# **CS3222 Linear Algebra: Supplementary Notes**

### **Matrices**

### **Matrix Multiplication**

In general, matrix multiplication is defined as follows.  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$ 

Matrix multiplication is not commutative, as shown.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

#### **Determinants**

$$\det\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc \text{ For example:} \begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix} = -(-2)$$

#### **Column Vectors**

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}^{T} = a \text{ typical vector in } \mathbb{C}^{3}. \text{ Addition of vectors works as follows.}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \text{ and multiplication by a scalar: } \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

#### **Linear Independence**

Are 
$$\begin{bmatrix} 1\\2\\-5 \end{bmatrix}$$
 and  $\begin{bmatrix} -1\\5\\6 \end{bmatrix}$  independent? Try:  $\lambda \begin{bmatrix} 1\\2\\-5 \end{bmatrix} + \mu \begin{bmatrix} -1\\5\\6 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \lambda - \mu\\2\lambda - 5\mu\\-5\lambda + 6\mu \end{bmatrix} = 0.$ 

The first element in the vector means that  $\lambda = \mu$ . This, with the second element shows that  $2\lambda + 5\lambda = 0 \Rightarrow \lambda = 0$  and  $\mu = 0$ . The third agrees with this result and so they ARE independent.

 $\begin{bmatrix} 1\\2\\-5 \end{bmatrix} \text{ and } \begin{bmatrix} -1\\5\\6 \end{bmatrix} \text{ span a 2-dimensional subspace of } \mathbb{C}^3. \text{ This consists of all vectors that can}$ be expressed as  $\lambda \begin{bmatrix} 1\\2\\-5 \end{bmatrix} + \mu \begin{bmatrix} -1\\5\\6 \end{bmatrix}. \text{ Is } \begin{bmatrix} 0\\7\\1 \end{bmatrix}$  one of these vectors? Yes, as shown below.  $\lambda - \mu = 0 \text{ and } 2\lambda - 5\mu = 7 \text{ and } -5\lambda + 6\mu = 1 \Rightarrow \lambda = 1 \text{ and } \mu = 1.$  0  $\begin{vmatrix} 5 \\ 5 \end{vmatrix}$  is not a vector of this type since the equations  $\lambda - \mu = 0$ ,  $2\lambda - 5\mu = 5$  and 1

 $-5\lambda + 6\mu = 1$  have no solution. There is a solution for any two but the third will not agree.

### **Coordinates**

The elements of an independent set SPAN a linear subspace (of the space of interest).

In the case of  $\begin{bmatrix} 1\\2\\-5 \end{bmatrix}$  and  $\begin{bmatrix} -1\\5\\6 \end{bmatrix}$  this is a 2-dimensional subspace of  $\mathbb{C}^3$ . The MULTIPLES of  $\begin{bmatrix} 1\\2\\-5 \end{bmatrix}$ and  $\begin{bmatrix} -1\\5\\6 \end{bmatrix}$  needed to represent any vector of this subspace as a linear combination of them are

the COORDINATES (with respect to that BASIS of the subspace).

For example 
$$\begin{bmatrix} 0\\7\\1 \end{bmatrix}$$
 is in the subspace above; and since we know that  $\begin{bmatrix} 0\\7\\1 \end{bmatrix} = 1 \begin{bmatrix} 1\\2\\-5 \end{bmatrix} + 1 \begin{bmatrix} -1\\5\\6 \end{bmatrix}$ 

we can say that the coordinates of the first vector with respect to the other two are (1,1).

# **Inner Products**

 $\langle | \rangle$  is defined on  $\mathbb{C}^3$  as  $\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} \rangle = x_1^* x_2 + y_1^* y_2 + z_1^* z_2$  and is an INNER PRODUCT, as

proved by the 3 rules below

1. 
$$\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \middle| \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \middle| \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right\rangle$$

2.  $\langle | \rangle$  is linear in the second argument and anti-linear in the first.

3. 
$$\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} \left| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = 0 = |x|^2 + |y|^2 + |z|^2 \Leftrightarrow x = y = z = 0$$

We could define another notion  $\langle \langle | \rangle \rangle$  as  $\langle \langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} \rangle = x_1^* x_2 + y_1^* y_2 + 3z_1^* z_2$  and this

would also be an inner product. (Check using the rules above.) So inner products are NOT unique. However we always use  $\langle | \rangle$ .

#### **Orthonormal Bases, Unitary Operators and Change of Basis**

A unitary operator (matrix) satisfies  $U^{\dagger}U = UU^{\dagger} = I$ . Let  $\{|b_0\rangle, |b_1\rangle, |b_2\rangle\}$  be a basis of  $\mathbb{C}^3$ . It is an ORTHONORMAL basis iff  $\langle b_0|b_0\rangle = \langle b_1|b_1\rangle = \langle b_2|b_2\rangle = 1$  and  $\langle b_0|b_1\rangle = \langle b_0|b_2\rangle = \langle b_1|b_0\rangle = \langle b_1|b_2\rangle = \langle b_2|b_0\rangle = \langle b_2|b_1\rangle = 0$ 

Let  $\{|c_0\rangle, |c_1\rangle, |c_2\rangle\}$  be another orthonormal basis. Suppose that

$$|b_{0}\rangle = \begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \end{bmatrix}, |b_{1}\rangle = \begin{bmatrix} b_{10} \\ b_{11} \\ b_{12} \end{bmatrix}, |b_{2}\rangle = \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix}, \text{ and } |c_{0}\rangle = \begin{bmatrix} c_{00} \\ c_{01} \\ c_{02} \end{bmatrix}, |c_{1}\rangle = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \end{bmatrix}, |c_{2}\rangle = \begin{bmatrix} c_{20} \\ c_{21} \\ c_{22} \end{bmatrix}$$

when written out in the standard basis.

Now consider an arbitrary vector  $|v\rangle \in C^3$  with components  $|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$  in the standard

basis. It will have components  $|v\rangle = v_0^b |b_0\rangle + v_1^b |b_1\rangle + v_2^b |b_2\rangle$  and  $|v\rangle = v_0^c |c_0\rangle + v_1^c |c_1\rangle + v_2^c |c_2\rangle$  in the  $|b_i\rangle$  and  $|c_i\rangle$  bases.

How are these things related??? We know that

$$v_{0}^{b}|b_{0}\rangle + v_{1}^{b}|b_{1}\rangle + v_{2}^{b}|b_{2}\rangle = \begin{bmatrix} v_{0} \\ v_{1} \\ v_{2} \end{bmatrix} = v_{0}^{c}|c_{0}\rangle + v_{1}^{c}|c_{1}\rangle + v_{2}^{c}|c_{2}\rangle = v_{0}|0\rangle + v_{1}|1\rangle + v_{2}|2\rangle .$$
  
Also  $|b_{i}\rangle = \sum_{j} b_{ij}|j\rangle$  and  $|c_{i}\rangle = \sum_{j} c_{ij}|j\rangle$  from the previous page.

The matrices  $B = [b_{ij}]$  and  $C = [c_{ij}]$  (where  $0 \le i, j \le 2$ ) are UNITARY (see main notes).

Now 
$$|\underline{c}\rangle = C|\underline{j}\rangle$$
, where  $|\underline{c}\rangle = \begin{vmatrix} |c_0\rangle \\ |c_1\rangle \\ |c_2\rangle \end{vmatrix}$  (i.e.  $|\underline{c}\rangle$  is just the  $|c\rangle$ 's in a column notation),

and  $|\underline{j}\rangle$  is the standard basis also in a column notation, implies  $|\underline{j}\rangle = C^{\dagger}|\underline{c}\rangle$ . Similarly  $|\underline{j}\rangle = B^{\dagger}|\underline{b}\rangle$ .

Now 
$$|v\rangle = v_0^c |c_0\rangle + v_1^c |c_1\rangle + v_2^c |c_2\rangle$$
  

$$= v_0^c \sum_j c_{0j} |j\rangle + v_1^c \sum_j c_{1j} |j\rangle + v_2^c \sum_j c_{2j} |j\rangle = \sum_{ij} v_i^c c_{ij} |j\rangle = [\underline{v}^c]^T C |\underline{j}\rangle \text{ where } \underline{v}^c = \begin{bmatrix} v_0^c \\ v_1^c \\ v_2^c \end{bmatrix}$$
(i.e.  $[\underline{v}^c]$  is the  $v_i^c$  in column notation).

Similarly  $|v\rangle = [v^b]^T B |j\rangle$ . So  $[\underline{v}^c]^T C = [\underline{v}^b]^T B$  or  $C^T [\underline{v}^c] = B^T [\underline{v}^b]$ . Since *C* is unitary,  $C^* C^T = [CC^{\dagger}]^* = I^* = I$ . So  $[\underline{v}^c] = C^* C^T [\underline{v}^c] = C^* B^T [\underline{v}^b]$ . That's how the COEFFICIENTS in the two bases relate.

From  $|\underline{c}\rangle = C|\underline{j}\rangle$  and  $|\underline{j}\rangle = B^{\dagger}|\underline{b}\rangle$  we get  $|\underline{c}\rangle = CB^{\dagger}|\underline{b}\rangle$ . That's how the two BASES relate.

Define  $CB^{\dagger} \equiv U$  and  $C^*B^T \equiv U^*$  where U and  $U^*$  are unitary. We can check that these give consistent expressions for  $|v\rangle$ .  $|v\rangle = v_0^c |c_0\rangle + v_1^c |c_1\rangle + v_2^c |c_2\rangle = [\underline{v}^c]^T |\underline{c}\rangle = (U^*[\underline{v}^b])^T (U|\underline{b}\rangle) = (C^*B^T[\underline{v}^b])^T (CB^{\dagger}|\underline{b}\rangle)$  $= [\underline{v}^b]^T BC^{\dagger}CB^{\dagger}|\underline{b}\rangle = [\underline{v}^b]^T BB^{\dagger}|\underline{b}\rangle = [\underline{v}^b]^T |\underline{b}\rangle = |v\rangle.$ 

N.B. Final Note:  $|b_i\rangle = \sum_j b_{ij} |j\rangle |b_i\rangle = \sum_j \langle j|b_i\rangle |j\rangle$ . Note the order of the indices.

The other choice of order in  $b_{ij}$  simplifies things a bit (in that you just end up with matrices and their adjoints in the derivation above, instead of matrices and their conjugates, transposes, *and* adjoints) but it looks very strange when you write  $|b_i\rangle = \sum_j b_{ji}|j\rangle$  right at the start without saying why (as happens in 80% or more of textbooks).

(It's even worse if you *don't notice* that the indeces are the wrong way round in  $|b_i\rangle = \sum_j b_{ji} |j\rangle$  and thus get screwed up later when something doesn't work out, and don't spot the source of the problem till after half a day's diligent detective work.)

## **Pauli Matrices**

Calculate  $\sigma_x \sigma_y \sigma_z \sigma_y \sigma_x$ . Do it left to right for example:

 $\sigma_x \sigma_y = i\sigma_z \text{ so } \sigma_x \sigma_y \sigma_z \sigma_y \sigma_x$  becomes  $i\sigma_z \sigma_z \sigma_y \sigma_x$ . Also  $\sigma_z^2 = I$  so  $i\sigma_z \sigma_z \sigma_y \sigma_x$  becomes  $i\sigma_y \sigma_x$  and  $\sigma_y \sigma_x = -\sigma_x \sigma_y = -i\sigma_z$  so  $i\sigma_y \sigma_x$  becomes  $i \times (-i)\sigma_z = \sigma_z$ .

All calculations with Pauli matrices can be done this way.

## **Exponentials**

Calculate 
$$e^{A}$$
 where  $A = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ .  
 $e^{A} = I + A + \frac{1}{2!}A^{2} + \frac{1}{3!}A^{3} + \frac{1}{4!}A^{4}$ ..  
 $A^{2} = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^{2n} = I, A^{2n+1} = A$ 

so 
$$e^{A} = \left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right)I + \left(1 + \frac{1}{3!} + \frac{1}{5!} + ...\right)A = \cosh(1)I + \sinh(1)A$$
  
=  $\begin{bmatrix} \cosh(1) + \sinh(1) & 5\sinh(1) \\ 0 & \cosh(1) - \sinh(1) \end{bmatrix}$ 

### **Eigenvalues and Eigenvectors**

Find the eigenvalues and eigenvectors of  $\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$ . The characteristic equation is det $[A - \lambda I] = 0$  so det  $\begin{bmatrix} 2 - \lambda & 1+i \\ 1-i & 3-\lambda \end{bmatrix}$ 

 $= (2 - \lambda)(3 - \lambda) - (1 + i)(1 - i) = 6 - 5\lambda + \lambda^2 - 2 = 0 \text{ which can be rewritten as}$   $\lambda^2 - 5\lambda + 4 = 0 = (\lambda - 1)(\lambda - 4).$ So  $\lambda = 1$  or  $\lambda = 4$ Eigenvector for  $\lambda = 1$  found by solving  $\begin{bmatrix} 2 - 1 & 1 + i \\ 1 - i & 3 - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 + i \\ 1 - i & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$   $a + (1 + i)b = 0 \Rightarrow \frac{a}{b} = \frac{-1 - i}{1} \Rightarrow |a_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = s \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} \text{ where } s \neq 0 \text{ can be any number.}$ Eigenvector for  $\lambda = 4$  can be found by solving  $\begin{bmatrix} 2 - 4 & 1 + i \\ 1 - i & 3 - 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 + i \\ 1 - i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$  $-2a + (1 + i)b = 0 \Rightarrow \frac{a}{b} = \frac{1 + i}{2} \Rightarrow |a_4\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = s \begin{bmatrix} 1 + i \\ 2 \end{bmatrix} \text{ where } s \neq 0 \text{ can be any number.}$ 

### **Direct Sums and Tensor Products**

Let *A* be as above and  $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ . Work out the eigenvalues and eigenvectors of  $A \oplus B$  and  $A \otimes B$ . *A* has eigenvalues/vectors  $1 \rightarrow \begin{bmatrix} -1 & -i \\ 1 \end{bmatrix}$  and  $4 \rightarrow \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$  (see above). *B* is  $i\sigma_x \cdot \sigma_x$  has eigenvalues/vectors  $1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $-1 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  (see eg. RHBnotes p. 41). So *B* has eigenvalues/vectors  $i \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $-i \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .  $A \oplus B$  is  $\begin{bmatrix} 2 & 1+i & 0 \\ 1-i & 3 & 0 \\ 0 & 0 & i \\ i & 0 \end{bmatrix}$  and has eigenvalues  $\{\lambda_A, \lambda_B\}$ , where  $\{\lambda_A\}$  are eigenvalues of *A* and  $\{\lambda_B\}$  are eigenvalues of *B* with corresponding eigenvectors. So the eigenvalues and eigenvectors of  $A \oplus B$  are:

$$1 \rightarrow \begin{bmatrix} -1-i \\ 1 \\ 0 \\ 0 \end{bmatrix}, 4 \rightarrow \begin{bmatrix} 1+i \\ 2 \\ 0 \\ 0 \end{bmatrix}, i \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -i \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$
 It's easy to see that this works.

 $A \otimes B$  is the array of numbers  $a_{ij}b_{kl}$ , normally arranged thus:

$$\begin{bmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{bmatrix} = \begin{bmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{bmatrix}$$
or, in our case 
$$\begin{bmatrix} 0 & 2i & 0 & -1+i \\ 2i & 0 & -1+i & 0 \\ 0 & 1+i & 0 & 3i \\ 1+i & 0 & 3i & 0 \end{bmatrix}.$$

Eigenvalues of  $A \otimes B$  are  $\{\lambda_A \cdot \lambda_B\}$ , where  $\{\lambda_A\}$  are the eigenvalues of A and  $\{\lambda_B\}$  are the eigenvalues of B. Eigenvectors of  $A \otimes B$  are tensor products of the corresponding eigenvectors of A and B.

So we get 
$$i \to \begin{bmatrix} -1 & -i \\ 1 & \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ -1 & -i \\ 1 \\ 1 \end{bmatrix}$$
,  $-i \to \begin{bmatrix} -1 & -i \\ 1 & \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ 1 + i \\ 1 \\ -1 \end{bmatrix}$   
$$4i \to \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+i \\ 1+i \\ 2 \\ 2 \end{bmatrix}$$
,  $-4i \to \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+i \\ -1-i \\ 2 \\ -2 \end{bmatrix}$ .

As you can see this is somewhat harder to check explicitly than in the direct sum case.

# **The Hadamard Transformation**

The general theory says that  $H^{\otimes 2}|00\rangle = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$ . We'll check in detail.

$$|0\rangle = \begin{bmatrix} 1\\0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0\\1 \end{bmatrix}. \text{ So .....}$$
$$|00\rangle = \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 1\\0 \end{bmatrix} = \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}; |01\rangle = \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 0\\1\\0 \end{bmatrix} = \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix};$$
$$|10\rangle = \begin{bmatrix} 0\\1\\0 \end{bmatrix} \otimes \begin{bmatrix} 1\\0\\0\\1\\0 \end{bmatrix} = \begin{bmatrix} 0\\0\\1\\0\\1 \end{bmatrix}; |11\rangle = \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix} \otimes \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix}.$$

So it all works as in the formula  $H^{\otimes n}|w\rangle = \frac{1}{\sqrt{2}^n}\sum_{z}(-1)^{w \cdot z}|z\rangle.$