of this subspace as a linear combination of them are the COORDINATES (with respect to that BASIS of the subspace).

For example $\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$ is in the subspace above; and since we know that $\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$ we can say that the coordinates of the first vector with respect to the other two are $(1,1)$.

Inner Products

$\langle | \rangle$ is defined on $\mathbb{C}^3$ as $\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} | \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \rangle = x_1{}^*x_2 + y_1{}^*y_2 + z_1{}^*z_2$ and is an INNER PRODUCT, as proved by the 3 rules below.

1. $\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} | \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \rangle = \langle \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} | \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \rangle^*$

2. $\langle | \rangle$ is linear in the second argument and anti-linear in the first.

3. $\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} | \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rangle = 0 = |x|^2 + |y|^2 + |z|^2 \Leftrightarrow x = y = z = 0$

We could define another notion $\langle\langle\ |\ \rangle\rangle$ as $\langle\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} | \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \rangle\rangle = x_1{}^* x_2 + y_1{}^* y_2 + 3z_1{}^* z_2$ and this would also be an inner product. (Check using the rules above.) So inner products are NOT unique. However we always use $\langle\ |\ \rangle$.

<u>Orthonormal Bases, Unitary Operators and Change of Basis</u>

A unitary operator (matrix) satisfies $U^\dagger U = UU^\dagger = I$.

Let $\{|b_0\rangle, |b_1\rangle, |b_2\rangle\}$ be a basis of $\mathbb{C}^3$.

It is an ORTHONORMAL basis iff $\langle b_0|b_0\rangle = \langle b_1|b_1\rangle = \langle b_2|b_2\rangle = 1$ and

$\langle b_0|b_1\rangle = \langle b_0|b_2\rangle = \langle b_1|b_0\rangle = \langle b_1|b_2\rangle = \langle b_2|b_0\rangle = \langle b_2|b_1\rangle = 0$

Let $\{|c_0\rangle, |c_1\rangle, |c_2\rangle\}$ be another orthonormal basis. Suppose that

$$|b_0\rangle = \begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \end{bmatrix},\ |b_1\rangle = \begin{bmatrix} b_{10} \\ b_{11} \\ b_{12} \end{bmatrix},\ |b_2\rangle = \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix}, \text{ and } |c_0\rangle = \begin{bmatrix} c_{00} \\ c_{01} \\ c_{02} \end{bmatrix},\ |c_1\rangle = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \end{bmatrix},\ |c_2\rangle = \begin{bmatrix} c_{20} \\ c_{21} \\ c_{22} \end{bmatrix},$$

when written out in the standard basis.

Now consider an arbitrary vector $|v\rangle \in C^3$ with components $|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$ in the standard basis. It will have components $|v\rangle = v_0^b|b_0\rangle + v_1^b|b_1\rangle + v_2^b|b_2\rangle$ and $|v\rangle = v_0^c|c_0\rangle + v_1^c|c_1\rangle + v_2^c|c_2\rangle$ in the $|b_i\rangle$ and $|c_i\rangle$ bases.

How are these things related??? We know that

$$v_0^b|b_0\rangle + v_1^b|b_1\rangle + v_2^b|b_2\rangle = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = v_0^c|c_0\rangle + v_1^c|c_1\rangle + v_2^c|c_2\rangle = v_0|0\rangle + v_1|1\rangle + v_2|2\rangle\ .$$

Also $|b_i\rangle = \sum_j b_{ij}|j\rangle$ and $|c_i\rangle = \sum_j c_{ij}|j\rangle$ from the previous page.

The matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ (where $0 \le i, j \le 2$) are UNITARY (see main notes).

Now $|\underline{c}\rangle = C|\underline{j}\rangle$, where $|\underline{c}\rangle = \begin{bmatrix} |c_0\rangle \\ |c_1\rangle \\ |c_2\rangle \end{bmatrix}$ (i.e. $|\underline{c}\rangle$ is just the $|c\rangle$'s in a column notation),

and $|\underline{j}\rangle$ is the standard basis also in a column notation, implies $|\underline{j}\rangle = C^\dagger|\underline{c}\rangle$.

Similarly $|\underline{j}\rangle = B^\dagger|\underline{b}\rangle$.

Now $|v\rangle = v_0^c|c_0\rangle + v_1^c|c_1\rangle + v_2^c|c_2\rangle$

$= v_0^c\sum_j c_{0j}|j\rangle + v_1^c\sum_j c_{1j}|j\rangle + v_2^c\sum_j c_{2j}|j\rangle = \sum_{ij} v_i^c c_{ij}|j\rangle = [\underline{v}^c]^T C|\underline{j}\rangle$ where $\underline{v}^c = \begin{bmatrix} v_0^c \\ v_1^c \\ v_2^c \end{bmatrix}$
(i.e. $[\underline{v}^c]$ is the v_i^c in column notation).

Similarly $|v\rangle = [\underline{v}^b]^T B|\underline{j}\rangle$.
So $[\underline{v}^c]^T C = [\underline{v}^b]^T B$ or $C^T[\underline{v}^c] = B^T[\underline{v}^b]$.
Since C is unitary, $C^*C^T = [CC^\dagger]^* = I^* = I$.
So $[\underline{v}^c] = C^*C^T[\underline{v}^c] = C^*B^T[\underline{v}^b]$. That's how the COEFFICIENTS in the two bases relate.

From $|\underline{c}\rangle = C|\underline{j}\rangle$ and $|\underline{j}\rangle = B^\dagger|\underline{b}\rangle$ we get $|\underline{c}\rangle = CB^\dagger|\underline{b}\rangle$. That's how the two BASES relate.

Define $CB^\dagger \equiv U$ and $C^*B^T \equiv U^*$ where U and U^* are unitary.

We can check that these give consistent expressions for $|v\rangle$.

$|v\rangle = v_0^c|c_0\rangle + v_1^c|c_1\rangle + v_2^c|c_2\rangle = [\underline{v}^c]^T|\underline{c}\rangle = (U^*[\underline{v}^b])^T(U|\underline{b}\rangle) = (C^*B^T[\underline{v}^b])^T(CB^\dagger|\underline{b}\rangle)$

$= [\underline{v}^b]^T BC^\dagger CB^\dagger|\underline{b}\rangle = [\underline{v}^b]^T BB^\dagger|\underline{b}\rangle = [\underline{v}^b]^T|\underline{b}\rangle = |v\rangle$.

N.B. Final Note: $|b_i\rangle = \sum_j b_{ij}|j\rangle$ $|b_i\rangle = \sum_j \langle j|b_i\rangle|j\rangle$. Note the order of the indices.
The other choice of order in b_{ij} simplifies things a bit (in that you just end up with matrices and their adjoints in the derivation above, instead of matrices and their conjugates, transposes, *and* adjoints) but it looks very strange when you write $|b_i\rangle = \sum_j b_{ji}|j\rangle$ right at the start without saying why (as happens in 80% or more of textbooks).

(It's even worse if you *don't notice* that the indeces are the wrong way round in $|b_i\rangle = \sum_j b_{ji}|j\rangle$ and thus get screwed up later when something doesn't work out, and don't spot the source of the problem till after half a day's diligent detective work.)

Pauli Matrices

Calculate $\sigma_x\sigma_y\sigma_z\sigma_y\sigma_x$. Do it left to right for example:
$\sigma_x\sigma_y = i\sigma_z$ so $\sigma_x\sigma_y\sigma_z\sigma_y\sigma_x$ becomes $i\sigma_z\sigma_z\sigma_y\sigma_x$. Also $\sigma_z^2 = I$ so $i\sigma_z\sigma_z\sigma_y\sigma_x$ becomes $i\sigma_y\sigma_x$ and $\sigma_y\sigma_x = -\sigma_x\sigma_y = -i\sigma_z$ so $i\sigma_y\sigma_x$ becomes $i\times(-i)\sigma_z = \sigma_z$.
All calculations with Pauli matrices can be done this way.

Exponentials

Calculate e^A where $A = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$.

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4..$$

$$A^2 = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^{2n} = I, A^{2n+1} = A$$

so $e^A = \left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots\right)I + \left(1 + \frac{1}{3!} + \frac{1}{5!} + \dots\right)A = \cosh(1)I + \sinh(1)A$

$$= \begin{bmatrix} \cosh(1) + \sinh(1) & 5\sinh(1) \\ 0 & \cosh(1) - \sinh(1) \end{bmatrix}$$

Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$.

The characteristic equation is $\det[A - \lambda I] = 0$ so $\det\begin{bmatrix} 2-\lambda & 1+i \\ 1-i & 3-\lambda \end{bmatrix}$

$= (2-\lambda)(3-\lambda) - (1+i)(1-i) = 6 - 5\lambda + \lambda^2 - 2 = 0$ which can be rewritten as

$\lambda^2 - 5\lambda + 4 = 0 = (\lambda - 1)(\lambda - 4)$.

So $\lambda = 1$ or $\lambda = 4$

Eigenvector for $\lambda = 1$ found by solving $\begin{bmatrix} 2-1 & 1+i \\ 1-i & 3-1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$.

$a + (1+i)b = 0 \Rightarrow \frac{a}{b} = \frac{-1-i}{1} \Rightarrow |a_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = s\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ where $s \neq 0$ can be any number.

Eigenvector for $\lambda = 4$ can be found by solving $\begin{bmatrix} 2-4 & 1+i \\ 1-i & 3-4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$.

$-2a + (1+i)b = 0 \Rightarrow \frac{a}{b} = \frac{1+i}{2} \Rightarrow |a_4\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = s\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ where $s \neq 0$ can be any number.

Direct Sums and Tensor Products

Let A be as above and $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Work out the eigenvalues and eigenvectors of $A \oplus B$ and $A \otimes B$.

A has eigenvalues/vectors $1 \to \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ and $4 \to \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ (see above).

B is $i\sigma_x$. σ_x has eigenvalues/vectors $1 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $-1 \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (see eg. RHBnotes p. 41).

So B has eigenvalues/vectors $i \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $-i \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$A \oplus B$ is $\begin{bmatrix} \begin{matrix} 2 & 1+i \\ 1-i & 3 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & i \\ i & 0 \end{matrix} \end{bmatrix}$ and has eigenvalues $\{\lambda_A, \lambda_B\}$, where $\{\lambda_A\}$ are eigenvalues of A and $\{\lambda_B\}$ are eigenvalues of B with corresponding eigenvectors.

So the eigenvalues and eigenvectors of $A \oplus B$ are:

$$1 \rightarrow \begin{bmatrix} -1-i \\ 1 \\ 0 \\ 0 \end{bmatrix}, 4 \rightarrow \begin{bmatrix} 1+i \\ 2 \\ 0 \\ 0 \end{bmatrix}, i \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -i \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$. It's easy to see that this works.

$A \otimes B$ is the array of numbers $a_{ij}b_{kl}$, normally arranged thus:

$$\begin{bmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{bmatrix} = \begin{bmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{bmatrix} \text{ or, in our case } \begin{bmatrix} 0 & 2i & 0 & -1+i \\ 2i & 0 & -1+i & 0 \\ 0 & 1+i & 0 & 3i \\ 1+i & 0 & 3i & 0 \end{bmatrix}.$$

Eigenvalues of $A \otimes B$ are $\{\lambda_A \cdot \lambda_B\}$, where $\{\lambda_A\}$ are the eigenvalues of A and $\{\lambda_B\}$ are the eigenvalues of B. Eigenvectors of $A \otimes B$ are tensor products of the corresponding eigenvectors of A and B.

So we get $i \rightarrow \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-i \\ -1-i \\ 1 \\ 1 \end{bmatrix}$, $-i \rightarrow \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1-i \\ 1+i \\ 1 \\ -1 \end{bmatrix}$,

$4i \rightarrow \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+i \\ 1+i \\ 2 \\ 2 \end{bmatrix}$, $-4i \rightarrow \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+i \\ -1-i \\ 2 \\ -2 \end{bmatrix}$.

As you can see this is somewhat harder to check explicitly than in the direct sum case.

The Hadamard Transformation

The general theory says that $H^{\otimes 2}|00\rangle = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$. We'll check in detail.

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So .....

$|00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $|01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$;

$|10\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $|11\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Now $H = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $H \otimes H = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

So $H \otimes H|00\rangle = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{1}{2}\left(1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ as required.

Now try $H^{\otimes 2}|01\rangle = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \dfrac{1}{2}\left(1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

$w = |01\rangle$

$z = |00\rangle$
$w \cdot z = 0$
so '+'
in sum

$z = |01\rangle$
$w \cdot z = 1$
so '–'
in sum

$z = |01\rangle$
$w \cdot z = 0$
so '+'
in sum

$z = |11\rangle$
$w \cdot z = 1$
so '–'
in sum

So it all works as in the formula $H^{\otimes n}|w\rangle = \dfrac{1}{\sqrt{2}^n}\sum_z (-1)^{w \cdot z}|z\rangle$.