Linear Algebra Quick Reference

Inner Product Space

$H = \mathbb{C}^n$

1. $|x\rangle + |y\rangle = |x+y\rangle = |y\rangle + |x\rangle \in H$ where $|x\rangle, |y\rangle \in H$.
2. $|\lambda x\rangle = \lambda|x\rangle \in H$ where $|x\rangle \in H, \lambda \in \mathbb{C}$.
3. $\langle y|x\rangle \in \mathbb{C}$ where $|x\rangle, |y\rangle \in H$.
4. $\langle x|y\rangle = \langle y|x\rangle^*$
5. $\langle z|(|x\rangle + |y\rangle)\rangle = \langle z|x\rangle + \langle z|y\rangle$ and $\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$
6. $\langle y|\mu x\rangle = \mu\langle y|x\rangle = \langle \mu^* y|x\rangle$
7. $\langle x|x\rangle = 0 \Rightarrow |x\rangle = \underline{0}$

Independent vectors	Set of vectors $|x_0\rangle \ldots |x_d\rangle$ where $\lambda_0|x_0\rangle + \lambda_1|x_1\rangle \ldots \lambda_d|x_d\rangle = \underline{0} \Rightarrow \lambda_0 = \lambda_1 = \ldots\lambda_d = 0$.
Basis	A maximal set of independent vectors.
Orthogonal Basis	A basis where $\langle x_j|x_k\rangle \neq 0 \Rightarrow j = k$.
Orthonormal Basis	An orthogonal basis where all vectors have norm 1.

Pauli Spin Matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x \;,\; \sigma_y\sigma_z = i\sigma_x = -\sigma_z\sigma_y \;,\; \sigma_z\sigma_x = i\sigma_y = -\sigma_x\sigma_z \;,$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \;.$$

Types of Operator

Adjoint $A^\dagger$	$\langle y|Ax\rangle = \langle A^\dagger y|x\rangle$ or $A^\dagger = (A^T)^*$.
Normal	$A^\dagger A = AA^\dagger$.
Unitary	$A^\dagger A = I = AA^\dagger$.
Hermitian	$A = A^\dagger$.

Bases

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;,\; |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;.$$

$$|0_H\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \;,\; |1_H\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \;.$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \;.$$

Direct Sums

1. $(|u_1\rangle, |v_1\rangle) + (|u_2\rangle, |v_2\rangle) = (|u_1\rangle + |u_2\rangle, |v_1\rangle + |v_2\rangle) \in H_1 \oplus H_2$ where $(|u_i\rangle, |v_i\rangle) \in H_1 \oplus H_2$
2. $\lambda(|u\rangle, |v\rangle) = (\lambda|u\rangle, \lambda|v\rangle)$ where $\lambda \in \mathbb{C}$.
3. $\langle(|u_1\rangle, |v_1\rangle)|(|u_2\rangle, |v_2\rangle)\rangle = \langle u_1|u_2\rangle + \langle v_1|v_2\rangle \in \mathbb{C}$.
4. $(A_1 \oplus A_2)(|u\rangle, |v\rangle) = (A_1|u\rangle, A_2|v\rangle)$.

Eigenvalues/vectors: $A_1|u\rangle = u|u\rangle$ and $A_2|v\rangle = v|v\rangle$ give eigenvalues and eigenvectors for each part, so $(A_1 \oplus A_2)(|u\rangle, \underline{0}) = u(|u\rangle, \underline{0})$ and $(A_1 \oplus A_2)(\underline{0}, |v\rangle) = v(\underline{0}, |v\rangle)$ give all the eigenvalues and vectors.

Tensor Products

General form of vector in $H_1 \otimes H_2$ is $\sum_{jk} a_{jk}|u_j\rangle \otimes |v_k\rangle$. Can be formed from two vectors in H. $|u\rangle = \sum_j a_j|u_j\rangle$ and $|v\rangle = \sum_k b_k|v_k\rangle$ means that $|u\rangle \otimes |v\rangle = \sum_{jk} a_j \cdot b_k|u_j\rangle \otimes |v_j\rangle$.

1. $\sum_{jk} a_{jk}|u_j\rangle \otimes |v_k\rangle + \sum_{jk} b_{jk}|u_j\rangle \otimes v_k = \sum_{jk} (a_{jk} + b_{jk})|u_j\rangle \otimes |v_k\rangle$
2. $\lambda\sum_{jk} a_{jk}|u_j\rangle \otimes |v_k\rangle = \sum_{jk} \lambda a_{jk}|u_j\rangle \otimes |v_k\rangle$ where $\lambda \in \mathbb{C}$.
3. $\langle(|u_1\rangle \otimes |v_1\rangle)|(|u_2\rangle \otimes |v_2\rangle)\rangle = \langle u_1|u_2\rangle \times \langle v_1|v_2\rangle$
4. $(A_1 \otimes A_2)\sum_{jk} a_{jk}|u_j\rangle \otimes |v_k\rangle = \sum_{jk} a_{jk}(A_1|u_j\rangle) \otimes (A_2|v_k\rangle)$

Eigenvalues/vectors: $A_1|u\rangle = u|u\rangle$ and $A_2|v\rangle = v|v\rangle$ give eigenvalues and eigenvectors for each part, so $(A_1 \otimes A_2)|u\rangle \otimes |v\rangle = u \cdot v|u\rangle \otimes |v\rangle$ gives all the eigenvalues and vectors.

Typical “wierd” equality: $(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = (|0_H\rangle \otimes |0_H\rangle + |1_H\rangle \otimes |1_H\rangle)$.

Notation: $|u\rangle \otimes |v\rangle = |u \otimes v\rangle = |u\rangle|v\rangle = |u, v\rangle = |uv\rangle$.

$\sum_z |z\rangle = (|0\ldots00\rangle + |0\ldots01\rangle + |0\ldots10\rangle + |0\ldots11\rangle + \ldots) =$ “the sum of all binary strings”.

$$H^{\otimes n}|w\rangle = \frac{1}{\sqrt{2}^n}\sum_z (-1)^{w \cdot z}|z\rangle.$$