1. Consider Grover’s algorithm. Show that:
   (i) \( (2|\psi\rangle\langle\psi| - I)(\sum_x \alpha_x |x\rangle) = \sum_x (-\alpha_x + 2\langle\alpha\rangle)|x\rangle \) where \( \langle\alpha\rangle = \sum_x \alpha_x / N \)
   (ii) Explain why the operation \( (2|\psi\rangle\langle\psi| - I) \) is called inversion about the mean.

2. Show that in Grover’s algorithm, the action of \( (2|\psi\rangle\langle\psi| - I) \) in the \(|\sigma\rangle \) and \(|\tau\rangle \) plane is reflection about \(|\psi\rangle \).

3. In Grover’s algorithm, show that if we change the computational basis so that \(|\sigma\rangle \) and \(|\tau\rangle \) are basis elements, then the matrix representation of the Grover transform \( G \) (restricted to the \(|\sigma\rangle, |\tau\rangle \) plane) will be:

   \[
   G_{\sigma,\tau} = \begin{bmatrix}
   \cos(\theta) & -\sin(\theta) \\
   \sin(\theta) & \cos(\theta)
   \end{bmatrix}
   \]

   where \( \theta \) is the angle between \(|\sigma\rangle \) and \(|\psi\rangle \). Show that the eigenvalues of \( G_{\sigma,\tau} \) are: \( e^{i\theta} \) with eigenvector \(|a⟩ = |\sigma⟩ - i|\tau⟩\), and \( e^{-i\theta} \) with eigenvector \(|b⟩ = |\sigma⟩ + i|\tau⟩\). Express the vector \(|\psi⟩ = N^{-1}\sum_x |x⟩\) in this basis, i.e. find \( d_a \) and \( d_b \) such that \(|\psi⟩ = d_a |a⟩ + d_b |b⟩\).

4. In Grover’s algorithm, let \( n = 2 \) so that \( N = 2^n = 4 \), and let \( M = 1 \). The oracle \( f \) with \( f(x) = 0 \) for all \( x \neq x_0 \) and \( f(x_0) = 1 \) can be chosen as one of the four control variants of Toffoli, (i.e. \( C^2 \cdot \oplus \) controlled on \(|00⟩, |01⟩, |10⟩, |11⟩\), respectively). Show that the circuit in Fig. 32 of the notes implements the operation \( G \). How many iterates of \( G \) are needed to determine \( x_0 \)?

5. What is the matrix for the quantum Fourier transform \( F \) when \( n = 1 \) and \( n = 2 \)? Show that \( F \) is unitary for any \( n \).

6. Work out the matrix for the quantum Fourier transform \( F \) and the network which implements it for \( n = 3 \).

7. Show that the inverse quantum Fourier transform is given by: \( F^\dagger : |j⟩ \rightarrow \sqrt{\frac{n}{2}} \sum_k e^{-2\pi ijk/2^n} |k⟩ \), where \( k \) ranges from 0 to \( 2^n - 1 \).

8. Work out the circuit for the inverse Fourier transform.

9. When \( n = 3 \), work out the Fourier transform of the state \((|000⟩ + |001⟩ + |010⟩ + |011⟩)/2\).

10. Repeat Q9 for the states \((|000⟩ + |001⟩ + |010⟩ + |011⟩ + |100⟩ + |101⟩ + |110⟩ + |111⟩)/2\sqrt{2} \) and \((|000⟩ + |001⟩ + |010⟩ + |011⟩ - |100⟩ - |101⟩ - |110⟩ - |111⟩)/2\sqrt{2}\).

11. A (unitary) quantum algorithm works on two registers \( R_1 \) and \( R_2 \), initialised to \(|0…0⟩\) and \(|u⟩\) respectively. Suppose that after completion, if the first register is measured in the standard basis, it produces a desired answer \(|φ_u⟩\) with probability no less than \((1-ε)\). Now suppose that there are several such \(|u⟩\)’s, with the \( u \)’s elements of a set \( T \), such that all the different \(|u⟩\)’s are orthogonal to each other, and such that for each \( u \), the probability of producing the desired answer \(|φ_u⟩\) is no less than \((1-ε)\). The algorithm is now run on the state \(|0…0⟩\otimes(∑_{u∈T} |d_u⟩|u⟩)\). Show that the probability that measurement of the first register yields the outcome \(|φ_u⟩\) is no less than \(|d_u|^2(1-ε)\). Relate this to Exercise 9.6 of the phase estimation algorithm.