## CS3222 Linear Algebra: Supplementary Notes

## Matrices

## Matrix Multiplication

In general, matrix multiplication is defined as follows. $\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\left[\begin{array}{ll}e & f \\ g & h\end{array}\right]=\left[\begin{array}{ll}a e+b g & a f+b h \\ c e+d g & c f+d h\end{array}\right]$
Matrix multiplication is not commutative, as shown.
$\left[\begin{array}{cc}1 & 2 \\ -1 & 0\end{array}\right]\left[\begin{array}{cc}1 & -1 \\ 0 & 1\end{array}\right]=\left[\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right] \neq\left[\begin{array}{cc}1 & -1 \\ 0 & 1\end{array}\right]\left[\begin{array}{cc}1 & 2 \\ -1 & 0\end{array}\right]=\left[\begin{array}{cc}2 & 2 \\ -1 & 0\end{array}\right]$

## Determinants

$\operatorname{det}\left(\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\right)=\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|=a d-b c$ For example: $\left|\begin{array}{rr}1 & 2 \\ -1 & 0\end{array}\right|=-(-2)$

## Column Vectors

$\left.\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{ll}x & y\end{array}\right]\right]^{\mathrm{T}}=$ a typical vector in $\mathbb{C}^{3}$. Addition of vectors works as follows.
$\left[\begin{array}{l}x_{1} \\ y_{1} \\ z_{1}\end{array}\right]+\left[\begin{array}{l}x_{2} \\ y_{2} \\ z_{2}\end{array}\right]=\left[\begin{array}{l}x_{1}+x_{2} \\ y_{1}+y_{2} \\ z_{1}+z_{2}\end{array}\right]$ and multiplication by a scalar: $\lambda\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}\lambda x \\ \lambda y \\ \lambda z\end{array}\right]$.

## Linear Independence

Are $\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]$ and $\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$ independent? Try: $\lambda\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]+\mu\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]=0 \Rightarrow\left[\begin{array}{c}\lambda-\mu \\ 2 \lambda-5 \mu \\ -5 \lambda+6 \mu\end{array}\right]=0$.
The first element in the vector means that $\lambda=\mu$. This, with the second element shows that $2 \lambda+5 \lambda=0 \Rightarrow \lambda=0$ and $\mu=0$. The third agrees with this result and so they ARE independent.
$\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]$ and $\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$ span a 2-dimensional subspace of $\mathbb{C}^{3}$. This consists of all vectors that can be expressed as $\lambda\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]+\mu\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$. Is $\left[\begin{array}{l}0 \\ 7 \\ 1\end{array}\right]$ one of these vectors? Yes, as shown below.
$\lambda-\mu=0$ and $2 \lambda-5 \mu=7$ and $-5 \lambda+6 \mu=1 \Rightarrow \lambda=1$ and $\mu=1$.
$\left[\begin{array}{l}0 \\ 5 \\ 1\end{array}\right]$ is not a vector of this type since the equations $\lambda-\mu=0,2 \lambda-5 \mu=5$ and
$-5 \lambda+6 \mu=1$ have no solution. There is a solution for any two but the third will not agree.

## Coordinates

The elements of an independent set SPAN a linear subspace (of the space of interest).
In the case of $\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]$ and $\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$ this is a 2-dimensional subspace of $\mathbb{C}^{3}$. The mULTIPLES of $\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]$
and $\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$ needed to represent any vector of this subspace as a linear combination of them are the COORDINATES (with respect to that BASIS of the subspace).

For example $\left[\begin{array}{l}0 \\ 7 \\ 1\end{array}\right]$ is in the subspace above; and since we know that $\left[\begin{array}{l}0 \\ 7 \\ 1\end{array}\right]=1\left[\begin{array}{c}1 \\ 2 \\ -5\end{array}\right]+1\left[\begin{array}{c}-1 \\ 5 \\ 6\end{array}\right]$
we can say that the coordinates of the first vector with respect to the other two are $(1,1)$.

## Inner Products

$\langle\mid\rangle$ is defined on $\mathbb{C}^{3}$ as $\left\langle\left[\begin{array}{l}x_{1} \\ y_{1} \\ z_{1}\end{array}\right]\left[\begin{array}{l}x_{2} \\ y_{2} \\ z_{2}\end{array}\right]\right\rangle=x_{1} * x_{2}+y_{1} * y_{2}+z_{1} * z_{2}$ and is an INNER PRODUCT, as proved by the 3 rules below.

1. $\left\langle\begin{array}{l}x_{1} \\ y_{1} \\ z_{1}\end{array}\right]\left|\left[\begin{array}{l}x_{2} \\ y_{2} \\ z_{2}\end{array}\right]\right\rangle=\left\langle\left[\begin{array}{l}x_{2} \\ y_{2} \\ z_{2}\end{array}\right]\right|\left[\begin{array}{l}x_{1} \\ y_{1} \\ z_{1}\end{array}\right]^{*}$
2. $\langle\mid\rangle$ is linear in the second argument and anti-linear in the first.
3. $\left\langle\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right\rangle=0=|x|^{2}+|y|^{2}+|z|^{2} \Leftrightarrow x=y=z=0$

We could define another notion $\langle\langle\mid\rangle\rangle$ as $\left\langle\left\langle\left[\begin{array}{l}x_{1} \\ y_{1} \\ z_{1}\end{array}\right]\left[\begin{array}{l}x_{2} \\ y_{2} \\ z_{2}\end{array}\right]\right\rangle\right\rangle=x_{1} * x_{2}+y_{1} * y_{2}+3 z_{1}{ }^{*} z_{2}$ and this would also be an inner product. (Check using the rules above.) So inner products are NOT unique. However we always use $\langle\mid\rangle$.

## Orthonormal Bases, Unitary Operators and Change of Basis

A unitary operator (matrix) satisfies $U^{\dagger} U=U U^{\dagger}=I$.
Let $\left\{\left|b_{0}\right\rangle,\left|b_{1}\right\rangle,\left|b_{2}\right\rangle\right\}$ be a basis of $\mathbb{C}^{3}$.
It is an ORTHONORMAL basis iff $\left\langle b_{0} \mid b_{0}\right\rangle=\left\langle b_{1} \mid b_{1}\right\rangle=\left\langle b_{2} \mid b_{2}\right\rangle=1$ and
$\left\langle b_{0} \mid b_{1}\right\rangle=\left\langle b_{0} \mid b_{2}\right\rangle=\left\langle b_{1} \mid b_{0}\right\rangle=\left\langle b_{1} \mid b_{2}\right\rangle=\left\langle b_{2} \mid b_{0}\right\rangle=\left\langle b_{2} \mid b_{1}\right\rangle=0$
Let $\left\{\left|c_{0}\right\rangle,\left|c_{1}\right\rangle,\left|c_{2}\right\rangle\right\}$ be another orthonormal basis. Suppose that
$\left|b_{0}\right\rangle=\left[\begin{array}{l}b_{00} \\ b_{01} \\ b_{02}\end{array}\right],\left|b_{1}\right\rangle=\left[\begin{array}{l}b_{10} \\ b_{11} \\ b_{12}\end{array}\right],\left|b_{2}\right\rangle=\left[\begin{array}{l}b_{20} \\ b_{21} \\ b_{22}\end{array}\right]$, and $\left|c_{0}\right\rangle=\left[\begin{array}{l}c_{00} \\ c_{01} \\ c_{02}\end{array}\right],\left|c_{1}\right\rangle=\left[\begin{array}{l}c_{10} \\ c_{11} \\ c_{12}\end{array}\right],\left|c_{2}\right\rangle=\left[\begin{array}{l}c_{20} \\ c_{21} \\ c_{22}\end{array}\right]$,
when written out in the standard basis.
Now consider an arbitrary vector $|v\rangle \in C^{3}$ with components $|v\rangle=\left[\begin{array}{l}v_{0} \\ v_{1} \\ v_{2}\end{array}\right]$ in the standard
basis. It will have components $|v\rangle=v_{0}^{b}\left|b_{0}\right\rangle+v_{1}^{b}\left|b_{1}\right\rangle+v_{2}^{b}\left|b_{2}\right\rangle$ and
$|v\rangle=v_{0}^{c}\left|c_{0}\right\rangle+v_{1}^{c}\left|c_{1}\right\rangle+v_{2}^{c}\left|c_{2}\right\rangle$ in the $\left|b_{i}\right\rangle$ and $\left|c_{i}\right\rangle$ bases.

How are these things related??? We know that
$v_{0}^{b}\left|b_{0}\right\rangle+v_{1}^{b}\left|b_{1}\right\rangle+v_{2}^{b}\left|b_{2}\right\rangle=\left[\begin{array}{c}v_{0} \\ v_{1} \\ v_{2}\end{array}\right]=v_{0}^{c}\left|c_{0}\right\rangle+v_{1}^{c}\left|c_{1}\right\rangle+v_{2}^{c}\left|c_{2}\right\rangle=v_{0}|0\rangle+v_{1}|1\rangle+v_{2}|2\rangle$.
Also $\left|b_{i}\right\rangle=\sum_{j} b_{i j}|j\rangle$ and $\left|c_{i}\right\rangle=\sum_{j} c_{i j}|j\rangle$ from the previous page.
The matrices $B=\left[b_{i j}\right]$ and $C=\left[c_{i j}\right]$ (where $0 \leq i, j \leq 2$ ) are UNITARY (see main notes).
Now $|\underline{c}\rangle=C|\underline{j}\rangle$, where $|\underline{c}\rangle=\left[\begin{array}{l}\left|c_{0}\right\rangle \\ \left|c_{1}\right\rangle \\ \left|c_{2}\right\rangle\end{array}\right]$ (i.e. $|\underline{c}\rangle$ is just the $|c\rangle$ 's in a column notation),
and $|\underset{\sim}{j}\rangle$ is the standard basis also in a column notation, implies $|\underset{\sim}{j}\rangle=C^{\dagger}|\underline{\underline{~}}\rangle$.
Similarly $|\underline{j}\rangle=B^{\dagger}|\underline{b}\rangle$.

Now $|v\rangle=v_{0}^{c}\left|c_{0}\right\rangle+v_{1}^{c}\left|c_{1}\right\rangle+v_{2}^{c}\left|c_{2}\right\rangle$
$\left.\begin{array}{l}=v_{0}^{c} \sum_{j} c_{0 j}|j\rangle+v_{1}^{c} \sum_{j} c_{1 j}|j\rangle+v_{2}^{c} \sum_{j} c_{2 j}|j\rangle=\sum_{i j} v_{i}^{c} c_{i j}|j\rangle=\left[\underline{v}^{c}\right]^{T} C|\underline{j}\rangle \text { where } \underline{v}^{c}=\left[\begin{array}{c}v_{0}^{c} \\ \text { (i.e. }\left[\underline{v}^{c}\right] \text { is the } v_{i}^{c} \text { in column notation). }\end{array} v_{1}^{c}\right. \\ v_{2}^{c}\end{array}\right]$
Similarly $|v\rangle=\left[v^{b}\right]^{T} B|j\rangle$.
So $\left[\underline{v}^{c}\right]^{T} C=\left[\underline{v}^{b}\right]^{T} B$ or $C^{T}\left[\underline{v}^{c}\right]=B^{T}\left[\underline{v}^{b}\right]$.
Since $C$ is unitary, $C^{*} C^{T}=\left[C C^{\dagger}\right]^{*}=I^{*}=I$.
So $\left[\underline{v}^{c}\right]=C^{*} C^{T}\left[\underline{v}^{c}\right]=C^{*} B^{T}\left[\underline{v}^{b}\right]$. That's how the COEFFICIENTS in the two bases relate.

From $|\underline{c}\rangle=C|\underline{j}\rangle$ and $|\underline{j}\rangle=B^{\dagger}|\underline{b}\rangle$ we get $|\underline{c}\rangle=C B^{\dagger}|\underline{b}\rangle$. That's how the two BASES relate.

Define $C B^{\dagger} \equiv U$ and $C^{*} B^{T} \equiv U^{*}$ where $U$ and $U^{*}$ are unitary.
We can check that these give consistent expressions for $|\nu\rangle$.
$|v\rangle=v_{0}^{c}\left|c_{0}\right\rangle+v_{1}^{c}\left|c_{1}\right\rangle+v_{2}^{c}\left|c_{2}\right\rangle=\left[\underline{v}^{c}\right]{ }^{T}|\underline{c}\rangle=\left(U^{*}\left[\underline{v}^{b}\right]\right)^{T}(U|\underline{b}\rangle)=\left(C^{*} B^{T}\left[\underline{v}^{b}\right]\right)^{T}\left(C B^{\dagger}|\underline{b}\rangle\right)$
$=\left[\underline{v}^{b}\right]^{T} B C^{\dagger} C B^{\dagger}|\underline{b}\rangle=\left[\underline{v}^{b}\right]^{T} B B^{\dagger}|\underline{b}\rangle=\left[\underline{v}^{b}\right]^{T}|\underline{b}\rangle=|v\rangle$.
N.B. Final Note: $\left|b_{i}\right\rangle=\sum_{j} b_{i j}|j\rangle\left|b_{i}\right\rangle=\sum_{j}\left\langle j \mid b_{i}\right\rangle|j\rangle$. Note the order of the indices.

The other choice of order in $b_{i j}$ simplifies things a bit (in that you just end up with matrices and their adjoints in the derivation above, instead of matrices and their conjugates, transposes, and adjoints) but it looks very strange when you write $\left|b_{i}\right\rangle=\sum_{j} b_{j i}|j\rangle$ right at the start without saying why (as happens in $80 \%$ or more of textbooks).
(It's even worse if you don't notice that the indeces are the wrong way round in $\left|b_{i}\right\rangle=\sum_{j} b_{j i}|j\rangle$ and thus get screwed up later when something doesn't work out, and don't spot the source of the problem till after half a day's diligent detective work.)

## Pauli Matrices

Calculate $\sigma_{x} \sigma_{y} \sigma_{z} \sigma_{y} \sigma_{x}$. Do it left to right for example:
$\sigma_{x} \sigma_{y}=i \sigma_{z}$ so $\sigma_{x} \sigma_{y} \sigma_{z} \sigma_{y} \sigma_{x}$ becomes $i \sigma_{z} \sigma_{z} \sigma_{y} \sigma_{x}$. Also $\sigma_{z}^{2}=I$ so $i \sigma_{z} \sigma_{z} \sigma_{y} \sigma_{x}$ becomes $i \sigma_{y} \sigma_{x}$ and $\sigma_{y} \sigma_{x}=-\sigma_{x} \sigma_{y}=-i \sigma_{z}$ so $i \sigma_{y} \sigma_{x}$ becomes $i \times(-i) \sigma_{z}=\sigma_{z}$.
All calculations with Pauli matrices can be done this way.

## Exponentials

Calculate $e^{A}$ where $A=\left[\begin{array}{cc}1 & 5 \\ 0 & -1\end{array}\right]$.
$e^{A}=I+A+\frac{1}{2!} A^{2}+\frac{1}{3!} A^{3}+\frac{1}{4!} A^{4} .$.
$A^{2}=\left[\begin{array}{cc}1 & 5 \\ 0 & -1\end{array}\right]\left[\begin{array}{cc}1 & 5 \\ 0 & -1\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right] \Rightarrow A^{2 n}=I, A^{2 n+1}=A$
so $e^{A}=\left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\right) I+\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\right) A=\cosh (1) I+\sinh (1) A$
$=\left[\begin{array}{cc}\cosh (1)+\sinh (1) & 5 \sinh (1) \\ 0 & \cosh (1)-\sinh (1)\end{array}\right]$

## Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\left[\begin{array}{cc}2 & 1+i \\ 1-i & 3\end{array}\right]$.
The characteristic equation is $\operatorname{det}[A-\lambda I]=0$ so $\operatorname{det}\left[\begin{array}{cc}2-\lambda & 1+i \\ 1-i & 3-\lambda\end{array}\right]$
$=(2-\lambda)(3-\lambda)-(1+i)(1-i)=6-5 \lambda+\lambda^{2}-2=0$ which can be rewritten as $\lambda^{2}-5 \lambda+4=0=(\lambda-1)(\lambda-4)$.
So $\lambda=1$ or $\lambda=4$
So $\lambda=1$ or $\lambda=4$
Eigenvector for $\lambda=1$ found by solving $\left[\begin{array}{ll}2-1 & 1+i \\ 1-i & 3-1\end{array}\right]\left[\begin{array}{l}a \\ b\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{cc}1 & 1+i \\ 1-i & 2\end{array}\right]\left[\begin{array}{l}a \\ b\end{array}\right]$. $a+(1+i) b=0 \Rightarrow \frac{a}{b}=\frac{-1-i}{1} \Rightarrow\left|a_{1}\right\rangle=\left[\begin{array}{l}a \\ b\end{array}\right]=s\left[\begin{array}{c}-1-i \\ 1\end{array}\right]$ where $s \neq 0$ can be any number.
Eigenvector for $\lambda=4$ can be found by solving $\left[\begin{array}{ll}2-4 & 1+i \\ 1-i & 3-4\end{array}\right]\left[\begin{array}{l}a \\ b\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{cc}-2 & 1+i \\ 1-i & -1\end{array}\right]\left[\begin{array}{l}a \\ b\end{array}\right]$. $-2 a+(1+i) b=0 \Rightarrow \frac{a}{b}=\frac{1+i}{2} \Rightarrow\left|a_{4}\right\rangle=\left[\begin{array}{l}a \\ b\end{array}\right]=s\left[\begin{array}{c}1+i \\ 2\end{array}\right]$ where $s \neq 0$ can be any number.

## Direct Sums and Tensor Products

Let $A$ be as above and $B=\left[\begin{array}{ll}0 & i \\ i & 0\end{array}\right]$. Work out the eigenvalues and eigenvectors of
$A \oplus B$ and $A \otimes B$.
$A$ has eigenvalues/vectors $1 \rightarrow\left[\begin{array}{c}-1-i \\ 1\end{array}\right]$ and $4 \rightarrow\left[\begin{array}{c}1+i \\ 2\end{array}\right]$ (see above).
$B$ is $i \sigma_{x} . \sigma_{x}$ has eigenvalues/vectors $1 \rightarrow\left[\begin{array}{l}1 \\ 1\end{array}\right]$ and $-1 \rightarrow\left[\begin{array}{c}1 \\ -1\end{array}\right]$ (see eg. RHBnotes p. 41).
So $B$ has eigenvalues/vectors $i \rightarrow\left[\begin{array}{l}1 \\ 1\end{array}\right]$ and $-i \rightarrow\left[\begin{array}{c}1 \\ -1\end{array}\right]$.
$A \oplus B$ is \(\left[\begin{array}{cccc}2 \& 1+i \& 0 <br>
1-i \& 3 \& <br>
\& \& 0 \& i <br>

0 \& i \& 0\end{array}\right] \quad\)| and has eigenvalues $\left\{\lambda_{A}, \lambda_{B}\right\}$, |
| :--- |
| where $\left\{\lambda_{A}\right\}$ are eigenvalues of $A$ |
| and $\left\{\lambda_{B}\right\}$ are eigenvalues of $B$ with |
| corresponding eigenvectors. |

So the eigenvalues and eigenvectors of $A \oplus B$ are:
$1 \rightarrow\left[\begin{array}{c}-1-i \\ 1 \\ 0 \\ 0\end{array}\right], 4 \rightarrow\left[\begin{array}{c}1+i \\ 2 \\ 0 \\ 0\end{array}\right], i \rightarrow\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right],-i \rightarrow\left[\begin{array}{c}0 \\ 0 \\ 1 \\ -1\end{array}\right]$. It's easy to see that this works.
$A \otimes B$ is the array of numbers $a_{i j} b_{k l}$, normally arranged thus:
$\left[\begin{array}{ll}a_{00} B & a_{01} B \\ a_{10} B & a_{11} B\end{array}\right]=\left[\begin{array}{lll}a_{00} b_{00} & a_{00} b_{01} & a_{01} b_{00}\end{array} a_{01} b_{01}\left[\begin{array}{cccc}0 & 2 i & 0 & -1+i \\ a_{00} b_{10} & a_{00} b_{11} & a_{01} b_{10} & a_{01} b_{11} \\ a_{10} b_{00} & a_{10} b_{01} & a_{11} b_{00} & a_{11} b_{01} \\ a_{10} b_{10} & a_{10} b_{11} & a_{11} b_{10} & a_{11} b_{11}\end{array}\right]\right.$ or, in our case $\left[\begin{array}{cccc}0 & 0 & -1+i & 0 \\ 2 i & 0 & 1+i & 0 \\ 0 i & 3 i \\ 1+i & 0 & 3 i & 0\end{array}\right]$.
Eigenvalues of $A \otimes B$ are $\left\{\lambda_{A} \cdot \lambda_{B}\right\}$, where $\left\{\lambda_{A}\right\}$ are the eigenvalues of $A$ and $\left\{\lambda_{B}\right\}$ are the eigenvalues of $B$. Eigenvectors of $A \otimes B$ are tensor products of the corresponding eigenvectors of $A$ and $B$.
So we get $i \rightarrow\left[\begin{array}{c}-1-i \\ 1\end{array}\right] \otimes\left[\begin{array}{l}1 \\ 1\end{array}\right]=\left[\begin{array}{c}-1-i \\ -1-i \\ 1 \\ 1\end{array}\right],-i \rightarrow\left[\begin{array}{c}-1-i \\ 1\end{array}\right] \otimes\left[\begin{array}{c}1 \\ -1\end{array}\right]=\left[\begin{array}{c}-1-i \\ 1+i \\ 1 \\ -1\end{array}\right]$,
$4 i \rightarrow\left[\begin{array}{c}1+i \\ 2\end{array}\right] \otimes\left[\begin{array}{l}1 \\ 1\end{array}\right]=\left[\begin{array}{c}1+i \\ 1+i \\ 2 \\ 2\end{array}\right],-4 i \rightarrow\left[\begin{array}{c}1+i \\ 2\end{array}\right] \otimes\left[\begin{array}{c}1 \\ -1\end{array}\right]=\left[\begin{array}{c}1+i \\ -1-i \\ 2 \\ -2\end{array}\right]$.
As you can see this is somewhat harder to check explicitly than in the direct sum case.

## The Hadamard Transformation

The general theory says that $H^{\otimes 2}|00\rangle=\frac{1}{2}[|00\rangle+|01\rangle+|10\rangle+|11\rangle]$. We'll check in detail.
$|0\rangle=\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and $|1\rangle=\left[\begin{array}{l}0 \\ 1\end{array}\right]$. So $\ldots .$.
$|00\rangle=\left[\begin{array}{l}1 \\ 0\end{array}\right] \otimes\left[\begin{array}{l}1 \\ 0\end{array}\right]=\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right] ;|01\rangle=\left[\begin{array}{l}1 \\ 0\end{array}\right] \otimes\left[\begin{array}{l}0 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 0\end{array}\right] ;$
$|10\rangle=\left[\begin{array}{l}0 \\ 1\end{array}\right] \otimes\left[\begin{array}{l}1 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 0\end{array}\right] ;|11\rangle=\left[\begin{array}{l}0 \\ 1\end{array}\right] \otimes\left[\begin{array}{l}0 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]$.

Now $H=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}1 & 1 \\ 1 & -1\end{array}\right]$ so $H \otimes H=\frac{1}{2}\left[\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1\end{array}\right]$.
So $H \otimes H|00\rangle=\frac{1}{2}\left[\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1\end{array}\right]\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right]=\frac{1}{2}\left[\begin{array}{l}1 \\ 1 \\ 1 \\ 1\end{array}\right]=\frac{1}{2}\left(\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right]+1\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 0\end{array}\right]+1\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 0\end{array}\right]+1\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]\right)$ as required.

Now try $H^{\otimes 2}|01\rangle=\left[\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1\end{array}\right]\left[\begin{array}{c}0 \\ 1 \\ 0 \\ 0\end{array}\right]=\frac{1}{2}\left[\begin{array}{c}1 \\ -1 \\ 1 \\ -1\end{array}\right]=\frac{1}{2}\left(\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right]-1\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 0\end{array}\right]+1\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 0\end{array}\right]-1\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]\right)$.


So it all works as in the formula $H^{\otimes n}|w\rangle=\frac{1}{\sqrt{2}^{n}} \sum_{z}(-1)^{w \cdot z}|z\rangle$.

