Algebras for Generalized Power Constructions

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Contents

Introduction

In order to give a denotational semantics for programming languages, quite a number of settings have been proposed: algebraic dcpo's, universal domains, topological spaces and locales, just to mention a few. This raises the question what these different approaches have in common and how they can be understood to evolve from the operational (or intended) behaviour of the programming language in question.

A suggestion how to attack this problem at a high level of abstraction has recently been made by E. Moggi in [Mog91]. He proposes that each connective of a programming language can be described by a (strong) monad on an underlying category of values. Programs then should be interpreted in the Kleisli category for that monad. This approach, if successful, makes it possible to give a semantics for a programming language in an incremental fashion. Step by step one adds a feature to the language resulting in a series of monads. Since it is not prescribed in which category this is carried out, one can hope to unearth the essential properties of those monads that are to model a particular construct. One of the invariants, as we may call them, ought to be found in the algebraic theories associated with the monads. We really expect them to essentially agree. A second benefit from such a meta-semantical approach can be that we may be able to extract a logic for reasoning about programs which will no longer refer to a particular denotational model. A first attempt in this direction has been made in [Pit91].

In the present work we deal with the classical problem of adding non-determinism to a sequential programming language. For most of the settings mentioned above, concrete power constructions have been introduced in order to capture the possibilities of multiple output of non-deterministic computations. Starting with the work of M. Hennessy and G. Plotkin in [HP79] it is well known that each of these constructions gives rise to a monad. The question that we are concerned with here is whether the equational characterization of power constructions given by Hennessy and Plotkin for algebraic dcpo's remains valid if the setting is changed. We extract the equational description in the usual way, that is, by characterizing the associated (Eilenberg-Moore) algebras.

The three settings that we will study are (arbitrary) dcpo's, topological spaces and locales. For all of them, power construction have been developed previously [Plo76, Smy78, Smy83, Joh82b, Rob88]. We examine the algebras for two power constructions, modeling non-determinism as *angelic* respectively *demonic*. It turns out that in all cases the algebras can be understood as semilattices in the respective categories, sometimes satisfying additional properties.

The first to define power constructions in order to model non-determinism were G. Plotkin [Plo76] and M. Smyth [Smy78] who did so for algebraic dcpo's. Later on, the so-called Hoare power domain was introduced. This construction is folklore and cannot be attributed to a single person. A study of possible generalizations for these (among other) constructions to arbitrary dcpo's has been carried out by R. Heckmann in [Hec90]. Using his definitions, we examine in how far these generalizations still yield the same algebraic theory as the original power domain functors.

The category **Top** of T_0-spaces as a setting for denotational semantics has been proposed by M. Smyth. In [Smy83] he presents the idea that an open set can be seen as a computable property. He illustrates this view by formulating concepts like 'specification', 'predicate transformer' and 'non-determinism' in topological terms. In order to model non-determinism in this setting, one also wants to use set-valued functions. M. Smyth gave definitions of Hoare, Smyth and Plotkin power spaces as subspaces of the power set. The topologies suggested by different views of non-determinism are known as lower, upper and Vietoris topology and therefore connect this approach with the study of power- or hyper-spaces in general topology. Such 'spaces of subsets' and multivalued functions have been examined by L. Vietoris [Vie21, Vie22] (see also [Mic51]).

Building on Smyth's definition we obtain monads for the Hoare and Smyth power space constructions on **Top**. We then proceed to examine the algebras for these monads. By doing this we obtain characterizations of several categories of domains as algebras for monads on subcategories of **Top** similar to those in [Day75, Wyl81, Tay90, Sch93a].

The idea of understanding open sets as properties of their elements can be further abstracted by using arbitrary frames instead of open set lattices. So one can view the elements of a frame as propositions, interpreting join as conjunction and meet as disjunction. These considerations lead to the *logic of finite observations* and, finally, to a presentation of denotational semantics in an axiomatic form, as a logic of programs [Abr87].

This approach is tightly related to the use of algebraic dcpo's in denotational semantics via Stone-type dualities. Rather than the category of frames one considers its opposite, the category of locales. In order to handle non-determinism one needs constructions for locales corresponding to power domains. The first power locale construction, the so-called Vietoris power locale, has been defined by P. Johnstone in [Joh82b]. His definition is based on the Vietoris power space and uses properties of the Vietoris topology. This construction yields a monad whose algebras are shown to carry a semilattice structure in the category of locales (see also [Joh85]). In [Vic], S. Vickers shows that for locally compact locales the reverse is also true, that is, every localic semilattice is an algebra for the monad.

The connection between this power locale construction and the axiomatic approach to semantics has been made in two steps. G. Winskel [Win85] described the classical power domains via modal assertions about the behaviour of a program. Carrying this point of view a step further, E. Robinson gave a definition of Hoare and Smyth power locale constructions [Rob88]. Moreover, he pointed out that Johnstone's Vietoris power locale corresponds to the Plotkin power domain.

We start our examinations of monads resulting from power construction by giving an account of some basic results on dcpo's, topological spaces and frames in Chapter 1. In the second chapter, we collect some results about locales. We then point out the connections between continuous dcpo's, locally compact spaces and locally compact locales in Chapter 3.

The fourth chapter is devoted to introducing some categorical notions, in particular that of monads and their algebras. We further give some general results on power monads for order-enriched categories. In Chapter 5 we direct our attention to the theory of generalized semilattices. In any order-enriched category there are two special cases of semilattices which correspond to $\vee$- and $\wedge$-semilattices in the category of posets: Any semilattice operation gives rise to an order. On a poset, this order or its dual might coincide with the intrinsic order. We speak of *inflationary* respective *deflationary* semilattices in these cases. These concepts are meaningful in every order-enriched category. We examine them for dcpo's, topological spaces and locales. We are also interested in retracts of (generalized) semilattices since, in most cases we study, the algebras for a monad are exactly the retracts of the free algebras. Since the free algebras can all be shown to carry semilattice structures, these results allow us to conclude that every algebra must be a semilattice. We also define unital generalized semilattices and show that the results about retracts of semilattices can be transferred to this case.

In Chapter 6, we study the Hoare power construction for all settings described above. Surprisingly enough, it turns out that in all cases the algebras are exactly the inflationary semilattices (with the slight restriction to sober spaces instead of all T_0-spaces). Furthermore, the restriction of the power functors to continuous dcpo's, locally compact spaces and locally compact locales are also well-behaved.

In the last chapter we give a similar account for the Smyth power construction. Here, however, things become more complicated. First of all, there are two different constructions for both, dcpo's and topological spaces, that generalize the Smyth power domain. Also, the algebras for the resulting monads are not characterized by being (deflationary) semilattices. For dcpo's and topological spaces we can indeed give examples of deflationary semilattices that do not have a structure map. In the cases of topological spaces and locales, we can prove that algebras for the monads must have small semilattices. However, we can give an example for a deflationary topological semilattice with small semilattices that fails to be an algebra for the monad. We have not been able to find a corresponding example for the category of locales. Nevertheless, for the subcategories of continuous dcpo's, locally compact spaces and locally compact locales we do get a nice characterization: the algebras are precisely those deflationary semilattices that have small semilattices. Furthermore, the two different Smyth power constructions coincide. In Chapters 6 and 7 we also give a short description of how to modify the definitions of our power constructions in order to make the algebras unital semilattices.

We have not tried to characterize the algebras for the generalized Plotkin (Vietoris) power construction. Hopefully, our results can be applied to solve this problem since this construction can be understood to be built from the respective Hoare and Smyth constructions.

The Appendix provides tables containing most of the names for categories and functors we use. In addition, our results are summarized there.

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Chapter 1

Some basic concepts

In this chapter we will introduce the basic notions, notations and results on dcpo's, topological spaces and frames that we are going to need later. We will not give all proofs. However, we try to develop the main ideas as we go along. If a proof is omitted it can be found in one of the references quoted. Most of the results are well-known with the exception of a few facts which are straightforward to prove and for which we do not know any reference.

1.1 Dcpo's and domains

All structures we shall study (dcpo's, topological spaces and locales) carry an order. Let us shortly introduce the relevant order-theoretic concepts. The reader who wishes to consult a more detailed introduction into domain theory and the theory of continuous posets may find [Plo81] or [GHK$^+$80] useful references.

1.1.1 Some order-theoretic notions

The *lower closure* of a subset S of a poset P is $\downarrow S := \{p \in P \mid \exists s \in S \,.\, p \leq s\}$. A set S that is lower closed (that is, $S = \downarrow S$) is called a *lower set*. An *upper set* and the *upper closure* are defined and denoted dually.

A poset P is a $\vee$*-semilattice* if for every pair of elements of P the join exists in P. A *morphism of* $\vee$*-semilattices* or $\vee$*-semilattice homomorphism* is a function that preserves binary joins. The dual notions are that of $\wedge$*-semilattice* and $\wedge$*-semilattice homomorphism.* A poset that at the same time is a $\vee$- and a $\wedge$-semilattice is called a *lattice*. If a $\vee$-semilattice S has a least element then this element is the unit for the binary join. We therefore say that S is a *unital* $\vee$*-semilattice*. A *morphism of unital* $\vee$*-semilattices* is a morphism of $\vee$-semilattices that preserves the least element. There is a dual notion for $\wedge$-semilattices.

A poset that has all joins is a *complete* $\vee$*-semilattice*. A *morphism of complete* $\vee$*-semilattices* is a function that preserves all joins. *Complete* $\wedge$*-semilattices* and their morphisms are defined analogously. Note that every complete $\vee$-semilattice has all meets as well and thus is a complete $\wedge$-semilattice or a *complete lattice*. We keep these different names, however, in order to make clear which morphisms we consider.

Functions between complete lattices that preserve all joins have an interesting property: Such a morphism $f\colon L \longrightarrow M$ can be used to define a morphism of complete $\wedge$-semilattices $g\colon M \longrightarrow L$ by setting $g(m) := \bigvee f^{-1}(\downarrow\{m\})$. The resulting function g is called the *upper adjoint* for f (and of course f is the *lower adjoint for* g). Note that in this case $f(l) \leq m$ iff $l \leq g(m)$ holds for all $l \in L$, $m \in M$. Analogously, given a morphism of complete $\wedge$-semilattices $g\colon M \longrightarrow L$ we can define a morphism of complete $\vee$-semilattices $f\colon L \longrightarrow M$ in the obvious way. Thus the category of complete $\wedge$-semilattices is dual to the category of complete $\vee$-semilattices.

More generally, let P and Q be two posets and $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ be functions where $f(p) \leq q$ iff $p \leq g(q)$. Then we say that (f, g) is an *adjoint pair* where f is the *lower adjoint* for g. The definite article may be used as there is at most one lower (or upper) adjoint for every given function. Note that every lower adjoint preserves the order and existing joins while every upper adjoint is also order-preserving but preserves existing meets. If $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ are order-preserving functions then (f, g) is an adjoint pair iff $f \circ g \leq id_Q$

and $g \circ f \geq id_P$. The proof of the following lemma is straightforward.

Lemma 1.1 *Let* $P \underset{g}{\overset{f}{\rightleftarrows}} Q \underset{g'}{\overset{f'}{\rightleftarrows}} R$ *be functions between posets where* f *and* f' *are the lower adjoints for* g *and* g'*, respectively. Then* $f' \circ f$ *is the lower adjoint for* $g \circ g'$. ■

We have already seen one example of adjoints: For any $\vee$-semilattice P, the binary join function from $P \times P$ to P is the lower adjoint for the diagonal $P \longrightarrow P \times P$ since $p \vee p' \leq p''$ iff $\langle p, p' \rangle \leq \langle p'', p'' \rangle$.

The following lemma states a well-known fact.

Lemma 1.2 *Let* P *be a retract of* Q *via* $P \underset{r}{\overset{e}{\rightleftarrows}} Q$*. Let* S *be a subset of* P*. If the join of* $e(S)$ *exists in* Q*, then the join of* S *exists in* P *and* $\bigvee S = r(\bigvee e(S))$.

Proof: Assume that $\bigvee e(S)$ exists in Q for some $S \subseteq P$. Since for all $s \in S$ we have $s = r(e(s)) \leq r(\bigvee e(S))$, we get that $r(\bigvee e(S))$ is an upper bound for S. If, on the other hand, $p \in P$ is any upper bound for S then $e(p)$ is an upper bound for $e(S)$ implying $p = r(e(p)) \geq r(\bigvee e(S))$. Hence $r(\bigvee e(S))$ is the least upper bound for S. ■

By considering the dual posets we may deduce that the above lemma also holds if join is replaced by meet.

Corollary 1.3 *Let* P *be a retract of* Q *via* $P \underset{r}{\overset{e}{\rightleftarrows}} Q$*. Let* S *be a subset of* P*. If the meet of* $e(S)$ *exists in* Q*, then the meet of* S *exists in* P *and* $\bigwedge S = r(\bigwedge e(S))$. ■

Hence a retract of a $\wedge$-semilattice, for example, is a $\wedge$-semilattice.

1.1.2 Directed complete partial orders

For a partially ordered set P we say that $S \subseteq P$ is *directed* if S is non-empty and if any two of its elements have a common upper bound in S. The dual notion is that of a *filtered* set.

Most posets we shall deal with have least upper bounds for all their directed subsets; they are called *directed complete partial orders* or *dcpo's* for short. Note that unlike other authors we do not assume a dcpo to have a least element.

We may think of the join $\bigvee^{\uparrow} S$ of a directed subset S (where the arrow is to indicate that S is supposed to be directed) as being approximated by the elements of S. This becomes precise if we consider the *Scott topology* on a dcpo D. A subset U of D is *Scott-open* if it has the following properties:

(i) It is upper closed, that is $U = \uparrow U$.

(ii) It is inaccessible by directed joins, that is if S is a directed subset of D with $\bigvee^{\uparrow} S \in U$ then S intersects U.

Any directed subset S of a dcpo D converges to $\bigvee^{\uparrow} S$ with respect to the Scott topology. Furthermore, $\bigvee^{\uparrow} S$ is the greatest limit point of the net S with respect to that topology. Since singleton closures are lower closures, the Scott topology of a dcpo is T_0.

The Scott-continuous functions (that is those that are continuous with respect to the Scott topology) can be characterized order-theoretically: A function $f: D \longrightarrow E$ between dcpo's is Scott-continuous iff it preserves directed joins. Note that a Scott-continuous function in particular is order-preserving. We denote the category of dcpo's with Scott-continuous functions by **DCPO**. Let T_σ denote the functor from **DCPO** to the category of T_0-spaces that arises if we consider a dcpo a topological space via its Scott topology. We set $\mathrm{T}_\sigma f = f$ for all morphisms f in **DCPO**.

1.1.3 Algebraic dcpo's

The usage of dcpo's in denotational semantics relies on the above idea of approximation. Of particular interest are dcpo's where every element is approximated by certain elements below it. The elements which cannot be approximated non-trivially play a special rôle. This concept is captured by the notion of *compact* elements of a dcpo. An element c of a dcpo D is *compact* if for every directed subset S of D with $c \leq \bigvee^{\uparrow} S$ there is $s \in S$ such that $c \leq s$. That means that a compact element c of D which is approximated by a directed set S is already reached by an element of S. Let the set of compact elements of a dcpo D be denoted by $\mathsf{C}D$. A dcpo D is *algebraic* if, for every element d of D, the set of compact elements below d is directed with join d, that is $d = \bigvee^{\uparrow}(\mathsf{C}D \cap \downarrow\{d\})$ for all $d \in D$. For an introduction into the use of algebraic dcpo's in denotational semantics see [Plo81].

Algebraic dcpo's have some nice properties. The Scott topology of an algebraic dcpo D, for example, has a basis of sets of the form $\uparrow\{c\}$ where c is a compact element of D. We can also recover an algebraic dcpo D from its set of compact elements $\mathsf{C}D$. A subset of a poset is an *ideal* if it is directed and lower closed. The dual notion of an ideal is a *filter*. We will denote the set of all ideals of a poset P ordered by inclusion by $\mathsf{Idl}\, P$. For any poset P, we thus get an algebraic dcpo $\mathsf{Idl}\, P$. The compact elements of $\mathsf{Idl}\, P$ are the ideals of the form $\downarrow\{p\}$ where $p \in P$ and are called *principal ideals*. Hence $P \cong \mathsf{C}\,\mathsf{Idl}\, P$. If D is an algebraic dcpo, then $D \cong \mathsf{Idl}\,\mathsf{C}D$. If P is a pre-ordered set (rather than a partially ordered one) then $\mathsf{C}\,\mathsf{Idl}\, P$ is isomorphic to the quotient $P/_{\equiv}$ where $\equiv$ is the equivalence relation defined by $p \equiv p'$ iff $p \leq p'$ and $p' \leq p$.

A Scott-continuous function f from an algebraic dcpo D to any dcpo E is uniquely determined by its values on the compact elements of D. More precisely, any order-preserving function $f\colon \mathsf{C}D \longrightarrow E$ can be uniquely extended to a Scott-continuous function $\hat{f}\colon D \longrightarrow E$ by setting $\hat{f}(d) := \bigvee^{\uparrow}(\mathsf{C}D \cap \downarrow\{d\})$. We denote the category of posets with order-preserving functions by **Pos**. Let **AlgD** be the full subcategory of **DCPO** whose objects are the algebraic dcpo's. There is an obvious way to extend Idl to a functor from the category **Pos** to the category **AlgD**. The functor thus defined is left adjoint to the forgetful functor from the category of posets to the category of algebraic domains.

1.1.4 Domains

The category **AlgD** of algebraic dcpo's with Scott-continuous functions, however, is not closed with respect to all constructions that one would wish to have. An example for such a construction is that of a retract.

Hence we weaken the notion of algebraicity. For elements d and d' of a dcpo D we say that d is *way below* d' if for every directed subset S of D with $d' \leq \bigvee^{\uparrow} S$ there is $s \in S$ such that $d \leq s$. In this situation, we write $d \ll d'$. One can understand the way below relation to be a notion of compactness relative to an element of the dcpo: An element $d \in D$ is compact iff it is way below itself. So if $d \ll d'$ we also say that d is *relatively compact* in d'. The set of all elements way below some $d \in D$ is denoted by $\twoheaddownarrow\{d\}$. If $d, d' \ll d''$ holds and if $d \vee d'$ exists in D then $d \vee d' \ll d''$.

A dcpo D is *continuous* if for all $d \in D$ the set $\twoheaddownarrow\{d\}$ is directed with join d. Continuous dcpo's have the property that every element is approximated by the elements which are relatively compact in that element. We use the generic term *domain* for them. In an algebraic dcpo, every element is even approximated by the compact elements below it. Therefore we also speak of *algebraic domains*. If we talk of continuous posets, then the word 'continuous' is to imply that the poset in question has directed joins, that is it is a domain. For a poset to be a *continuous lattice* we require that it is a complete lattice. If D is a domain then the sets $\twoheaduparrow\{d\} := \{d' \in D \mid d \ll d'\}$ where $d \in D$ form a basis for the Scott topology. Furthermore, the *interpolation property* holds in D: Given $d \ll d'$ in D there is $d'' \in D$ with $d \ll d'' \ll d'$. Note that a dcpo D is continuous iff $\bigvee^{\uparrow}\colon \mathsf{Idl}\, D \longrightarrow D$ has a lower adjoint. If it exists, this lower adjoint is $d \longmapsto \twoheaddownarrow\{d\}\colon D \longrightarrow \mathsf{Idl}\, D$.

Above we have seen that retracts of posets (via order-preserving functions) inherit many order-theoretic properties. A retract of a dcpo, for example, is a dcpo. Continuity, however, is not such a property. It is only inherited by images of Scott-continuous retractions. So in the category of posets (with order-preserving functions) a retract of a domain need not be domain. In the category of dcpo's with Scott-continuous functions, however, continuity is known to be a property that retracts inherit.

Lemma 1.4 *In* **DCPO***, a retract of a continuous dcpo is continuous.* ■

Note that a retract of an algebraic dcpo in **DCPO** need not be algebraic.

1.1.5 Order-consistent topologies

The Scott topology of a dcpo allows us to regard any dcpo as a topological space. Sometimes we shall do so without explicitly mentioning the topology. In these cases we always mean the Scott topology.

There is also another topology on posets that we will need. The *weak topology* on a poset P is generated by all sets of the form $P \setminus \downarrow\{p\}$ where $p \in P$. Let D be a dcpo. As the Scott closure of a singleton $\{d\}$ is its lower closure $\downarrow\{d\}$, the Scott topology is finer than the weak topology on D. Both topologies share the property that the closure of a singleton is its lower closure. Such topologies (on posets) are called *order-consistent*. Note that in [GHK$^+$80], order consistency is defined differently. The weak topology is the weakest order-consistent topology on any given poset.

We have seen above that directed joins play a special rôle for the Scott topology. A connection between joins and order-consistent topologies arises from the following result.

Lemma 1.5 *Let P be a poset with an order-consistent topology. For any subset S of P the join of S exists in P iff the join of the closure of S exists in P. In this case $\bigvee S = \bigvee \mathsf{cl}\, S$.*

Proof: If $\bigvee S$ exists then because of $\mathsf{cl}\, S \subseteq \downarrow\{\bigvee S\}$ we know that $\bigvee S$ is an upper bound for $\mathsf{cl}\, S$. Obviously, it is the least such element of P. If $\bigvee \mathsf{cl}\, S$ exists, then it clearly is an upper bound for S. If p is any upper bound for S then $\mathsf{cl}\, S \subseteq \downarrow\{p\}$ which proves that $\bigvee \mathsf{cl}\, S$ is the least upper bound for S. ■

An *up-complete $\vee$-semilattice* is a dcpo that also is a $\vee$-semilattice, that is a poset that has joins of all non-empty subsets.

Corollary 1.6 *Let P and Q be posets with order-consistent topologies and further let $f\colon P \longrightarrow Q$ be a continuous order-preserving function.*

(i) Let S be a subset of P. The join of $f(S)$ exists in Q iff the join of $f(\mathsf{cl}_P\, S)$ exists in Q. In this case, $\bigvee f(S) = \bigvee f(\mathsf{cl}_P\, S)$.

(ii) If P and Q additionally are up-complete $\vee$-semilattices then f preserves all non-empty joins iff it preserves joins of non-empty closed sets.

Proof: (i) By the preceeding lemma, $\bigvee f(S)$ exists iff $\bigvee \mathsf{cl}_Q\, f(S)$ exists. Furthermore, the two must be equal. As $f(S) \subseteq f(\mathsf{cl}_P\, S) \subseteq \mathsf{cl}_Q\, f(\mathsf{cl}_P\, S) = \mathsf{cl}_Q\, f(S)$ by continuity of f, this implies the assertion.

(ii) If f preserves joins of non-empty closed sets then $\bigvee f(\mathsf{cl}_P\, S) = f(\bigvee \mathsf{cl}_P\, S)$ for any non-empty subset S of P. By (i) and Lemma 1.5 we get $\bigvee f(S) = \bigvee f(\mathsf{cl}_P\, S) = f(\bigvee \mathsf{cl}_P\, S) = f(\bigvee S)$ for all non-empty S in P. ∎

1.2 Topological spaces

Another setting (besides dcpo's) where we study power constructions is that of topological spaces. A source for most result we gather here is [Joh82a]. We denote the category of T_0-spaces with continuous functions by **Top**. As we have seen before, we may equip a poset with a topology. Here we want to go in the opposite direction. The *specialization order* on a T_0-space X is defined by setting $x \leq x'$ iff $x \in \mathsf{cl}\{x'\}$. (If X is not T_0 this only defines a pre-order.) Clearly the closure of $\{x'\}$ is the lower closure of $\{x'\}$ with respect to the specialization order on X. Hence the topology given on X is consistent with the specialization order on X.

If P is a poset with an order-consistent topology then the intrinsic order on P and the specialization order on P coincide. Hence an order-consistent topology on a poset contains all the information about the order.

We will think of any T_0-space as being equipped with its specialization order and thus refer to order-theoretic concepts as upper set, ideal etc. Every open subset of a T_0-space is an upper set (and thus of course every closed subset is a lower set). Continuous functions are well-behaved with respect to the specialization order in the sense that they are order-preserving. Hence **Top** can be understood as a subcategory of **Pos**.

In order to ensure that the specialization order makes a given T_0-space not just a poset but a dcpo we introduce the following concept. A non-empty closed subset of a topological space is *irreducible* if it is not the union of any two of its closed proper subsets. Clearly, every singleton closure is irreducible. If the sets of this form are the only irreducible closed subsets of a T_0-space X then we say that X is *sober*. Hence a T_0-space X is sober iff for any non-empty irreducible closed subset A of X there is $x \in X$ with $A = \mathsf{cl}\{x\}$.

An alternative way to understand the concept of sobriety is the following: Let ΩX denote the complete lattice of open subsets of X (ordered by inclusion). Every element x of X gives rise to a filter in ΩX, namely to the filter of its open neighbourhoods $\mathcal{N}(x) := \{U \in \Omega X \mid x \in U\}$. The filters thus arising are inaccessible by arbitrary unions. Such filters are also called *completely prime*. More formally, a filter F in a complete lattice L is *completely prime* iff for any subset S of L with $\bigvee S \in F$ there is $s \in S$ such that $s \in F$. A T_0-space X is sober iff the only completely prime filters in ΩX are the filters of open neighbourhoods of elements of X. There is yet another equivalent way of defining sobriety which we will encounter when we introduce locales.

Any domain D, for example, is sober with respect to its Scott topology: We already know that D is T_0. Let A be a non-empty irreducible closed subset of D. We show that A has a greatest element. For that consider the set $\Downarrow A := \bigcup_{a \in A} \Downarrow\{a\}$. This set is directed. Otherwise there are $d, d' \in \Downarrow A$ such that $\Uparrow\{d\} \cap \Uparrow\{d'\} \cap A = \emptyset$. But then $A \subseteq (D \setminus \Uparrow\{d\}) \cup (D \setminus \Uparrow\{d'\})$ which by irreducibility of A implies $A \subseteq D \setminus \Uparrow\{d\}$ or $A \subseteq D \setminus \Uparrow\{d'\}$. But $D \setminus \Uparrow\{d\}$ and $D \setminus \Uparrow\{d'\}$ both contain an element of A which is a contradiction. Hence $\bigvee^{\uparrow} \Downarrow A$ exists and is an element of A. But every $a \in A$ is the join of the elements way below it which proves that $\bigvee^{\uparrow} \Downarrow A$ is an upper bound for A. Hence $A = \downarrow \{\bigvee^{\uparrow} \Downarrow A\}$.

Other examples of sober spaces arise from the following proposition.

Proposition 1.7 *Every up-complete $\vee$-semilattice is sober with respect to the weak topology.*

Proof: Let L be an up-complete $\vee$-semilattice and let $A \subseteq L$ be non-empty, irreducible and weakly closed. If we have $A \subseteq \downarrow\{l_1\} \cup \cdots \cup \downarrow\{l_n\}$ for some $l_1, \ldots, l_n \in L$ then by irreducibility of A there is $i \in \{1, \ldots, n\}$ such that $A \subseteq \downarrow\{l_i\}$. Since A is closed with respect to the weak topology it is the intersection of the finitely generated lower sets containing it. Hence, by the above consideration, we have $A = \bigcap\{\downarrow\{l\} \mid A \subseteq \downarrow\{l\}\}$. Since A is non-empty, the set of upper bounds for A is also non-empty. Therefore,

$$A = \bigcap\{\downarrow\{l\} \mid A \subseteq \downarrow\{l\}\} = \downarrow\{\bigwedge\{l \mid l \text{ upper bound for } A\}\} = \downarrow\{\bigvee A\}.$$

Hence A is the closure of the singleton $\{\bigvee A\}$. It is obvious that L is T_0 and thus must be sober with respect to the weak topology. ■

If X is a sober space then the specialization order makes X a dcpo. In this case the topology given on X is coarser than the Scott topology (and finer than the weak topology). If $f \colon X \longrightarrow Y$ is a continuous function between sober spaces then f preserves directed joins. Thus any continuous function between sober spaces is Scott-continuous.

Sobriety is a property that is inherited by retracts in **Top**. This is a rather well-known fact. However, we do not know of any (published) reference. In order to prove it we need the following property of continuous functions. If $f\colon X \longrightarrow Y$ is a continuous function and S a subset of X then

$$\mathsf{cl}_Y\, f(S) = \mathsf{cl}_Y\, f(\mathsf{cl}_X\, S).$$

Furthermore, we need the following lemma.

Lemma 1.8 *Let $f\colon X \longrightarrow Y$ be a continuous function between topological spaces. If A is an irreducible closed set in X then so is $\mathsf{cl}_Y\, f(A)$ in Y.*

Proof: Let B and B' be closed subsets of Y such that $\mathsf{cl}_Y\, f(A) \subseteq B \cup B'$. Then $A \subseteq f^{-1}(\mathsf{cl}_Y\, f(A)) \subseteq f^{-1}(B) \cup f^{-1}(B')$ and, by irreducibility of A, either $A \subseteq f^{-1}(B)$ implying $\mathsf{cl}_Y\, f(A) \subseteq f\Big(f^{-1}(B)\Big) \subseteq B$ or $A \subseteq f^{-1}(B')$ and thus $\mathsf{cl}_Y\, f(A) \subseteq B'$. ■

Proposition 1.9 *Retracts of sober spaces are sober.*

Proof: Let Y be a sober space and assume that X is a retract of Y via $X \underset{r}{\overset{e}{\rightleftarrows}} Y$. Let A be an irreducible closed non-empty subset of X. We have to show that there is a unique $x \in X$ with $A = \mathsf{cl}_X\{x\}$. Because of Lemma 1.8, the closure of the image of A under e is irreducible in Y. By sobriety of Y there is a unique $y \in Y$ satisfying $\mathsf{cl}_Y\, e(A) = \mathsf{cl}_Y\{y\}$. We will show that $r(y)$ has the desired property. By the above property of continuous functions,

$$A = r(e(A)) \subseteq r(\mathsf{cl}_Y\, e(A)) \subseteq \mathsf{cl}_X\, r\Big(\mathsf{cl}_Y\{y\}\Big) = \mathsf{cl}_X\{r(y)\}.$$

On the other hand, $r(y) \in r(\mathsf{cl}_Y\, e(A)) \subseteq \mathsf{cl}_X\, r(e(A)) = A$ again by the above property. Thus $A = \mathsf{cl}_X\{r(y)\}$. If we assume that X satisfies the T_0-separation property then $r(y)$ is unique with that property. Otherwise, assume $\mathsf{cl}_X\{x\} = \mathsf{cl}_X\{r(y)\}$ for some $x \in X$. Hence

$$\mathsf{cl}_Y\{e(x)\} = \mathsf{cl}_Y\, e\Big(\{\mathsf{cl}_X\{x\}\Big) = \mathsf{cl}_Y\, e\Big(\mathsf{cl}_X\{r(y)\}\Big) = \mathsf{cl}_Y\{e(r(y))\}$$

is an irreducible closed subset of Y which by sobriety of Y implies $e(x) = e(r(y))$. Consequently, $x = r(e(x)) = r(e(r(y))) = r(y)$. ■

1.3 Frames

In this section we shall look for another connection between topological spaces and ordered sets. We have already seen that there is a natural order on the points of a T_0-space (which is determined by its topology).

Given a topological space X there is also an order on the set of open subsets of X, namely subset inclusion. Recall that we denote this poset by ΩX. It is obvious that ΩX is in fact a complete lattice.

We can say more about those complete lattices that can arise as the open set lattice of a topological space. In such a poset finite meets distribute over arbitrary joins. For if $\mathcal{U} \subseteq \Omega X$ and $U' \in \Omega X$ then $U' \cap \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} (U' \cap U)$ for any topological space X.

Definition. A *frame* is a poset that has all joins and all finite meets where finite meets distribute over arbitrary joins.

But note that for any frame of the form ΩX where X is a topological space a much stronger distributivity law holds. Assume that $U^i_j \in \Omega X$ for all $i \in I$ and all $j \in \{1, \ldots, n_i\}$. Let M be the set of all choice functions $f\colon I \longrightarrow \bigcup_{i \in I} \{1, \ldots, n_i\}$, that is $f(i) \in \{1, \ldots, n_i\}$ holds for all $i \in I$. Then $\bigcup_{i \in I} (U^i_1 \cap \cdots \cap U^i_{n_i}) = \bigcap_{f \in M} \bigcup_{i \in I} U^i_{f(i)}$.

Let us investigate which morphisms fit with frames. For any continuous function $f\colon X \longrightarrow Y$ between topological spaces we get a function between the corresponding open set lattices if we take the inverse image function $f^{-1}\colon \Omega Y \longrightarrow \Omega X$. (Note the contravariance!) It is straightforward that this function preserves arbitrary unions as well as finite intersections. Hence a *morphism of frames* is a function that preserves arbitrary joins and finite meets. We will denote the category of frames with these morphisms by **Frm**. The following results can be found in [Joh82a] or [Vic89].

1.3.1 The free frame

Let us first consider finitary algebraic theories, that is theories with finitely many operators of finite arities. Then the forgetful functor from the category of models for that theory to **Set** has a left adjoint [Man76]. The free unital semilattice, for example, over any given set S is isomorphic to the set of finite subsets of S which we denote by $\mathsf{P}_{\mathsf{fin}}S$. In this case, the semilattice operation is union.

In general, the process of constructing the free model for a finitary algebraic theory can be described as follows:

(1) For every element of the given set take a generator.

(2) Form all well-formed expressions involving the operators of the theory and the generators from step (1).

(3) From the equations of the theory, deduce when two expressions resulting from step (2) have to be equal and form the corresponding equivalence classes.

If we try to apply this process to theories that are not finitary we notice that it need not work. The free complete lattice on three generators, for example, does not exist, that is it is a proper class (see Section I.4.7 in [Joh82a]). The theory of frames is another example of a non-finitary theory. So it is quite remarkable that the free frame on every given set does indeed exist. The following result is a consequence of Lemma I.4.4 and Theorem II.1.2 in [Joh82a].

Proposition 1.10 *The free frame on any set S is given by the set $\mathsf{P}_L(\mathsf{P}_{\mathsf{fin}}S, \supseteq)$ of lower subsets of $(\mathsf{P}_{\mathsf{fin}}S, \supseteq)$ ordered by inclusion.* ■

Note that we actually build the free frame on any set S in two steps: First we take the free semilattice over S. In the second step, we construct the joins for the frame, considering the semilattice from the first step a $\wedge$-semilattice. Furthermore, we may think of every $L \in \mathsf{P}_L(\mathsf{P}_{\mathsf{fin}} S, \supseteq)$ as the join of finite meets of generators since

$$L = \bigvee \{\downarrow \{\{s_1\}\} \wedge \cdots \wedge \downarrow \{\{s_n\}\} \mid \{s_1, \ldots, s_n\} \in L\}.$$

1.3.2 Presentations of frames by generators and relations

Given a set G of generators, we often do not wish to construct the free frame over the set G but rather 'build in' some relations between these generators. For example, we might wish to build the free frame over a poset P. In this case we want the generators to keep the given order. In this section we will describe how to do this.

We may think of the relations given on the generators as equations of the form

$$\bigvee_{i \in I} g_1^i \wedge \cdots \wedge g_{n_i}^i = \bigvee_{j \in J} g_1^j \wedge \cdots \wedge g_{m_j}^j$$

where the g_k^i and the g_l^j are generators. So assume we are given a set of generators G and a set R of relations of the above form. As for the free frame, we will proceed in two steps by first constructing finite meets and then adding arbitrary joins. Doing this, the relations from R have to be taken into consideration. As we cannot have expressions like $\bigvee_{i \in I} g_1^i \wedge \cdots \wedge g_{n_i}^i$ in a semilattice, we use a trick. For every relation in R, say $\bigvee_{i \in I} g_1^i \wedge \cdots \wedge g_{n_i}^i = \bigvee_{j \in I} g_1^j \wedge \cdots \wedge g_{m_j}^j$, we create a new generator, say x. The above relation are replaced by

$$g_1^i \wedge \cdots \wedge g_{n_i}^i,\ g_1^j, \ldots, g_{m_j}^j \leq x \qquad \text{for all } i \in I,\ j \in J.$$

We then construct the free unital semilattice, say L, for these generators G' and these new relations R'.

The frame we want is a subset of $\mathsf{P}_L L$. In general we will not take all lower subsets of S because we then would end up with something too big. Once more, we want to regard a lower set of L as the join of its elements. For that we have to make sure that the relations from R are built in properly. To describe those lower sets that we do admit we make the following definition.

We say that a subset I of L is a *C-ideal* for the relations from R' if the following hold:

(i) I is a lower set.

(ii) For all $l \in L$ and all $x \in G'$: If $g_1 \wedge \ldots \wedge g_n \wedge l \in I$ for all relations $g_1 \wedge \ldots \wedge g_n \leq x$ in R' then $x \wedge l \in I$.

Theorem 1.11 (Theorem 4.4.2 in [Vic89]) *The set of all C-ideals for R' of L ordered by inclusion is a frame A that has the following universal property: Given a map f from the set G of generators to any frame B such that*

$$\bigvee_{i \in I} f(g_1^i) \wedge \cdots \wedge f(g_{n_i}^i) = \bigvee_{j \in J} f(g_1^j) \wedge \cdots \wedge f(g_{m_j}^j)$$

for all equations $\bigvee_{i \in I} g_1^i \wedge \cdots \wedge g_{n_i}^i = \bigvee_{j \in J} g_1^j \wedge \cdots \wedge g_{m_j}^j$ from R there is a unique frame homomorphism $\hat{f}\colon A \longrightarrow B$ such that $\hat{f}\mid_G = f$. ■

We denote the frame thus constructed by $Fr\langle G \mid R\rangle$. We will see concrete examples for this construction later when we construct the co-product in **Frm**.

Chapter 2

Locales

In the last chapter we have given a contravariant functor from **Top** to **Frm**. This functor is defined by $X \longmapsto \Omega X$ and $f \longmapsto f^{-1}$ for every morphism $f\colon X \longrightarrow Y$ in **Top**. But we would rather make this a covariant functor, that is we want to view the function f^{-1} as a morphism from ΩX to ΩY. The reason for this is that we want to understand constructions like products, co-products etc. given for topological spaces in terms of the corresponding open set lattices.

We define the category **Loc** to be $\mathbf{Frm}^{\mathrm{op}}$. In order to be able to clearly distinguish between the two we shall refer to the objects of **Loc** as *locales* and the morphisms as *continuous functions* (between locales). Every continuous function $f\colon A \longrightarrow B$ between locales corresponds to a unique morphism of frames which we shall denote by $f'\colon B \longrightarrow A$. We define the functor $\Omega\colon \mathbf{Top} \longrightarrow \mathbf{Loc}$ by setting $(\Omega f)' := f^{-1}$ for every continuous function $f\colon X \longrightarrow Y$. Most of the following results for locales can be found either in [Joh82a] or in [Vic89].

2.1 Locales and topological spaces

2.1.1 Points for a locale

We now try to answer the following question: Given a locale A, how can we define a topological space X such that $A \cong \Omega X$?

The points of a topological space X can be understood as the set of all functions from a given one-point space (that is a terminal object in **Top**) to X. As all these functions are continuous, we get a frame homomorphism $\Omega X \longrightarrow \mathit{2}$ for each of them. Here $\mathit{2}$ denotes the two-point lattice with elements 0_{2} and 1_{2} which is initial in **Frm** and terminal in **Loc**. Given a locale A we define the set $\mathsf{Pt}A$ of points of A to be the set of all continuous functions $\mathit{2} \longrightarrow A$. It remains to give a topology on this set. If $x \in X$ and $p_x\colon \{\star\} \longrightarrow X$ is the function that maps $\star$ to x, then $x \in U$ holds iff $p_x^{-1}(U) = \{\star\}$ for any open subset U of X. We thus take as open sets of $\mathsf{Pt}A$ the sets of the form $U_a := \{p \in \mathsf{Pt}A \mid p'(a) = 1_{\mathit{2}}\}$ where $a \in A$. It is not hard to check that this gives a topology on $\mathsf{Pt}A$ and we refer to the topological space thus defined as $\mathsf{Sp}A$. For every locale A the space $\mathsf{Sp}A$ is sober. The specialization order on $\mathsf{Sp}A$ is the pointwise order for the corresponding frame homomorphisms $A \longrightarrow \mathit{2}$. From now on, by $\mathsf{Pt}A$ we mean the ordered set of points for A thus obtained. Note that $\mathsf{Pt}A$ is a dcpo since it has all directed joins.

Given a continuous function $f\colon A \longrightarrow B$ between locales we want to define a continuous function $\mathsf{Sp}f\colon \mathsf{Sp}A \longrightarrow \mathsf{Sp}B$ between topological spaces. Note that every morphism of locales $f\colon A \longrightarrow B$ induces a Scott-continuous function $\mathsf{Pt}f\colon \mathsf{Pt}A \longrightarrow \mathsf{Pt}B$ which is given by $\mathsf{Pt}f(p) := f \circ p$ for all $p \in \mathsf{Pt}A$. Since

$$(\mathsf{Pt}f)^{-1}(\{q \in \mathsf{Pt}B \mid q'(b) = 1_{\mathit{2}}\}) = \{p \in \mathsf{Pt}A \mid p'(f'(a)) = 1_{\mathit{2}}\},$$

for all $b \in B$, this function is also continuous with respect to the topologies on $\mathsf{Pt}A$ respective $\mathsf{Pt}B$ as introduced above. Considered as a function from $\mathsf{Sp}A$ to $\mathsf{Sp}B$ we denote it by $\mathsf{Sp}f$ and thus obtain a functor from **Loc** to **Top**. Let **Sob** be the full subcategory of **Top** whose objects are all sober spaces. Then Pt is the composition of Sp with the forgetful functor $\mathbf{Sob} \longrightarrow \mathbf{DCPO}$ that maps a sober space X to the set X with the specialization order.

Another way to obtain points for a locale arises as follows: The filter of open neighbourhoods $\mathcal{N}(x)$ of every $x \in X$ is a completely prime filter in ΩX. So we might as well have taken the set of completely prime filters of a locale A to be the set of points of A. As $x \in U$ iff $U \in \mathcal{N}(x)$, the open sets in this setting should be the sets of the form

$$\{F \subseteq A \mid F \text{ completely prime filter and } a \in F\}$$

where $a \in A$. How does this fit with $\mathsf{Sp}A$ as defined above? If p is a point of the locale A then $\{a \in A \mid p'(a) = 1_{\mathit{2}}\}$ is a completely prime filter. If, on the other hand, F is a completely prime filter in A then $p'(a) := 1$ iff $a \in F$ defines a continuous function $p\colon \mathit{2} \longrightarrow A$. This gives in fact a homeomorphism between the topological space just given and $\mathsf{Sp}A$.

There is a third description of the points for a locale, namely as the *prime* elements of the locale. An element p of a $\wedge$-semilattice S is *prime* if it is not the greatest element of S and if $s \wedge t \leq p$ implies $s \leq p$ or $t \leq p$ for all $s, t \in S$. This corresponds to associating with each point x of a topological space X the set $X \setminus \{x\}$ which is prime in the open set lattice of X. If p is a prime element of a locale A then $A \setminus \downarrow\{p\}$ is a completely prime filter. On the other hand, if $F \subseteq A$ is a completely prime filter then $\bigvee(A \setminus F)$ is a prime element. This idea is in fact the oldest of the three. Hence we have several equivalent ways of understanding the space $\mathsf{Sp}A$. For reasons of convenience we will freely switch between these points of view.

2.1.2 Properties for points

For a topological space X, we interpret the membership relation $x \in U$ to mean that x satisfies the property given by U. If A is a locale and $a \in A$ then we say that a point p of A *satisfies* a if, understood as an element of $\mathsf{Sp}A$, it is an element of the open set U_a generated by a. In this case we write $p \models a$. The topology of $\mathsf{Sp}A$ consists of all sets of the form $U_a := \{p \in \mathsf{Sp}A \mid p \models a\}$ where $a \in A$. The idea of viewing the elements of a locale as the properties of its points can be traced to M. Smyth. For a more detailed account on this approach, see [Vic89]. In computer science, these ideas give rise to a presentation of denotational semantics in axiomatic form, that is as a logic of programs [Abr87].

Assume we have a mapping m from the points of A to the set of points of some other locale B. Under which conditions can we assure that $m(p)$ satisfies some $b \in B$? Let us consider the set $S \subseteq A$ of all properties $a \in A$ with $p \models a \implies m(p) \models b$ for all points p of A. Note that if $p \models \bigvee S$ for some $p \in \mathsf{Pt}A$ then there is $s \in S$ with $p \models s$ which implies $m(p) \models b$. Hence $\bigvee S$ is the greatest element of S. This gives us a mapping from B to A which we will denote by $\overline{m}$. If it is the case that $p \models \overline{m}(b) \iff m(p) \models b$ for all $p \in \mathsf{Pt}A$ then $\overline{m}$ is a frame homomorphism. If, on the other hand, there is a morphism of locales $f\colon A \longrightarrow B$ with $p \models f'(b)$ iff $m(p) \models b$ for all $b \in B$ and all $p \in \mathsf{Pt}A$ then $f'(b)$ is the weakest property a such that if p satisfies a then $m(p)$ satisfies b. This approach corresponds to the idea of a predicate having a weakest precondition [Smy83].

2.1.3 Sobrification

If X is a sober space then the filters of the form $\mathcal{N}(x)$ where $x \in X$ are the only completely prime filters in ΩX. In this case, the mapping $x \longmapsto \mathcal{N}(x)$ is a homeomorphism between X and $\mathsf{Sp}\Omega X$: A subset U' of $\mathsf{Sp}\Omega X$ is open iff there is an open subset U of X such that

$$U' = \{F \subseteq \Omega X \mid F \text{ completely prime filter and } U \in F\} = \{\mathcal{N}(x) \mid x \in U\}.$$

One can show that a topological space X is sober iff $X \cong \mathsf{Sp}\Omega X$ via $x \longmapsto \mathcal{N}(x)$. Furthermore, the family of these isomorphisms $X \longrightarrow \mathsf{Sp}\Omega X$, where X is a sober space, is a natural isomorphism from the identity functor on **Sob** to $\mathsf{Sp}\Omega$.

Since the space $\mathsf{Sp}A$ is sober for any locale A the above ensures that the image of **Loc** under Sp is **Sob**. This also gives us the possibility to assign a sober space to any topological space. The *sobrification* of a topological space X is $\mathsf{Sp}\Omega X$. The open set lattice of $\mathsf{Sp}\Omega X$ is isomorphic to that of X. The functor Sp is right adjoint to Ω.

2.1.4 Spatialization

The analogue of sobriety for locales is *spatiality*: A locale A is called *spatial* if $A \cong \Omega\,\mathsf{Sp}A$ via $a \longmapsto \{p \in \mathsf{Sp}A \mid p'(a) = 1_2\}$. We also say that such a locale *has enough points* as it has enough points to separate its elements: A locale is spatial iff for every $a, b \in A$ with $a \not\leq b$ there is a point p of A with $p'(a) = 1_2$ and $p'(b) = 0_2$.

For every sober space X the locale ΩX is spatial: $\mathsf{Sp}\Omega X \cong X$ as mentioned above. If A is a locale then we refer to $\Omega\,\mathsf{Sp}A$ as the *spatialization* of A. A locale A is spatial iff there is a topological space X such that $A \cong \Omega X$. This answers the above question: We can only expect to find a topological space X with $A \cong \Omega X$ if A is a spatial locale. In this case we may choose $X = \mathsf{Sp}A$. Hence the image of **Top** under Ω is the full subcategory of spatial locales. The family of isomorphisms $A \longrightarrow \Omega\,\mathsf{Sp}A$ is a natural isomorphism from the identity functor on the category of spatial locales to $\Omega\,\mathsf{Sp}$. Furthermore, the full subcategory **Sob** of sober spaces is equivalent to the full subcategory of spatial locales. This equivalence has been well-known for a long time. For details of the proofs we omitted see [Joh82a].

2.1.5 Adjunctions

We have already seen that every T_0-space is equipped with an order, namely the specialization order. Hence it makes sense to talk about adjoint pairs of continuous functions. But if (f, g) is an adjoint pair of continuous functions where $f\colon X \longrightarrow Y$ then the frame homomorphisms (g^{-1}, f^{-1}) form an adjoint pair between the frames

ΩY and ΩX. (Note that lower and upper adjoint interchanged places!) The reason for this is the following: Let U be an open subset of X. For all $x \in U$ we have $g(f(x)) \in U$ as $g(f(x)) \geq x$. Hence

$$f^{-1}\Big(g^{-1}(U)\Big) = \{x \in X \mid g(f(x)) \in U\} \supseteq U.$$

This proves $f^{-1} \circ g^{-1} \geq id_X^{-1} = id'_{\Omega X}$. The other inequality needed to prove our claim, namely $g^{-1} \circ f^{-1} \leq id'_{\Omega Y}$, can be shown analogously. Hence we have shown the following lemma.

Lemma 2.1 *Let X and Y be topological spaces and let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be continuous functions. Furthermore, let f be the lower adjoint for g. Then $f^{-1}\colon \Omega Y \longrightarrow \Omega X$ is the upper adjoint for g^{-1}.* ■

On the other hand, we may consider pairs of continuous functions $A \underset{g}{\overset{f}{\rightleftarrows}} B$ between locales such that the corresponding frame homomorphisms (g', f') form an adjoint pair. This raises the question whether a result corresponding to the last lemma holds for the order-preserving functions $\mathsf{Sp}f\colon \mathsf{Sp}A \longrightarrow \mathsf{Sp}B$ and $\mathsf{Sp}g\colon \mathsf{Sp}B \longrightarrow \mathsf{Sp}A$.

Lemma 2.2 *Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be continuous functions between locales such that g' is the lower adjoint for f'. Then $\mathsf{Sp}g\colon \mathsf{Sp}B \longrightarrow \mathsf{Sp}A$ is the upper adjoint for $\mathsf{Sp}f$.*

Proof: We have to show that $\mathsf{Sp}f \circ \mathsf{Sp}g \leq id_{\mathsf{Sp}\,B}$ and $\mathsf{Sp}g \circ \mathsf{Sp}f \geq id_{\mathsf{Sp}\,A}$ with respect to the order or specialization. Think of the points of the locales (and thus the points of the corresponding spaces) as continuous functions from *2* to the respective locale. The order of specialization is just the pointwise order on the corresponding frame homomorphisms. Assume that $p\colon 2 \longrightarrow B$ is a point of $\mathsf{Sp}B$. We calculate

$$\begin{aligned}
\Big((\mathsf{Sp}f \circ \mathsf{Sp}g)(p)\Big)' &= \Big(\mathsf{Sp}f(g \circ p)\Big)' && \text{(Definition of } \mathsf{Sp}\text{)} \\
&= (f \circ g \circ p)' && \text{(Definition of } \mathsf{Sp}\text{)} \\
&= p' \circ g' \circ f' \\
&\leq p' && (g' \circ f' \leq id'_B).
\end{aligned}$$

In the same manner we can prove $\mathsf{Sp}g \circ \mathsf{Sp}f \geq id_{\mathsf{Sp}\,A}$. ■

2.2 Products

2.2.1 Products in the category of locales

In the first chapter we have seen how to build the frame presented by a given set of generators with relations between them. Now we shall see a concrete example, namely the product in the category of locales (which is the co-product in **Frm**). Since we think of locales as generalized topological spaces we will first see what we can say about the open set lattice of the product of two topological spaces. Let X and Y be topological spaces. The topology on $X \times Y$ is generated by sets of the form $U \times V$ where $U \in \Omega X$ and $V \in \Omega Y$. Let us examine the equations that have to hold for such open sets in $X \times Y$. If $\mathcal{S}$ is a subset of ΩX and $V \in \Omega Y$ then $\bigcup_{S \in \mathcal{S}} (S \times V) = \left(\bigcup \mathcal{S}\right) \times V$ and the corresponding equation holds for all $U \in \Omega X$ and $\mathcal{T} \subseteq \Omega Y$, namely $\bigcup_{T \in \mathcal{T}} (U \times T) = U \times \left(\bigcup \mathcal{T}\right)$. Also, if $U, U' \in \Omega X$ and $V, V' \in \Omega Y$ then $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. Furthermore, $X \times Y$ is the greatest element of $\Omega(X \times Y)$.

For locales A and B we define $A \otimes B$ to be the frame presented by generators $a \otimes b$, where $a \in A$, $b \in B$, subject to the following relations:

(i) $1_A \otimes 1_B = 1_{A \otimes B}$

(ii) $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) = (a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$ $\qquad (a_1, a_2 \in A,\ b_1, b_2 \in B)$

(iii) $\bigvee_{s \in S} (s \otimes b) = (\bigvee S) \otimes b$ $\qquad (S \subseteq A,\ b \in B)$

(iv) $\bigvee_{t \in T} (a \otimes t) = a \otimes (\bigvee T)$ $\qquad (a \in A,\ T \subseteq B)$

We may understand the construction of $A \otimes B$ as follows: Let $S(A, B)$ be the semi-lattice $\{a \otimes b \mid a \in A, b \in B\}$ with meets calculated componentwise and with $a \otimes 0_B = 0_{A \otimes B} = 0_A \otimes b$ for all $a \in A$, $b \in B$. We say that $I \subseteq S(A, B)$ is a $\otimes$-*ideal* if

(i) I is a lower set,

(ii) for all $S \subseteq A, b \in B$: If $s \otimes b \in I$ for all $s \in S$ then $(\bigvee S) \otimes b \in I$,

(iii) for all $a \in A, T \subseteq B$: If $a \otimes t \in I$ for all $t \in T$ then $a \otimes (\bigvee T) \in I$.

A $\otimes$-ideal is just a special instance of a C-ideal for the above relations as defined in Section 1.3. Hence by Theorem 1.11, $A \otimes B$ is isomorphic to the set of all $\otimes$-ideals of $S(A, B)$ ordered by inclusion. We can define continuous functions $\pi_1: A \otimes B \longrightarrow A$ and $\pi_2: A \otimes B \longrightarrow B$ by setting $\pi_1'(a) := a \otimes 1_B$ and $\pi_2'(b) := 1_A \otimes b$.

Proposition 2.3 (Proposition 6.4.2 in [Vic89]) *The product of locales A and B is $A \otimes B$ together with the projections π_1 and π_2. Therefore the category of locales has finite products.* ■

If $f: C \longrightarrow A$ and $g: C \longrightarrow B$ are continuous functions then $\langle f, g \rangle: C \longrightarrow A \otimes B$ is given by $\langle f, g \rangle'(a \otimes b) := f'(a) \wedge g'(b)$. The diagonal $\Delta_A = \langle id_A, id_B \rangle: A \longrightarrow A \otimes A$, for example, is defined by $\Delta_A'(a_1 \otimes a_2) := a_1 \wedge a_2$.

2.2.2 Products of locales and products of topological spaces

Let us examine the connection between products in **Loc** and **Top**. Since the product of spatial locales need not be spatial, $\Omega X \otimes \Omega Y$ is not necessarily isomorphic to $\Omega(X \times Y)$ for arbitrary topological spaces X and Y. For a counterexample, see Section II.2.14 in [Joh82a]. However, there is a well-known connection between the two (see for example Section II.2 in [Joh82a]).

Proposition 2.4 *The functor* Sp: **Loc** $\longrightarrow$ **Top** *preserves products.*

Proof: Since Sp has a left adjoint, it preserves all limits. ■

Hence the poset $\mathsf{Pt}(A \otimes B)$ is isomorphic to $\mathsf{Pt}A \times \mathsf{Pt}B$ for arbitrary locales A and B. Furthermore, this proposition allows us to conclude that if $\Omega X \otimes \Omega Y$ is a spatial locale for sober spaces X and Y then

$$\Omega X \otimes \Omega Y = \Omega\, \mathsf{Sp}(\Omega X \otimes \Omega Y) \cong \Omega(\mathsf{Sp}\Omega X \times \mathsf{Sp}\Omega Y) \cong \Omega(X \times Y).$$

As an immediate corollary we obtain the following result.

Proposition 2.5 *Arbitrary products of sober spaces are sober.*

Proof: If $(X_i)_{i \in I}$ is an arbitrary family of sober spaces then, by the last proposition, we have

$$\prod_{i \in I} X_i \cong \prod_{i \in I} \mathsf{Sp}\Omega X_i \cong \mathsf{Sp}\left(\bigotimes_{i \in I} \Omega X_i\right).$$

Since $\mathsf{Sp}A$ is sober for every locale A we are done. ∎

Chapter 3

Local compactness and continuity

When studying power constructions, we will also direct our attention to domains. Locally compact spaces and locally compact locales correspond to domains in a way we will explain in this chapter.

3.1 Locally compact spaces and domains

The notion of compactness is one of the most important concepts in general topology. A topological space X is *compact* if, for every family $\mathcal{U}$ of open subsets of X with $X \subseteq \bigcup \mathcal{U}$, there are $U_1, \ldots, U_n \in \mathcal{U}$ such that $X \subseteq U_1 \cup \cdots \cup U_n$.

Note that we do not require a compact space to satisfy the Hausdorff separation property since most topologies we encounter are non-Hausdorff ones. A subset K of a topological space X is *compact* if it is compact as a subspace of X. Note that a subset K of a topological space X is compact iff it satisfies the following property: If $\mathcal{U}$ is a directed family of open subsets of X such that $K \subseteq \bigcup^{\uparrow} \mathcal{U}$ then there is $U \in \mathcal{U}$ with $K \subseteq U$.

The properties of compact sets which we will need later are the following: Finite unions of compact sets are compact as are continuous images of compact sets. Note, however, that the intersection of two compact sets need not be compact. Furthermore, every finite set is compact.

Another possibility to get a compact set from a given compact set is to take its *saturation*: For a subset S of a topological space X let $\bigcap\{U \subseteq X \mid U \text{ open}, U \supseteq S\}$ be the *saturation* of S. For a T_0-space X, the saturation of any subset of X is the upper closure with respect to the specialization order. Therefore, we will denote the saturation of some subset S of a T_0-space by $\uparrow S$. Note that arbitrary unions and intersections of saturated sets are saturated.

A property of topological spaces that is closely associated with continuity of dcpo's is the following.

Definition. A topological space X is *locally compact* if for every $x \in X$ there is a basis for the neighbourhood filter of x consisting of compact subsets of X. In other words, for every open neighbourhood U of some element x of X there is a compact neighbourhood of x that is contained in U.

Hence local compactness guarantees the existence of arbitrarily small compact sets with non-empty interior. The full subcategory of **Top** whose objects are locally compact spaces will be denoted by **LCompT**.

Every algebraic domain D is locally compact with respect to the Scott topology: A basis for the neighbourhood filter of any $d \in D$ is given by the sets of the form $\uparrow\{c\}$ where c is a compact element below d in D. Therefore a neighbourhood filter for some $d \in D$ has a basis consisting of compact open sets. An arbitrary domain is still locally compact with respect to its Scott topology: If U is a Scott-open subset of a domain D then for all $d \in U$ there is $d' \in U$ with $d' \ll d$. Consequently, $\uparrow\{d'\}$ is a compact neighbourhood of d in U.

Therefore we can restrict and co-restrict the functor T_σ that maps a dcpo D to the underlying set with the Scott topology to a functor from the category of domains to the category of locally compact T_0-spaces.

Another connection between local compactness and continuity is the following: For a topological space X consider the complete lattice ΩX of open subsets of X. We will study the way below relation in ΩX. If U and U' are open subsets of X such that there is a compact set K with $U \subseteq K \subseteq U'$ then $U \ll U'$ in ΩX. Indeed, if $\mathcal{U}$ is a directed subset of ΩX with $U' \subseteq \bigcup^{\uparrow} \mathcal{U}$ then, by compactness of K, there is $U'' \in \mathcal{U}$ with $U \subseteq K \subseteq U''$.

If X is a locally compact space then the reverse is also true: If $U \ll U'$ in ΩX then there is a compact subset K of X with $U \subseteq K \subseteq U'$. For if $U \ll U'$ in ΩX then, by local compactness of X, there is a compact neighbourhood K_x in U' for every $x \in U'$. Then $U' = \bigcup^{\uparrow}\{\mathsf{int}\, K_{x_1} \cup \cdots \cup \mathsf{int}\, K_{x_n} \mid x_1, \ldots, x_n \in U\}$, and since $U \ll U'$ we can find $x_1, \ldots, x_n \in U'$ with $U \subseteq \mathsf{int}\, K_{x_1} \cup \cdots \cup \mathsf{int}\, K_{x_n}$. Consequently, $K_{x_1} \cup \cdots \cup K_{x_n}$ is a compact superset of U in U'.

Proposition 3.1 (Theorem II.1.7 in [GHK$^+$80]) *If X is a locally compact space then ΩX is a continuous lattice.*

Proof: By the above considerations, $\mathsf{int}\, K \ll U$ for any compact set K in U. As X is locally compact,

$$U = \bigcup\{\mathsf{int}\, K \mid K \text{ compact subset of } U\} \subseteq \bigcup \mathord{\downarrow\!\!\downarrow}\{U\} \subseteq U.$$

Since ΩX is a complete lattice, $\mathord{\downarrow\!\!\downarrow}\{U\}$ has to be directed and we are done. ■

For sober spaces, local compactness is even characterized by continuity of ΩX.

Proposition 3.2 (Theorem V.5.6 in [GHK$^+$80]) *A sober space is locally compact iff its open set lattice is continuous.* ■

3.2 Locally compact locales

Definition. A locale is *locally compact* if, as a complete lattice, it is continuous. The full subcategory of locally compact locales will be denoted by **LCompL**.

The following two propositions were first proved by Hofmann and Lawson [HL78].

Proposition 3.3 *A locally compact locale is spatial.*

Proof: Let $a \not\leq b$ in a locally compact locale A. First we show that there is a Scott-open filter in A containing a but not b. By continuity of A there is $a' \ll a$ with $a' \not\leq b$. By the interpolation property we can find a sequence $(d_n)_{n\in\mathbb{N}}$ such that $a' \ll d_{n+1} \ll d_n \ll a$ for all $n \in \mathbb{N}$. Then $F := \bigcup_{n\in\mathbb{N}} \uparrow\{d_n\} = \bigcup_{n\in\mathbb{N}} \Uparrow\{d_n\}$ is a Scott-open filter containing a but not b. By Zorn's Lemma, there is a maximal Scott-open filter G containing F but not b. We prove that G is completely prime. Suppose there are c and c' in $A \setminus G$ with $c \vee c' \in G$. Then $G' := \{d \in A \mid d \vee c \in G\}$ is a Scott-open filter containing G. Since $c' \in G' \setminus G$, we must have $b \in G'$ by maximality of G. This implies $b \vee c \in G$. Similarly, we can show that $b \vee c' \in G$. Therefore, $b \vee (c \wedge c') = (b \vee c) \wedge (b \vee c') \in G$. This implies $b \in G$ and thus contradicts the definition of G. Consequently, G is a completely prime filter in A containing a but not b. Hence A has enough points and thus is spatial. ■

Proposition 3.4 *(i) If X is a locally compact space then ΩX is a locally compact locale.*

(ii) If A is a locally compact locale then $\mathsf{Sp}A$ *is a locally compact space.*

Proof: (i) Follows from Proposition 3.2.

(ii) We already know that $\mathsf{Sp}A$ is a sober space with $\mathsf{Sp}\,\Omega\,\mathsf{Sp}A \cong \mathsf{Sp}A$. Since A is spatial according to the last lemma, $\Omega\,\mathsf{Sp}A \cong A$. Hence $\mathsf{Sp}A$ is a sober space with continuous open set lattice and therefore is locally compact by Proposition 3.2. ■

Furthermore, the full subcategory of locally compact locales is equivalent to the category **LCompS** of locally compact sober spaces. The equivalences between these categories are given by the restrictions and co-restrictions of Ω and Sp.

3.3 Products

Above, we have seen that the category of locales has finite products. The same is true for the categories of dcpo's and topological spaces. Our study of power constructions for domains, locally compact spaces and locally compact locales makes it necessary to consider finite products for these categories. The following two propositions are well-known.

Proposition 3.5 *The product of finitely many continuous dcpo's is continuous. The category* **Dom** *of domains is closed under finite products formed in* **DCPO***.* ∎

Proposition 3.6 *Finite products of locally compact spaces are locally compact.* ∎

There are two ways of proving that the category **LCompL** of locally compact locales is also closed with respect to finite products. If A and B are locally compact locales, then they are spatial by Proposition 3.3. Hence, because of Proposition 3.4, $\mathsf{Sp}A$ and $\mathsf{Sp}B$ are locally compact spaces with $A \cong \Omega\,\mathsf{Sp}A$ and $B \cong \Omega\,\mathsf{Sp}B$. It is sufficient to show that $A \otimes B$ is spatial, as, by Proposition 2.4, $\mathsf{Sp}(A \otimes B) \cong \mathsf{Sp}A \times \mathsf{Sp}B$ which implies $A \otimes B \cong \Omega\,\mathsf{Sp}(A \otimes B) \cong \Omega(\mathsf{Sp}A \times \mathsf{Sp}B)$. Since $\mathsf{Sp}A \times \mathsf{Sp}B$ is a locally compact space by the last proposition, this proves that $\Omega(\mathsf{Sp}A \times \mathsf{Sp}B)$ is a locally compact locale. This is the way our Proposition 3.8 is usually shown (see e.g. II.2.13 in [Joh82a]). Since we are interested in the way below relation for $A \otimes B$ we will prove the desired result differently.

Lemma 3.7 *Let A and B be locales. If $a' \ll a$ in A and $b' \ll b$ in B then $a' \otimes b'$ is way below $a \otimes b$ in $A \otimes B$.*

Proof: Let $a \in A$ and $b \in B$ and let further R be a directed subset of $A \otimes B$ with $a \otimes b \leq \bigvee^{\uparrow} R$. We set $R' := \{a' \otimes b' \in A \otimes B \mid \forall a'' \ll a',\, b'' \ll b'.\exists r \in R\,.\, a'' \otimes b'' \leq r\}$. Obviously, if $a'' \leq a'$ and $b'' \leq b'$ for some $a' \otimes b' \in R'$ then $a'' \otimes b'' \in R'$. Let S be a subset of A and let $b' \in B$ such that $s \otimes b' \in R'$ for all $s \in S$. We show that this implies $\bigvee S \otimes b' \in R'$. If $a'' \ll \bigvee S$ for some $a'' \in A$ then there are $s_1, \ldots, s_n \in X$ with $a'' \leq s_1 \vee \cdots \vee s_n$. If $b'' \ll b'$ then there are $r_1, \ldots, r_n \in R$ with $s_i \otimes b'' \leq r_i$ for all $i \in \{1, \ldots, n\}$. Since R is directed, there is an upper bound r for $r_1, \ldots, r_n$ in R. Hence

$$a'' \otimes b'' \leq (s_1 \vee \cdots \vee s_n) \otimes b'' = s_1 \otimes b'' \vee \cdots \vee s_n \otimes b'' \leq r_1 \vee \cdots \vee r_n \leq r$$

which proves $\bigvee S \otimes b' \in R'$. Similarly we can show that $a' \otimes t \in R'$ for all elements t of some $T \subseteq B$ implies $a' \otimes \bigvee T \in R'$. Therefore R' is a $\otimes$-ideal as defined in Section 2.2. The join of R' can be calculated as follows: If $r \in R$ then $r = \bigvee\limits_{i \in I} a_i \otimes b_i$ for some generators $a_i \otimes b_i$ of $A \otimes B$. Clearly, all $a_i \otimes b_i$ are in R' which proves that the join of R' is the join of R. Since R' is a $\otimes$-ideal, this implies $a \otimes b \in R'$. Hence if $a' \ll a$ in A and $b' \ll b$ in B then, by definition of R', there is $r' \in R$ with $a' \otimes b' \leq r'$ which proves $a' \otimes b' \ll a \otimes b$ in $A \otimes B$. ■

Proposition 3.8 *The category of locally compact locales* **LCompL** *is a subcategory of* **Loc** *that is closed under finite products.*

Proof: Obviously, the two-element locale *2* serves as terminal object in **Loc**. Let A and B be locally compact locales. We want to show that $A \otimes B$ is locally compact. Since $A \otimes B$ is a complete lattice, the set $\twoheaddownarrow\{c\}$ is directed for all $c \in A \otimes B$. By the last lemma, $\{a' \otimes b' \mid a' \ll a,\, b' \ll b\}$ is a subset of $\twoheaddownarrow\{a \otimes b\}$ for all $a \otimes b$ in $A \otimes B$. Hence

$$\begin{aligned}
a \otimes b &\geq \bigvee^{\uparrow} \twoheaddownarrow\{a \otimes b\} \\
&\geq \bigvee \{a' \otimes b' \mid a' \ll a,\, b' \ll b\} \\
&= \bigvee_{a' \ll a} \bigvee_{b' \ll b} a' \otimes b' \\
&= \Big(\bigvee_{a' \ll a} a'\Big) \otimes \Big(\bigvee_{b' \ll b} b'\Big) \\
&= a \otimes b.
\end{aligned}$$

Since any element of $A \otimes B$ is the join of elements of the form $a \otimes b$, we are done. ■

Chapter 4

Monads and their Algebras

In this chapter we introduce a notion that is important for our studies of power constructions. A property that all these constructions have in common is that they can be understood as the functor part of a monad. Our main task will be to characterize the algebras for the arising monads. Besides introducing this concept and related notions we will prove some results for monads on categories that, in a way, behave like the category of posets. The sources for most of the material presented in the first two sections of this chapter are [Mac71] and [BW84].

4.1 Monads

Consider a pair of functors $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ where F is left adjoint to G. Then there are natural transformations $\eta\colon id_{\mathbf{C}} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow id_{\mathbf{D}}$ such that the following diagrams of natural transformations commute:

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "id_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "id_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd}$$

We can define another natural transformation, namely $\mu := G\varepsilon F\colon (GF)^2 \longrightarrow GF$. It is not difficult to check that the following diagrams are commutative:

$$\begin{tikzcd} GF \arrow[r, "GF\eta"] \arrow[dr, "id_{GF}"'] & (GF)^2 \arrow[d, "\mu"] & GF \arrow[l, "\eta GF"'] \arrow[dl, "id_{GF}"] \\ & GF & \end{tikzcd} \qquad \begin{tikzcd} (GF)^3 \arrow[r, "\mu GF"] \arrow[d, "GF\mu"'] & GF \arrow[d, "\mu"] \\ (GF)^2 \arrow[r, "\mu"] & GF \end{tikzcd}$$

Generalizing this to a new notion we define a *monad* on a category $\mathbf{C}$ to be a triple (T, η, μ) where

(i) T is an endofunctor on $\mathbf{C}$,

(ii) $\eta\colon id_{\mathbf{C}} \longrightarrow T$ and $\mu\colon T^2 \longrightarrow T$ are natural transformations such that the following diagrams of natural transformations commute

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "id_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "id_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

We have already seen that if $(F, G, \eta, \varepsilon)$ is an adjunction then the triple $(GF, \eta, G\varepsilon F)$ is a monad on $\mathbf{C}$.

Here is a concrete example of a monad. Define $\mathsf{Wd}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ as follows: For any set S, let $\mathsf{Wd}\, S$ be the set of all words over the alphabet S. For every function $f\colon S \longrightarrow T$ we map the word $s_1 \cdots s_n$ over S to the word $f(s_1) \cdots f(s_n)$ over T. This definition makes $\mathsf{Wd}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ an endofunctor. A natural transformation $\eta\colon id_{\mathbf{Set}} \longrightarrow \mathsf{Wd}$ can be defined in the obvious way: An element s of S is mapped to the one-letter-word s over S. In order to get a monad we still need a natural transformation $\mu\colon \mathsf{Wd}^2 \longrightarrow \mathsf{Wd}$. The elements of $\mathsf{Wd}^2\, S$ are words over the alphabet $\mathsf{Wd}\, S$, that is words consisting of words over S which we can understand to be words over S and thus elements of $\mathsf{Wd}\, S$ if we remove whatever separates different words. We shall employ the monad (Wd, η, μ) again in order to illustrate further concepts.

Let us introduce a concept stating when two monads on a category are 'the same':

Definition. We say that two monads (T, η, μ) and (T', η', μ') on a category $\mathbf{C}$ are *equivalent* if there is a natural isomorphism $\iota: T \longrightarrow T'$ such that $\eta' = \iota \bullet \eta$ and such that $\mu' \bullet (\iota \circ \iota) = \iota \bullet \mu$. Here $\bullet$ denotes the vertical composite while $\circ$ means the horizontal composite of natural transformations.

Proposition 4.1 *Let (T, η, μ) be a monad on $\mathbf{C}$. Further, let T' be an endofunctor on $\mathbf{C}$ such that there is a natural isomorphism $\iota: T \longrightarrow T'$. If $\eta': id_{\mathbf{C}} \longrightarrow T'$ and $\mu: (T')^2 \longrightarrow T'$ are natural transformations such that $\eta' = \iota \bullet \eta$ and $\mu' \bullet (\iota \circ \iota) = \iota \bullet \mu$ then (T', η', μ') is a monad on $\mathbf{C}$ which is equivalent to (T, η, μ).*

Proof: We just have to show that (T', η', μ') is a monad. Since ι is a natural isomorphism, we have

$$\mu'_C = \iota_C \circ \mu_C \circ T\iota_C^{-1} \circ \iota_{T'C}^{-1} = \iota_C \circ \mu_C \circ \iota_{TC}^{-1} \circ T'\iota_C^{-1}$$

for any $\mathbf{C}$-object C. Let C be an arbitrary $\mathbf{C}$-object. Then

$$\begin{aligned}
\mu'_C \circ T'\eta'_C &= \iota_C \circ \mu_C \circ \iota_{TC}^{-1} \circ T'\iota_C^{-1} \circ T'(\iota_C \circ \eta_C) \\
&= \iota_C \circ \mu_C \circ \iota_{TC}^{-1} \circ T'(\iota_C^{-1} \circ \iota_C) \circ T'\eta_C \\
&= \iota_C \circ \mu_C \circ T\eta_C \circ \iota_C^{-1} && (\text{Naturality of } \iota^{-1}: T' \longrightarrow T) \\
&= \iota_C \circ \iota_C^{-1} && (\mu_C \circ T\eta_C = id_{TC}) \\
&= id_{T'C}\,.
\end{aligned}$$

Further,

$$\begin{aligned}
\mu'_C \circ \eta'_{T'C} &= \iota_C \circ \mu_C \circ T\iota_C^{-1} \circ \iota_{T'C}^{-1} \circ \iota_{T'C} \circ \eta_{T'C} \\
&= \iota_C \circ \mu_C \circ T\iota_C^{-1} \circ \eta_{T'C} \\
&= \iota_C \circ \mu_C \circ \eta_{TC} \circ \iota_C^{-1} && (\text{Naturality of } \eta: id_{\mathbf{C}} \longrightarrow T) \\
&= \iota_C \circ \iota_C^{-1} && (\mu_C \circ \eta_{TC} = id_{TC}) \\
&= id_{T'C}\,.
\end{aligned}$$

Finally,

$$
\begin{aligned}
& \mu'_C \circ T'\mu'_C \\
= \;& \iota_C \circ \mu_C \circ \iota^{-1}_{TC} \circ T'\iota^{-1}_C \circ T'(\iota_C \circ \mu_C \circ T\iota^{-1}_C \circ \iota^{-1}_{T'C}) \\
= \;& \iota_C \circ \mu_C \circ \iota^{-1}_{TC} \circ T'\mu_C \circ T'T\iota^{-1}_C \circ T'\iota^{-1}_{T'C} \\
= \;& \iota_C \circ \mu_C \circ T\mu_C \circ \iota^{-1}_{T^2C} \circ T'T\iota^{-1}_C \circ T'\iota^{-1}_{T'C} && (\text{Naturality of } \iota^{-1}\colon T' \longrightarrow T) \\
= \;& \iota_C \circ \mu_C \circ \mu_{TC} \circ \iota^{-1}_{T^2C} \circ T'T\iota^{-1}_C \circ T'\iota^{-1}_{T'C} && (\mu_C \circ T\mu_C = \mu_C \circ \mu_{TC}) \\
= \;& \iota_C \circ \mu_C \circ \mu_{TC} \circ T^2\iota^{-1}_C \circ \iota^{-1}_{TT'C} \circ T'\iota^{-1}_{T'C} && (\text{Naturality of } \iota^{-1}\colon T' \longrightarrow T) \\
= \;& \iota_C \circ \mu_C \circ T\iota^{-1}_C \circ \mu_{T'C} \circ \iota^{-1}_{TT'C} \circ T'\iota^{-1}_{T'C} && (\text{Naturality of } \mu\colon T^2 \longrightarrow T) \\
= \;& \iota_C \circ \mu_C \circ T\iota^{-1}_C \circ \iota^{-1}_{T'C} \circ \iota_{T'C} \circ \mu_{T'C} \circ \iota^{-1}_{TT'C} \circ T'\iota^{-1}_{T'C} \\
= \;& \mu'_C \circ \mu'_{T'C}. && \blacksquare
\end{aligned}
$$

Hence, given a monad (T, η, μ) and an endofunctor T' on a category $\mathbf{C}$ that is naturally isomorphic to T we can define a monad for T' that is equivalent to that for T.

4.2 Algebras

Every adjunction defines a monad. A natural question is whether every monad arises like that. Suppose we are given a monad (T, η, μ) on $\mathbf{C}$. Let us consider all adjunctions $(F, G, \eta, \varepsilon)$ such that $(GF, \eta, G\varepsilon F) = (T, \eta, \mu)$. Note that the unit for all these adjunctions is η while the co-unit might vary. We take these adjunctions to be the objects of a category which we denote by $Ad(T)$. Let $(F, G, \eta, \varepsilon)$ and $(F', G', \eta, \varepsilon')$ be two objects of $Ad(T)$ where $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $F'\colon \mathbf{C} \longrightarrow \mathbf{D}'$. A morphism from $(F, G, \eta, \varepsilon)$ to $(F', G', \eta, \varepsilon')$ is given by a functor $M\colon \mathbf{D} \longrightarrow \mathbf{D}'$ such that the two diagrams

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[dr, "F'"'] & \mathbf{D} \arrow[d, "M"] \\
& \mathbf{D}'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{C} & \mathbf{D} \arrow[l, "G"'] \arrow[d, "M"] \\
& \mathbf{D}' \arrow[ul, "G'"]
\end{tikzcd}
$$

commute. Note that for every such functor M an equation relating the natural transformations ε and ε' holds, namely

$$M\varepsilon = \varepsilon' M.$$

We will now describe the terminal object of $Ad(T)$.

Definition. We say that a pair (C, ξ) is an *algebra for a monad* (T, η, μ) on a category $\mathbf{C}$ or a *T-algebra* if $\xi\colon TC \longrightarrow C$ is a morphism in $\mathbf{C}$ such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "id_C"'] & TC \arrow[d, "\xi"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} T^2C \arrow[r, "\mu_C"] \arrow[d, "T\xi"'] & TC \arrow[d, "\xi"] \\ TC \arrow[r, "\xi"] & C \end{tikzcd}$$

In this case we say that ξ is a *structure map* for C. The *category* T-**Alg** *of T-algebras* (or *Eilenberg-Moore category*) is defined as follows: Its objects are all T-algebras. A morphism from (C, ξ) to (C', ξ') in T-**Alg** is given by a morphism $f\colon C \longrightarrow C'$ in $\mathbf{C}$ such that the following diagram is commutative.

$$\begin{tikzcd} TC \arrow[r, "\xi"] \arrow[d, "Tf"'] & C \arrow[d, "f"] \\ TC' \arrow[r, "\xi'"] & C' \end{tikzcd}$$

This definition is due to S. Eilenberg and J. C. Moore [EM65] who also were the first to prove that there is an adjunction, say (F^A, G^A, η, μ^A), corresponding to the category of T-algebras that is terminal in $Ad(T)$: $F^A\colon \mathbf{C} \longrightarrow T$-**Alg** is obtained by setting $F^A C := (TC, \mu_C)$ and $F^A f := Tf$. The functor $G^A\colon T$-**Alg** $\longrightarrow \mathbf{C}$ results from defining $G^A(C, \xi) := C$ and $G^A f := f$. In particular, for any monad (T, η, μ) there is indeed an adjunction $(F, G, \eta, \varepsilon)$ with $(T, \eta, \mu) = (GF, \eta, G\varepsilon\eta)$.

Note the similarity of the diagrams defining a monad with those defining an algebra for the monad. In the above example, the category of Wd-algebras is (isomorphic to) the category of monoids. The functor Wd can be decomposed into the functor that maps a set S to the free monoid over S and the forgetful functor to **Set**.

An adjunction $(F, G, \eta, \varepsilon)$, where the target of F is $\mathbf{D}$, is called *monadic* if the functor $M\colon \mathbf{D} \longrightarrow T\text{-}\mathbf{Alg}$ that defines the unique morphism from $(F, G, \eta, \varepsilon)$ to $(F^A, G^A, \eta, \varepsilon^A)$ is an equivalence of the categories $\mathbf{D}$ and $T\text{-}\mathbf{Alg}$.

The category of algebras for a monad (T, η, μ) helps to determine in what respect the functor T can be understood to be free. The fact that in the above example the category of Wd-algebras is the category of all monoids means that Wd gives the free monoid over **Set**. We can also understand the category of T-algebras to be the largest category $\mathbf{D}$ for which there is an adjunction $(F, G, \eta, \varepsilon)$ defining the given monad (T, η, μ) such that $F\colon \mathbf{C} \longrightarrow \mathbf{D}$. Since G^A is a forgetful functor (that is F^A is more or less equal to T), this gives the largest category $\mathbf{D}$ for which T can be understood to assign the free $\mathbf{D}$-object to every $\mathbf{C}$-object. To see this, note the following: The universal property for the adjunction $(F^A, G^A, \eta, \varepsilon^A)$ yields that for all $\mathbf{C}$-morphisms $f\colon C \longrightarrow G^A A$, where A is an object of $T\text{-}\mathbf{Alg}$, there is a unique $\hat{f}\colon F^A C \longrightarrow A$ in $T\text{-}\mathbf{Alg}$ such that $G^A \hat{f} \circ \eta_C = f$. Let $(F, G, \eta, \varepsilon)$ be any adjunction in $Ad(T)$ where the target of F is $\mathbf{D}$. Every object of the form GD where D is a $\mathbf{D}$-object is of the form $G^A A$ for the T-algebra $A := MD = (GD, G\varepsilon_D)$. Hence the 'for all' in the universal property for the adjunction $(F^A, G^A, \eta, \varepsilon^A)$ has the greatest possible domain of variation.

We have just seen that the algebras of the form (TC, μ_C), where C is a $\mathbf{C}$-object, satisfy a universal property. This is the reason that they are also called *free algebras* for the monad. They are the objects of the subcategory of $T\text{-}\mathbf{Alg}$ which we obtain if we take the image of F^A in $T\text{-}\mathbf{Alg}$. For this category we can find an adjunction which is initial in $Ad(T)$.

Let us now investigate algebras for equivalent monads.

Proposition 4.2 *Let (T, η, μ) and (T', η', μ') be equivalent monads on the cate-*

gory $\mathbf{C}$. *Then the category of* T*-algebras is isomorphic to that of* T'*-algebras.*

Proof: Let the natural isomorphism from T to T' be ι. Let (C, ξ') be a T'-algebra. We claim that $(C, \xi' \circ \iota_C)$ is a T-algebra. We calculate $(\xi' \circ \iota_C) \circ \eta_C = \xi' \circ \eta'_C = id_C$. Furthermore,

$$\begin{aligned}
(\xi' \circ \iota_C) \circ T(\xi' \circ \iota_C) &= \xi' \circ \iota_C \circ T\xi' \circ T\iota_C \\
&= \xi' \circ T'\xi' \circ \iota_{T'C} \circ T\iota_C && (\text{Naturality of } \iota\colon T \longrightarrow T') \\
&= \xi' \circ \mu'_C \circ \iota_{T'C} \circ T\iota_C && (\xi' \circ T\xi' = \xi' \circ \mu'_C) \\
&= \xi' \circ \iota_C \circ \mu_C && (\iota \bullet \mu = \mu' \bullet (\iota \circ \iota)) \\
&= \xi' \circ \mu'_C.
\end{aligned}$$

If $f\colon (C, \xi') \longrightarrow (D, \zeta')$ is a morphism of T'-algebras then it is also a morphism of T-algebras from $(C, \xi' \circ \iota_C)$ to $(D, \zeta' \circ \iota_D)$ since $f \circ \xi' \circ \iota_C = \zeta' \circ T'f \circ \iota_C = \zeta' \circ \iota_D \circ Tf$.■

4.3 Monads on the category of posets

Let **Pos** denote the category of posets and order-preserving functions. Then the **Pos**-morphisms $P \longrightarrow Q$ also form a poset: They can be ordered pointwise.

Assume that $\mathbf{C}$ is a subcategory of **Pos** and that T is an endofunctor on $\mathbf{C}$ which is order-preserving on morphisms. This means if $f, g\colon C \longrightarrow C'$ in $\mathbf{C}$ such that $f \leq g$ with respect to the pointwise order then $Tf \leq Tg$. Suppose that T is the functor part of a monad (T, η, μ). What can we say about T-algebras in this setting?

By definition, every T-algebra C is a retract of TC in the category of posets and order-preserving functions. Hence if all posets of the form TC are $\wedge$-semilattices, for example, then all T-algebras must be $\wedge$-semilattices as well. We will study, among others, monads on **DCPO** and **Top**. Since these categories can be considered subcategories of **Pos**, the following three lemmata can be applied to them. For most monads we consider we can understand the corresponding functor to map any object to a subset of its power set; the order is either subset inclusion or reverse inclusion.

These two cases are dual, therefore we may concentrate on the first of them. Let $\mathsf{P}_L C$ denote the subposet of $\mathsf{P}C$ consisting of all lower subsets of C, that is those subsets S of C where $S = {\downarrow} S$, ordered by subset inclusion. The set of all upper subsets of C ordered by reverse inclusion will be denoted by $\mathsf{P}_U C$.

Lemma 4.3 *Let (T, η, μ) be a monad on a subcategory $\mathbf{C}$ of $\mathbf{Pos}$. Furthermore, suppose that TC is a subset of $\mathsf{P}_L C$ and that $\eta_C(x) = {\downarrow}\{x\}$ for all $x \in C$ and all $\mathbf{C}$-objects C.*

(i) If (C, ξ) is a T-algebra then ξ is the lower adjoint for η_C and $\xi(S) = \bigvee S$ holds for all $S \in TC$. Hence in C the joins of all sets belonging to TC exist.

(ii) Let (C, ξ) and (C', ξ') be T-algebras. Additionally, let $f\colon C \longrightarrow C'$ be a $\mathbf{C}$-morphism such that $\bigvee Tf(S) = \bigvee f(S)$ for all $S \in TC$. Then f is a morphism of the T-algebras (C, ξ) and (C', ξ') iff it preserves joins of all sets belonging to TC.

Proof: (i) Let $S \in TC$ where (C, ξ) is a T-algebra. As ${\downarrow}\{s\} \subseteq S$ for all $s \in S$ we have $s = \xi(\eta_C(s)) = \xi({\downarrow}\{s\}) \leq \xi(S)$. Thus $\xi(S)$ is an upper bound for S. If, on the other hand, p is an upper bound for S then we have $S \subseteq {\downarrow}\{p\}$ implying $\xi(S) \leq \xi({\downarrow}\{p\}) = \xi(\eta_C(p)) = p$. Hence $\xi(S)$ is the least upper bound for S. For every T-algebra (C, ξ) we have $\xi \circ \eta_C = id_C$ by definition. For all $S \in TC$ we know that $\eta_C(\xi(S)) = \eta_C(\bigvee S) = {\downarrow}\{\bigvee S\} \supseteq S$ which proves $\eta_C \circ \xi \geq id_{TC}$. This implies that ξ is the lower adjoint for η_C.

(ii) According to (i), the structure maps for C and C' have to be the functions mapping subsets of TC respective TC' to their joins. If $f\colon C \longrightarrow C'$ is a morphism of T-algebras then $\bigvee f(S) = \bigvee Tf(S) = \xi'(Tf(S)) = f(\xi(S)) = f(\bigvee S)$ for all $S \in TC$. If f preserves the joins of all sets belonging to TC then $f(\xi(S)) = f(\bigvee S) = \bigvee f(S) = \bigvee Tf(S) = \xi'(Tf(S))$. ∎

Lemma 4.4 *Let (T, η, μ) be a monad on a subcategory $\mathbf{C}$ of $\mathbf{Pos}$. Furthermore, suppose that TC is a subset of $\mathsf{P}_U C$ and that $\eta_C(x) = \uparrow\{x\}$ for all $x \in C$ and all $\mathbf{C}$-objects C.*

(i) If (C, ξ) is a T-algebra then ξ is the upper adjoint for η_C and $\xi(S) = \bigwedge S$ holds for all $S \in TC$. Hence in C the meets of all sets belonging to TC exist.

(ii) Let C and C' be T-algebras and let $f: C \longrightarrow C'$ be a $\mathbf{C}$-morphism such that $\bigwedge f(S) = \bigwedge Tf(S)$ for all $S \in TC$. Then f is a morphism of T-algebras iff it preserves the meets of all sets belonging to TC.

Proof: Dual to that for Lemma 4.3. ∎

So in the category of posets we can show that in certain cases the structure maps have to be adjoints for the components of the natural transformation η. We were able to deduce this property by only making use of the triangular diagram for algebras. It turns out that in these cases the commutativity of the rectangular diagram already follows from the fact that the triangular diagram commutes.

Lemma 4.5 *Let (T, η, μ) be a monad on a subcategory $\mathbf{C}$ of $\mathbf{Pos}$ such that T is order-preserving on $\mathbf{C}$-morphisms and such that the structure map for every $\mathbf{C}$-object C necessarily is the lower (upper) adjoint for η_C. Furthermore, let $\xi: TC \longrightarrow C$ be the lower (upper) adjoint for η_C. If $\xi \circ \eta_C = id_C$ then (C, ξ) is an algebra for the monad.*

Proof: We shall only prove the assertion for the case of the lower adjoints since the other part can be shown analogously. For that it remains to show that for $\xi: TC \longrightarrow C$ with $\xi \circ \eta_C = id_C$ the following diagram commutes:

$$\begin{tikzcd} T^2C \arrow[r, "\mu_C"] \arrow[d, "T\xi"'] & TC \arrow[d, "\xi"] \\ TC \arrow[r, "\xi"] & C \end{tikzcd}$$

As μ_C is a structure map for TC, it has to be the lower adjoint for η_{TC} by assumption. By Lemma 1.1 this implies that $\xi \circ \mu_C$ is the lower adjoint for $\eta_{TC} \circ \eta_C$. But $\xi \circ T\xi$ is the lower adjoint for $T\eta_C \circ \eta_C$:

$$\begin{aligned}
(\xi \circ T\xi) \circ (T\eta_C \circ \eta_C) &= \xi \circ T(\xi \circ \eta_C) \circ \eta_C \\
&= \xi \circ T id_C \circ \eta_C && (\xi \circ \eta_C = id_C) \\
&= \xi \circ id_{TC} \circ \eta_C \\
&= id_C && (\xi \circ \eta_C = id_C)
\end{aligned}$$

as well as

$$\begin{aligned}
(T\eta_C \circ \eta_C) \circ (\xi \circ T\xi) &= T\eta_C \circ (\eta_C \circ \xi) \circ T\xi \\
&\geq T\eta_C \circ id_{TC} \circ T\xi && (\eta_C \circ \xi \geq id_{TC}) \\
&= T(\eta_C \circ \xi) \\
&\geq T\, id_{TC} && (\text{Monotonicity of } T) \\
&= id_{T^2C}\,.
\end{aligned}$$

Since η is a natural transformation $id_{\mathbf{C}} \longrightarrow T$, the diagram

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & TC \arrow[d, "T\eta_C"] \\
TC \arrow[r, "\eta_{TC}"] & T^2C
\end{tikzcd}$$

commutes. Hence both, $\xi \circ \mu_C$ and $\xi \circ T\xi$, are lower adjoints for $T\eta_C \circ \eta_C = \eta_{TC} \circ \eta_C$ and thus must coincide. ■

4.4 Monads for O-categories

The proof of the last lemma also works in a more general setting: This proof mainly uses the fact that the hom-sets in the category of posets are themselves posets. Hence we can define a class of categories for which the above result is still true.

Definition. An *order-enriched* category, or, for short, an *O-category* is a category where the hom-sets are partially ordered sets such that composition is order-preserving. An *O-functor* is a functor between O-categories that preserves the order on hom-sets. This notion is due to M. Wand [Wan79] and has been employed in [HP79].

If $\mathbf{C}$ is an O-category and $f, g: C \longrightarrow C'$ as well as $f', g': C' \longrightarrow C''$ are $\mathbf{C}$-morphisms such that $f \leq g$ and $f' \leq g'$ then $f' \circ f \leq g' \circ g$. For O-categories we still have the concept of an adjoint pair. If $C \underset{g}{\overset{f}{\rightleftarrows}} C'$ are morphisms in an O-category $\mathbf{C}$ then (f, g) is an *adjoint pair* if $f \circ g \leq id_{C'}$ and $g \circ f \geq id_C$. In this case we say that f is a *lower adjoint* for g and that g is an *upper adjoint* for f.

Obviously, **DCPO** is an O-category. Since T_0-spaces carry an order that is preserved by continuous functions, we can order hom-sets in **Top** pointwise and thus make **Top** an O-category. The category **Loc** of locales can also be understood to be an O-category. For continuous functions $f, g: A \longrightarrow B$ in **Loc** we set $f \leq g$ iff $f' \leq g'$ holds for the corresponding frame homomorphisms. We choose this order on hom-sets of **Loc** because it makes Sp and Ω O-functors (compare also Lemmata 2.1 and 2.2). Note that with respect to this ordering, (f, g) is an adjoint pair for $A \underset{g}{\overset{f}{\rightleftarrows}} B$ in **Loc** iff (g', f') is an adjoint pair in **Frm**: $g \circ f \leq id_A$ iff $(g \circ f)' \leq id'_A$ iff $f' \circ g' \leq id'_A$.

The above proof of Lemma 4.5 works in arbitrary O-categories if we know that adjoints are unique. We say that an O-category $\mathbf{C}$ has *unique adjoints* if the following two conditions are satisfied:

(i) If $f, f': C \longrightarrow C'$ both are lower adjoints for some $g: C' \longrightarrow C$ then $f = f'$.

(ii) If $g, g': C' \longrightarrow C$ both are upper adjoints for some $f: C \longrightarrow C'$ then $g = g'$.

The three O-categories mentioned above, that is **DCPO**, **Top** and **Loc**, all have unique adjoints.

Lemma 4.6 *Let* $\mathbf{C}$ *be an* O*-category with unique adjoints and let* (T, η, μ) *be a monad on* $\mathbf{C}$ *where* T *is an* O*-functor. Further, suppose that the structure map for every* $\mathbf{C}$*-object* C *necessarily is the lower (upper) adjoint for* η_C*. If the lower (upper) adjoint* $\xi\colon TC \longrightarrow C$ *for* η_C *satisfies* $\xi \circ \eta_C = id_C$ *then* (C, ξ) *is a* T*-algebra.*

Proof: Exactly as that for Lemma 4.5. ∎

The introduction of O-categories saves us the introduction of co-monads. Indeed, in order to get a result like Lemma 4.5 for the category of locales we have to pass to the concrete category of frames. But a monad on **Loc** yields a co-monad on **Frm** and so Lemma 4.5 does not apply to this case.

In [Koc92b], A. Kock studies what he calls 2-doctrines. This is a concept similar to that of a monad for 2-categories. He then shows a result similar to the last lemma. He also gives a constructive theory for the lift monad on posets [Koc92a] which can be understood to be of the type of the monads considered in Lemma 4.5.

Chapter 5

Generalized semilattices

5.1 Semilattices in arbitrary categories

Until now, we have only considered semilattices in the category **Pos** of posets and monotone functions. Depending on whether the posets in question have binary joins or binary meets we speak of $\vee$- or $\wedge$-semilattices. We can carry the concept of semilattices to other categories. For that, let us look at the algebraic characterization of semilattices by means of equations.

Let P be a $\vee$-semilattice and let $\vee\colon P \times P \longrightarrow P$ denote the binary join function. This function has the following properties:

(i) It is *idempotent*, that is $\vee\langle p, p\rangle = p \vee p = p$ for all $p \in P$.

(ii) It is *symmetric*, that is $\vee\langle p, p'\rangle = p \vee p' = p' \vee p = \vee\langle p', p\rangle$ for all $p, p' \in P$.

(iii) It is *associative*, that is $\vee\langle \vee\langle p, p'\rangle, p''\rangle = (p \vee p') \vee p'' = p \vee (p' \vee p'') = \vee\langle p, \vee\langle p', p''\rangle\rangle$ for all $p, p', p'' \in P$.

These properties are also satisfied by the binary meet function on a $\wedge$-semilattice. These conditions are actually enough to characterize semilattices: If S is a set and

$v: S \times S \longrightarrow S$ an idempotent, symmetric and associative function, then $s \leq s'$ iff $v\langle s, s'\rangle = s'$ defines an order on S such that v is the binary join with respect to that order. Note that we could have made S a $\wedge$-semilattice if we had chosen to define the dual order $s \leq s'$ iff $v\langle s, s'\rangle = s$.

The above properties make sense in an arbitrary category with binary products. Idempotency for a morphism $v: C \times C \longrightarrow C$, for example, means that $v \circ \Delta_C = id_C$. Here $\Delta_C: C \longrightarrow C \times C$ denotes the diagonal from C to the product $C \times C$, that is $\Delta_C = \langle id_C, id_C\rangle$.

Definition. A *semilattice structure* for an object C of a category $\mathbf{C}$ with binary products is a $\mathbf{C}$-morphism $v: C \times C \longrightarrow C$ that renders the following diagrams commutative:

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[dr, "id_C"'] & C \times C \arrow[d, "v"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} C \times C \arrow[r, "\langle \pi_2, \pi_1 \rangle"] \arrow[dr, "v"'] & C \times C \arrow[d, "v"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} C \times C \times C \arrow[r, "id_C \times v"] \arrow[d, "v \times id_C"'] & C \times C \arrow[d, "v"] \\ C \times C \arrow[r, "v"] & C \end{tikzcd}$$

Idempotency Symmetry Associativity

In this case we also say that (C, v) is a semilattice in $\mathbf{C}$.

Note that we do not distinguish between $(C \times C) \times C$ and $C \times (C \times C)$ as the two are naturally isomorphic. Here are some properties of the morphisms from the above diagram that we are going to need later: In a category $\mathbf{C}$ with binary products there is an endofunctor on $\mathbf{C}$ defined by $C \longmapsto C \times C$ and $f \longmapsto f \times f$. The diagonal on objects of $\mathbf{C}$ is natural from the identity functor on $\mathbf{C}$ to this functor, that is for all morphisms $f: C \longrightarrow D$ in $\mathbf{C}$ the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "f"'] & C \times C \arrow[d, "f \times f"] \\ D \arrow[r, "\Delta_D"] & D \times D \end{tikzcd}$$

We can also understand $\langle \pi_2, \pi_1 \rangle$ as a natural transformation: For the category $\mathbf{C}$ consider the functor from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C} \times \mathbf{C}$ that interchanges factors. Then $\langle \pi_2, \pi_1 \rangle$ is natural from the identity on $\mathbf{C} \times \mathbf{C}$ to this functor, that is the following diagram is commutative for all morphisms $f \times g\colon C \times D \longrightarrow C' \times D'$ in $\mathbf{C} \times \mathbf{C}$:

$$\begin{tikzcd}
C \times D \arrow[r, "\langle \pi_2, \pi_1 \rangle"] \arrow[d, "f \times g"'] & D \times C \arrow[d, "g \times f"] \\
C' \times D' \arrow[r, "\langle \pi_2, \pi_1 \rangle"] & D' \times C'
\end{tikzcd}$$

Note further that all $\langle \pi_2, \pi_1 \rangle$ are isomorphisms and their own inverses. Hence the family of all $\langle \pi_2, \pi_1 \rangle$ is a natural isomorphism between the two functors from above.

Definition. We say that a $\mathbf{C}$-morphism $f\colon C \longrightarrow D$, where (C, v) and (D, w) are semilattices in the category $\mathbf{C}$, is a *morphism of semilattices* if the following diagram commutes:

$$\begin{tikzcd}
C \times C \arrow[r, "v"] \arrow[d, "f \times f"'] & C \arrow[d, "f"] \\
D \times D \arrow[r, "w"] & D
\end{tikzcd}$$

Let $\mathbf{C}^{\mathbf{SLat}}$ denote the category of semilattices with semilattice morphisms as a subcategory of $\mathbf{C}$.

Let $\mathbb{N}$ denote the set of positive integers. From the 'binary' semilattice structure $v\colon C \times C \longrightarrow C$ we can get an 'n-ary' semilattice structure v_n from the n-fold product of C to C for any $n \in \mathbb{N}$ by the following inductive definition:

$$\begin{aligned}
&v_1 := id_C, \\
&v_{n+1} := v \circ (v_n \times id_C) \qquad\qquad \text{for } n \geq 1.
\end{aligned}$$

By associativity of v we get $v_{n+1} = v \circ (id_C \times v_n)$. We can also show that $v_n \circ \Delta_C^n = id_C$ for all $n \in \mathbb{N}$, where Δ_C^n is the n-ary diagonal $\langle id_C, \ldots, id_C \rangle\colon C \longrightarrow C \times \cdots \times C$: For

$n = 1$ the claim is trivial. By induction we get

$$\begin{aligned} v_{n+1} \circ \Delta_C^{n+1} &= v \circ (v_n \times id_C) \circ \langle id_C, \ldots, id_C \rangle \\ &= v \circ \langle v_n \circ \langle id_C, \ldots, id_C \rangle, id_C \rangle \\ &= v \circ \langle id_C, id_C \rangle \\ &= v \circ \Delta_C \\ &= id_C \,. \end{aligned}$$

Furthermore, v_n is invariant under permutations of factors in $C \times \cdots \times C$.

Also, if $f\colon C \longrightarrow D$ is a morphism of the semilattices (C, v) and (D, w) then the n-ary product $f \times \cdots \times f\colon C \times \cdots \times C \longrightarrow D \times \cdots \times D$ is a morphism of the 'n-ary' semilattices (C, v_n) and (D, w_n), that is $w_n \circ (f \times \cdots \times f) = f \circ v_n$. The proof is by induction on n:

$$\begin{aligned} w_{n+1} \circ (f \times \cdots \times f) &= w \circ (w_n \circ id_C) \circ (f \times \cdots \times f) \\ &= w \circ ((w_n \circ (f \times \cdots \times f)) \times f) \\ &= w \circ (f \times f) \circ (v_n \times id_C) \\ &= f \circ v \circ (v_n \times id_C) \\ &= f \circ v_{n+1}. \end{aligned}$$

5.2 Inflationary and deflationary semilattices

In the category **Set** we may think of semilattices as either $\vee$- or $\wedge$-semilattices since we may introduce an appropriate order on such a set. In the category of posets, however, every object already carries an order. Hence there are two special cases for a semilattice (P, v) in this category: The order on P that we obtain if we interpret v as join or its opposite might coincide with the intrinsic order on P. In the first case, P is a $\vee$-semilattice as defined in Section 1.1 and the binary join is given by v. In the second case, P is a $\wedge$-semilattice and the binary meet is given by v.

A semilattice structure $v\colon P \times P \longrightarrow P$ makes a poset P a $\vee$-semilattice (with respect to the intrinsic order on P) iff $\Delta_P \circ v \geq id_{P\times P}$: Obviously, this condition is satisfied if v is the binary join for some poset P. On the other hand, given a poset P with semilattice structure $v: P \times P \longrightarrow P$ such that $\Delta_P \circ v \geq id_{P\times P}$ we know that v must be the lower adjoint for Δ_P because $v \circ \Delta_P = id_P$ holds by idempotency of v. Hence $\langle p, p' \rangle \leq \Delta_P(v\langle p, p' \rangle) = \langle v\langle p, p' \rangle, v\langle p, p' \rangle \rangle$ which proves that $v\langle p, p' \rangle$ is an upper bound for p and p'. For all $p, p', p'' \in P$ we must have $v\langle p, p' \rangle \leq p''$ iff $\langle p, p' \rangle \leq \Delta_P(p'') = \langle p'', p'' \rangle$ iff $p, p' \leq p''$. Therefore $v\langle p, p' \rangle$ is the least upper bound for p and p'. Analogously, one can show that v makes P a $\wedge$-semilattice iff $\Delta_P \circ v \leq id_{P\times P}$ iff v is the upper adjoint for Δ_P.

So all we need in order to introduce an analogue of the concepts of $\vee$- and $\wedge$-semilattice to more general settings is a partial order on the hom-sets.

Definition. A semilattice (C, v) in an O-category $\mathbf{C}$ is *inflationary* iff (v, Δ_C) is an adjoint pair and *deflationary* iff this is true for (Δ_C, v). If $\mathbf{C}$ is an arbitrary O-category then we denote by $\mathbf{C}^\vee$ the category of inflationary semilattices in $\mathbf{C}$. The name for the category of deflationary semilattices in $\mathbf{C}$ is $\mathbf{C}^\wedge$.

5.3 Examples of generalized semilattices

In the category of posets, an inflationary semilattice is a $\vee$-semilattice and a semilattice morphism between inflationary semilattices is a function that preserves binary joins. Analogous statements hold for deflationary semilattices.

5.3.1 Dcpo's

In the category **DCPO** of dcpo's with Scott-continuous functions we order morphisms pointwise. An inflationary semilattice in **DCPO** is nothing but a poset that has all non-empty joins: A dcpo D has all directed joins. If it additionally is an inflationary semilattice then it has binary joins as well. Recall that we refer to these posets as up-complete $\vee$-semilattices. A *homomorphism of up-complete $\vee$-semilattices* is a function that preserves non-empty joins. A semilattice homomorphism for inflationary semilattices in **DCPO** is a morphism of up-complete $\vee$-semilattices since a function preserves all nonempty joins iff it preserves binary and directed joins. The condition that the semilattice structure $\vee \colon D \times D \longrightarrow D$ be a morphism of the category does not mean anything in this case for it is equivalent to finite joins distributing over directed ones.

A deflationary semilattice D in the category of dcpo's is what we call a $\wedge$-*continuous* $\wedge$-*semilattice*. By this we mean a dcpo that has binary meets such that directed joins distribute over binary meets, that is $d \wedge \bigvee^{\uparrow} S = \bigvee^{\uparrow}_{s \in S}(d \wedge s)$ for all $d \in D$ and all directed subsets S of D. Such a poset is also referred to as a *preframe* in [JV91]. We will not use this name, however, because it does not fit with how we wish to think about it, namely as a dcpo with an additional structure. A morphism of deflationary semilattices in **DCPO** is a function that preserves binary meets and directed joins.

5.3.2 Topological spaces

In the category **Top** of T_0-spaces, a semilattice is also referred to as *topological semilattice*. Recall that T_0-spaces carry an order, namely the specialization order, and that every continuous function preserves that order. To make this category an O-category, we also take the pointwise order on the hom-sets. An inflationary topological semilattice has binary joins with respect to the specialization order and the binary join function is continuous. Note that an inflationary topological semilattice with a sober topology is an up-complete $\vee$-semilattice because any sober space has directed joins. A morphism of inflationary topological semilattices is a continuous function that preserves binary joins.

A deflationary topological semilattice has binary meets with respect to the specialization order and the binary meet function has to be continuous. Considered a poset, a deflationary topological semilattice with sober topology is a $\wedge$-continuous

$\wedge$-semilattice. A morphism of deflationary topological semilattices is a continuous function that preserves binary meets.

5.3.3 Locales

Finally, let us have a look at semilattices in the category of locales. We call them *localic semilattices* after [Joh85].

Lemma 5.1 *If A is a localic semilattice with semilattice structure $v: A \otimes A \longrightarrow A$ then v induces a semilattice structure on $\mathsf{Sp}A$ making it a topological semilattice.*

Proof: By Lemma 2.4, $\mathsf{Sp}: \mathbf{Loc} \longrightarrow \mathbf{Top}$ is a functor preserving binary products. Hence $v: A \otimes A \longrightarrow A$ induces a continuous function from $\mathsf{Sp}A \times \mathsf{Sp}A$ to $\mathsf{Sp}A$. From the same lemma we get that this function satisfies the semilattice equations. ■

We give another motivation for the order on hom-sets of **Loc** we introduced above to make the category of locales an O-category. Because of the last lemma we know that every semilattice structure on A induces a semilattice structure on $\mathsf{Sp}A$. So the question is whether the topological semilattice corresponding to an inflationary localic semilattice is inflationary as well. Recall that for continuous functions $f, g: A \longrightarrow B$ between locales we have $f \leq g$ iff $f' \leq g'$ holds for the corresponding frame homomorphisms with respect to the pointwise order.

Lemma 5.2 *If $v: A \otimes A \longrightarrow A$ is a semilattice structure making the locale A an inflationary (a deflationary) semilattice then the induced structure makes $\mathsf{Sp}A$ an inflationary (a deflationary) topological semilattice.*

Proof: We will only prove the assertion for the case of inflationary semilattices. If $v: A \otimes A \longrightarrow A$ makes A an inflationary semilattice then Δ_A is the upper adjoint for v which means that Δ'_A is the lower adjoint for v'. By Lemma 2.2 this implies that $\mathsf{Sp}\Delta_A$ is the upper adjoint for $\mathsf{Sp}v$. Since Sp preserves products by Proposition 2.4, the semilattice structure induced by $\mathsf{Sp}v$ is the upper adjoint for $\Delta_{\mathsf{Sp}\,A}$ and therefore we are done. ■

Note that if (X, v) is a topological semilattice then ΩX need not be a localic semilattice since Ωv is a continuous function from $\Omega(X \times X)$ to ΩX and therefore does not necessarily define a function from $\Omega X \otimes \Omega X$ to ΩX.

If (A, v) is an inflationary localic semilattice then the frame homomorphism corresponding to v must be the upper adjoint for Δ'_A, the frame homomorphism corresponding to the diagonal on A. Using the way the product $A \otimes A$ in **Loc** is constructed via generators and relations, we can explicitly say what v has to look like: For any $a \in A$ we must have

$$v'(a) := \bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \wedge b_2 \leq a\}.$$

Let us now consider semilattice structures for some deflationary semilattice (A, v) in **Loc**. For any $a \in A$ and any family $(b_1^i \otimes b_2^i)_{i \in I}$ of generators of $A \otimes A$ we have $v'(a) \leq \bigvee\limits_{i \in I} b_1^i \otimes b_2^i$ iff $a \leq \Delta'_A(\bigvee\limits_{i \in I} b_1^i \otimes b_2^i) = \bigvee\limits_{i \in I} b_1^i \wedge b_2^i$. So we get

$$\begin{aligned} v'(a) &:= \bigwedge\{\bigvee_{i \in I} b_1^i \otimes b_2^i \mid a \leq \bigvee_{i \in I} b_1^i \wedge b_2^i\} \\ &= \bigvee\{b_1 \otimes b_2 \in A \otimes A \mid a \leq \bigvee_{i \in I} b_1^i \otimes b_2^i \implies b_1 \otimes b_2 \leq \bigvee_{i \in I} b_1^i \otimes b_2^i\}. \end{aligned}$$

5.4 Retracts of generalized semilattices

We have already seen several properties that are inherited by retracts. In this section we will explore what can be said about retracts of generalized semilattices. The category of posets, for example, is a category where a retract of an inflationary or a deflationary semilattice carries a corresponding structure.

Let $\mathbf{C}$ be a category with binary products and let C be an object of $\mathbf{C}$ with a semilattice structure, say $v: C \times C \longrightarrow C$. What can we say about a $\mathbf{C}$-object D that is a retract of C via $D \underset{r}{\overset{e}{\rightleftarrows}} C$? If, from this data, we can define a semilattice structure on D at all then it should be $r \circ v \circ (e \times e) \colon D \times D \longrightarrow D$. For this is the only morphism of 'the right type' that we can build from the given data in a natural way:

$$\begin{array}{ccc} D \times D & \xrightarrow{e \times e} & C \times C \\ & & \big\downarrow v \\ D & \xleftarrow{\;r\;} & C \end{array}$$

In fact, this is the semilattice structure we used to prove that in **Pos** a retract of an inflationary semilattice is an inflationary semilattice. So let us see what we can show for this structure. First for idempotency. We calculate

$$\begin{aligned} r \circ v \circ (e \times e) \circ \Delta_C &= r \circ v \circ \Delta_C \circ e && \text{(Naturality of } \Delta) \\ &= r \circ e && \text{(Idempotency of } v) \\ &= id_D \,. \end{aligned}$$

Next for symmetry. We get

$$\begin{aligned} r \circ v \circ (e \times e) \circ \langle \pi_2, \pi_1 \rangle &= r \circ v \circ \langle \pi_2, \pi_1 \rangle \circ (e \times e) && \text{(Naturality of } \langle \pi_2, \pi_1 \rangle) \\ &= r \circ v \circ (e \times e) && \text{(Symmetry of } v). \end{aligned}$$

So this worked fine so far. But problems arise when we try to prove associativity

$$\begin{aligned} & \big(r \circ v \circ (e \times e)\big) \circ \big((r \circ v \circ (e \times e)) \times id_D\big) \\ = \; & \big(r \circ v \circ (e \times e)\big) \circ \big((r \circ v \circ (e \times e)) \times (r \circ id_C \circ e)\big) \\ = \; & r \circ v \circ (e \times e) \circ (r \times r) \circ (v \times id_C) \circ (e \times e \times e) \end{aligned}$$

for here we get stuck.

To see that in general it is not possible to go on, consider the following example: Let $S := \{a, b, c\}$ and let $\bot, l, r, \top$ be the elements of the four element lattice L that is not a chain. We consider a $\vee$-semilattice L with semilattice structure $\vee_L \colon L \times L \longrightarrow L$. We define an embedding of S into L by setting $e(a) := \bot$, $e(b) := l$ and $e(c) := r$. In order to define a retraction $s \colon L \longrightarrow S$ we just have to specify what the image of $\top$

should be. We set $s(\top) := a$ thus making S a retract of L in **Set**. If we try to define a semilattice structure on S as described above, namely by setting $\vee_S := s \circ \vee_L \circ (e \times e)$, then we end up with a function from $S \times S$ to S that is not associative for

$$\begin{aligned} b \vee_S (b \vee_S c) &= b \vee_S \Big(s(e(b) \vee_L e(c))\Big) = b \vee_S \Big(s(l \vee_L r)\Big) \\ &= b \vee_S s(\top) = b \vee_S a = s(l \vee_L \bot) = s(l) = b \end{aligned}$$

while

$$(b \vee_S b) \vee_S c = b \vee_S c = a.$$

So we need an additional condition to prove that a retract of a semilattice is again a semilattice. If we knew, for example, that

$$r \circ v \circ (e \times e) \circ (r \times r) = r \circ v \qquad (*)$$

is satisfied then we could continue the above calculations:

$$\begin{aligned} & \Big(r \circ v \circ (e \times e)\Big) \circ \Big((r \circ v \circ (e \times e)) \times id_D\Big) \\ = \;& r \circ v \circ (e \times e) \circ (r \times r) \circ (v \times id_C) \circ (e \times e \times e) && \text{(By above calculations)} \\ = \;& r \circ v \circ (v \times id_C) \circ (e \times e \times e) && \text{(By } (*)) \\ = \;& r \circ v \circ (id_C \times v) \circ (e \times e \times e) && \text{(Associativity of } v) \\ = \;& r \circ v \circ (e \times e) \circ (r \times r) \circ (id_C \times v) \circ (e \times e \times e) && \text{(By } (*)) \\ = \;& \Big(r \circ v \circ (e \times e)\Big) \circ \Big(id_C \times (r \circ v \circ (e \times e))\Big) && (r \circ e = id_D). \end{aligned}$$

Note that if $\Big(D, r \circ v \circ (e \times e)\Big)$ is a semilattice then $(*)$ implies that r is a morphism of the semilattices C and D. Thus we obtain the following lemma.

Lemma 5.3 *Let* $\mathbf{C}$ *be a category with binary products. Further, let* C *be a* $\mathbf{C}$*-object with a semilattice structure* $v\colon C \times C \longrightarrow C$*. If* D *is a retract of* C *via* $D \underset{r}{\overset{e}{\rightleftarrows}} C$ *such that* $r \circ v \circ (e \times e) \circ (r \times r) = r \circ v$ *then* $r \circ v \circ (e \times e)\colon D \times D \longrightarrow D$ *is a semilattice structure for* D*. Moreover,* $r\colon C \longrightarrow D$ *is a semilattice homomorphism.* ∎

In **Pos**, the retract of an inflationary semilattice is an inflationary semilattice. This suggests that an analogue could hold for semilattices in O-categories. Although we cannot show it in general it is possible to prove a weaker version.

Lemma 5.4 *Let* $\mathbf{C}$ *be an* O*-category with binary products such that the endofunctor on* $\mathbf{C}$ *given by* $C \longmapsto C \times C$, $f \longmapsto f \times f$ *is an* O*-functor. Further, let* D *be a retract of* C *in* $\mathbf{C}$ *via* $D \underset{r}{\overset{e}{\rightleftarrows}} C$. *Let* $v\colon C \times C \longrightarrow C$ *make* (C, v) *an inflationary semilattice.*

(i) If (r, e) *is an adjoint pair then* $\big(D, r \circ v \circ (e \times e)\big)$ *is a semilattice.*

(ii) If $\big(D, r \circ v \circ (e \times e)\big)$ *is a semilattice then it is inflationary as well.*

Proof: (i) By Lemma 5.3 it is enough to show that $r \circ v \circ (e \times e) \circ (r \times r) = r \circ v$. As by assumption $e \circ r \geq id_C$ and thus $(e \circ r) \times (e \circ r) \geq id_{C \times C}$, we have $r \circ v \circ (e \times e) \circ (r \times r) = r \circ v \circ ((e \circ r) \times (e \circ r)) \geq r \circ v$. On the other hand,

$$\begin{array}{rcll} r \circ v \circ (e \times e) \circ (r \times r) & \leq & r \circ v \circ (e \times e) \circ (r \times r) \circ \Delta_C \circ v & (id_{C \times C} \leq \Delta_C \circ v) \\ & = & r \circ v \circ \Delta_C \circ e \circ r \circ v & (\text{Naturality of } \Delta) \\ & = & r \circ e \circ r \circ v & (v \circ \Delta_C = id_C) \\ & = & r \circ v & (r \circ e = id_D). \end{array}$$

(ii) We have to show that $\Delta_D \circ r \circ v \circ (e \times e) \geq id_{D \times D}$. Since $\Delta_C \circ v \geq id_{C \times C}$ we may calculate

$$\begin{array}{rcl} \Delta_D \circ r \circ v \circ (e \times e) & = & (r \times r) \circ \Delta_C \circ v \circ (e \times e) \geq (r \times r) \circ (e \times e) \\ & = & (r \circ e) \times (r \circ e) \\ & = & id_{D \times D}\,. \end{array}$$

■

The corresponding result for deflationary semilattices can be shown analogously.

Lemma 5.5 *Let* $\mathbf{C}$ *be a category that satisfies the conditions from Lemma* 5.4 *and let* D *be a retract of* C *in* $\mathbf{C}$ *via* $D \underset{r}{\overset{e}{\rightleftarrows}} C$. *Further, let* $v\colon C \times C \longrightarrow C$ *make* (C, v) *a deflationary semilattice.*

(i) If (e, r) *is an adjoint pair then* $\big(D, r \circ v \circ (e \times e)\big)$ *is a semilattice.*

(ii) If $\big(D, r \circ v \circ (e \times e)\big)$ *is a semilattice then it is deflationary as well.* ■

Note that Lemmata 5.4 and 5.5 can be applied to the categories **DCPO**, **Top** and **Loc**. It is obvious that the product functor mentioned there is indeed an O-functor for **DCPO** and **Top**. For the category of locales, this follows from the following considerations: Let $f, g\colon A \longrightarrow B$ be morphisms in **Loc** such that $f \leq g$. By definition, this is equivalent to $f' \leq g'$. As

$$(f \otimes f)'(a \otimes b) = f'(a) \otimes f'(b) \leq g'(a) \otimes g'(b) = (g \otimes g)'(a \otimes b)$$

for all $a \in A$, $b \in B$, we conclude that $f \otimes f \leq g \otimes g$.

For the categories **DCPO** and **Top** we have the following: There is a faithful O-functor to **Pos** that preserves products. For such O-categories we can conclude the desired result from Lemma 1.2.

Proposition 5.6 *Let* $\mathbf{C}$ *be an* O*-category with binary products such that there is a faithful and product-preserving* O*-functor* $U\colon \mathbf{C} \longrightarrow \mathbf{Pos}$*. If* D *is a retract of the inflationary (deflationary) semilattice* (C, v) *via* $D \underset{r}{\overset{e}{\rightleftarrows}} C$ *then* $\big(D, r \circ v \circ (e \times e)\big)$ *is an inflationary (a deflationary) semilattice in* $\mathbf{C}$*.*

Proof: We will only prove the result for the case where C is an inflationary semilattice. For simplicity we assume that $U(C \times C) = UC \times UC$. It is not hard to show that (UC, Uv) is an inflationary semilattice in **Pos**. For example,

$$Uv \circ \Delta_{UC} = Uv \circ \langle id_{UC}, id_{UC} \rangle = Uv \circ \langle U\, id_C, U\, id_C \rangle = Uv \circ U\, \Delta_C = U(v \circ \Delta_C) = id_{UC}\,.$$

Symmetry and associativity of Uv also follow from the fact that U preserves binary products. Hence (UC, Uv) is a $\vee$-semilattice and Uv is the binary join function. As UD is a retract of UC it must also be a $\vee$-semilattice by Lemma 1.2. Furthermore, the binary join for UD is given by $Ur \circ Uv \circ (Ue \times Ue) = U(r \circ v \circ (e \times e))$. Hence $U(r \circ v \circ (e \times e))$ is idempotent, symmetric and associative. By faithfulness of U we may conclude that $r \circ v \circ (e \times e)$ has the same properties: In order to prove symmetry, for example, we need $(r \circ v \circ (e \times e)) \circ \langle \pi_2, \pi_1 \rangle = r \circ v \circ (e \times e)$. By symmetry of $U(r \circ v \circ (e \times e))$ we have

$$\begin{aligned} U(r \circ v \circ (e \times e) \circ \langle \pi_2, \pi_2 \rangle) &= U(r \circ v \circ (e \times e)) \circ U\langle \pi_2, \pi_1 \rangle \\ &= U(r \circ v \circ (e \times e)) \circ \langle \pi_2, \pi_1 \rangle \\ &= U(r \circ v \circ (e \times e)). \end{aligned}$$

Hence $r \circ v \circ (e \times e) \circ \langle \pi_2, \pi_1 \rangle$ and $r \circ v \circ (e \times e)$ are both mapped to $U(r \circ v \circ (e \times e))$ and thus by faithfulness of U must be equal. Therefore $\big(D, r \circ v \circ (e \times e)\big)$ is a semilattice in $\mathbf{C}$. By Lemma 5.4 (ii) it must be an inflationary semilattice. ■

Note that this result does not apply to the category of locales though there are two obvious functors to the category of posets. Neither of these, however, satisfies the above requirements: One of these functors is the contravariant functor to the category of frames. But contravariant functors do not transport notions like idempotency, symmetry etc. The second functor, namely Pt, that maps a locale A to the poset $\mathsf{Pt}A$ of points of A, is not faithful.

5.5 Unital semilattices

In this section we will study the analogue of semilattices with unit. Recall that a unit for a semilattice (S, v) over **Set** is an element s_0 of the set S such that $v\langle s, s_0 \rangle = s = v\langle s_0, s \rangle$ for all $s \in S$. (Note that by symmetry of v it is only necessary to demand one of these equations.) In this case we say that the semilattice is *unital*. A unital inflationary semilattice in **Pos**, for example, is a $\vee$-semilattice with a least element. To generalize the notion of unital semilattices to a category $\mathbf{C}$ with finite products, recall that an element of a $\mathbf{C}$-object C is given by a morphism from *1* to C. Here *1* denotes the empty product, that is the terminal object in $\mathbf{C}$. So when can we say that an element $o\colon \mathit{1} \longrightarrow C$ is the unit for a semilattice structure $v\colon C \times C \longrightarrow C$? If we have another look at the above equation we see that we can reformulate it via the following diagram:

$$
\begin{tikzcd}
1 \times C \arrow[r, "o \times id_C"] \arrow[dr, "\pi_2"'] & C \times C \arrow[d, "v"] \\
 & C
\end{tikzcd}
$$

Note that here $\pi_2: 1 \times C \longrightarrow C$ is an isomorphism.

Definition. A *unital semilattice* (C, v, o) in a category $\mathbf{C}$ with finite products is given by a semilattice structure $v: C \times C \longrightarrow C$ for C and a morphism $o: 1 \longrightarrow C$ such that $v \circ (o \times id_C) = \pi_2$. A *morphism of unital semilattices* (C, v, o) and (D, w, p) is a morphism $f: C \longrightarrow D$ of semilattices such that $f \circ o = p$.

Note that the analogues of Lemma 5.3 to Proposition 5.6 for unital semilattices are true for categories that have finite products. However, to apply Proposition 5.6 we additionally have to make sure that U preserves the empty product. It is sufficient to prove the next proposition as the other results can easily be deduced from it.

Lemma 5.7 *Let* $\mathbf{C}$ *be a category with finite products. Further, let* (C, v, o) *be a unital semilattice in* $\mathbf{C}$. *If* D *is a retract of* C *via* $D \underset{r}{\overset{e}{\rightleftarrows}} C$ *such that* $r \circ v \circ (e \times e) \circ (r \times r) = r \circ v$ *then* $(r \circ v \circ (e \times e), r \circ o)$ *is a unital semilattice structure for* D. *Moreover,* $r: C \longrightarrow D$ *is a homomorphism of unital semilattices.*

Proof: Because of Lemma 5.3 it remains to prove that $r \circ o$ is a unit for the semilattice structure $r \circ v \circ (e \times e)$ on D. We calculate

$$
\begin{aligned}
(r \circ v \circ (e \times e)) \circ ((r \circ o) \times id_D) &= r \circ v \circ (e \times e) \circ (r \times r) \circ (o \times e) \\
&= r \circ v \circ (o \times id_C) \circ (id_1 \times e) \\
&= r \circ \pi_2 \circ (id_1 \times e) \\
&= r \circ e \circ \pi_2 \\
&= \pi_2
\end{aligned}
$$

which proves the desired equation. ■

In **Pos**, **DCPO** and **Top**, the unit for an inflationary semilattice has to be the least element with respect to the order given for the respective objects. Hence a unital inflationary topological semilattice, for example, is a $\vee$-semilattice with least element with respect to the specialization order such that the binary join is a continuous function. The requirement that the unit for the semilattice structure be a morphism of the category is mute for these categories. Analogous statements hold for deflationary semilattices in these categories. In the category of locales, we can also characterize units for inflationary or deflationary semilattices.

Lemma 5.8 *Let (A, v) be a semilattice in* **Loc***.*

(i) If A is an inflationary semilattice then it is unital iff a continuous function from 2 to A is defined by $o'(a) := 1_2 \iff a = 1_A$. In this case, the unit for (A, v) is given by o.

(ii) If A is a deflationary semilattice then it is unital iff a continuous function from 2 to A is defined by $o'(a) := 0_2 \iff a = 0_A$. In this case, the unit for (A, v) is given by o.

Proof: (i) We have seen that $v'(a) = \bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \wedge b_2 \leq a\}$ for all $a \in A$ in Section 5.3. So if the above definition makes o' a frame homomorphism then

$$
\begin{aligned}
(o \otimes id_A)'(v'(a)) &= (o \otimes id_A)'\left(\bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \wedge b_2 \leq a\}\right) \\
&= 1_2 \otimes \left(\bigvee\{b_2 \in A \mid b_2 \leq a\}\right) \\
&= 1_2 \otimes a \\
&= \pi_2'(a)
\end{aligned}
$$

for all $a \in A$. Now assume that (A, v, o) is a unital inflationary semilattice in **Loc**. Then

$$
\begin{aligned}
1_2 \otimes a &= \pi_2'(a) \\
&= (o \otimes id_A)'(v'(a)) \\
&= (o \otimes id_A)'\Big(\bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \wedge b_2 \leq a\}\Big) \\
&= \bigvee\{o'(b_1) \otimes b_2 \in A \otimes A \mid b_1 \wedge b_2 \leq a\}
\end{aligned}
$$

for all $a \in A$. Hence $o'(a) \otimes 1_A \leq 1_{\mathit{2}} \otimes a$ for all $a \in A$ which proves $o'(a) = 0_{\mathit{2}}$ for all $a \in A \setminus \{1_A\}$.

(ii) Assume first that o' as defined above is indeed a frame homomorphism. Since $v'(a) \leq a \otimes a$ we have $(o \otimes id_A)'(v'(a)) \leq (o \otimes id_A)'(a \otimes a) \leq \pi_2'(a)$ for all $a \in A$. By the above considerations, $v'(a) = \bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \otimes b_2 \leq v'(a)\}$. We have

$$\begin{aligned} a &= \Delta_A'(v'(a)) \\ &= \Delta_A'(\bigvee\{b_1 \otimes b_2 \in A \otimes A \mid b_1 \otimes b_2 \leq v'(a)\}) \\ &= \bigvee\{b_1 \wedge b_2 \mid b_1 \otimes b_2 \leq v'(a)\} \\ &\leq \bigvee\{b_2 \in A \mid \exists b_1 \in A \setminus \{0_A\} \,.\, b_1 \otimes b_2 \leq v'(a)\} \end{aligned}$$

and therefore also

$$\begin{aligned} (o \otimes id_A)'(v'(a)) &= \bigvee\{o'(b_1) \otimes b_2 \mid b_1 \otimes b_2 \leq v'(a)\} \\ &= 1_{\mathit{2}} \otimes \bigvee\{b_2 \in A \mid \exists b_1 \in A \setminus \{0_A\} \,.\, b_1 \otimes b_2 \leq v'(a)\} \\ &\geq 1_{\mathit{2}} \otimes a \\ &= \pi_2'(a) \end{aligned}$$

for all $a \in A$.

On the other hand, let o be a unit for v. Then

$$1_{\mathit{2}} \otimes a = \pi_2'(a) = (o \otimes id_A)'(v'(a)) \leq (o \otimes id_A)'(a \otimes a) = o'(a) \otimes a$$

for all $a \in A$. Hence if $a \in A \setminus \{0_A\}$ then $1_{\mathit{2}} \leq o'(a)$ which proves the assertion. ∎

The idea of carrying algebraic concepts like that of a semilattice to arbitrary categories is well-established. Our concrete results on retracts of semilattices in order-enriched categories appear to be new.

Chapter 6

The Hoare power construction

6.1 Algebraic domains

Considering non-deterministic programming languages, there are several possibilities how to think of non-determinism. We shall only deal with bounded non-determinism here, that is at any given time a program can only have finitely many choices.

In a deterministic programming language any terminating program (deterministically) produces one final state for each initial state. If we consider non-deterministic programming languages, however, we lose this property. So, starting from one initial state, there typically are several possible final states any given program might end up with.

Suppose we already have a denotational semantics for the deterministic part (like variable assignments, loops and conditionals) of such a non-deterministic language in the setting of algebraic domains. A program is denoted by a Scott-continuous function between algebraic domains and the obvious idea is to denote a non-deterministic program by a set-valued function. But should we allow all sets of states and how should we order the sets we do allow? In this chapter we shall consider so-called 'angelic' non-determinism, that is we say that a program satisfies a certain condition if at least one of its branches does.

We may assume that for any given program p we already know how to denote all of its branches, namely by Scott-continuous functions between the algebraic domains D and E, and we want to denote p by a function from D to some domain that lives in the power set of E. Given another program q that is to be denoted by a function between the same domains, when should the denotation of q be greater than or equal to the denotation of p? Because of the above philosophy we want this to be the case iff for any branch of p there is one of q with greater or equal denotation.

Hence for any algebraic domain D we (pre-)order $\mathsf{P}_{\mathsf{fne}}\mathsf{C}D$, the set of non-empty finite sets of compact elements of D, via $F \leq^{\flat} F'$ iff for all c in F there is c' in F' with $c \leq c'$. The desired domain is now obtained by ideal completion of the poset $\mathsf{P}_{\mathsf{fne}}\mathsf{C}D$ (see e.g. [Plo81]).

Definition. The *Hoare power domain* $\mathsf{P}^{\flat}_{\mathsf{A}}D$ of an algebraic domain D is the ideal completion of $(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^{\flat})$.

There is another description of the Hoare power domain, namely as a subposet of the power set. It can be shown that $\mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^{\flat})$ is isomorphic to $\mathsf{A}_{\mathsf{D}}D$, the set of non-empty Scott-closed subsets of D ordered by inclusion: We map every non-empty finite set of compact elements to its lower closure in D. Let $i\colon \mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^{\flat}) \longrightarrow \mathsf{A}_{\mathsf{D}}D$ be the unique Scott-continuous extension of this function which is defined by setting $i(\mathcal{I}) := \bigvee^{\uparrow}_{F \in \mathcal{I}} {\downarrow} F = \mathsf{cl}(\bigcup_{i \in I} {\downarrow} F)$. One easily sees that $A \longmapsto \{F \in \mathsf{P}_{\mathsf{fne}}\mathsf{C}D \mid F \subseteq A\}$ defines a function from $\mathsf{A}_{\mathsf{D}}D$ to $\mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^{\flat})$ that is the inverse for i.

From what we have said before it follows that $\mathsf{P}^{\flat}_{\mathsf{A}}D$ is an algebraic domain. Furthermore, $\mathsf{P}^{\flat}_{\mathsf{A}}D$ has all non-empty joins. Recall that we called such a poset an up-complete $\vee$-semilattice. In the algebraic case, we also say that such a poset is an *algebraic $\vee$-semilattice* since we assume 'algebraic' to imply that the poset in question has directed joins. Algebraic $\vee$-semilattices are also referred to as *Scott domains* [Sco82].

Let us now consider a slightly different approach to the problem of giving a denotational semantics for a non-deterministic programming language. How is non-determinism introduced to a language? Often one considers a deterministic language to which a command is added that is responsible for all non-deterministic effects. Assume once more that we already have an interpretation for a deterministic programming language and that we now want to add the command $\underline{\mathsf{or}}$ (with the usual meaning) to it. Hence we have to decide how to denote a program like $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$. But because of the above way of interpreting non-determinism we would like that denotation to be greater than those of the programs p and q since everything one of the two can do, $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ can do as well, but only just as well: the set of possible final states for $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ is the union of the corresponding sets for p and q. Hence we want the denotation of $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ to be the least upper bound of those for p and q. Consequently, the domain used to model p and q should have binary joins which of course need not exist in an algebraic domain. Thus we would like to somehow add finite joins to the domains we used to model the deterministic part of our language in a way that allows us to extend functions in a unique way.

It turns out that we can also do this using the Hoare power domain. To make this precise, let us extend the Hoare power construction to a functor. For every Scott-continuous function $f\colon D \longrightarrow E$ we define a function $\mathsf{P}^\flat_\mathsf{A} f\colon \mathsf{P}^\flat_\mathsf{A} D \longrightarrow \mathsf{P}^\flat_\mathsf{A} E$ by setting $\mathsf{P}^\flat_\mathsf{A} f(A) := \mathsf{cl}\, f(A)$ for every non-empty Scott-closed subset A of D. Thus we obtain a functor $\mathsf{P}^\flat_\mathsf{A}$ from the category $\mathbf{AlgD}$ of algebraic domains (with Scott-continuous functions) to the category $\mathbf{AlgD}^\vee$ of algebraic $\vee$-semilattices with functions that preserve non-empty joins. Recall that we can understand $\mathbf{AlgD}^\vee$ to be the category of inflationary semilattices in $\mathbf{AlgD}$.

The following result by M. Hennessy and G. Plotkin [HP79] implies that the two approaches we described in fact lead to the same definition.

Theorem 6.1 *The functor* $\mathsf{P}^\flat_\mathsf{A}\colon \mathbf{AlgD} \longrightarrow \mathbf{AlgD}^\vee$ *is left adjoint to the forgetful functor* $\mathbf{AlgD}^\vee \longrightarrow \mathbf{AlgD}$. ■

Hence the Hoare power domain gives the free inflationary semilattice for **AlgD**.

Remark. If we wish the power domain to contain the empty set (which serves as its least element), an analogous result holds. The construction we thus obtain is left adjoint to the forgetful functor from the category of algebraic lattices with functions that preserve all joins to **AlgD**.

6.2 Directed complete partial orders

There is an obvious generalization of the Hoare power domain construction to arbitrary dcpo's.

Definition. The Hoare power dcpo $\mathsf{P}^\flat_{\mathsf{D}}D$ for a dcpo D is the set of all non-empty Scott-closed subsets of D ordered by inclusion.

In the remainder of this section we will prove a result that is the analogue of Theorem 6.1. First of all, note that, for any dcpo, the Hoare power dcpo is a poset that has all non-empty joins, that is it is an up-complete $\vee$-semilattice. We define $\mathbf{DCPO}^\vee$ to be the category of up-complete $\vee$-semilattices with functions that preserve non-empty joins. Note that every up-complete $\vee$-semilattice becomes a complete $\vee$-semilattice if a least element is adjoined to it. Alternatively, we can understand $\mathbf{DCPO}^\vee$ to be the category of inflationary semilattices in **DCPO**.

We can make $\mathsf{P}^\flat_{\mathsf{D}}$ a functor from **DCPO** to $\mathbf{DCPO}^\vee$. Let $f\colon D \longrightarrow E$ be a Scott-continuous function between dcpo's. We define $\mathsf{P}^\flat_{\mathsf{D}}f$ to be the function that maps any non-empty Scott-closed subset A of D to the Scott-closure of $f(A)$ in E. In order to see that $\mathsf{P}^\flat_{\mathsf{D}}f$ indeed preserves all non-empty joins note that

$$\mathsf{P}^\flat_{\mathsf{D}}f\Big(\mathsf{cl}\bigcup\mathcal{S}\Big) = \mathsf{cl}\, f\Big(\mathsf{cl}\bigcup\mathcal{S}\Big) = \mathsf{cl}\, f\Big(\bigcup\mathcal{S}\Big) = \mathsf{cl}\bigcup_{A\in\mathcal{S}} f(A) = \mathsf{cl}\bigcup_{A\in\mathcal{S}}\mathsf{cl}\, f(A) = \mathsf{cl}\bigcup\mathsf{P}^\flat_{\mathsf{D}}f(A)$$

for any non-empty subset $\mathcal{S}$ of $\mathsf{P}^\flat_{\mathsf{D}}D$. Now for functoriality of $\mathsf{P}^\flat_{\mathsf{D}}$. Obviously, $\mathsf{P}^\flat_{\mathsf{D}}$ preserves identities. If $f\colon D \longrightarrow E$ and $g\colon E \longrightarrow F$ are Scott-continuous functions between dcpo's then $\mathsf{P}^{\flat}_{\mathsf{D}}(g \circ f)(A) = \mathsf{cl}\, g(f(A)) = \mathsf{cl}\, g\Big(\mathsf{cl}\, f(A)\Big) = \mathsf{P}^{\flat}_{\mathsf{D}} f\Big(\mathsf{P}^{\flat}_{\mathsf{D}} g(A)\Big)$ for any $A \in \mathsf{P}^{\flat}_{\mathsf{D}} D$. We can embed D into $\mathsf{P}^{\flat}_{\mathsf{D}} D$ via the Scott-continuous function $\downarrow_D : d \longmapsto \downarrow\{d\}$. The following theorem can be easily deduced from Theorem 4.4 in [HM85].

Theorem 6.2 *The functor* $\mathsf{P}^{\flat}_{\mathsf{D}} : \mathbf{DCPO} \longrightarrow \mathbf{DCPO}^{\vee}$ *is left adjoint to the forgetful functor* $U : \mathbf{DCPO}^{\vee} \longrightarrow \mathbf{DCPO}$.

Proof: We have to show that for any dcpo D the Hoare power dcpo $\mathsf{P}^{\flat}_{\mathsf{D}} D$ has the universal property given by the following diagram:

$$
\begin{array}{ccc}
\mathbf{DCPO} & \qquad & \mathbf{DCPO}^{\vee} \\
\begin{tikzcd}
D \arrow[r, "\downarrow_D"] \arrow[dr, "\forall f"'] & U\, \mathsf{P}^{\flat}_{\mathsf{D}} D \arrow[d, "U\hat{f}"] \\
& UL
\end{tikzcd}
& &
\begin{tikzcd}
\mathsf{P}^{\flat}_{\mathsf{D}} D \arrow[d, dashed, "\exists!\hat{f}"] \\
L
\end{tikzcd}
\end{array}
$$

Let $f : D \longrightarrow UL$ be a Scott-continuous function where D is a dcpo and L an up-complete $\vee$-semilattice. Clearly, $A = \bigcup_{a \in A} \downarrow\{a\} = \bigvee_{a \in A} \downarrow_D(a)$ for any non-empty Scott-closed subset A of D. There is at most one $\hat{f} : \mathsf{P}^{\flat}_{\mathsf{D}} D \longrightarrow L$ rendering the above diagram commutative because $\hat{f}(A) = \bigvee_{a \in A} f(a)$ has to hold for all $A \in \mathsf{P}^{\flat}_{\mathsf{D}} D$. We use this as the definition of $\hat{f}$. In order to see that $\hat{f}$ is a $\mathbf{DCPO}^{\vee}$-morphism, let $\mathcal{S}$ be a non-empty subset of $\mathsf{P}^{\flat}_{\mathsf{D}} D$. Then

$$
\begin{aligned}
\hat{f}\Big(\mathsf{cl} \bigcup \mathcal{S}\Big) &= \bigvee f\Big(\mathsf{cl} \bigcup \mathcal{S}\Big) \\
&= \bigvee f\Big(\bigcup \mathcal{S}\Big) && \text{(By Corollary 1.6 (i))} \\
&= \bigvee \bigcup_{A \in \mathcal{S}} f(A) \\
&= \bigvee \{\bigvee f(A) \mid A \in \mathcal{S}\} \\
&= \bigvee \{\hat{f}(A) \mid A \in \mathcal{S}\}.
\end{aligned}
$$

Note that this is where Scott continuity of f is used. If we had chosen $\mathsf{P}^\flat_\mathsf{D} D$ to be the non-empty closed subsets of D with respect to another topology then we could not have employed Corollary 1.6. Obviously, $\hat{f} \circ \downarrow_D = f$ is satisfied. ■

The above adjunction is monadic: Note that the co-unit $\bigvee_L \colon \mathsf{P}^\flat_\mathsf{D} U \longrightarrow id_{\mathbf{DCPO}^\vee}$ of $\mathsf{P}^\flat_\mathsf{D} \dashv U$ is $A \longmapsto \bigvee A$ for every non-empty Scott-closed subset A of an up-complete $\vee$-semilattice L. In order to prove our claim we have to show that every algebra for the monad $(U\,\mathsf{P}^\flat_\mathsf{D}, \downarrow, U \bigvee \mathsf{P}^\flat_\mathsf{D})$ is an up-complete $\vee$-semilattice. But since every $(U\,\mathsf{P}^\flat_\mathsf{D})$-algebra (D, ξ) is a retract of the up-complete $\vee$-semilattice $U\,\mathsf{P}^\flat_\mathsf{D} D$ it must be another such. As $\mathbf{DCPO}^\vee$ is the category of inflationary semilattices for **DCPO** we can understand $\mathsf{P}^\flat_\mathsf{D}$ to give the free inflationary semilattice for **DCPO**.

6.2.1 Unital semilattices

Let us now investigate what happens if we allow the empty set as an element of the power dcpo. For any dcpo D we denote the resulting poset by $\dot{\mathsf{P}}^\flat_\mathsf{D} D$. Obviously, $\dot{\mathsf{P}}^\flat_\mathsf{D} D$ is a complete lattice.

Theorem 6.3 *The free complete $\vee$-semilattice in* **DCPO** *is given by* $\dot{\mathsf{P}}^\flat_\mathsf{D}$.

Proof: Let D be a dcpo and L a complete $\vee$-semilattice. We can embed D into $\dot{\mathsf{P}}^\flat_\mathsf{D} D$ via $\downarrow_D$. Let U denote the forgetful functor from the category of complete $\vee$-semilattices to **DCPO**. For every $f \colon D \longrightarrow UL$ in **DCPO** and every $A \in \dot{\mathsf{P}}^\flat_\mathsf{D} D$ we set $\tilde{f}(A) := \bigvee f(A)$. We have to prove that $\tilde{f}$ is the unique morphism of complete $\vee$-semilattices satisfying $U\tilde{f} \circ \downarrow_D = f$. Since $\tilde{f}$ extends $\hat{f}$ as defined in the proof of the previous theorem it remains to show that $\hat{f}$ maps the least element of $\dot{\mathsf{P}}^\flat_\mathsf{D} D$, that is the empty set, to the least element $\perp_L$ of L. We calculate $\tilde{f}(\emptyset) = \bigvee f(\emptyset) = \bigvee \emptyset = \perp_L$. ■

Alternatively, for any dcpo D we can understand $\dot{\mathsf{P}}^\flat_\mathsf{D}$ to arise from adjoining a least element to $\mathsf{P}^\flat_\mathsf{D} D$. Hence $\mathsf{P}^\flat_\mathsf{D}$ gives the free unital inflationary semilattice for **DCPO**.

6.2.2 Domains

In this section we want to show that the category of domains **Dom** is closed with respect to the Hoare power construction for dcpo's. This allows us to conclude that $\mathsf{P}^\flat_\mathsf{D}$ also gives the free inflationary semilattice for **Dom**. For that we have to prove that $\mathsf{P}^\flat_\mathsf{D} D$ is a continuous $\vee$-semilattice for every domain D. In order to do this we need the following well-known lemma.

Lemma 6.4 *The Scott closure of an arbitrary subset S of a domain D is*

$$\{\bigvee^{\uparrow} S' \mid S' \text{ directed subset of } \downarrow S\}.$$

Proof: As every Scott-closed subset is lower closed and also closed with respect to directed joins, we must have $\mathsf{cl}\, S \supseteq \{\bigvee^{\uparrow} S' \mid S' \text{ directed subset of } \downarrow S\} =: \overline{S}$. Since S is a subset of $\overline{S}$ it remains to prove that $\overline{S}$ is Scott-closed. First we will show that $\overline{S}$ is lower closed. If $d \in \downarrow \overline{S}$ then there is a directed subset S' of $\downarrow S$ with $d \leq \bigvee^{\uparrow} S'$. Hence for all $d' \ll d$ there is $s' \in S'$ with $d' \leq s'$. Therefore $\twoheaddownarrow\{d\}$ is a directed subset of $\downarrow S$. By definition of $\overline{S}$, this implies $d = \bigvee^{\uparrow} \twoheaddownarrow\{d\} \in \overline{S}$. It remains to show that $\overline{S}$ is closed with respect to directed joins. If T is a directed subset of $\overline{S}$ then $\bigcup^{\uparrow}\{\twoheaddownarrow\{t\} \mid t \in T\}$ is also directed and has the same join as T. If we can show that $\bigcup^{\uparrow}\{\twoheaddownarrow\{t\} \mid t \in T\}$ is a subset of $\downarrow S$ then we are done. Let $d \ll t \in T$. There is a directed subset S' of $\downarrow S$ such that $t \leq \bigvee^{\uparrow} S'$. Hence there is $s' \in S$ such that $d \leq s'$ which proves $d \in \downarrow S$. ∎

The following proposition also follows from Corollary 5.10 in [Law79].

Proposition 6.5 *If D is a domain so is $\mathsf{P}^\flat_\mathsf{D} D$.*

Proof: For $d \ll d'$ in D we have $\downarrow\{d\} \ll \downarrow\{d'\}$ in $\mathsf{P}^\flat_\mathsf{D}$: Let $\downarrow\{d'\} \subseteq \mathsf{cl} \bigcup^{\uparrow}_{i\in I} A_i$ for a directed family $(A_i)_{i\in I}$ of non-empty Scott-closed subsets of D. By Lemma 6.4,

$$\mathsf{cl} \bigcup_{i\in I} A_i = \{\bigvee^{\uparrow} S \mid S \text{ directed subset of } \downarrow \bigcup_{i\in I} A_i\}.$$

Hence there is a directed subset S of $\downarrow \bigcup_{i\in I} A_i = \bigcup_{i\in I} A_i$ such that $d' \leq \bigvee^{\uparrow} S$. As d is way below d' there is $s \in S$ with $d \leq s$. Therefore $\downarrow\{d\} \subseteq \downarrow A_i = A_i$ for some $i \in I$.
We have to show that for any $A \in \mathsf{P}^{\flat}_{\mathsf{D}}D$ the set of elements way below A is directed with join A. As $\mathsf{P}^{\flat}_{\mathsf{D}}D$ has binary joins, it is enough to show that $\bigvee \twoheaddownarrow\{A\} = A$ holds in $\mathsf{P}^{\flat}_{\mathsf{D}}D$. By the above, if $d \ll a \in A$ then $\downarrow\{d\} \ll A$ in $\mathsf{P}^{\flat}_{\mathsf{D}}D$. As

$$A = \mathsf{cl} \bigcup^{\uparrow} \{\downarrow\{d_1\} \cup \ldots \cup \downarrow\{d_n\} \mid \exists a_1, \ldots, a_n \in A. \forall i \in \{1, \ldots, n\} \,.\, d_i \ll a_i\}$$

we are done. ∎

From the results of Section 6.2 and the last proposition we immediately get the following result.

Theorem 6.6 *The free inflationary semilattice for* **Dom** *is given by* $\mathsf{P}^{\flat}_{\mathsf{D}}$. ∎

6.3 Topological spaces

The definition of the Hoare power dcpo in the last section suggests a definition for a Hoare power space, namely the set of non-empty closed subsets of a T_0-space. However, it is not quite obvious which topology on this set we should choose. Hence we will split the construction of the Hoare power space into two steps. For a T_0-space X let $\mathsf{A}X$ be the set of non-empty closed subsets of X ordered by inclusion. This yields an up-complete $\vee$-semilattice for every T_0-space X. We can extend A to a functor from **Top** to $\mathbf{DCPO}^{\vee}$ in the way suggested from the Hoare power construction for dcpo's. For a continuous function $f\colon X \longrightarrow Y$ between T_0-spaces let $\mathsf{A}f\colon \mathsf{A}X \longrightarrow \mathsf{A}Y$ be defined by $\mathsf{A}f(A) := \mathsf{cl}\, f(A)$ for every non-empty closed subset A of X. The proof that $\mathsf{A}f$ is a morphism in $\mathbf{DCPO}^{\vee}$ and that A indeed is a functor is exactly the same as for $\mathsf{P}^{\flat}_{\mathsf{D}}\colon \mathbf{DCPO} \longrightarrow \mathbf{DCPO}^{\vee}$.

Now for the topology on $\mathsf{A}X$. We want the function $x \longmapsto \mathsf{cl}\{x\}$ to be continuous in order to get a natural transformation as in the preceding section. The idea is to use the topology on X and thus demand that all the sets $\{A \in \mathsf{A}X \mid A \subseteq B\}$ be closed for every closed subset B of X. These sets are those of the form $\downarrow\{B\}$ where $B \in \mathsf{A}X$ if we exclude $B = \emptyset$ for a moment. Hence we equip $\mathsf{A}X$ with the weak topology which ensures that the closed sets in $\mathsf{A}X$ are generated by the sets of the form $\{A \in \mathsf{A}X \mid A \subseteq B\}$ where B is a closed subset of X.

For any up-complete $\vee$-semilattice L let $\mathsf{T}_\omega L$ be L with the weak topology. If we set $\mathsf{T}_\omega f(l) := f(l)$ for any $\mathbf{DCPO}^\vee$-morphism $f\colon L \longrightarrow M$ we obtain a continuous function $\mathsf{T}_\omega f\colon \mathsf{T}_\omega L \longrightarrow \mathsf{T}_\omega M$: We have $(\mathsf{T}_\omega f)^{-1}(\downarrow m) = \{l \in L \mid f(l) \leq m\}$ for all $m \in M$ and this set is either empty or equal to $\downarrow\{\bigvee\{l \in L \mid f(l) \leq m\}\}$ and therefore weakly closed in L. Obviously, T_ω:$\mathbf{DCPO}^\vee \longrightarrow \mathbf{Top}$ is a functor. We can show that $\mathsf{cl}_X\colon x \longmapsto \mathsf{cl}\{x\}$ is a continuous function from X to $\mathsf{T}_\omega\mathsf{A}X$ for every T_0-space X. If A is an element of $\mathsf{T}_\omega\mathsf{A}X$ then

$$\mathsf{cl}_X^{-1}(\downarrow\{A\}) = \{x \in X \mid \mathsf{cl}\{x\} \subseteq A\} = \{x \in X \mid x \in A\} = A.$$

Note that in general cl_X is not continuous with respect to the Scott topology on $\mathsf{A}X$.

Proposition 6.7 *The functor* $\mathsf{A}\colon \mathbf{Top} \longrightarrow \mathbf{DCPO}^\vee$ *is left adjoint to* T_ω.

Proof: Exactly as for Theorem 6.2. ■

We therefore formulate the following definition which is due to M. Smyth [Smy83].

Definition. For any T_0-space X we define the *Hoare power space* $\mathsf{P}_\mathsf{T}^\flat X$ to be the up-complete $\vee$-semilattice of non-empty closed subsets of X with the weak topology. The *Hoare power space functor* $\mathsf{P}_\mathsf{T}^\flat$ is given by $\mathsf{T}_\omega\mathsf{A}$.

The co-unit for the adjunction $\mathsf{A} \dashv \mathsf{T}_\omega$ is $\bigvee_L\colon \mathsf{A}\mathsf{T}_\omega L \longrightarrow L$. It is our aim to characterize the algebras for the monad $(\mathsf{P}_\mathsf{T}^\flat, \mathsf{cl}, \mathsf{T}_\omega\bigvee\mathsf{A})$ in the remainder of this section. Note that the above adjunction is not monadic since, for example, every continuous $\vee$-semilattice equipped with its Scott topology is an algebra for the monad. Hence not every $\mathsf{P}_\mathsf{T}^\flat$-algebra is an up-complete $\vee$-semilattice with its weak topology.

First of all note that for any T_0-space X we can apply Lemma 4.3 (i) since we can view X as a poset. Thus we know that if a structure map $\xi\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow X$ exists at all then it has to be the lower adjoint for η_X with respect to the specialization order. Therefore $\xi(A) = \bigvee A$ must hold for every non-empty closed subset A of X. Consequently, for every T_0-space X there is at most one structure map for X that we will refer to as ξ_X from now on. By Lemma 4.6 we also know that if $A \longmapsto \bigvee A\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow X$ is a continuous function then it is indeed the structure map for X.

This yields a preliminary characterization of $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebras: A T_0-space X has a structure map iff $A \longmapsto \bigvee A\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow X$ is a continuous function. We can do more than that since we can describe the $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebras without referring to $\mathsf{P}_{\mathsf{T}}^{\flat}X$ at all. For that purpose, note that every space of the form $\mathsf{P}_{\mathsf{T}}^{\flat}X$ is an inflationary topological semilattice. The semilattice structure on $\mathsf{P}_{\mathsf{T}}^{\flat}X$ is given by binary union $\cup\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \times \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow \mathsf{P}_{\mathsf{T}}^{\flat}X$. As $\cup$ is the binary join with respect to the specialization order on $\mathsf{P}_{\mathsf{T}}^{\flat}X$, that is subset inclusion, it is the lower adjoint for the diagonal $\Delta_{\mathsf{P}_{\mathsf{T}}^{\flat}X}\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow \mathsf{P}_{\mathsf{T}}^{\flat}X \times \mathsf{P}_{\mathsf{T}}^{\flat}X$. In order to show that $\cup\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \times \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow \mathsf{P}_{\mathsf{T}}^{\flat}X$ is continuous we calculate

$$\cup^{-1}\Big(\downarrow\{A\}\Big) = \{\langle B, B'\rangle \in \mathsf{P}_{\mathsf{T}}^{\flat}X \times \mathsf{P}_{\mathsf{T}}^{\flat}X \mid B \cup B' \subseteq A\} = \downarrow\{A\} \times \downarrow\{A\}$$

for every non-empty closed subset A of X. Consequently, every $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebra is an inflationary semilattice by Proposition 5.6. Furthermore, every T_0-space of the form $\mathsf{P}_{\mathsf{T}}^{\flat}X$ is sober because of Proposition 1.7. Hence, by Proposition 1.9, we know that every $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebra, as a retract of a sober space, is sober as well. In order to show that every sober inflationary topological semilattice yields a $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebra we need the following result.

Lemma 6.8 *The function $\xi_X\colon A \longmapsto \bigvee A\colon \mathsf{P}_{\mathsf{T}}^{\flat}X \longrightarrow X$ is continuous for every sober inflationary topological semilattice X.*

Proof: As X is an inflationary topological semilattice it has binary joins with respect to the specialization order. By sobriety, it also has directed joins which proves that ξ_X

is well-defined. Now for continuity. Let B be a closed subset of X. We have to show that $\xi_X^{-1}(B) = \{A \in \mathsf{P}_\mathsf{T}^\flat X \mid \bigvee A \in B\}$ is closed in $\mathsf{P}_\mathsf{T}^\flat X$. Since X is sober, B is Scott-closed implying that $\bigvee A = \bigvee^\uparrow \{a_1 \vee \cdots \vee a_n \mid a_1, \ldots, a_n \in A\} \in B$ for some $A \in \mathsf{P}_\mathsf{T}^\flat X$ iff $a_1 \vee \cdots \vee a_n \in B$ for all $a_1, \ldots, a_n \in A$. We show that $\xi_X^{-1}(B)$ is weakly closed in the up-complete $\vee$-semilattice $\mathsf{A}X$: Let $A \in \mathsf{P}_\mathsf{T}^\flat X \setminus \xi_X^{-1}(B)$. By the above, there are $a_1, \ldots, a_n \in A$ such that $a_1 \vee \cdots \vee a_n \in X \setminus B$. By continuity of $\vee_n : X \times \cdots \times X \longrightarrow X$ we can find open subsets $O_1, \ldots, O_n$ of X such that $\langle a_1, \ldots, a_n \rangle \in O_1 \times \cdots \times O_n \subseteq \vee_n^{-1}(X \setminus B)$. Consequently,

$$O_1 \cap \cdots \cap O_n = \vee_n(O_1 \times \cdots \times O_n) \subseteq X \setminus B.$$

Hence $B \subseteq (X \setminus O_1) \cup \cdots \cup (X \setminus O_n)$. We prove that $\downarrow\{X \setminus O_1\} \cup \cdots \cup \downarrow\{X \setminus O_n\}$ is a (weakly closed) superset of $\xi_X^{-1}(B)$ in $\mathsf{A}X$ that does not contain A: If A' is an element of $\mathsf{P}_\mathsf{T}^\flat X$ such that $\bigvee A' \in B$ then $\bigvee A'$ is an element of one of the $X \setminus O_i$. Therefore, $A' \subseteq X \setminus O_i$ for some $i \in \{1, \ldots, n\}$ and thus $A' \in \downarrow\{X \setminus O_1\} \cup \ldots \cup \downarrow\{X \setminus O_n\}$. This proves $\xi_X^{-1}(B) \subseteq \downarrow\{X \setminus O_1\} \cup \ldots \cup \downarrow\{X \setminus O_n\}$. On the other hand, we know that for every $i \in \{1, \ldots, n\}$ there is $a_i \in A \cap O_i$. So A is not a subset of any of the $X \setminus O_i$. Consequently, $A \in \mathsf{K}_\mathsf{T} X \setminus (\downarrow\{X \setminus O_1\} \cup \ldots \cup \downarrow\{X \setminus O_n\})$. ∎

Let us now investigate the morphisms of $\mathsf{P}_\mathsf{T}^\flat$-algebras. We claim that $f : X \longrightarrow Y$, where X and Y are inflationary topological semilattices with sober topologies, is a morphism of $\mathsf{P}_\mathsf{T}^\flat$-algebras iff it is a morphism of topological semilattices. Because $\bigvee \mathsf{P}_\mathsf{T}^\flat f(A) = \bigvee \mathsf{cl}\, f(A) = \bigvee f(A)$ for all $A \in \mathsf{P}_\mathsf{T}^\flat X$ by Lemma 1.5, we may apply Lemma 4.3 (ii). Hence f is a morphism of $\mathsf{P}_\mathsf{T}^\flat$-algebras iff it preserves joins of closed sets. By Corollary 1.6 (ii) this is true iff f preserves non-empty joins. As X and Y are sober spaces, any continuous function between them preserves directed joins anyway. Hence f is a morphism of $\mathsf{P}_\mathsf{T}^\flat$-algebras iff it preserves binary joins iff it is a morphism of inflationary topological semilattices. Therefore we have shown the following result.

Theorem 6.9 *The category of $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebras is the subcategory of inflationary topological semilattices in the category of sober spaces.* ∎

Consequently, $\mathsf{P}_{\mathsf{T}}^{\flat}$ gives the free inflationary semilattice for the category of sober spaces. Alternatively, we can understand the category of $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebras as the category of up-complete $\vee$-semilattices with an order-consistent topology that is weaker than the Scott topology. The morphisms in this setting are continuous functions that preserve non-empty joins.

6.3.1 Unital semilattices

If we allow the empty set to be an element of the Hoare power space of some T_0-space X then we obtain a slightly different power space functor, say $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$. Obviously, $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}X$ is a complete lattice with respect to the specialization order for every T_0-space X. Therefore, $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}X$ is a unital inflationary semilattice in **Top**. Furthermore, it is still a sober space. We can make $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$ an endofunctor on **Top** if we use the same definition as for $\mathsf{P}_{\mathsf{T}}^{\flat}$, that is if we set $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}f(A) := \mathsf{cl}\, f(A)$ for every continuous function $f\colon X \longrightarrow Y$. Put differently, $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}f$ is the unique extension of $\mathsf{P}_{\mathsf{T}}^{\flat}f$ making it a morphism of unital semilattices in **Top**. As $\mathsf{P}_{\mathsf{T}}^{\flat}X$ can be embedded into $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}X$ we can re-use the definition of the natural transformation cl from above. Also, $\mathsf{T}_\omega \bigvee \mathsf{A}$ from above can be extended so that we get a monad, say $(\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}, \eta, \mu)$. Every $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$-algebra is a unital inflationary topological semilattice with sober topology. It can be shown in the same way as the last theorem that every such topological semilattice is a $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$-algebra and that morphisms of such semilattices are exactly the morphisms of $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$-algebras.

Theorem 6.10 *The category of $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$-algebras is the subcategory of unital inflationary semilattices in the category of sober spaces.* ∎

This category is isomorphic to the category whose objects are complete lattices with an order consistent topology that is weaker than the Scott topology and whose morphisms are continuous functions that preserve all joins.

6.3.2 Locally compact spaces

We have seen above that, in some way, continuous dcpo's correspond to locally compact spaces. Let us investigate the restriction of $\mathsf{P}^\flat_\mathsf{T}$ to the category **LCompT** of locally compact T_0-spaces. We can prove that **LCompT** is closed with respect to $\mathsf{P}^\flat_\mathsf{T}$. Let X be a locally compact T_0-space. In order to prove that in this case $\mathsf{P}^\flat_\mathsf{T}X$ is locally compact as well we have to study the open sets of $\mathsf{P}^\flat_\mathsf{T}X$. As the closed sets of $\mathsf{P}^\flat_\mathsf{T}X$ are generated by all the sets $\downarrow\{B\} = \{A \in \mathsf{P}^\flat_\mathsf{T}X \mid A \subseteq B\}$, where B is a non-empty closed subset of X, the open sets in $\mathsf{P}^\flat_\mathsf{T}X$ are generated by the sets of the form $\mathsf{P}^\flat_\mathsf{T}X \setminus \downarrow\{B\} = \{A \in \mathsf{P}^\flat_\mathsf{T}X \mid A \cap (X \setminus B) \neq \emptyset\}$. Hence the topology of $\mathsf{P}^\flat_\mathsf{T}X$ is generated by the sets $\Diamond U := \{A \in \mathsf{P}^\flat_\mathsf{T}X \mid A \cap U \neq \emptyset\}$ where U is an open subset of X. Note that $\mathsf{P}^\flat_\mathsf{T}X$ need not be continuous even if X is locally compact.

Proposition 6.11 *If X is a locally compact space then so is $\mathsf{P}^\flat_\mathsf{T}X$.*

Proof: Let $A \in \Diamond U_1 \cap \cdots \cap \Diamond U_n$ where the U_i are open subsets of X. In order to prove $\mathsf{P}^\flat_\mathsf{T}X$ locally compact we have to find a compact neighbourhood of A in $\Diamond U_1 \cap \cdots \cap \Diamond U_n$. As $A \in \Diamond U_1 \cap \cdots \cap \Diamond U_n$, we can find some $x_i \in A \cap U_i$ for all $i \in \{1, \ldots, n\}$. Also for all $i \in \{1, \ldots, n\}$ there is an open subset V_i of X and a compact subset K_i of X such that $x \in V_i \subseteq K_i \subseteq U_i$ by local compactness of X. Let $\mathcal{K} := \{B \in \mathsf{P}^\flat_\mathsf{T}X \mid \forall i \in \{1, \ldots, n\} \,.\, B \cap K_i \neq \emptyset\}$. We will show that $\mathcal{K}$ is compact in $\mathsf{P}^\flat_\mathsf{T}X$ and that $A \in \Diamond V_1 \cap \cdots \cap \Diamond V_n \subseteq \mathcal{K} \subseteq \Diamond U_1 \cap \cdots \cap \Diamond U_n$. By choice of the V_i we have $A \in \Diamond V_1 \cap \cdots \cap \Diamond V_n \subseteq \mathcal{K}$. If $B \in \mathcal{K}$ then B intersects all K_i and hence all U_i. Consequently, $\mathcal{K} \subseteq \Diamond U_1 \cap \cdots \cap \Diamond U_n$. It remains to show compactness of $\mathcal{K}$. Using Alexander's Subbase Lemma, it is enough to find a finite subcover in any open cover $\mathcal{O}$ of $\mathcal{K}$ where the elements of $\mathcal{O}$ are of the form $\Diamond O$ for some open subset O of X. So assume we are given such a cover $\mathcal{O}$ for $\mathcal{K}$. As $\mathsf{cl}\{y_1, \ldots, y_n\}$ is in $\mathcal{K}$ for any choice of $y_1 \in K_1, \ldots, y_n \in K_n$ we have $K_1 \cup \cdots \cup K_n \subseteq \bigcup_{\Diamond O \in \mathcal{O}} O$. By compactness of

$K_1 \cup \cdots \cup K_n$ there are $\Diamond O_1, \ldots, \Diamond O_m \in \mathcal{O}$ such that $K_1 \cup \ldots \cup K_n \subseteq O_1 \cup \cdots \cup O_m$. Obviously, $\mathcal{K} \subseteq \Diamond O_1 \cup \cdots \cup \Diamond O_m$. ■

Theorem 6.12 *The functor $\mathsf{P}_{\mathsf{T}}^{\flat}$ gives the free sober inflationary semilattice for the category of locally compact spaces* **LCompT**. ■

Remark. These considerations also apply to $\dot{\mathsf{P}}_{\mathsf{T}}^{\flat}$.

6.3.3 Hoare power space versus Hoare power dcpo

What is the connection between the Hoare power space and the Hoare power dcpo? Any dcpo D gives rise to a T_0-space $\mathsf{T}_\sigma D$ if equipped with its Scott topology. Therefore, we get a Hoare power space and a Hoare power dcpo for D. By definition, the two have the same points and if we regard the Hoare power space $\mathsf{P}_{\mathsf{T}}^{\flat}\mathsf{T}_\sigma D$ as a poset via the specialization order, then it is isomorphic to $\mathsf{P}_{\mathsf{D}}^{\flat} D$. When studying universal properties, however, we considered Scott-continuous functions for $\mathsf{P}_{\mathsf{D}}^{\flat} D$ while we equipped $\mathsf{P}_{\mathsf{T}}^{\flat}\mathsf{T}_\sigma D$ with the weak topology.

Nevertheless, for domains the two coincide: We already know that the weak topology is coarser than the Scott topology on $\mathsf{P}_{\mathsf{T}}^{\flat}\mathsf{T}_\sigma D$. So let $\mathcal{U}$ be a Scott-open subset of $\mathsf{P}_{\mathsf{T}}^{\flat}\mathsf{T}_\sigma D$ where D is a domain. Further, let A be an element of $\mathcal{U}$. By continuity of D we have that $A = \mathsf{cl} \bigcup^{\uparrow}\{\downarrow\{d_1\} \cup \cdots \cup \downarrow\{d_n\} \mid \exists a_1, \ldots, a_n \in A. \forall i \in \{1, \ldots, n\} \,.\, d_i \ll a_i\}$. By Scott-openness of $\mathcal{U}$ there are $a_1, \ldots, a_n \in A$ such that for any $i \in \{1, \ldots, n\}$ there is $d_i \ll a_i$ with $\downarrow\{d_1\} \cup \cdots \cup \downarrow\{d_n\} \in \mathcal{U}$. We will show that $\Diamond \Uparrow\{d_1\} \cap \cdots \cap \Diamond \Uparrow\{d_n\}$ is a neighbourhood of A in $\mathcal{U}$. By choice of the d_i we know that $a_i \in A \cap \Uparrow\{d_i\}$ for all $i \in \{1, \ldots, n\}$. Hence $A \in \Diamond \Uparrow\{d_1\} \cap \cdots \cap \Diamond \Uparrow\{d_n\}$. If $B \in \Diamond \Uparrow\{d_1\} \cap \cdots \cap \Diamond \Uparrow\{d_n\}$, on the other hand, then $B \supseteq \downarrow\{d_1\} \cup \cdots \cup \downarrow\{d_n\} \in \mathcal{U}$ implying $B \in \mathcal{U}$. Because $\Diamond \Uparrow\{d_1\} \cap \cdots \cap \Diamond \Uparrow\{d_n\}$ is weakly open in $\mathsf{A}D$ this proves our claim.

Proposition 6.13 *The functors $\mathsf{T}_\sigma \mathsf{P}_{\mathsf{D}}^{\flat}$ and $\mathsf{P}_{\mathsf{T}}^{\flat}\mathsf{T}_\sigma$ coincide on* **Dom**. ■

6.4 Locales

We motivated the introduction of the Hoare power domain for (algebraic) domains by considering semantics for non-deterministic programming languages. A similar approach works for locales. The general idea behind the use of locales in semantics is to consider the elements of a locale A to be properties that some program p might have. If a program p has the property $a \in A$ we write $\mathsf{p} \models a$ and say that p satisfies a. We think of the property $a \wedge b$, where a and b are elements of A, as the property of satisfying both, a and b. Hence we want $\mathsf{p} \models a \wedge b$ to hold iff both, $\mathsf{p} \models a$ and $\mathsf{p} \models b$. Similarly, p has the property given by $\bigvee S$, where S is a subset of A, iff p satisfies (at least) one of the properties from S. This leads to $\mathsf{p} \models \bigvee S$ iff $\mathsf{p} \models s$ for some $s \in S$. If we think of the greatest element 1_A of A to be a property that trivially is satisfied by any program, then $\mathsf{p} \models 1_A$ for all programs p. Analogously we can understand the least element 0_A of A to be a property that no program ever satisfies. Hence every program p gives rise to a point p for the locale A if we set $p'(a) := 1$ iff $\mathsf{p} \models a$.

As we did for domains, we assume that we already know how to model the deterministic part of our language. Suppose that p is a non-deterministic program and that we know how to model the properties of the branches of p via the locale A. The elements of the locale A should give rise to new properties that p might have. For every element a of A we will denote this new property by $\diamond a$. This yields a new locale $\mathsf{P}_{\mathsf{L}}^{\flat}A$ generated by the $\diamond a$. But we want to model our view of non-determinism with this locale and therefore require certain relations to be satisfied in $\mathsf{P}_{\mathsf{L}}^{\flat}A$. We introduced angelic non-determinism by stating that a non-deterministic program satisfies a property iff one of its branches does. So we define $\mathsf{p} \models \diamond a$ to mean that at least one of the branches of p satisfies a. When does $\mathsf{p} \models \bigvee\limits_{s \in S} \diamond s$ hold for some $S \subseteq A$? This is the case iff there is some $s \in S$ with $\mathsf{p} \models \diamond s$. It is equivalent to the existence of a branch of p satisfying s, that is the existence of a branch of p satisfying $\bigvee S$. Hence we want $\bigvee\limits_{s \in S} \diamond s = \diamond \bigvee S$ to hold in $\mathsf{P}_{\mathsf{L}}^{\flat}A$ for all $S \subseteq A$. We thought of 1_A as being satisfied by every deterministic program modelled by A. Hence $\mathsf{p} \models \diamond 1_A$ holds for any non-deterministic program p that can be modelled using $\mathrm{P}_{\mathrm{L}}^{\flat}A$. So it seems superfluous to have another element $1_{\mathrm{P}_{\mathrm{L}}^{\flat}A}$ that is also satisfied by any such program p. Therefore we require $\diamond 1_A = 1_{\mathrm{P}_{\mathrm{L}}^{\flat}A}$. These are the only relations we want to hold in $\mathrm{P}_{\mathrm{L}}^{\flat}A$ since it does not make sense to demand similar equations for finite meets: A program p is to satisfy $\diamond(a \wedge b)$ iff one of its branches satisfies both, a and b, while $\mathsf{p} \models \diamond a \wedge \diamond b$ iff there are two (possibly different) branches of p, one satisfying a and one satisfying b.

In Section 1.3 we described the construction of the free frame via generators and relations, so we can define a locale as follows.

Definition. The Hoare power locale $\mathrm{P}_{\mathrm{L}}^{\flat}A$ for a locale A is the frame presented by

$$Fr\left\langle \diamond a : a \in A \;\mid\; \diamond 1_A = 1_{\mathrm{P}_{\mathrm{L}}^{\flat}A}\,,\; \bigvee_{s\in S} \diamond s = \diamond \bigvee S \;(S \subseteq A)\right\rangle.$$

The first power construction for locales, the so-called Vietoris or Plotkin power construction, is due to P. Johnstone [Joh82b]. The connection between power locales and power domains as well as their application in denotational semantics may be traced to G. Winskel and E. Robinson. In [Win85], Winskel introduces two modal operators, $\diamond$ and $\Box$, and shows how the three power domain constructions can be built up from assertions using these operators. Building on this work, Robinson gives the above definition of a Hoare power locale in [Rob88].

Let us once more consider the special case where the only non-deterministic command is $\underline{\mathsf{or}}$. By the above philosophy we want $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ to satisfy $\diamond a$ iff either p satisfies $\diamond a$ or q does. But this can be expressed using products in **Loc**. If we set $\langle p, q\rangle'(a \otimes b) := 1_{\mathit{2}}$ iff $\mathsf{p} \models a$ and $\mathsf{q} \models b$ then we obtain a point for the locale $\mathrm{P}_{\mathrm{L}}^{\flat}A \otimes \mathrm{P}_{\mathrm{L}}^{\flat}A$. Hence $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q} \models \diamond a$ iff either $\langle p, q\rangle \models \diamond a \otimes 1_{\mathrm{P}_{\mathrm{L}}^{\flat}A}$ or $\langle p, q\rangle \models 1_{\mathrm{P}_{\mathrm{L}}^{\flat}A} \otimes \diamond a$ which is the case iff $\langle p, q\rangle \models (\diamond a \otimes 1_{\mathrm{P}_{\mathrm{L}}^{\flat}A}) \vee (1_{\mathrm{P}_{\mathrm{L}}^{\flat}A} \otimes \diamond a)$. This gives us a connection between $\mathrm{P}_{\mathrm{L}}^{\flat}A$ and its product $\mathrm{P}_{\mathrm{L}}^{\flat}A \otimes \mathrm{P}_{\mathrm{L}}^{\flat}A$. If p and q are programs then we can map the corresponding point $\langle p, q\rangle$ to the program $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ which we see as a point of $\mathrm{P}_{\mathrm{L}}^{\flat}A$. In this case, $(\diamond a \otimes 1_A) \vee (1_A \otimes \diamond a)$ is the weakest property such that for every $\langle p, q\rangle$, where p and q are programs, the program $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ must satisfy $\diamond a$. By our considerations in Section 2.1.1 we would expect $\diamond a \longmapsto 1_A \otimes \diamond a \vee \diamond a \otimes 1_A$ to define a frame homomorphism from $\mathbb{P}_{\mathsf{L}}^{\flat} A$ to $\mathbb{P}_{\mathsf{L}}^{\flat} A \otimes \mathbb{P}_{\mathsf{L}}^{\flat} A$ and we shall see that this is indeed the case for the above definition of $\mathbb{P}_{\mathsf{L}}^{\flat} A$. Furthermore, we can axiomatize the properties of this frame homomorphism such that we can understand $\mathbb{P}_{\mathsf{L}}^{\flat} A$ to be the free frame for which such a morphism exists.

There is another way to motivate this definition for the Hoare power locale, namely from purely mathematical considerations. From the Hoare power space one would like to pass to a Hoare power locale. One can do so by finding connections between the locales ΩX and $\Omega\,\mathbb{P}_{\mathsf{T}}^{\flat} X$ where X is a T_0-space. In Section 6.3 we obtained a subbasis for $\Omega\,\mathbb{P}_{\mathsf{T}}^{\flat} X$, namely the sets of the form $\Diamond U = \{A \in \mathbb{P}_{\mathsf{T}}^{\flat} X \mid A \cap U \neq \emptyset\}$ where U is an open subset of X. These sets have the following property: Given an arbitrary family $\mathcal{U}$ of open subsets of X we have

$$\bigcup_{U \in \mathcal{U}} \Diamond U = \bigcup_{U \in \mathcal{U}} \{A \in \mathbb{P}_{\mathsf{T}}^{\flat} X \mid A \cap U \neq \emptyset\} = \{A \in \mathbb{P}_{\mathsf{T}}^{\flat} X \mid \exists U \in \mathcal{U} \,.\, A \cap U \neq \emptyset\} = \Diamond \bigcup \mathcal{U}.$$

Furthermore, $\Diamond X = \{A \in \mathbb{P}_{\mathsf{T}}^{\flat} X \mid A \cap X \neq \emptyset\} = \mathbb{P}_{\mathsf{T}}^{\flat} X$ which means that the greatest element in ΩX, namely X, generates the greatest element in $\Omega\,\mathbb{P}_{\mathsf{T}}^{\flat} X$, that is $\mathbb{P}_{\mathsf{T}}^{\flat} X$.

For $U, U' \in \Omega X$ we can only prove that $\Diamond(U \cap U') \subseteq \Diamond U \cap \Diamond U'$ and it is obvious that in general these two sets are not equal. Hence the above equations are the only relations we can show to hold between the sets of the form $\Diamond U$ where $U \in \Omega X$. This suggests the same definition for a Hoare power locale as the one we introduced above: For every element a of some locale A, take a generator $\diamond a$. Construct the free frame for these generators subject to the relations $\diamond 1_A = 1_{\mathbb{P}_{\mathsf{L}}^{\flat} A}$ and $\bigvee_{s \in S} \diamond s = \diamond \bigvee S$ for all $S \subseteq A$.

6.4.1 The monad

As for dcpo's and T_0-spaces, we can extend the Hoare power construction for locales to a functor. Let $f \colon A \longrightarrow B$ be a continuous function between locales. In order to define a continuous function $\mathbb{P}_\mathsf{L}^\flat f \colon \mathbb{P}_\mathsf{L}^\flat A \longrightarrow \mathbb{P}_\mathsf{L}^\flat B$ we have to define a function $(\mathbb{P}_\mathsf{L}^\flat f)'$ on the generators $\diamond b$ of $\mathbb{P}_\mathsf{L}^\flat B$ that preserves the relations from the above definition. If we set $(\mathbb{P}_\mathsf{L}^\flat f)'(\diamond b) := \diamond f'(b)$ then obviously $(\mathbb{P}_\mathsf{L}^\flat f)'(\diamond 1_B) = \diamond f'(1_B) = \diamond 1_A = 1_{\mathbb{P}_\mathsf{L}^\flat A}$. Furthermore, for any subset T of B we have

$$(\mathbb{P}_\mathsf{L}^\flat f)'(\diamond \bigvee T) = \diamond f'(\bigvee T) = \diamond \bigvee_{t \in T} f'(t) = \bigvee_{t \in T} \diamond f'(t) = \bigvee_{t \in T} (\mathbb{P}_\mathsf{L}^\flat f)'(\diamond t).$$

Obviously, $\mathbb{P}_\mathsf{L}^\flat$ preserves identities. To see that it also preserves composition, note that if $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ are continuous functions then

$$\left(\mathbb{P}_\mathsf{L}^\flat (g \circ f)\right)'(\diamond c) = \diamond (g \circ f)'(c) = \diamond f'(g'(c)) = (\mathbb{P}_\mathsf{L}^\flat f)'(\diamond g'(c)) = (\mathbb{P}_\mathsf{L}^\flat f)'\left((\mathbb{P}_\mathsf{L}^\flat g)'(\diamond c)\right)$$

for all $c \in C$. Hence $(\mathbb{P}_\mathsf{L}^\flat (g \circ f))'$ and $(\mathbb{P}_\mathsf{L}^\flat g \circ \mathbb{P}_\mathsf{L}^\flat f)'$ are frame homomorphisms that agree on the generators of $\mathbb{P}_\mathsf{L}^\flat C$ and thus must coincide.

The functor $\mathbb{P}_\mathsf{L}^\flat$ is part of a monad on **Loc** we are going to define now. For that we require a natural transformation $\eta \colon id_{\mathbf{Loc}} \longrightarrow \mathbb{P}_\mathsf{L}^\flat$. So we need a continuous function $\eta_A \colon A \longrightarrow \mathbb{P}_\mathsf{L}^\flat A$ for every locale A. Again, we will give a function on the generators of $\mathbb{P}_\mathsf{L}^\flat A$. Set $\eta'_A(\diamond a) := a$ for any $a \in A$. It is not difficult to check that η'_A respects the relations from the above definition; therefore we omit the proof. In order to see that this assignment makes η the desired natural transformation, let $f \colon A \longrightarrow B$ be a morphism in **Loc**. For all $b \in B$ we have

$$(\eta_B \circ f)'(\diamond b) = f'(\eta'_B(\diamond b)) = f'(b) = \eta'_A(\diamond f'(b)) = \eta'_A\left((\mathbb{P}_\mathsf{L}^\flat f)'(\diamond b)\right) = (\mathbb{P}_\mathsf{L}^\flat f \circ \eta_A)'(\diamond b)$$

which proves the commutativity of the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathbb{P}_\mathsf{L}^\flat A \arrow[d, "\mathbb{P}_\mathsf{L}^\flat f"] \\
B \arrow[r, "\eta_B"] & \mathbb{P}_\mathsf{L}^\flat B
\end{tikzcd}$$

Now for the second natural transformation needed, $\mu \colon (\mathbb{P}_\mathsf{L}^\flat)^2 \longrightarrow \mathbb{P}_\mathsf{L}^\flat$. We obtain a continuous function from $(\mathbb{P}_\mathsf{L}^\flat)^2 A$ to $\mathbb{P}_\mathsf{L}^\flat A$ by setting $\mu'_A(\diamond a) := \diamond \diamond a$. In order to prove naturality of μ we have to check the commutativity of the following diagram for any continuous function $f\colon A \longrightarrow B$:

$$\begin{tikzcd}
(\mathsf{P}_{\mathsf{L}}^{\flat})^2 A \arrow[r, "\mu_A"] \arrow[d, "(\mathsf{P}_{\mathsf{L}}^{\flat})^2 f"'] & \mathsf{P}_{\mathsf{L}}^{\flat} A \arrow[d, "\mathsf{P}_{\mathsf{L}}^{\flat} f"] \\
(\mathsf{P}_{\mathsf{L}}^{\flat})^2 B \arrow[r, "\mu_B"] & \mathsf{P}_{\mathsf{L}}^{\flat} B
\end{tikzcd}$$

But for any such f and all $b \in B$ we calculate

$$\begin{aligned}
\Big((\mathsf{P}_{\mathsf{L}}^{\flat})^2 f\Big)'\Big(\mu'_B(\diamond b)\Big) &= \Big((\mathsf{P}_{\mathsf{L}}^{\flat})^2 f\Big)'(\diamond\diamond b) = \diamond(\mathsf{P}_{\mathsf{L}}^{\flat} f)'(\diamond b) \\
&= \diamond\diamond f'(b) = \mu'_A(\diamond f'(b)) = \mu'_A\Big((\mathsf{P}_{\mathsf{L}}^{\flat} f)'(\diamond b)\Big).
\end{aligned}$$

Since $(\mu_B \circ (\mathsf{P}_{\mathsf{L}}^{\flat})^2 f)'$ and $(\mathsf{P}_{\mathsf{L}}^{\flat} f \circ \mu_A)'$ are frame homomorphisms agreeing on the generators of $\mathsf{P}_{\mathsf{L}}^{\flat} B$, they must coincide.

Proposition 6.14 *The above definition makes* $(\mathsf{P}_{\mathsf{L}}^{\flat}, \eta, \mu)$ *a monad on* **Loc**.

Proof: It remains to check the commutativity of the diagrams for a monad (see the beginning of Section 4.1). First for the two triangular ones. If A is any locale then for all $a \in A$ we have

$$\eta'_{\mathsf{P}_{\mathsf{L}}^{\flat} A}\Big(\mu'_A(\diamond a)\Big) = \eta'_{\mathsf{P}_{\mathsf{L}}^{\flat} A}(\diamond\diamond a) = \diamond a = \diamond\,\eta'_A(\diamond a) = (\mathsf{P}_{\mathsf{L}}^{\flat}\eta_A)'(\diamond\diamond a) = (\mathsf{P}_{\mathsf{L}}^{\flat}\eta_A)'\Big(\mu'_A(\diamond a)\Big)$$

which proves $\mu_A \circ \eta_{\mathsf{P}_{\mathsf{L}}^{\flat} A} = id_{\mathsf{P}_{\mathsf{L}}^{\flat} A} = \mu_A \circ \mathsf{P}_{\mathsf{L}}^{\flat}\eta_A$. Now let us have a look at the rectangular diagram for monads. For all $a \in A$ we calculate

$$\mu'_{\mathsf{P}_{\mathsf{L}}^{\flat} A}\Big(\mu'_A(\diamond a)\Big) = \mu'_{\mathsf{P}_{\mathsf{L}}^{\flat} A}(\diamond\diamond a) = \diamond\diamond\diamond a = \diamond\,\mu'_A(\diamond a) = (\mathsf{P}_{\mathsf{L}}^{\flat}\mu_A)'(\diamond\diamond a) = (\mathsf{P}_{\mathsf{L}}^{\flat}\mu_A)'\Big(\mu'_A(\diamond a)\Big)$$

which proves our claim. ∎

Note the similarities between the monads arising from the Hoare power construction for locales and spaces. We constructed the Hoare power locale $\mathsf{P}_{\mathsf{L}}^{\flat} A$ for any locale A such that its generators satisfy the equations shown for the generators of $\Omega\,\mathsf{P}_{\mathsf{T}}^{\flat} X$ where

X is a T_0-space. If $f\colon X \longrightarrow Y$ is a continuous function then $(\mathsf{P}_{\mathsf{T}}^\flat f)^{-1}(\Diamond U) = \Diamond f^{-1}(U)$ for every $U \in \Omega X$. Compare this equation to the definition of $\mathsf{P}_{\mathsf{L}}^\flat f$ where $f\colon A \longrightarrow B$ is a continuous function between locales. Furthermore, $\eta_X^{-1}(\Diamond U) = U$ for any $U \in \Omega X$ which resembles the equation that defines η_A, namely $\eta'_A(\diamond a) = a$. A similar analogy holds for μ. We will further investigate the connections between these two Hoare power constructions in a separate paragraph.

6.4.2 The algebras for the monad

Next we want to characterize the algebras for the monad $(\mathsf{P}_{\mathsf{L}}^\flat, \eta, \mu)$. So let (A, ξ) be a $\mathsf{P}_{\mathsf{L}}^\flat$-algebra. First of all we show that (ξ, η_A) has to be an adjoint pair in the O-category **Loc** as defined in Section 4.4. By definition, this is true iff ξ' is the upper adjoint for η'_A in the usual sense. Since ξ is a structure map for A we have $\xi \circ \eta_A = id_A$, which is equivalent to $\eta'_A \circ \xi' = id'_A$. Therefore it is enough to show that $\xi' \circ \eta'_A \geq id'_{\mathsf{P}_{\mathsf{L}}^\flat A}$. For all $a \in A$, let $\bigvee_{i\in I} \diamond a_1^i \wedge \cdots \wedge \diamond a_{n(i)}^i := \xi'(a)$. For arbitrary $a \in A$ this implies

$$a = \eta'_A\left(\xi'(a)\right) = \eta'_A\left(\bigvee_{i\in I} \diamond a_1^i \wedge \cdots \wedge \diamond a_{n(i)}^i\right) = \bigvee_{i\in I} \eta'_A(\diamond a_1^i \wedge \cdots \wedge \diamond a_{n(i)}^i) = \bigvee_{i\in I} a_1^i \wedge \cdots \wedge a_{n(i)}^i.$$

Consequently, $\diamond a = \diamond \bigvee_{i\in I} a_1^i \wedge \cdots \wedge a_{n(i)}^i = \bigvee_{i\in I} \diamond(a_1^i \wedge \cdots \wedge a_{n_i}^i) \leq \bigvee_{i\in I} \diamond a_1^i \wedge \cdots \wedge \diamond a_{n_i}^i$ which suffices to prove $\xi' \circ \eta'_A \geq id'_{\mathsf{P}_{\mathsf{L}}^\flat A}$. There is at most one structure map for any locale A which we will refer to as ξ_A from now on. Also, the conditions for Lemma 4.6 are satisfied; this is a fact we will exploit later.

Recall that for **DCPO** and **Top** the algebras for the monads resulting from the respective Hoare power constructions are inflationary semilattices for the respective category. The same is true for the category of locales. To define a semilattice structure on locales of the form $\mathsf{P}_{\mathsf{L}}^\flat A$ let us have another look at the semilattice structure for $\mathsf{P}_{\mathsf{T}}^\flat X$ where X is an arbitrary T_0-space. The semilattice structure making $\mathsf{P}_{\mathsf{T}}^\flat X$ an inflationary semilattice is binary union. Hence we can hope for an idea how to define a semilattice structure for $\mathsf{P}_{\mathsf{L}}^\flat A$, where A is an arbitrary locale, if we look at the corresponding inverse image function. Let U be an open subset of some T_0-space X. Then

$$\begin{aligned}
\cup^{-1}(\Diamond U) &= \{\langle A, B\rangle \in \mathrm{P}^\flat_{\mathrm{T}} X \times \mathrm{P}^\flat_{\mathrm{T}} X \mid (A \cup B) \cap U \neq \emptyset\} \\
&= \{\langle A, B\rangle \in \mathrm{P}^\flat_{\mathrm{T}} X \times \mathrm{P}^\flat_{\mathrm{T}} X \mid A \cap U \neq \emptyset \text{ or } B \cap U \neq \emptyset\} \\
&= \{\langle A, B\rangle \in \mathrm{P}^\flat_{\mathrm{T}} X \times \mathrm{P}^\flat_{\mathrm{T}} X \mid A \cap U \neq \emptyset\} \\
&\qquad\qquad \cup \{\langle A, B\rangle \in \mathrm{P}^\flat_{\mathrm{T}} X \times \mathrm{P}^\flat_{\mathrm{T}} X \mid B \cap U \neq \emptyset\} \\
&= (\Diamond U \times \mathrm{P}^\flat_{\mathrm{T}} X) \cup (\mathrm{P}^\flat_{\mathrm{T}} X \times \Diamond U).
\end{aligned}$$

Therefore we try to define a continuous function $v\colon \mathrm{P}^\flat_{\mathrm{L}} A \otimes \mathrm{P}^\flat_{\mathrm{L}} A \longrightarrow \mathrm{P}^\flat_{\mathrm{L}} A$ by setting $v'(\diamond a) := (\diamond a \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \diamond a)$ for all $a \in A$. Compare this definition with our motivations for the Hoare power locale. Obviously,

$$\begin{aligned}
v'(\diamond 1_A) &= (\diamond 1_A \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \diamond 1_A) \\
&= (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) = 1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A} = 1_{\mathrm{P}^\flat_{\mathrm{L}} A \otimes \mathrm{P}^\flat_{\mathrm{L}} A}.
\end{aligned}$$

For any subset S of A we have

$$\begin{aligned}
v'(\diamond \bigvee S) &= (\diamond \bigvee S \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \diamond \bigvee S) \\
&= (\bigvee_{s \in S} \diamond s \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \bigvee_{s \in S} \diamond s) \\
&= \bigvee_{s \in S} (\diamond s \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee \bigvee_{s \in S} (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \diamond s) \\
&= \bigvee_{s \in S} \big((\diamond s \otimes 1_{\mathrm{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathrm{P}^\flat_{\mathrm{L}} A} \otimes \diamond s)\big) \\
&= \bigvee_{s \in S} v'(\diamond s)
\end{aligned}$$

which proves that the above does define a continuous function as claimed. This function is strongly related to the semilattice structure P. Johnstone introduces for the Vietoris power locale in [Joh82b].

Lemma 6.15 *The continuous function $v\colon \mathrm{P}^\flat_{\mathrm{L}} A \otimes \mathrm{P}^\flat_{\mathrm{L}} A \longrightarrow \mathrm{P}^\flat_{\mathrm{L}} A$ makes $\mathrm{P}^\flat_{\mathrm{L}} A$ an inflationary localic semilattice.*

Proof: First for idempotency of v. We calculate

$$\begin{aligned}
(v \circ \Delta_{\mathsf{P}_{\mathsf{L}}^{\flat} A})'(\diamond a) &= \Delta'_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \Big(v'(\diamond a)\Big) \\
&= \Delta'_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \Big((\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a)\Big) \\
&= (\diamond a \wedge 1_A) \vee (1_A \wedge \diamond a) \\
&= \diamond a
\end{aligned}$$

for any $a \in A$. Next for symmetry. For any $a \in A$ we have

$$\begin{aligned}
\Big(v \circ \langle \pi_2, \pi_1 \rangle\Big)'(\diamond a) &= \langle \pi_2, \pi_1 \rangle' \Big((\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a)\Big) \\
&= (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a) \vee (\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \\
&= (\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \wedge \diamond a) \\
&= v'(\diamond a).
\end{aligned}$$

Similarly, we can show associativity since

$$\begin{aligned}
& \Big(v \circ (v \otimes id_{\mathsf{P}_{\mathsf{L}}^{\flat} A})\Big)'(\diamond a) \\
= \;& (v \otimes id_{\mathsf{P}_{\mathsf{L}}^{\flat} A})'(v'(\diamond a)) \\
= \;& (v \otimes id_{\mathsf{P}_{\mathsf{L}}^{\flat} A})'\Big((\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a)\Big) \\
= \;& \Big(((\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a)) \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}\Big) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a) \\
= \;& (\diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond a) \\
= \;& \Big(v \circ (id_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes v)\Big)'(\diamond a)
\end{aligned}$$

holds for all $a \in A$. We also claimed that v makes $\mathsf{P}_{\mathsf{L}}^{\flat} A$ an inflationary semilattice, that is $\Delta_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \circ v \geq id_{\mathsf{P}_{\mathsf{L}}^{\flat} A \otimes \mathsf{P}_{\mathsf{L}}^{\flat} A}$ holds with respect to the order we introduced on hom-sets of **Loc** in Section 4.4. Hence we have to show that $(\Delta_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \circ v)' \geq id'_{\mathsf{P}_{\mathsf{L}}^{\flat} A \otimes \mathsf{P}_{\mathsf{L}}^{\flat} A}$. If $\diamond a \otimes \diamond b$ is a generator of $\mathsf{P}_{\mathsf{L}}^{\flat} A \otimes \mathsf{P}_{\mathsf{L}}^{\flat} A$ then

$$\begin{aligned}
(\Delta_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \circ v)'(\diamond a \otimes \diamond b) &= v'(\diamond a \wedge \diamond b) \\
&= v'(\diamond a) \wedge v'(\diamond b) \\
&= \Big((\diamond a \otimes 1_A) \vee (1_A \otimes \diamond a)\Big) \wedge \Big((\diamond b \otimes 1_A) \vee (1_A \otimes \diamond b)\Big) \\
&\geq \diamond a \otimes \diamond b.
\end{aligned}$$

Therefore $(\mathsf{P}_{\mathsf{L}}^{\flat} A, v)$ is an inflationary semilattice in **Loc**. ■

Proposition 6.16 *Every $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebra is an inflationary localic semilattice.*

Proof: By the above, (ξ_A, η_A) has to be an adjoint pair for any $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebra (A, ξ_A). Hence, because of Lemma 5.4, we may conclude from the last lemma that every $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebra is an inflationary localic semilattice. Its semilattice structure is given by $\xi \circ v \circ (\eta_A \otimes \eta_A)$ where v denotes the semilattice structure on $\mathsf{P}_{\mathsf{L}}^{\flat} A$ from Lemma 6.15.■

Let us now consider morphisms of $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebras. For that, assume that $f\colon A \longrightarrow B$ is a morphism for the $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebras (A, ξ_A) and (B, ξ_B), that is $\xi_B \circ \mathsf{P}_{\mathsf{L}}^{\flat} f = f \circ \xi_A$. By the above we know that A and B inherit their semilattice structure from $\mathsf{P}_{\mathsf{L}}^{\flat} A$ and $\mathsf{P}_{\mathsf{L}}^{\flat} B$, respectively. Let us denote the semilattice structure making $\mathsf{P}_{\mathsf{L}}^{\flat} A$ an inflationary semilattice by v and the corresponding structure for B by w. Then $\big(A, \xi_A \circ v \circ (\eta_A \otimes \eta_A)\big)$ and $\big(B, \xi_B \circ w \circ (\eta_B \otimes \eta_B)\big)$ are inflationary localic semilattices by Lemma 5.4.

Lemma 6.17 *Every morphism of the $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebras (A, ξ_A) and (B, ξ_B) is a morphism of semilattices from $(A, \xi_A \circ v \circ (\eta_A \otimes \eta_A))$ to $(B, \xi_B \circ w \circ (\eta_B \otimes \eta_B))$.*

Proof: Let $f\colon A \longrightarrow B$ be a morphism of $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebras. We first show that $\mathsf{P}_{\mathsf{L}}^{\flat} f$ is a semilattice homomorphism from $(\mathsf{P}_{\mathsf{L}}^{\flat} A, v)$ to $(\mathsf{P}_{\mathsf{L}}^{\flat} B, w)$, that is $\mathsf{P}_{\mathsf{L}}^{\flat} f \circ v = w \circ (\mathsf{P}_{\mathsf{L}}^{\flat} f \otimes \mathsf{P}_{\mathsf{L}}^{\flat} f)$. If $b \in B$ then

$$
\begin{aligned}
v'\Big((\mathsf{P}_{\mathsf{L}}^{\flat} f)'(\diamond b)\Big) &= v'(\diamond f'(b)) \\
&= (\diamond f'(b) \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes \diamond f'(b)) \\
&= ((\mathsf{P}_{\mathsf{L}}^{\flat} f)'(\diamond b) \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} A}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} A} \otimes (\mathsf{P}_{\mathsf{L}}^{\flat} f)'(\diamond b)) \\
&= (\mathsf{P}_{\mathsf{L}}^{\flat} f \otimes \mathsf{P}_{\mathsf{L}}^{\flat} f)'\Big((\diamond b \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\flat} B}) \vee (1_{\mathsf{P}_{\mathsf{L}}^{\flat} B} \otimes \diamond b)\Big) \\
&= (\mathsf{P}_{\mathsf{L}}^{\flat} f \otimes \mathsf{P}_{\mathsf{L}}^{\flat} f)'\Big(w'(\diamond b)\Big).
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
& f \circ \xi_A \circ v \circ (\eta_A \otimes \eta_A) \\
= \;& \xi_B \circ \mathsf{P}_{\mathsf{L}}^{\flat} f \circ v \circ (\eta_A \otimes \eta_A) && (f \text{ morphism of } \mathsf{P}_{\mathsf{L}}^{\flat}\text{-algebras}) \\
= \;& \xi_B \circ w \circ (\mathsf{P}_{\mathsf{L}}^{\flat} f \otimes \mathsf{P}_{\mathsf{L}}^{\flat} f) \otimes (\eta_A \otimes \eta_A) && (\mathsf{P}_{\mathsf{L}}^{\flat} f \text{ morphism of semilattices}) \\
= \;& \xi_B \circ w \circ (\eta_B \otimes \eta_B) \circ (f \otimes f) && (\text{Naturality of } \eta)
\end{aligned}
$$

which proves that f is a morphism of semilattices as claimed. ■

Thus we can understand the category of $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebras as a subcategory of the category of inflationary localic semilattices with semilattice homomorphisms. We claim that every inflationary semilattice in **Loc** has a structure map $\xi_A\colon \mathsf{P}_{\mathsf{L}}^{\flat}A \longrightarrow A$ making (A, ξ_A) a $\mathsf{P}_{\mathsf{L}}^{\flat}$-algebra. For that let us first note that, because of Lemma 4.6, it is enough to prove that for every inflationary localic semilattice (A, v) the continuous function η_A has a lower adjoint making A a retract of $\mathsf{P}_{\mathsf{L}}^{\flat}A$ in **Loc**. Once more we will obtain an idea how to prove this from the corresponding considerations for topological spaces.

If X is a sober inflationary semilattice in **Top** then it has a structure map ξ_X making it a $\mathsf{P}_{\mathsf{T}}^{\flat}$-algebra. The inverse image of an open set U in X under ξ_X is

$$\begin{aligned}
& \xi_X^{-1}(U) \\
= \ & \{A \in \mathsf{P}_{\mathsf{T}}^{\flat}X \mid \bigvee A \in U\} \\
= \ & \{A \in \mathsf{P}_{\mathsf{T}}^{\flat}X \mid \exists U_1, \ldots, U_n \in \Omega X \,.\, A \in \Diamond U_1 \cap \cdots \cap \Diamond U_n \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{and } \vee_n (U_1 \times \cdots \times U_n) \subseteq U\} \\
= \ & \bigcup\{\Diamond U_1 \cap \cdots \cap \Diamond U_n \mid n \in \mathbb{N}, U_1, \ldots, U_n \in \Omega X \text{ and } U_1 \times \cdots \times U_n \subseteq \vee_n^{-1}(U)\}.
\end{aligned}$$

Let (A, v) be an inflationary localic semilattice. We set

$$\xi'_A(a) := \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A \text{ and } a_1 \otimes \cdots \otimes a_n \leq v'_n(a)\}.$$

We have to show that ξ'_A is a frame homomorphism, that ξ'_A is the upper adjoint for $\eta_A{}'$, and that $\xi_A \circ \eta_A = id_A$. We start with the last point. Since v'_n is the upper adjoint for Δ'_n we have $a_1 \otimes \cdots \otimes a_n \leq v'_n(a)$ iff $a_1 \wedge \cdots \wedge a_n = \Delta'_n(a_1 \otimes \cdots \otimes a_n) \leq a$ for all $a, a_1, \ldots, a_n \in A$. For all $a \in A$ we calculate

$$\begin{aligned}
\eta'_A\left(\xi'_A(a)\right) &= \eta'_A\left(\bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathsf{P}_{\mathsf{L}}^{\flat}A \mid a_1 \otimes \cdots \otimes a_n \leq v'_n(a)\}\right) \\
&= \bigvee\{a_1 \wedge \cdots \wedge a_n \mid a_1 \wedge \cdots \wedge a_n \leq a\} \\
&= a.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\xi'_A(a) &= \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathrm{P}^{\flat}_{\mathsf{L}} A \mid a_1 \otimes \cdots \otimes a_n \leq v'_n(a)\} \\
&= \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathrm{P}^{\flat}_{\mathsf{L}} A \mid a_1 \wedge \cdots \wedge a_n = \Delta'_n(a_1 \otimes \cdots \otimes a_n) \leq a\} \\
&= \bigvee\{\bigvee_{i \in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n_i} \in \mathrm{P}^{\flat}_{\mathsf{L}} A \mid \eta'_A\Big(\bigvee_{i \in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_n\Big) \leq a\} \\
&= \bigvee\{\mathsf{a} \in \mathrm{P}^{\flat}_{\mathsf{L}} A \mid \eta'_A(\mathsf{a}) \leq a\}.
\end{aligned}$$

Hence ξ'_A is indeed the upper adjoint for η'_A and thus preserves all meets, in particular finite ones. It remains to prove that ξ'_A preserves all joins. For that we need the following result.

Lemma 6.18 *Let S and T be two sets of generators of the n-fold product of a locale A where n is an arbitrary positive integer. If $\bigvee S = \bigvee T$ in $A \otimes \cdots \otimes A$ then $\bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid a_1 \otimes \cdots \otimes a_n \in S\} = \bigvee\{\diamond b_1 \wedge \cdots \wedge \diamond b_n \mid b_1 \otimes \cdots \otimes b_n \in T\}$ in $\mathrm{P}^{\flat}_{\mathsf{L}} A$.*

Proof: Consider the mapping m from $A \otimes \cdots \otimes A$ to $\mathrm{P}^{\flat}_{\mathsf{L}} A$ which is defined by setting $m(a_1 \otimes \cdots \otimes a_n) := \diamond a_1 \wedge \cdots \wedge \diamond a_n$. Note that m is not a morphism, neither in **Loc** nor in **Frm**. We want to employ the constructions of $\mathrm{P}^{\flat}_{\mathsf{L}} A$ and $A \otimes \cdots \otimes A$ via C-ideals as in Theorem 1.11. So let $\mathcal{I}$ be a set of generators of $\mathrm{P}^{\flat}_{\mathsf{L}} A$ that is closed with respect to finite meets satisfying the following condition: If $R \subseteq A$ and $a_1, \ldots, a_n \in A$ such that $\diamond r \wedge \diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathcal{I}$ for all $r \in R$ then $\diamond \bigvee R \wedge \diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathcal{I}$. Assume that $\mathcal{I} \supseteq \{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid a_1 \otimes \cdots \otimes a_n \in S\} = m(S)$. It is enough to show that $\mathcal{I} \supseteq \{\diamond b_1 \wedge \cdots \wedge \diamond b_n \mid b_1 \otimes \cdots \otimes b_n \in T\} = m(T)$ in this case. For that, note that $m^{-1}(\mathcal{I})$ is a $\otimes$-ideal in $A \otimes \cdots \otimes A$ containing S: If R is an arbitrary subset of A and if $a_2, \ldots, a_n \in A$ such that $r \otimes a_2 \otimes \cdots \otimes a_n \in m^{-1}(\mathcal{I})$ for all $r \in R$ then $\diamond r \wedge \diamond a_2 \wedge \cdots \wedge \diamond a_n \in \mathcal{I}$. Hence $\diamond \bigvee R \wedge \diamond a_2 \wedge \cdots \wedge \diamond a_n \in \mathcal{I}$ implying $\bigvee R \otimes a_2 \otimes \cdots \otimes a_n \in m^{-1}(\mathcal{I})$. Similarly we can show that analogous results hold for the other components. Since $\bigvee S = \bigvee T$ in $A \otimes \cdots \otimes A$, the $\otimes$-ideal $m^{-1}(\mathcal{I})$ must contain T. This implies $m(T) \subseteq m(m^{-1}(\mathcal{I})) \subseteq \mathcal{I}$. ■

To prove that ξ'_A preserves all joins let S be an arbitrary subset of A. As v'_n preserves all joins,

$$
\begin{aligned}
& \bigvee\{a_1 \otimes \cdots \otimes a_n \mid a_1 \otimes \cdots \otimes a_n \leq v_n'(\bigvee S)\} \\
= \; & v_n'\Big(\bigvee S\Big) \\
= \; & \bigvee_{s \in S} v_n'(s) \\
= \; & \bigvee\{a_1 \otimes \cdots \otimes a_n \mid \exists s \in S \,.\, a_1 \otimes \cdots \otimes a_n \leq v_n'(s)\}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Hence by the last lemma

$$
\begin{aligned}
\xi_A'(\bigvee S) &= \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N},\, a_1 \otimes \cdots \otimes a_n \leq v_n'(\bigvee S)\} \\
&= \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N},\, \exists s \in S \,.\, a_1 \otimes \ldots \otimes a_n \leq v_n'(s)\} \\
&= \bigvee_{s \in S} \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N},\, a_1 \otimes \cdots \otimes a_n \leq v_n'(s)\} \\
&= \bigvee_{s \in S} \xi_A'(s).
\end{aligned}
$$

We have shown the following result.

Proposition 6.19 *Every inflationary localic semilattice gives rise to a $\mathbb{P}_{\mathsf{L}}^{\flat}$-algebra.* ■

It remains to characterize the morphisms of $\mathbb{P}_{\mathsf{L}}^{\flat}$-algebras. We already know that every morphism of $\mathbb{P}_{\mathsf{L}}^{\flat}$-algebras is a semilattice homomorphism (see Lemma 6.17). The following lemma gives us the reverse.

Lemma 6.20 *If $f\colon (A, v) \longrightarrow (B, w)$ is a morphism of inflationary localic semilattices then f is a morphism from (A, ξ_A) to (B, ξ_B) in the category of $\mathbb{P}_{\mathsf{L}}^{\flat}$-algebras.*

Proof: First of all note that $f \circ v_n = w_n \circ (f \otimes \cdots \otimes f)$ for all $n \in \mathbb{N}$ by the considerations from Section 5.1. Hence for all $n \in \mathbb{N}$ and all $b \in B$ we have

$$
\begin{aligned}
& \bigvee\{a_1 \otimes \cdots \otimes a_n \mid a_1 \otimes \cdots \otimes a_n \leq v_n'(f'(b))\} \\
= \; & v_n'(f'(b)) \\
= \; & (f \otimes \cdots \otimes f)'(w_n'(b)) \\
= \; & (f \otimes \cdots \otimes f)'\left(\bigvee\{b_1 \otimes \cdots \otimes b_n \mid b_1 \otimes \cdots \otimes b_n \leq w_n'(b)\}\right) \\
= \; & \bigvee\{f'(b_1) \otimes \cdots \otimes f'(b_n) \mid b_1 \otimes \cdots \otimes b_n \leq w_n'(b)\}.
\end{aligned}
$$

Because of Lemma 6.18 this yields

$$\bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid a_1 \otimes \cdots \otimes a_n \leq v'_n(f'(b))\}$$
$$= \bigvee\{\diamond f'(b_1) \wedge \cdots \wedge \diamond f'(b_n) \mid b_1 \otimes \cdots \otimes b_n \leq w'_n(b)\}$$

for all $b \in B$ and all $n \in \mathbb{N}$. Consequently, for all $b \in B$ we finally get

$$\begin{aligned}
\xi'_A\big(f'(b)\big) &= \bigvee\{\diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N},\, a_1 \otimes \cdots \otimes a_n \leq v'_n(f'(b))\} \\
&= \bigvee\{\diamond f'(b_1) \wedge \cdots \wedge \diamond f'(b_n) \mid n \in \mathbb{N},\, b_1 \otimes \cdots \otimes b_n \leq w'_n(b)\} \\
&= \bigvee\{(\mathsf{P}^\flat_\mathsf{L} f)'(\diamond b_1) \wedge \cdots \wedge (\mathsf{P}^\flat_\mathsf{L} f)'(\diamond b_n) \mid n \in \mathbb{N},\, b_1 \otimes \cdots \otimes b_n \leq w'_n(b)\} \\
&= (\mathsf{P}^\flat_\mathsf{L} f)'\left(\bigvee\{\diamond b_1 \wedge \cdots \wedge \diamond b_n \mid n \in \mathbb{N},\, b_1 \otimes \cdots \otimes b_n \leq w'_n(b)\}\right) \\
&= (\mathsf{P}^\flat_\mathsf{L} f)'\big(\xi'_B(b)\big).
\end{aligned}$$

■

We have shown the following theorem.

Theorem 6.21 *The category of $\mathsf{P}^\flat_\mathsf{L}$-algebras is the category of inflationary localic semilattices* $\mathbf{Loc}^\vee$. ■

6.4.3 Unital semilattices

When we examined the Hoare power space we also asked ourselves what happens if we allow the empty set as one of its elements. It turned out that the resulting Hoare power space construction $\dot{\mathsf{P}}^\flat_\mathsf{T}$ gives the free unital inflationary semilattice for **Sob**. Thus the question arises whether one can modify the Hoare power locale construction such that it also gives the free unital inflationary semilattice for **Loc**. In order to find such a construction, we examine $\Omega\dot{\mathsf{P}}^\flat_\mathsf{T}X$, the open set lattice of $\dot{\mathsf{P}}^\flat_\mathsf{T}X$, for an arbitrary T_0-space X. In $\Omega\,\mathsf{P}^\flat_\mathsf{T}X$, the greatest element, that is the whole space $\mathsf{P}^\flat_\mathsf{T}X$, is one of the generators of $\Omega\,\mathsf{P}^\flat_\mathsf{T}X$, namely $\Diamond X = \{A \in \mathsf{P}^\flat_\mathsf{T}X \mid A \cap X \neq \emptyset\} = \mathsf{P}^\flat_\mathsf{T}X$. For $\Omega\dot{\mathsf{P}}^\flat_\mathsf{T}X$, however, this is no longer true. The reason for this is that the empty set, which is an element of $\dot{\mathsf{P}}^\flat_\mathsf{T}X$, is not in $\Diamond X$ anymore. Therefore $\Diamond 1_{\Omega X} \neq 1_{\Omega\dot{\mathsf{P}}^\flat_\mathsf{T}X}$ in $\dot{\mathsf{P}}^\flat_\mathsf{T}X$.

Hence, for every locale A, we define

$$\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A := Fr\left\langle \diamond a : a \in A \mid \bigvee_{s\in S} \diamond s = \diamond \bigvee S \quad (S \subseteq A)\right\rangle.$$

If $f\colon A \longrightarrow B$ is a continuous function in **Loc** we set $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}f)'(\diamond b) := \diamond f'(b)$ for all $b \in B$ to obtain a continuous function from $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A$ to $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}B$. The proof that this does define a morphism in **Loc** and that this assignment makes $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$ an endofunctor on **Loc** is the same as that for $\mathsf{P}^{\flat}_{\mathsf{L}}$. We can also define a natural transformations $\eta\colon id_{\mathbf{Loc}} \longrightarrow \dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$ and $\mu\colon (\dot{\mathsf{P}}^{\flat}_{\mathsf{L}})^2 \longrightarrow \dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$ the same way as we did for $\mathsf{P}^{\flat}_{\mathsf{L}}$, that is by setting $\eta'_A(\diamond a) := a$ and $\mu'_A(\diamond a) := \diamond\diamond a$ for all $a \in A$ where A is a locale. The proof that $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}, \eta, \mu)$ is a monad is the same as for Proposition 6.14.

Theorem 6.22 *The category of $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$-algebras is the category of unital inflationary semilattices in* **Loc**.

Proof: We can show in the same way as for the monad $(\mathsf{P}^{\flat}_{\mathsf{L}}, \eta, \mu)$ that every $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$-algebra is an inflationary localic semilattice. To prove that these semilattices are unital let us first show this for locales of the form $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A$. We define $o\colon \mathit{2} \longrightarrow \dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A$ by setting $o'(\diamond a) := 0_{\mathit{2}}$ for all $a \in A$. Obviously, o' preserves the relations defining $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A$ and thus defines a morphism in **Loc** as desired. By Lemma 5.8 (i), o makes $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A$ a unital inflationary semilattice. As Lemma 5.4 also holds for unital semilattices, every $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$-algebra is a unital inflationary semilattice in **Loc**. Furthermore, every morphism in the category of $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$-algebras is a morphism of semilattices. The proof for this is exactly as that for Lemma 6.17. To see that every such $f\colon A \longrightarrow B$ also preserves the respective units, let $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A, v, o)$ and $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}B, w, p)$ be the unital inflationary semilattices corresponding to the $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}$-algebras $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}A, \mu_A)$ and $(\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}B, \mu_B)$, respectively. First of all we will show that $\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}f$ preserves the units of these semilattices. For all $b \in B$ we have $o'\Big((\dot{\mathsf{P}}^{\flat}_{\mathsf{L}}f)'(\diamond b)\Big) = o'(\diamond f'(b)) = 0_{\mathit{2}} = p'(\diamond b)$. The semilattice structures on A and B making them inflationary semilattices must be given by $(\xi_A \circ v \circ (\eta_A \times \eta_A), \xi_A \circ o)$ and $(\xi_B \circ w \circ (\eta_B \times \eta_B), \xi_B \circ p)$, respectively. Consequently, $f \circ \xi_A \circ o = \xi_B \circ \mathsf{P}^{\flat}_{\mathsf{L}}f \circ o = \xi_B \circ p$ which proves our claim.

It remains to prove that every unital inflationary semilattice in **Loc** gives rise to a $\dot{\mathrm{P}}_{\mathrm{L}}^{\flat}$-algebra and that every morphism of such semilattices is a morphism of the corresponding $\dot{\mathrm{P}}_{\mathrm{L}}^{\flat}$-algebras. But if (A, v, o) is a unital inflationary semilattice in **Loc** then, by Lemma 5.8 (i), $o'(a) = 1_2$ iff $a = 1_A$. Hence 1_A is inaccessible by arbitrary joins in A. This allows us to define a morphism $\xi_A \colon \dot{\mathrm{P}}_{\mathrm{L}}^{\flat} A \longrightarrow A$ by setting

$$
\begin{aligned}
\xi'_A(1_A) &:= 1_{\dot{\mathrm{P}}_{\mathrm{L}}^{\flat} A}, \\
\xi'_A(a) &:= \bigvee \{ \diamond a_1 \wedge \cdots \wedge \diamond a_n \mid n \in \mathbb{N},\ a_1 \otimes \cdots \otimes a_n \leq v'_n(a) \} \quad (a \in A \setminus \{1_A\}).
\end{aligned}
$$

The proof that (A, ξ_A) is a $\dot{\mathrm{P}}_{\mathrm{L}}^{\flat}$-algebra works as the corresponding proof for $\mathrm{P}_{\mathrm{L}}^{\flat}$-algebras. Moreover, to show that every morphism of unital inflationary semilattices in **Loc** is a morphism of the corresponding $\dot{\mathrm{P}}_{\mathrm{L}}^{\flat}$-algebras we can employ the proof of Lemma 6.20.■

6.4.4 Locally compact locales

We already examined the Hoare power construction for domains and locally compact T_0-spaces and now do so for locally compact locales. Since locally compact locales correspond to locally compact sober spaces one might expect that we can get similar results for them. The proof of Proposition 6.11 suggests that if $b_1 \ll a_1, \ldots, b_n \ll a_n$ in a locale A then $\diamond b_1 \ll \diamond a_1, \ldots, \diamond b_n \ll \diamond a_n$ in $\mathrm{P}_{\mathrm{L}}^{\flat} A$.

Lemma 6.23 *If $b \ll a$ in a locale A then $\diamond b \ll \diamond a$ in $\mathrm{P}_{\mathrm{L}}^{\flat} A$.*

Proof: Let $\mathcal{S}$ be a directed subset of $\mathrm{P}_{\mathrm{L}}^{\flat} A$ such that $\diamond a \leq \bigvee^{\uparrow} \mathcal{S}$. We can replace $\mathcal{S}$ by a directed set $\mathcal{S}'$ of elements of the form $\diamond a_1 \wedge \cdots \wedge \diamond a_n$, where $a_1, \ldots, a_n \in A$, such that $\bigvee^{\uparrow} \mathcal{S} = \bigvee^{\uparrow} \mathcal{S}'$ and such that for any $\diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathcal{S}'$ there is $s \in \mathcal{S}$ with $\diamond a_1 \wedge \cdots \wedge \diamond a_n \leq s$: Let $\mathcal{S}' := \{ \diamond a_1 \wedge \cdots \wedge \diamond a_n \in \mathrm{P}_{\mathrm{L}}^{\flat} A \mid \exists s \in \mathcal{S} \,.\, \diamond a_1 \wedge \cdots \wedge \diamond a_n \leq s \}$. It is enough to show that $\mathcal{S}'$ is directed as the other properties we claimed it to have are easily seen to be satisfied. Let $\diamond a_1 \wedge \cdots \wedge \diamond a_n, \diamond a'_1 \wedge \cdots \wedge \diamond a'_m \in \mathcal{S}'$. There are s and s' in $\mathcal{S}$ with $\diamond a_1 \wedge \cdots \wedge \diamond a_n \leq s$ and $\diamond a'_1 \wedge \cdots \wedge \diamond a'_m \leq s'$. By directedness of $\mathcal{S}$ there is an upper bound s'' for s and s' in $\mathcal{S}$. Therefore,

$$\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m}\diamond(a_i \vee a'_j) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m}(\diamond a_i \vee \diamond a'_j) = (\diamond a_1 \wedge \cdots \wedge \diamond a_n) \vee (\diamond a'_1 \wedge \cdots \wedge \diamond a'_m) \leq s''$$

which proves that $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m}\diamond(a_i \vee a'_j)$ is a common upper bound for $\diamond a_1 \wedge \cdots \wedge \diamond a_n$ and $\diamond a'_1 \wedge \cdots \wedge \diamond a'_n$ in $\mathcal{S}'$. Hence it is sufficient to prove that if $\diamond a \leq \bigvee^{\uparrow}_{i\in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}$ then there is $i \in I$ such that $\diamond b \leq \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}$. But if $\diamond a \leq \bigvee^{\uparrow}_{i\in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}$ for a directed subset $\{\diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)} \mid i \in I\}$ of $\mathbb{P}^\flat_{\mathrm{L}} A$ then $a \leq \bigvee^{\uparrow}_{i\in I} a^i_1 \wedge \cdots \wedge a^i_{n(i)}$. Hence by the assumption there is $i \in I$ such that $b \leq a^i_1 \wedge \cdots \wedge a^i_{n(i)}$. Consequently, $\diamond b \leq \diamond(a^i_1 \wedge \cdots \wedge a^i_{n(i)}) \leq \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}$. ■

Lemma 6.24 *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of a locale A. If $\diamond b_1 \ll \diamond a_1, \ldots, \diamond b_n \ll \diamond a_n$ in $\mathbb{P}^\flat_{\mathrm{L}} A$. Then $\diamond b_1 \wedge \cdots \wedge \diamond b_n \ll \diamond a_1 \wedge \cdots \wedge \diamond a_n$.*

Proof: By Lemma 6.15, $\mathbb{P}^\flat_{\mathrm{L}} A$ is an inflationary localic semilattice where the semilattice structure is given by $v' \colon \diamond a \longmapsto (\diamond a \otimes 1_{\mathbb{P}^\flat_{\mathrm{L}} A}) \vee (1_{\mathbb{P}^\flat_{\mathrm{L}} A} \otimes \diamond a)$ for all $a \in A$. Assume that $\{\diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)} \mid i \in I\}$ is a directed subset of $\mathbb{P}^\flat_{\mathrm{L}} A$ such that

$$\diamond a_1 \wedge \cdots \wedge \diamond a_n \leq \bigvee^{\uparrow}_{i\in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}.$$

As v'_n is the upper adjoint for Δ'_n, where $\Delta_n \colon A \longrightarrow A \otimes \cdots \otimes A$ is the n-ary diagonal, this is equivalent to

$$\diamond a_1 \otimes \cdots \otimes \diamond a_n \leq v'_n\Big(\bigvee^{\uparrow}_{i\in I} \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}\Big) = \bigvee^{\uparrow}_{i\in I} v'_n(\diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}).$$

By Lemma 3.7, $\diamond b_1 \otimes \cdots \otimes \diamond b_n \ll \diamond a_1 \otimes \cdots \otimes \diamond a_n$. Hence there is $i \in I$ such that $\diamond b_1 \otimes \cdots \otimes \diamond b_n \leq v'_n(\diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)})$ which is equivalent to $\diamond b_1 \wedge \cdots \wedge \diamond b_n = \Delta'_n(\diamond b_1 \otimes \cdots \wedge \diamond b_n) \leq \diamond a^i_1 \wedge \cdots \wedge \diamond a^i_{n(i)}$. ■

Proposition 6.25 *If A is a locally compact locale then so is $\mathbb{P}^\flat_{\mathrm{L}} A$.*

Proof: As $\mathbb{P}^\flat_{\mathrm{L}} A$ is a complete lattice, we already know that $\twoheaddownarrow\{s\}$ is directed for all $s \in \mathbb{P}^\flat_{\mathrm{L}} A$. By Lemma 6.23 we know that $\{\diamond b \mid b \ll a\} \subseteq \twoheaddownarrow\{\diamond a\}$ for all $a \in A$. Hence because of Lemma 6.24 we get

$$\{\diamond b_1 \wedge \cdots \wedge \diamond b_n \mid b_1 \ll a_1, \ldots, b_n \ll a_n \text{ in } A\} \subseteq \twoheaddownarrow\{\diamond a_1 \wedge \cdots \wedge \diamond a_n\}$$

in $\mathsf{P}_\mathsf{L}^\flat A$ for all $a_1, \ldots, a_n \in A$. Consequently,

$$
\begin{aligned}
\diamond a_1 \wedge \cdots \wedge \diamond a_n &\geq \bigvee{}^{\uparrow} \twoheaddownarrow\{\diamond a_1 \wedge \cdots \wedge \diamond a_n\} \\
&\geq \bigvee{}^{\uparrow}\{\diamond b_1 \wedge \cdots \wedge \diamond b_n \mid b_1 \ll a_1, \ldots, b_n \ll a_n\} \\
&= \bigvee{}^{\uparrow}\{\diamond b_1 \in A \mid b_1 \ll a_1\} \wedge \cdots \wedge \bigvee{}^{\uparrow}\{\diamond b_n \in A \mid b_n \ll a_n\} \\
&= \diamond \bigvee{}^{\uparrow} \twoheaddownarrow\{a_1\} \wedge \cdots \wedge \diamond \bigvee{}^{\uparrow} \twoheaddownarrow\{a_n\} \\
&= \diamond a_1 \wedge \cdots \wedge \diamond a_n.
\end{aligned}
$$

■

Therefore $\mathsf{P}_\mathsf{L}^\flat$ gives the free inflationary localic semilattice for the full subcategory of **Loc** whose objects are the locally compact locales.

Remark. The same considerations apply to $\dot{\mathsf{P}}_\mathsf{L}^\flat$.

6.4.5 Power space versus power locale

We have already seen a connection between the Hoare power space $\mathsf{P}_\mathsf{T}^\flat X$ and the Hoare power locale $\mathsf{P}_\mathsf{L}^\flat \Omega X$ for some T_0-space X: As the generators for the topology on $\mathsf{P}_\mathsf{T}^\flat X$ satisfy the equations for the Hoare power locale we get a frame homomorphism from $\mathsf{P}_\mathsf{L}^\flat \Omega X$ to $\Omega\, \mathsf{P}_\mathsf{T}^\flat X$ from the assignment $\diamond U \longmapsto \Diamond\, U$. The question is when this defines an isomorphism.

Let us first examine the points for $\mathsf{P}_\mathsf{L}^\flat A$ where A is an arbitrary locale. The points of $\mathsf{P}_\mathsf{L}^\flat A$ correspond to frame homomorphisms from $\mathsf{P}_\mathsf{L}^\flat A$ to $\mathit{2}$ which in turn correspond to functions from A to $\mathit{2}$ that preserve all joins and 1. But these can be understood as being given by the elements of $A \setminus \{1_A\}$: For any $a \in A \setminus \{1_A\}$ the function $p_a\colon A \longrightarrow \mathit{2}$ that maps $b \in A$ to 0_{2} iff b is below a is such a homomorphism. On the other hand, if any such homomorphism is mapped to the largest element mapped to 0_{2} then this assignment gives us the inverse for $a \longmapsto p_a$. Note that this bijection is order-reversing and hence $\mathsf{Pt}\mathsf{P}_\mathsf{L}^\flat A$ with respect to the specialization order is isomorphic to $(A \setminus \{1_A\})^{op}$ (see also [Vic89]). If A is spatial then $(A \setminus \{1_A\})^{op}$ can be understood to be the up-complete $\vee$-semilattice of nonempty closed subsets of $\mathsf{Sp}A$ ordered by inclusion. Thus for any spatial locale A the points of $\mathsf{P}^{\flat}_{\mathsf{L}}A$ agree with the points of $\mathsf{P}^{\flat}_{\mathsf{T}}\mathsf{Sp}A$. Put differently, for every T_0-space X the points of $\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X$ are just the points of $\mathsf{P}^{\flat}_{\mathsf{T}}X$, that is the non-empty closed subsets of X. This is a homeomorphism $\iota_X\colon \mathsf{P}^{\flat}_{\mathsf{T}}X \longrightarrow \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X$: An open subset $\Diamond U = \{A \in \mathsf{P}^{\flat}_{\mathsf{T}}X \mid A \cap U \neq \emptyset\}$ is mapped to the set

$$\begin{aligned}\{p \in \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X \mid \exists A \in \Diamond U \,.\, p'(\diamond U') = 0_{\mathit{2}} \text{ iff } U' \subseteq X \setminus A\}& \\ = \{p \in \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X \mid p'(\diamond U) = 1_{\mathit{2}}\} &= \{p \in \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X \mid p \models \diamond U\}\end{aligned}$$

of points of $\mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X$.

The family of all ι_X where X is a T_0-space is a natural isomorphism $\mathsf{P}^{\flat}_{\mathsf{T}} \longrightarrow \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega$. To see this, let $f\colon X \longrightarrow Y$ be a continuous function. The isomorphism ι_X maps a non-empty closed subset A of X to the point $\mathit{2} \longrightarrow \mathsf{P}^{\flat}_{\mathsf{L}}\Omega X$ in $\mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega X$ that is defined by $p'_A(\diamond U) = 0_{\mathit{2}}$ iff $A \cap U = \emptyset$. We want to show that $\mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega f \circ \iota_X = \iota_Y \circ \mathsf{P}^{\flat}_{\mathsf{T}}f$. We do this by proving that for all $A \in \mathsf{P}^{\flat}_{\mathsf{T}}X$ the resulting continuous functions of locales from $\mathit{2}$ to $\mathsf{P}^{\flat}_{\mathsf{L}}\Omega Y$ coincide. For every $A \in \mathsf{P}^{\flat}_{\mathsf{T}}X$ we have

$$(\mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega f)(\iota_X(A)) = \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega f(p_A) = \mathsf{P}^{\flat}_{\mathsf{L}}\Omega f \circ p_A.$$

For all $V \in \Omega Y$ we calculate $p'_A((\mathsf{P}^{\flat}_{\mathsf{L}}\Omega f)'(\diamond V)) = p'_A(\diamond(\Omega f)'(V)) = p_A(\diamond f^{-1}(V)) = 0_{\mathit{2}}$ iff $A \cap f^{-1}(V) = \emptyset$. Since $\iota_Y(\mathsf{P}^{\flat}_{\mathsf{T}}f(A)) = p_{\mathsf{cl}\, f(A)}$ and since in addition $p_{\mathsf{cl}\, f(A)}(\diamond V) = 0_{\mathit{2}}$ iff $\mathsf{cl}\, f(A) \cap V = \emptyset$ for all $V \in \Omega Y$ we have shown naturality of ι.

This gives rise to the idea that the monad $(\mathsf{P}^{\flat}_{\mathsf{T}}, \mathsf{cl}, \bigcup)$ is equivalent to a monad we can build from $(\mathsf{P}^{\flat}_{\mathsf{L}}, \eta, \mu)$. We cannot take $\mathsf{Sp}\eta\Omega$ as the first natural transformation since $\mathsf{Sp}\eta\Omega$ is a natural transformation $\mathsf{Sp}\Omega \longrightarrow \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega$ while we want the first functor to be the identity on **Top**. But there is a natural isomorphism $id_{\mathbf{Sob}} \longrightarrow \mathsf{Sp}\Omega$ which we denote by α (see Section 2.1.3). We thus get the natural transformation $\overline{\eta} := \mathsf{Sp}\eta\Omega \bullet \alpha\colon id_{\mathbf{Sob}} \longrightarrow \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega$. Moreover, $\overline{\mu} := \mathsf{Sp}\circ(\mu \bullet (\mathsf{P}^{\flat}_{\mathsf{L}}\circ\beta\circ\mathsf{P}^{\flat}_{\mathsf{L}}))\circ\Omega$ is a natural transformation $(\mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega)^2 \longrightarrow \mathsf{Sp}\mathsf{P}^{\flat}_{\mathsf{L}}\Omega$ where $\beta\colon \Omega\,\mathsf{Sp} \longrightarrow id_{\mathbf{Loc}}$ is the natural transformation from Section 2.1.4.

Proposition 6.26 *There is a natural isomorphism* $\mathsf{P}^\flat_\mathsf{T} \longrightarrow \mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\Omega$ *for the functors* $\mathsf{P}^\flat_\mathsf{T}, \mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\Omega\colon \mathbf{Sob} \longrightarrow \mathbf{Sob}$. *Furthermore, the monads* $(\mathsf{P}^\flat_\mathsf{T}, \mathsf{cl}, \bigcup)$ *and* $(\mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\Omega, \overline{\eta}, \overline{\mu})$ *on* $\mathbf{Sob}$ *are equivalent.* ∎

If $\mathsf{P}^\flat_\mathsf{L}\Omega X$ is spatial then $\mathsf{P}^\flat_\mathsf{L}\Omega X \cong \Omega\,\mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\Omega X \cong \Omega\,\mathsf{P}^\flat_\mathsf{T} X$. Since for every locally compact space X we get a locally compact space ΩX, we know by Proposition 6.25 that $\mathsf{P}^\flat_\mathsf{L}\Omega X$ is locally compact as well and therefore a spatial locale. As the above isomorphisms all are natural we obtain the following result.

Proposition 6.27 *(i) The functors* $\Omega\,\mathsf{P}^\flat_\mathsf{T}\colon \mathbf{LCompT} \longrightarrow \mathbf{LCompL}$ *and* $\mathsf{P}^\flat_\mathsf{L}\Omega$ *are naturally isomorphic.*

(ii) The functors $\mathsf{P}^\flat_\mathsf{T}\mathsf{Sp}, \mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\colon \mathbf{LCompL} \longrightarrow \mathbf{LCompT}$ *are naturally isomorphic.* ∎

Chapter 7

The Smyth power construction

7.1 Algebraic domains

We will now present a different way to view non-determinism. Before, we only asked for one of the branches of a non-deterministic program to give the desired result. But, of course, if we run such a program we cannot be sure that we really end up with this result. In the worst case, the program might even be non-terminating. Now we only consider a non-deterministic program to have a certain property if *all* its branches satisfy that property. We also speak of 'demonic' non-determinism in this case. The denotation of a program whose branches are denoted by Scott-continuous functions with source D and target E is to be a function from D which takes its values in the power set of E. This time, however, we want the denotation of a program p to be greater than or equal to that of q iff for each branch of p there is one of q with lesser (or equal) denotation. For any algebraic domain D we thus get the following preorder on non-empty finite subsets F and F' of $\mathsf{C}D$: We set $F \leq^\sharp F'$ iff for all $c' \in F'$ there is $c \in F$ with $c \leq c'$. The following definition is due to M. Smyth [Smy78].

Definition. The *Smyth power domain* $\mathsf{P}^\sharp_\mathsf{A} D$ of an algebraic domain D is the ideal completion of $(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^\sharp)$.

As for the Hoare power domain, there is another way of trying to capture non-determinism. Once again, given a denotational semantics for a deterministic programming language, we would like to add the command $\underline{\mathsf{or}}$ to that language. The way we have interpreted non-determinism, we want $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q}$ to be denoted by the greatest lower bound of the denotations for p and q. This approach calls for the free deflationary semilattice for **AlgD**. As for the Hoare power construction, this point of view also leads to the Smyth power domain. Let $\mathbf{AlgD}^{\wedge}$ denote the category of deflationary semilattices in **AlgD**. We will refer to these posets as *algebraic* $\wedge$*-semilattices*. The following result is also due to Hennessy and Plotkin [HP79].

Theorem 7.1 *The free deflationary semilattice on* **AlgD** *is given by* $\mathsf{P}_{\mathsf{A}}^{\sharp}$. ■

Put differently, this means that the forgetful functor from $\mathbf{AlgD}^{\wedge}$ to **AlgD** has a left adjoint. Furthermore, this adjunction is monadic.

7.1.1 $\mathsf{P}_{\mathsf{A}}^{\sharp}D$ as a subset of $\mathsf{P}D$

In order to define a Smyth power construction for arbitrary dcpo's, we would like to be able to view $\mathsf{P}_{\mathsf{A}}^{\sharp}D$ as a subset of the power set of D. If we go back to how we accomplished the corresponding task for the Hoare power construction, we find that we mapped a non-empty finite subset of $\mathsf{C}D$ to its lower closure in D. As $\mathsf{P}_{\mathsf{fne}}\mathsf{C}D$ is ordered differently for the Smyth construction, we map a finite subset of $\mathsf{C}D$ to its upper closure in D. But in order to make this an order-preserving function we have to order $\mathsf{P}D$ by reverse inclusion.

This yields an embedding of $(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \supseteq)$ into $(\mathsf{P}D, \supseteq)$. There is a unique Scott-continuous extension e of this embedding to $\mathsf{P}_{\mathsf{A}}^{\sharp}D = \mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \supseteq)$. The image of $\mathsf{P}_{\mathsf{A}}^{\sharp}D$ under e is the set of filtered intersections of sets of the form $\uparrow\{c_1\}\cup\cdots\cup\uparrow\{c_n\}$ where $c_1, \ldots, c_n \in \mathsf{C}D$. There is a nicer characterization for this subset of $\mathsf{P}D$.

First we observe that all sets in the image of $\mathsf{P}_{\mathsf{A}}^{\sharp}D$ are upper closed. We claim that the image of $\mathsf{P}^\sharp_{\mathsf{A}} D$ in $\mathsf{P} D$ is the set of non-empty saturated Scott-compact subsets of D. In order to prove this we need some results for sober spaces which can be found in [HM81] where they are shown with the help of spectral theory of complete lattices. A direct proof of Lemma 7.2 and Theorem 7.4 has first been given in [KP93].

Recall that an open subset of a topological space is prime (in the open set lattice of X) if it is not the intersection of two of its proper open supersets. As prime open sets obviously correspond to irreducible closed sets (via complementation), a T_0-space X is sober iff the only prime elements of ΩX are complements of singleton closures.

Lemma 7.2 (Proposition 2.19 in [HM81]) *Let X be a sober space and let $\mathcal{S}$ be a filtered family of compact saturated subsets of X.*

(i) If $\bigcap_{\downarrow} \mathcal{S} \subseteq U$ for some open subset U of X then there is $K \in \mathcal{S}$ with $K \subseteq U$.

(ii) The set $\bigcap_{\downarrow} \mathcal{S}$ is also compact in X. If, in addition, all elements of $\mathcal{S}$ are non-empty then $\bigcap_{\downarrow} \mathcal{S}$ is non-empty as well.

Proof: (i) Let $\mathcal{R} := \{U \in \Omega X \mid \bigcap_{\downarrow} \mathcal{S} \subseteq U,\ K \not\subseteq U \text{ for all } K \in \mathcal{S}\}$. Suppose that $\mathcal{R}$ is non-empty since otherwise (i) is true. Then $\mathcal{R}$ is an inductive system: If $\mathcal{C}$ is a chain in $\mathcal{R}$ then, by compactness of K, we have $K \subseteq \bigcup \mathcal{C}$ iff $K \subseteq U$ for some $U \in \mathcal{C}$ for all $K \in \mathcal{S}$. Hence $K \not\subseteq \bigcup \mathcal{C}$ for all $K \in \mathcal{S}$ which proves that $\bigcup \mathcal{C} \in \mathcal{R}$. By Zorn's Lemma there is a maximal element, say U, in $\mathcal{R}$. We show that U is prime. Let $U = V \cap V'$ for open subsets V and V' of X with $U \subset V, V'$. By maximality of U in $\mathcal{R}$ there are K and K' in $\mathcal{S}$ with $K \subseteq V$ and $K' \subseteq V'$. As $\mathcal{S}$ is filtered, there is $K'' \in \mathcal{S}$ with $K'' \subseteq K \cap K' \subseteq U$ which contradicts $U \in \mathcal{R}$. By sobriety of X there is $x \in X$ with $U = X \setminus \mathsf{cl}\{x\}$. Since $x \in V$ holds for all open V with $U \subset V$ and because all elements of $\mathcal{S}$ are saturated we must have $x \in K$ for all $K \in \mathcal{S}$. As this contradicts $\bigcap_{\downarrow} \mathcal{S} \subseteq U$ we conclude that $\mathcal{R}$ is empty.

(ii) Follows from (i). ∎

Obviously, the sets of the form $\uparrow\{d_1, \ldots, d_n\}$, where $d_1, \ldots, d_n \in D$, are Scott-compact saturated subsets of D. Hence, according to Lemma 7.2, all sets in the image of $\mathsf{P}^\sharp_{\mathsf{A}} D$ under $e\colon \mathsf{P}^\sharp_{\mathsf{A}} D \longrightarrow \mathsf{P} D$ are non-empty, Scott-compact and saturated. Therefore it is sufficient to prove that all non-empty saturated compact subsets of D can be written as filtered intersections of sets of the form $\uparrow\{c_1, \ldots, c_n\}$ where $c_1, \ldots, c_n \in \mathsf{C} D$.

Lemma 7.3 *If K is a non-empty Scott-compact saturated subset of a domain D then $K = \bigcap^{\downarrow}\{\uparrow\{d_1, \ldots, d_n\} \mid K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}\}$.*

Proof: Obviously, $K \subseteq \bigcap^{\downarrow}\{\uparrow\{d_1, \ldots, d_n\} \mid K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}\}$. On the other hand we know that every saturated set is the intersection of its open neighbourhoods. By compactness of K this yields

$$K = \bigcap^{\downarrow}\{\twoheaduparrow\{d_1, \ldots, d_n\} \mid K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}\}.$$

But if $K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}$ for some $d_1, \ldots, d_n \in D$ then, by the interpolation property, for all $k \in K$ we can find $e_k \in \twoheaduparrow\{d_1, \ldots, d_n\}$ with $e_k \ll k$. Again by compactness of K there are $d'_1, \ldots, d'_m \in \twoheaduparrow\{d_1, \ldots, d_n\}$ such that K is a subset of $\twoheaduparrow\{d'_1, \ldots, d'_m\}$. Hence

$$K \subseteq \twoheaduparrow\{d'_1, \ldots, d'_m\} \subseteq \uparrow\{d'_1, \ldots, d'_m\} \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}$$

and we may conclude

$$\begin{array}{ll} & \bigcap^{\downarrow}\{\uparrow\{d_1, \ldots, d_n\} \mid K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}\} \\ \subseteq & \bigcap^{\downarrow}\{\twoheaduparrow\{d_1, \ldots, d_n\} \mid K \subseteq \twoheaduparrow\{d_1, \ldots, d_n\}\} \\ = & K. \end{array}$$

∎

For a T_0-space X let $\mathsf{K}X$ denote the set of non-empty compact saturated subsets of X ordered by reverse inclusion. For an algebraic domain D we have just seen that $\mathsf{P}^\sharp_{\mathsf{A}} D$ is isomorphic to $\mathsf{K}_{\mathsf{D}} D := \mathsf{K}\mathsf{T}_\sigma D$.

7.2 Directed complete partial orders

7.2.1 A Smyth power construction

The observations from the last section allow us to define a Smyth power construction for sober dcpo's. For a non-sober domain D we cannot guarantee $\mathsf{K}_{\mathsf{D}}D$ to have directed joins, that is filtered intersections. Hence we search for another way to understand the Smyth power construction. For that, let $\Omega_{ne}X$ denote the set of non-empty open subsets of a topological space X. The following result is also known as the Hofmann-Mislove Theorem.

Theorem 7.4 (Theorem 2.16 in [HM81]) *If X is a sober space then the set $\mathsf{K}X$ of non-empty compact saturated subsets of X ordered by reverse inclusion is isomorphic to the set of Scott-open filters in $\Omega_{ne}X$ ordered by inclusion. The isomorphisms are given by*

$$\begin{aligned} K &\longmapsto \mathcal{F}(K) := \{U \in \Omega_{ne}X \mid K \subseteq U\} && \textit{where } K \in \mathsf{K}X \textit{ and} \\ \varphi &\longmapsto \textstyle\bigcap^{\downarrow} \varphi && \textit{where } \varphi \subseteq \Omega_{ne}X \textit{ is a Scott-open filter.} \end{aligned}$$ ■

For an arbitrary dcpo D let $\mathsf{F}_{\mathsf{D}}D$ be the set of Scott-open filters in $\Omega_{ne}D$ ordered by reverse inclusion. Recall that by default the topology we consider for a dcpo is the Scott topology; hence $\Omega_{ne}D$ is to denote the lattice of non-empty Scott-open subsets of D. As directed unions of Scott-open filters in $\Omega_{ne}D$ again are Scott-open filters, $\mathsf{F}_{\mathsf{D}}D$ is indeed a dcpo. Later on it will become obvious that this construction does not satisfy our expectations. For this reason, we introduce another Smyth power dcpo construction. This power construction has been studied by R. Heckmann in [Hec92b]. His goal, however, is different from ours; he does not consider the monad associated with this construction.

We can extend F_{D} to an endofunctor on **DCPO** that yields a monad. From a Scott-continuous function $f\colon D \longrightarrow E$ between dcpo's we obtain another Scott-continuous function $\mathsf{F}_{\mathsf{D}}f\colon \mathsf{F}_{\mathsf{D}}D \longrightarrow \mathsf{F}_{\mathsf{D}}E$ if we set

$$\mathsf{F_D} f(\varphi) := \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \in \varphi\}$$

for all $\varphi \in \mathsf{F_D} D$. We show that $\mathsf{F_D} f$ is well-defined: If $\bigcup^{\uparrow}\mathcal{V} \in \mathsf{F_D} f(\varphi)$ for some directed family $\mathcal{V}$ in $\Omega_{ne}\, E$ then $\bigcup^{\uparrow}\{f^{-1}(V) \mid V \in \mathcal{V}\} = f^{-1}(\bigcup^{\uparrow}\mathcal{V}) \in \varphi$ which, by Scott-openness of φ, implies the existence of some $V \in \mathcal{V}$ with $f^{-1}(V) \in \varphi$ and thus $V \in \mathsf{F_D} f(\varphi)$. We now show that $\mathsf{F_D} f$ is indeed Scott-continuous. So let $\mathcal{S}$ be a directed family of Scott-open filters in $\Omega_{ne}\, D$. Then

$$\begin{aligned}
\mathsf{F_D} f\Big(\bigcup^{\uparrow}\mathcal{S}\Big) &= \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \in \bigcup^{\uparrow}\mathcal{S}\} \\
&= \{V \in \Omega_{ne}\, E \mid \exists\, \varphi \in \mathcal{S} \,.\, f^{-1}(V) \in \varphi\} \\
&= \bigcup^{\uparrow}\{\mathsf{F_D} f(\varphi) \mid \varphi \in \mathcal{S}\}.
\end{aligned}$$

Lemma 7.5 *The above definition makes $\mathsf{F_D}$ an endofunctor on* **DCPO**.

Proof: It is obvious that $\mathsf{F_D}$ preserves identities. Let $f\colon D \longrightarrow E$ and $g\colon E \longrightarrow F$ be morphisms in **DCPO**. For any Scott-open filter φ in $\Omega_{ne}\, D$ we get

$$\begin{aligned}
\Big(\mathsf{F_D} g \circ \mathsf{F_D} f\Big)(\varphi) &= \mathsf{F_D} g\Big(\{V \in \Omega_{ne}\, E \mid f^{-1}(V) \subseteq \varphi\}\Big) \\
&= \{W \in \Omega_{ne}\, F \mid f^{-1}\Big(g^{-1}(W)\Big) \in \varphi\} \\
&= \mathsf{F_D}(g \circ f)(\varphi).
\end{aligned}$$

∎

7.2.2 The monad

In order to define a monad we still need two natural transformations. We can embed any dcpo D into $\mathsf{F_D} D$ if we map any element x of D to the Scott-open filter of its open neighbourhoods $\mathcal{N}_D(x)$. This defines a function $\mathcal{N}_D\colon D \longrightarrow \mathsf{F_D} D$ which is Scott-continuous since for any directed subset S of D and any $U \in \Omega_{ne}\, D$ we have $\bigvee^{\uparrow} S \in U$ iff there is $s \in S$ with $s \in U$. This assignment makes $\mathcal{N}$ a natural transformation from the identity on **DCPO** to $\mathsf{F_D}$. To see this, let $f\colon D \longrightarrow E$ be a Scott-continuous function. For any $d \in D$ we have

$$\mathsf{F_D} f\Big(\mathcal{N}_D(d)\Big) = \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \ni d\} = \{V \in \Omega_{ne}\, E \mid V \ni f(d)\} = \mathcal{N}_D\,\Big(f(d)\Big).$$

We can also define a natural transformation $(\mathsf{F}_{\mathsf{D}})^2 \longrightarrow \mathsf{F}_{\mathsf{D}}$. For every dcpo D we set $\mu_D(\Phi) := \{U \in \Omega_{ne}\, D \mid \{\varphi \mid U \in \varphi\} \in \Phi\}$ for every Φ in $(\mathsf{F}_{\mathsf{D}})^2 D$. First of all we show that μ_D is well-defined, that is $\{U \in \Omega_{ne}\, D \mid \{\varphi \mid U \in \varphi\} \in \Phi\}$ is a Scott-open filter in $\Omega_{ne}\, D$ for all $\Phi \in (\mathsf{F}_{\mathsf{D}})^2 D$.

Lemma 7.6 *If U is a Scott-open subset of a dcpo D then $\Box U := \{\varphi \in \mathsf{F}_{\mathsf{D}} D \mid U \in \varphi\}$ is Scott-open in $\mathsf{F}_{\mathsf{D}} D$.*

Proof: Obviously, $\Box U$ is an upper subset of $\mathsf{F}_{\mathsf{D}} D$. If $\mathcal{S}$ is a directed family in $\mathsf{F}_{\mathsf{D}} D$ with $\bigcup^{\uparrow} \mathcal{S} \in \Box U$ then $U \in \varphi$ must hold for some $\varphi \in \mathcal{S}$. Hence there is $\varphi \in \mathcal{S}$ with $\varphi \in \Box U$ which proves Scott-openness of $\Box U$. ∎

Obviously, $\mu_D(\Phi) = \{U \in \Omega_{ne}\, D \mid \Box U \in \Phi\}$ is an upper set in $\mathsf{F}_{\mathsf{D}} D$. It is closed with respect to binary intersections since $\Box U \cap \Box U' = \Box(U \cap U')$ for all $U, U' \in \Omega D$. Furthermore, this set is Scott-open since $\bigcup^{\uparrow}\{\Box U \mid U \in \mathcal{U}\} = \Box \bigcup^{\uparrow} \mathcal{U}$ for all directed subsets $\mathcal{U}$ of $\Omega_{ne}\, D$. We also have to show that for all dcpo's D the function μ_D is Scott-continuous. If $\mathfrak{S}$ is a directed family in $(\mathsf{F}_{\mathsf{D}})^2 D$ then

$$
\begin{aligned}
\mu_D\left(\textstyle\bigcup^{\uparrow} \mathfrak{S}\right) &= \{U \in \Omega_{ne}\, D \mid \Box U \in \textstyle\bigcup^{\uparrow} \mathfrak{S}\} \\
&= \{U \in \Omega_{ne}\, D \mid \exists \Phi \in \mathfrak{S} \,.\, \Box U \in \Phi\} \\
&= \textstyle\bigcup^{\uparrow}\{\mu_D(\Phi) \mid \Phi \in \mathfrak{S}\}.
\end{aligned}
$$

Let $f\colon D \longrightarrow E$ be a morphism in **DCPO**. Note that, for any open subset V of E, we get $\Box f^{-1}(V) = \{\varphi \in \mathsf{F}_{\mathsf{D}} D \mid f^{-1}(V) \in \varphi\} = (\mathsf{F}_{\mathsf{D}} f)^{-1}(\Box V)$. Now for all $\Phi \in (\mathsf{F}_{\mathsf{D}})^2 D$ we calculate

$$
\begin{aligned}
\mu_E\left((\mathsf{F}_{\mathsf{D}})^2 f(\Phi)\right) &= \mu_E\left(\{\mathcal{V} \in \Omega_{ne}\, \mathsf{F}_{\mathsf{D}} E \mid (\mathsf{F}_{\mathsf{D}} f)^{-1}(\mathcal{V}) \in \Phi\}\right) \\
&= \{V \in \Omega_{ne}\, E \mid (\mathsf{F}_{\mathsf{D}} f)^{-1}(\Box V) \in \Phi\} \\
&= \{V \in \Omega_{ne}\, E \mid \Box f^{-1}(V) \in \Phi\} \\
&= \mathsf{F}_{\mathsf{D}} f(\mu_D(\Phi)),
\end{aligned}
$$

which shows that μ is indeed a natural transformation $(\mathsf{F}_{\mathsf{D}})^2 \longrightarrow \mathsf{F}_{\mathsf{D}}$.

Proposition 7.7 *The triple* $(\mathsf{F_D}, \mathcal{N}, \mu)$ *is a monad on* **DCPO**.

Proof: It remains to prove commutativity of the diagrams defining a monad. For any dcpo D and any $\varphi \in \mathsf{F_D} D$ we have

$$\begin{aligned}
\mu_D\left(\mathcal{N}_{\mathsf{F_D} D}(\varphi)\right) &= \mu_D\left(\{\mathcal{U} \in \Omega_{ne}\, \mathsf{F_D} D \mid \varphi \in \mathcal{U}\}\right) \\
&= \{U \in \Omega_{ne}\, D \mid \varphi \in \Box U\} \\
&= \varphi \\
&= \{U \in \Omega_{ne}\, D \mid \mathcal{N}_D^{-1}(\Box U) \in \varphi\} \\
&= \mu_D\left(\{\mathcal{U} \in \Omega_{ne}\, \mathsf{F_D} D \mid \mathcal{N}_D^{-1}(\mathcal{U}) \in \varphi\}\right) \\
&= \mu_D\left(\mathsf{F_D} \mathcal{N}_D(\varphi)\right)
\end{aligned}$$

which proves the commutativity of the two triangular diagrams. In order to show that the rectangular diagram also commutes, let $\boldsymbol{\Phi}$ be an element of $(\mathsf{F_D})^3 D$. Because $\mu_D^{-1}(\Box U) = \Box\Box U$ for all $U \in \Omega_{ne}\, D$, we have

$$\begin{aligned}
\mu_D\left(\mu_{\mathsf{F_D} D}(\boldsymbol{\Phi})\right) &= \{U \in \Omega_{ne}\, D \mid \Box\Box U \in \boldsymbol{\Phi}\} \\
&= \{U \in \Omega_{ne}\, D \mid \mu_D^{-1}(\Box U) \in \boldsymbol{\Phi}\} \\
&= \mu_D\left(\mathsf{F_D} \mu_D(\boldsymbol{\Phi})\right).
\end{aligned}$$

■

7.2.3 The algebras for the monad

In the remainder of this section we will examine whether $\mathsf{F_D}$ gives the free deflationary semilattice for **DCPO**. Recall that a deflationary semilattice in **DCPO** is a dcpo with binary meets that distribute over directed joins. We call such a poset a $\wedge$-continuous $\wedge$-semilattice.

Note that every dcpo of the form $\mathsf{F_D} D$ is a $\wedge$-semilattice since the intersection of two Scott-open filters is again a Scott-open filter. Also, binary meets distribute over directed joins in $\mathsf{F_D} D$. Hence any $\mathsf{F_D} D$ is a deflationary semilattice in **DCPO**.

Lemma 7.8 *Every* $\mathsf{F_D}$*-algebra is a deflationary semilattice in* **DCPO**.

Proof: This follows from Proposition 5.6. ■

Moreover, every morphism of F_{D}-algebras preserves the semilattice structure, that is binary meets.

Lemma 7.9 *If $f\colon D \longrightarrow E$ is a Scott-continuous function between dcpo's then the function $\mathsf{F}_{\mathsf{D}} f\colon \mathsf{F}_{\mathsf{D}} D \longrightarrow \mathsf{F}_{\mathsf{D}} E$ preserves binary meets.*

Proof: Let φ and φ' be elements of $\mathsf{F}_{\mathsf{D}} D$. We calculate

$$\begin{aligned}
\mathsf{F}_{\mathsf{D}} f(\varphi) \cap \mathsf{F}_{\mathsf{D}} f(\varphi') &= \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \in \varphi\} \cap \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \in \varphi'\} \\
&= \{V \in \Omega_{ne}\, E \mid f^{-1}(V) \in \varphi \cap \varphi'\} \\
&= \mathsf{F}_{\mathsf{D}} f(\varphi \cap \varphi').
\end{aligned}$$

∎

Lemma 7.10 *Every morphism of F_{D}-algebras preserves binary meets.*

Proof: The semilattice structure for an F_{D}-algebra (D, ξ) is $\wedge = \xi \circ \cap \circ (\mathcal{N}_D \times \mathcal{N}_D)$. Hence for every morphism of F_{D}-algebras $f\colon (D, \xi) \longrightarrow (E, \zeta)$ we have to show that $f \circ \xi \circ \cap \circ (\mathcal{N}_D \times \mathcal{N}_D) = \zeta \circ \cap \circ (\mathcal{N}_E \times \mathcal{N}_E) \circ (f \times f)$. We calculate

$$\begin{aligned}
& f \circ \xi \circ \cap \circ (\mathcal{N}_D \times \mathcal{N}_D) \\
= \;& \zeta \circ \mathsf{F}_{\mathsf{D}} f \circ \cap \circ (\mathcal{N}_D \times \mathcal{N}_D) && (f \text{ morphism of } \mathsf{F}_{\mathsf{D}}\text{-algebras}) \\
= \;& \zeta \circ \cap \circ (\mathsf{F}_{\mathsf{D}} f \times \mathsf{F}_{\mathsf{D}} f) \circ (\mathcal{N}_D \times \mathcal{N}_D) && (\mathsf{F}_{\mathsf{D}} f \text{ preserves binary meets}) \\
= \;& \zeta \circ \cap \circ (\mathcal{N}_E \times \mathcal{N}_E) \circ (f \times f) && (\text{Naturality of } \mathcal{N}).
\end{aligned}$$

∎

Thus we achieve the following result.

Proposition 7.11 *The category of F_{D}-algebras is a subcategory of the category of deflationary semilattices in* **DCPO**. ∎

However, the reverse is not true. The reason for this is that any F_{D}-algebra has more than just binary meets.

Lemma 7.12 *Every F_D-algebra has meets of non-empty compact subsets.*

Proof: Let (D, ξ) be an F_D-algebra. Each non-empty compact subset K of D gives rise to a Scott-open filter in $\Omega_{ne}\, D$, namely the filter $\mathcal{F}(K)$ of its open neighbourhoods. We will show that $\xi(\mathcal{F}(K)) = \bigwedge K$. For any $k \in K$ the set $\mathcal{N}_D(k)$ of open neighbourhoods of k contains $\mathcal{F}(K)$. Therefore $k = \xi(\mathcal{N}_D(k)) \geq \xi(\mathcal{F}(K))$ for all $k \in K$ which proves that $\xi(\mathcal{F}(K))$ is a lower bound for K. If d is a lower bound for K then every open neighbourhood of d is a neighbourhood of K. Hence $d = \xi(\mathcal{N}_D(d)) \leq \xi(\mathcal{F}(K))$. This proves that $\xi(\mathcal{F}(K))$ is indeed the greatest lower bound for K. ∎

We will now give an example that shows that F_D does not give the free deflationary semilattice for **DCPO**.

7.2.4 Why F_D does not give the free deflationary semilattice for DCPO

We construct a dcpo which is a deflationary semilattice and which has lots of compact subsets that do not have a meet. Let D be the dcpo of non-empty subsets of the unit interval $[0, 1]$ which have countable complements, the order on D being subset inclusion. First of all we will show that D is a deflationary semilattice in **DCPO**. For that purpose we have to prove that the binary meet function from $D \times D$ to D is Scott-continuous. This is equivalent to binary meets distributing over directed joins. Since in D these operations are given by binary intersection and directed union, respectively, D is indeed a deflationary semilattice in **DCPO**.

We will show that D contains a non-empty Scott-compact subset that does not have a meet in D. By the last lemma this implies that D is not an F_D-algebra. Let $K := \{[0, 1] \setminus \{x\} \mid x \in [0, 1/2]\}$. To show that K is compact (with respect to the Scott topology), let U be a non-empty Scott-open subset of D. We claim that all but finitely many of the elements of K belong to U. Assume there are infinitely many

$x \in [0,1/2]$ such that $[0,1] \setminus \{x\} \in D \setminus U$. We can therefore find a sequence $(z_n)_{n\in\mathbb{N}}$ in $[0,1/2]$ with $[0,1] \setminus \{z_n\} \in D \setminus U$ for all $n \in \mathbb{N}$ such that $z_n = z_m$ iff $n = m$. But

$$[0,1] = [0,1] \setminus \emptyset = [0,1] \setminus \bigcap_{m\in\mathbb{N}}^{\downarrow} \{z_n \mid n \geq m\} = \bigcup_{m\in\mathbb{N}}^{\uparrow} ([0,1] \setminus \{z_n \mid n \geq m\})$$

is an element of U. Hence by Scott-openness of U we can find $m \in \mathbb{N}$ such that $[0,1] \setminus \{z_n \mid n \geq m\} \in U$. This implies $X \setminus \{z_n\} \in U$ for all $n \geq m$ which contradicts the choice of the z_n. Consequently, any non-empty Scott-open subset of D contains all but finitely many elements of K. This proves the compactness of K in D. But K does not have any lower bound in D and thus we are done.

It can also be deduced from R. Heckmann's results that F_D does not give the free deflationary semilattice for **DCPO** since he shows that this functor does not have the necessary universal property [Hec92a]. In [HP79] it is shown that the forgetful functor from $\mathbf{DCPO}^{\vee}$ to **DCPO** does have a left adjoint. However, this left adjoint is not explicitly given. It seems that this functor does not map a dcpo D to a subset of its power set.

7.2.5 Domains

The reason that F_D does not give the free deflationary semilattice for **DCPO** is that it is too big. For if we take all Scott-open filters in $\Omega_{ne}\, D$ then we automatically get the filters of open neighbourhoods of compact subsets of D. This implies that any F_D-algebra must have meets of compact sets while we only require meets of non-empty finite sets. We therefore will give an alternative definition for a generalization of the Smyth power construction to arbitrary dcpo's. As this alternative construction is easier to handle and closer to what we want, we will not examine the monad $(\mathsf{F}_\mathsf{D}, \mathcal{N}, \mu)$ in this general setting any longer. Instead we will investigate a subcategory of **DCPO** where the F_D does give the free deflationary semilattice.

We restrict the monad $(\mathsf{F}_\mathsf{D}, \mathcal{N}, \mu)$ to the category **Dom** of domains. As any domain is sober we may consider the set of compact saturated subsets $\mathsf{K}_\mathsf{D} D$ instead of taking

Scott-open filters of non-empty open sets. First of all we show that **Dom** is closed with respect to the application of $\mathsf{K_D}$. Because of the Hofmann-Mislove Theorem this result is a consequence of the Lawson duality for domains [Law79].

Proposition 7.13 *If D is a continuous dcpo so is $\mathsf{K_D}D$.*

Proof: By Lemma 7.3, $K = \bigcap\!\!\downarrow\{\uparrow\{d_1,\dots,d_n\} \mid K \subseteq \Uparrow\{d_1,\dots,d_n\}\}$ if K is a non-empty compact saturated subset of D. Note that any set of the form $\uparrow\{d_1,\dots,d_n\}$ with $\Uparrow\{d_1,\dots,d_n\} \supseteq K$ is way below K in $\mathsf{K_D}D$: If $K \supseteq \bigcap\!\!\downarrow \mathcal{S}$ for some directed family $\mathcal{S}$ in $\mathsf{K_D}D$ then, by Lemma 7.2 (i), there is $K' \in \mathcal{S}$ with

$$K' \subseteq \Uparrow\{d_1,\dots,d_n\} \subseteq \uparrow\{d_1,\dots,d_n\}$$

since $\Uparrow\{d_1,\dots,d_n\}$ is an open neighbourhood for K. Continuity of $\mathsf{K_D}D$ follows from the fact that $\{\uparrow\{d_1,\dots,d_n\} \mid K \subseteq \Uparrow\{d_1,\dots,d_n\}\}$ is a directed subset of $\Downarrow\{K\}$ in $\mathsf{K_D}D$ with join (that is intersection) K. ■

We therefore get a monad on **Dom** that is equivalent to $(\mathsf{F_D}, \mathcal{N}, \mu)$: As a functor, take $\mathsf{K_D}$. A natural transformation from the identity on **Dom** to $\mathsf{K_D}$ is $\uparrow$ where $\uparrow_D(d) := \uparrow\{d\}$. For the natural transformation $(\mathsf{K_D})^2 \longrightarrow \mathsf{K_D}$ take $\bigcup$ where the components of $\bigcup$ are defined by $\bigcup_D(\mathcal{K}) = \bigcup \mathcal{K}$. Since the resulting monad $(\mathsf{K_D}, \uparrow, \bigcup)$ is equivalent to $(\mathsf{F_D}, \mathcal{N}, \mu)$ on **Dom** it is enough to characterize the $\mathsf{K_D}$-algebras. For this it is sufficient to show that every deflationary semilattice in **Dom** gives rise to a $\mathsf{K_D}$-algebra and that every morphism of such semilattices is a morphism of the corresponding $\mathsf{K_D}$-algebras because of Proposition 7.11.

Lemma 7.14 *Every deflationary semilattice in* **Dom** *has a unique structure map making it a $\mathsf{K_D}$-algebra. This structure map maps every non-empty compact and saturated set to its meet.*

Proof: Let D be a deflationary semilattice in **Dom**. This means that D is a domain with binary meets that distribute over directed joins. For every $K \in \mathsf{K_D}D$

set $\xi_D(K) := \bigvee^{\uparrow}\{d_1 \wedge \cdots \wedge d_n \mid \Uparrow\{d_1, \ldots, d_n\} \supseteq K\}$. To see that ξ_D is Scott-continuous, let $\mathcal{S}$ be a directed family in $\mathsf{K_D}D$. Obviously, $\bigvee^{\uparrow}_{K \in \mathcal{S}} \xi_D(K) \leq \xi_D(\bigcap_{\downarrow} \mathcal{S})$. But if $d \ll \xi_D(\bigcap_{\downarrow} \mathcal{S})$ then $\Uparrow\{d\}$ is an open neighbourhood of $\bigcap_{\downarrow} \mathcal{S}$. Sobriety of D allows us to apply Lemma 7.2 (i), so there exists $K \in \mathcal{S}$ with $\Uparrow\{d\} \supseteq K$. Therefore, $d \leq \xi_D(K) \leq \xi_D(\bigcap_{\downarrow} \mathcal{S})$. Furthermore, $\xi_D(K) = \bigwedge K$ for all non-empty compact and saturated $K \subseteq D$: It is obvious that $\xi_D(K)$ is a lower bound for K. If $d \in D$ is any lower bound for K then $\Uparrow\{d'\}$ is a neighbourhood of K for all $d' \ll d$. So we can find $d_1, \ldots, d_n \in D$ with $\Uparrow\{d_1, \ldots, d_n\} \supseteq K$ and $\Uparrow\{d'\} \supseteq \uparrow\{d_1, \ldots, d_n\}$. Hence $d' \leq d_1 \wedge \cdots \wedge d_n \leq \xi_D(K)$. By Lemma 4.4, ξ_D is the unique structure map for D which completes the proof. ■

Lemma 7.15 *Let $f \colon D \longrightarrow E$ be a morphism of the deflationary semilattices D and E in* **Dom**. *Then f preserves meets of non-empty compact sets.*

Proof: By the previous lemma we know that meets of non-empty compact sets do exist in D and E. Let $K \in \mathsf{K_D}D$. Obviously, $f(\bigwedge K) \leq \bigwedge f(K)$. If $e \ll \bigwedge f(K)$ then $f^{-1}(\Uparrow\{e\})$ is an open neighbourhood of K. Hence, by Lemmata 7.2 (i) and 7.3, there are $d_1, \ldots, d_n \in D$ with $\Uparrow\{d_1, \ldots, d_n\} \supseteq K$ and $f^{-1}(\Uparrow\{e\}) \supseteq \uparrow\{d_1, \ldots, d_n\}$. Therefore, $e \leq f(d_1) \wedge \cdots \wedge f(d_n) = f(d_1 \wedge \cdots \wedge d_n) \leq f(\bigwedge K)$. ■

Theorem 7.16 *The category of $\mathsf{K_D}$-algebras in* **Dom** *is the category of deflationary semilattices in* **Dom**.

Proof: Because of Lemma 7.14 and by Proposition 7.11 it remains to prove that any morphism of deflationary semilattices is a morphism of the corresponding $\mathsf{K_D}$-algebras. This follows from Lemma 4.4 (ii) and from the last lemma. ■

These results for $\mathsf{F_D}$ restricted to **Dom** have already been published in [Sch93a].

7.2.6 Another Smyth power construction

The above observations call for a smaller Smyth power domain. Following R. Heckmann [Hec92b], we introduce the following notion.

Definition. A filter φ in $\Omega_{ne}X$, where X is a T_0-space, is called *strong* if for every $U \in \varphi$ there is a finite subset F of U such that the filter of open neighbourhoods of F is a subset of φ. By $\mathsf{F}^s_{\mathsf{D}}D$ we denote the set of strong filters in $\Omega_{ne}D$ ordered by inclusion. Note that the filter of open neighbourhoods of $\{d_1, \ldots, d_n\} \subseteq D$ is $\mathcal{N}_D(d_1) \cap \cdots \cap \mathcal{N}_D(d_n)$.

Obviously, any strong filter is Scott-open in $\Omega_{ne}X$. The reverse, however, is not true. For an example, see Section 7.3.3. It is not hard to show that the set of strong filters in ΩX is closed with respect to binary intersection and directed union. Hence $\mathsf{F}^s_{\mathsf{D}}D$ is a substructure of the $\wedge$-continuous $\wedge$-semilattice $\mathsf{F}_{\mathsf{D}}D$.

If X is a sober space then strong filters in $\Omega_{ne}X$ correspond to certain compact sets. A subset K of a sober space X is *strongly compact* iff every neighbourhood of K contains a finitely generated upper set that includes K. The use of strongly compact sets for defining a Smyth power construction is due to Smyth [Smy78]. It is an easy consequence of the Hofmann-Mislove Theorem 7.4 that, for any sober space X, the dcpo of strong filters in $\Omega_{ne}X$ is isomorphic to the dcpo of strongly compact subsets of X ordered by reverse inclusion.

There are dcpo's which have compact sets that are not strongly compact: Consider once more the example from Section 7.2.4. A subset K of the dcpo D from this example is strongly compact iff $\bigcup\{A \subseteq [0,1] \mid [0,1] \setminus A \in K\}$ is countable. Hence, if K is a strongly compact subset of D then its meet exists and is the intersection of all open sets in K. Therefore the compact set we used to show that D is not an F_{D}-algebra is not strongly compact. Another example for a compact subset of a dcpo that is not strongly compact is given in Chapter 7 of [Hec92b].

Proposition 7.17 *In a domain, every Scott-open filter is strong.*

Proof: Let U be an element of the Scott-open filter φ in $\Omega_{ne}\,D$ where D is a domain. Since $U = \bigcup\{\Uparrow\{d_1,\ldots,d_n\} \mid d_1,\ldots,d_n \in U\}$ there are $d_1,\ldots,d_n \in U$ such that $\Uparrow\{d_1,\ldots,d_n\} \in \varphi$ as φ is Scott-open. Obviously, the filter of open neighbourhoods of $\uparrow\{d_1,\ldots,d_n\}$ is a subset of φ. ∎

The preceding proposition tells us that for a domain D we have $\mathsf{F}_{\mathsf{D}} D = \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$, so by considering the construction $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$ instead of F_{D} we do not lose the nice properties we had for F_{D} restricted to **Dom**.

In order to make $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$ a functor, we show that the restriction to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$ and co-restriction to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} E$ of $\mathsf{F}_{\mathsf{D}} f$ is well-defined for any $f\colon D \longrightarrow E$ in **DCPO**. We have to prove that $\{V \in \Omega_{ne}\,E \mid f^{-1}(V) \in \varphi\}$ is a strong filter in $\Omega_{ne}\,E$ for any $\varphi \in \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$. If $f^{-1}(V) \in \varphi$ for some open subset V of E then there is a finite set $F \subseteq f^{-1}(V)$ such that the filter of open neighbourhoods of F is a subset of φ. Consequently, $f(F)$ is a finite subset of V which satisfies

$$\{V \in \Omega_{ne}\,E \mid f(F) \subseteq V\} \subseteq \{V \in \Omega_{ne}\,E \mid f^{-1}(V) \in \varphi\}.$$

Hence we can make $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$ an endofunctor on **DCPO** if we let $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} f$ be the restriction and co-restriction of $\mathsf{F}_{\mathsf{D}} f$ as just described.

Furthermore, the co-restriction of $\mathcal{N}_D\colon D \longrightarrow \mathsf{F}_{\mathsf{D}} D$ to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$ is also well-defined; we will denote this function by $\mathcal{N}_D$ as well. Therefore, $\mathcal{N}$ is a natural transformation from the identity on **DCPO** to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$.

In order to obtain a monad it remains to give a natural transformation $(\mathsf{F}^{\mathsf{s}}_{\mathsf{D}})^2 \longrightarrow \mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$. For that we define $\boxed{\scriptstyle s}U := \{\varphi \in \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D \mid U \in \varphi\} = \Box U \cap \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$ for any open subset U of some dcpo D. If Φ is an element of $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$ we set $\mu_D(\Phi) := \{U \in \Omega_{ne}\,D \mid \boxed{\scriptstyle s}U \in \Phi\}$. The proof that this defines a filter in $\Omega_{ne}\,D$ is the same as the corresponding proof for the monad $(\mathsf{F}_{\mathsf{D}}, \mathcal{N}, \mu)$. To see that $\mu_D(\Phi)$ is indeed a strong filter in $\Omega_{ne}\,D$ let $U \in \mu_D(\Phi)$. This implies $\boxed{\scriptstyle s}U \in \Phi$. Hence, by strongness of Φ, there are $\varphi_1,\ldots,\varphi_n \in \boxed{\scriptstyle s}U$ with $\mathcal{N}_{\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D}(\varphi_1) \cap \cdots \cap \mathcal{N}_{\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D}(\varphi_n) \subseteq \Phi$. As $\varphi_1,\ldots,\varphi_n \in \boxed{\scriptstyle s}U$ we have $U \in \varphi_1,\ldots,\varphi_n$ by definition of $\boxed{\scriptstyle s}U$. By strongness of the φ_i, there are finite subsets $F_1,\ldots,F_n$ of U

such that the filter of open neighbourhoods of some F_i is contained in φ_i for all $i \in \{1, \ldots, n\}$. Consequently, $F := F_1 \cup \cdots \cup F_n$ is a finite subset of U such that the filter of open neighbourhoods of F is a subset of $\varphi_1 \cap \cdots \cap \varphi_n$. If U' is an open subset of D with $U' \supseteq F$ then $U' \in \varphi_1, \ldots, \varphi_n$ or $\varphi_1, \ldots, \varphi_n \in \boxdot U'$. By Scott openness of $\boxdot U'$ we may conclude $\boxdot U' \in \mathcal{N}_{\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D}(\varphi_1) \cap \cdots \cap \mathcal{N}_{\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D}(\varphi_n)$ which yields $\boxdot U' \in \Phi$ and therefore $U' \in \mu_D(\Phi)$. Hence the filter of open neighbourhoods of F is a subset of $\mu_D(\Phi)$ which proves strongness of $\mu_D(\Phi)$. The proof that μ_D is a Scott-continuous function from $(\mathsf{F}^{\mathsf{s}}_{\mathsf{D}})^2 D$ to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$ and that this definition makes μ a natural transformation $(\mathsf{F}^{\mathsf{s}}_{\mathsf{D}})^2 \longrightarrow \mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$ is the same as for the monad $(\mathsf{F}_{\mathsf{D}}, \mathcal{N}, \mu)$. We have shown:

Proposition 7.18 *The triple* $(\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}, \mathcal{N}, \mu)$ *is a monad on* **DCPO**. ■

We next show that every F_{D}-algebra is an $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$-algebra. Let (D, ξ) be an F_{D}-algebra. Obviously, the restriction of ξ to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$, say ζ, is a left inverse for $\mathcal{N}_D \colon D \longrightarrow \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D$. Furthermore,

$$
\begin{aligned}
\zeta\Big(\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}\zeta(\Phi)\Big) &= \zeta\Big(\{U \in \Omega_{ne}\, D \mid \zeta^{-1}(U) \in \Phi\}\Big) \\
&= \xi\Big(\{U \in \Omega_{ne}\, D \mid (\xi^{-1}(U) \cap \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D) \in \Phi\}\Big) \\
&= \xi\Big(\{U \in \Omega_{ne}\, D \mid (\Box U \cap \mathsf{F}^{\mathsf{s}}_{\mathsf{D}} D) \in \Phi\}\Big) && (\xi \circ \mathsf{F}_{\mathsf{D}}\xi = \xi \circ \mu_D) \\
&= \zeta\Big(\{U \in \Omega_{ne}\, D \mid \boxdot U \in \Phi\}\Big) \\
&= \zeta\Big(\mu_D(\Phi)\Big)
\end{aligned}
$$

for all $\Phi \in (\mathsf{F}^{\mathsf{s}}_{\mathsf{D}})^2 D$.

Consequently, every morphism of F_{D}-algebras is a morphism of $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$-algebras. Hence, from Proposition 7.11, we obtain the following result.

Proposition 7.19 *The category of* $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$*-algebras is a subcategory of the category of deflationary semilattices in* **DCPO**. ■

However, $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$ does not give the free deflationary semilattice for **DCPO** either as is shown on page 18 in [Hec92a].

7.2.7 Unital semilattices

If we want every Smyth power dcpo to have a greatest element then we have to consider Scott-open (or strong) filters in ΩD instead of $\Omega_{ne}\, D$. The greatest element for both of these changed power constructions, say $\dot{\mathsf{F}}_{\mathsf{D}}$ and $\dot{\mathsf{F}}^{\mathsf{s}}_{\mathsf{D}}$, is the whole set ΩD. Therefore $\dot{\mathsf{F}}_{\mathsf{D}} D$ and $\dot{\mathsf{F}}^{\mathsf{s}}_{\mathsf{D}} D$ both are unital deflationary semilattices in **DCPO**. Note that ΩD is inaccessible by arbitrary joins in both, $\dot{\mathsf{F}}_{\mathsf{D}} D$ and $\dot{\mathsf{F}}^{\mathsf{s}}_{\mathsf{D}} D$, since it is the only filter in ΩD containing the empty set. Hence we get $\dot{\mathsf{F}}_{\mathsf{D}} D$ from $\dot{\mathsf{F}}_{\mathsf{D}} D$ by adjoining a greatest element and the corresponding statement is true for the second Smyth power construction. For every $f\colon D \longrightarrow E$ in **DCPO** we define $\dot{\mathsf{F}}_{\mathsf{D}} f$ to be the unique extension of $\mathsf{F}_{\mathsf{D}}\, f$ that maps the greatest element in $\dot{\mathsf{F}}_{\mathsf{D}} D$ to the greatest element in $\dot{\mathsf{F}}_{\mathsf{D}} E$. Therefore any $\dot{\mathsf{F}}_{\mathsf{D}} f$ is a morphism of unital deflationary semilattices in **DCPO**. In the same manner as Proposition 7.11 we can show that the category of $\dot{\mathsf{F}}_{\mathsf{D}}$-algebras is a subcategory of the category of unital deflationary semilattices in **DCPO**. The same is true for the functor $\dot{\mathsf{F}}^{\mathsf{s}}_{\mathsf{D}}$ which is defined in the obvious way.

7.2.8 Remarks on the Smyth power dcpo constructions

Unlike the Hoare power construction it is not obvious how a Smyth power dcpo should be defined for non-continuous dcpo's. Furthermore, we cannot characterize the algebras for either of the two constructions for any subcategory of **DCPO** larger than **Dom**. In a way the reason for this is that we cannot say much about the Scott topology on the Smyth power dcpo's. This seems to be a phenomenon that appears for non-continuous domains in general. For example, there is no order-theoretic characterization of sobriety for dcpo's. The difficulties in solving the open problems of Section 4.6 in [Hec92b] also seem to originate from this fact. In the next section we will generalize both constructions to arbitrary T_0-spaces. The topologies we choose for the different power spaces make it possible to prove much stronger results than what we could show for dcpo's.

Let us briefly consider the naïve approach to obtaining the free deflationary semilattice for **DCPO**: Take all finitely generated upper subsets of D and close the resulting set under filtered intersections. If D is continuous then this process is finished after adding filtered intersections of sets of the form $\uparrow\{d_1, \ldots, d_n\}$ where $d_1, \ldots, d_n \in D$. By that, we obtain $\mathsf{K}_{\mathsf{D}} D = \mathsf{K}^{\mathsf{s}}_{\mathsf{D}} D$. If D is not continuous, however, then we might need to add filtered intersections of filtered intersections of ... of finitely generated upper subsets of D. This process has to end sometime and we can prove properties of the resulting poset, say $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}} D$, by transfinite induction. If D is any dcpo then there is at most one structure map for D with respect to $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}}$ by Lemma 4.4. This structure map has to map every element of $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}} D$ to its meet. In order to prove that this function is Scott-continuous we need an analogue to Lemma 7.2:

Let $\mathcal{S}$ be a filtered family in $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}} D$. If $\bigcap_{\downarrow} \mathcal{S} \subseteq U$ for some Scott-open set $U \subseteq D$ then there is $K \in \mathcal{S}$ with $K \subseteq U$.

So we can only hope to succeed for dcpo's that satisfy this property. However, $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}} D$ need not have this property even if D does. Hence we cannot make $\mathsf{K}^{\mathsf{f}}_{\mathsf{D}}$ an endofunctor on a reasonable subcategory of **DCPO** that is larger than **Dom**.

7.3 Topological spaces

The above Smyth-type power constructions for dcpo's suggest several possible definitions for a Smyth power space. We considered the domain of Scott-open filters of non-empty open subsets rather than that of non-empty compact saturated sets because the latter does not necessarily have directed joins. For topological spaces, we can drop this requirement. As our studies of power constructions for topological spaces are to help us examining power locale constructions we only consider constructions that preserve sobriety.

7.3.1 A naïve approach

Let us first examine the obvious candidate to give the free deflationary semilattice for the category of T_0-spaces. We define $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ to be the set

$$\{\uparrow\{x_1,\ldots,x_n\} \mid x_1,\ldots,x_n \in X\}$$

with the topology generated by the sets of the form $\{\uparrow\{x_1,\ldots,x_n\} \mid x_1,\ldots,x_n \in U\}$ where U is an open subset of X. The specialization order for $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ is reverse inclusion. It is not hard to check that $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ is a deflationary topological semilattice; the corresponding semilattice operation for $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ is (binary) union. Any T_0-space X is embedded into $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ by $\uparrow_X: x \longmapsto \uparrow\{x\}$. This function is continuous because

$$\uparrow_X^{-1}(\{\uparrow\{x_1,\ldots,x_n\} \mid x_1,\ldots,x_n \in U\}) = \{x \in X \mid \uparrow_X(x) \in U\} = U$$

for any open subset U of X. We can make $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$ a functor that yields a monad. If we set $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}f(\uparrow\{x_1,\ldots,x_n\}) := \uparrow\{f(x_1),\ldots,f(x_n)\}$ for any $f: X \longrightarrow Y$ in **Top** and any $x_1,\ldots,x_n \in X$ then we obtain a continuous function $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}f: \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X \longrightarrow \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}Y$. One easily checks that $\uparrow: id_{\mathbf{Top}} \longrightarrow \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$ is a natural transformation. To get a natural transformation $(\mathsf{K}^{\mathsf{f}}_{\mathsf{T}})^2 \longrightarrow \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$, let X be a T_0-space and let $\mathcal{K}$ be an element of $(\mathsf{K}^{\mathsf{f}}_{\mathsf{T}})^2X$. Then $\bigcup_X(\mathcal{K}) := \bigcup\mathcal{K}$ is a continuous function from $(\mathsf{K}^{\mathsf{f}}_{\mathsf{T}})^2X$ to $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$: As $\mathcal{K}$ is of the form $\uparrow\{K_1,\ldots,K_n\}$ for some $K_1,\ldots,K_n \in \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ and since $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ is closed with respect to binary union, $\bigcup_X(\mathcal{K}) = K_1 \cup \cdots \cup K_n$ is an element of $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$. It is easy to see that this defines a continuous function for any T_0-space X and that $\bigcup$ is a natural transformation as desired. Furthermore, $(\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}, \uparrow, \bigcup)$ is a monad.

By Lemma 4.4, a T_0-space X is an algebra for the monad iff it has meets of sets of the form $\uparrow\{x_1,\ldots,x_n\}$, that is meets of non-empty finite sets, such that

$$\bigwedge_X: \uparrow\{x_1,\ldots,x_n\} \longmapsto x_1 \wedge \cdots \wedge x_n$$

is a continuous function from $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ to X. Hence any $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$-algebra is a deflationary semilattice. But if X is a deflationary semilattice then $\bigwedge_X$ need not be continuous:

Let U be an open neighbourhood of some $x \in X$ and let n be an arbitrary positive integer. Then, by continuity of the n-ary meet function from $X \times \cdots \times X$ to X, we can find neighbourhoods $U_1, \ldots, U_n$ of x such that $\wedge_n(U_1 \times \cdots \times U_n) \subseteq U$. Hence $U' := U_1 \cap \cdots \cap U_n$ is an open neighbourhood of x such that $x_1 \wedge \cdots \wedge x_n \in U$ for all $x_1, \ldots, x_n \in U'$. However, to get continuity of $\bigwedge_X$ we need an open neighbourhood U' of x such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in U'$ we have $x_1 \wedge \cdots \wedge x_n = \bigwedge_X(\uparrow\{x_1, \ldots, x_n\}) \in U$. This is equivalent to requesting the existence of a semilattice S in U with $x \in \mathsf{int}\, S$.

Hence by transferring the Smyth power constructions for dcpo's to topological spaces, we cannot expect to get the free deflationary semilattice for **Top**. Rather, the algebras of the corresponding monads all have the following property:

Definition. A topological semilattice is said to have *small semilattices* if each of its points has a neighbourhood basis consisting of semilattices.

It is not difficult to show that the $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$-algebras are the deflationary topological semilattices with small semilattices. We are not able to use this functor in order to define a Smyth power locale since it does not give us suitable equations for the open subsets of $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$. Therefore we will not consider this functor any further.

We will rather study the power construction resulting from taking all non-empty compact saturated subsets of a topological space X. After examining this definition for the category of locally compact sober spaces, we proceed to study the power construction corresponding to $\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$.

7.3.2 A Smyth power construction

Recall that $\mathsf{K}X$ denotes the set of non-empty compact saturated subsets of a T_0-space X ordered by $\supset$. By $\mathsf{K}_{\mathsf{T}}X$ we denote the set $\mathsf{K}X$ with the topology generated by the sets of the form $\Box U := \{K \in \mathsf{K}X \mid K \subseteq U\}$ where U is an open subset of X. This definition of a Smyth power space can be traced to M. Smyth [Smy83].

Note that the specialization order for $\mathsf{K_T}X$ is reverse inclusion and that $\mathsf{K_T}X$ has binary meets which are given by union. Furthermore, the sets of the form $\Box U$, where U is an open subset of X, form a basis for the topology on $\mathsf{K_T}X$ because $\Box U \cap \Box U' = \Box(U \cap U')$ for all $U, U' \in \Omega X$. Also, $\bigcup^{\uparrow}\{\Box U \mid U \in \mathcal{U}\} = \Box \bigcup^{\uparrow} \mathcal{U}$ for any directed subset $\mathcal{U}$ of $\Omega_{ne} X$.

Lemma 7.20 *If X is a sober space then so is $\mathsf{K_T}X$.*

Proof: Let Φ be a completely prime filter of open subsets of $\mathsf{K_T}X$. For such a filter, we define $K := \bigcap^{\downarrow}\{U \in \Omega_{ne} X \mid \Box U \in \Phi\}$. We will show that K is an element of $\mathsf{K_T}X$ and that Φ is the filter of open neighbourhoods of K in $\mathsf{K_T}X$. To see that K is indeed a non-empty compact saturated subset of X note that $\{U \in \Omega_{ne} X \mid \Box U \in \Phi\}$ is a Scott-open filter in $\Omega_{ne} X$. Hence, because of the Hofmann-Mislove Theorem 7.4, K is an element of $\mathsf{K_T}X$. Let us show that Φ is the filter of all open neighbourhoods of K in $\mathsf{K_T}X$. Assume that $\mathcal{U}$ is a subset of ΩX. If $\bigcup_{U \in \mathcal{U}} \Box U$ is a neighbourhood of K in $\mathsf{K_T}X$ then there is $U \in \mathcal{U}$ with $K \in \Box U$. Hence $U \in \{U' \in \Omega_{ne} X \mid \Box U' \in \Phi\}$, again by the Hofmann-Mislove Theorem 7.4, and therefore $\bigcup_{U \in \mathcal{U}} \Box U \in \Phi$. On the other hand, if $\bigcup_{U \in \mathcal{U}} \Box U \in \Phi$ then there is $U \in \mathcal{U}$ with $\Box U \in \Phi$ by complete primeness of Φ. Then $\Box U$ and therefore $\bigcup_{U \in \mathcal{U}} \Box U$ is an open neighbourhood of K. ∎

Note that the sobrification for $\mathsf{K_T}X$ is the space of all Scott-open filters on $\Omega_{ne} X$. To make $\mathsf{K_T}$ an endofunctor on **Top**, set $\mathsf{K_T}f(K) := \uparrow f(K)$ for any continuous function $f\colon X \longrightarrow Y$ and any $K \in \mathsf{K_T}X$. As $f(K)$ is non-empty and compact if K is and since the saturation of a compact set is compact as well, $\mathsf{K_T}f$ is well-defined. Continuity of $\mathsf{K_T}f$ follows from the fact that

$$\begin{aligned}
(\mathsf{K_T}f)^{-1}(\Box V) &= \{K \in \mathsf{K_T}X \mid \uparrow f(K) \subseteq V\} \\
&= \{K \in \mathsf{K_T}X \mid f(K) \subseteq V\} \\
&= \{K \in \mathsf{K_T}X \mid K \subseteq f^{-1}(V)\} \\
&= \Box f^{-1}(V)
\end{aligned}$$

for any open subset V of Y.

We can embed X into $\mathsf{K_T}X$ by setting $\Uparrow_X(x) := \uparrow\{x\}$ for all $x \in X$. Because $\Uparrow_X^{-1}(\Box U) = \{x \in X \mid \uparrow\{x\} \in U\} = U$ for any open subset U of X, the function $\Uparrow_X$ is continuous. Thus we obtain a natural transformation $\Uparrow\colon id_{\mathbf{Top}} \longrightarrow \mathsf{K_T}$ since for all $f\colon X \longrightarrow Y$ in **Top** and all $x \in X$ we have

$$\mathsf{K_T}f(\Uparrow_X(x)) = \uparrow f(\uparrow\{x\}) = \uparrow\{f(x)\} = \Uparrow_Y(f(x)).$$

Furthermore, we can define a natural transformation $\bigcup\colon (\mathsf{K_T})^2 \longrightarrow \mathsf{K_T}$. For any $\mathrm{T_0}$-space X set $\bigcup_X(\mathcal{K}) := \bigcup\mathcal{K}$ for all $\mathcal{K} \in (\mathsf{K_T})^2 X$. To show that $\bigcup_X$ is well-defined, we have to prove that $\bigcup\mathcal{K}$ is non-empty, compact, and saturated if $\mathcal{K}$ is. Obviously, $\bigcup\mathcal{K}$ is non-empty if $\mathcal{K}$ is. Furthermore, if $x \in \uparrow\bigcup\mathcal{K}$ then there is $K \in \mathcal{K}$ with $x \in \uparrow K = K \subseteq \bigcup\mathcal{K}$. Therefore it remains to prove that $\bigcup\mathcal{K}$ is compact. Let $\mathcal{U}$ be an open cover of $\bigcup\mathcal{K}$. For all $K \in \mathcal{K}$ there are finitely many elements of $\mathcal{U}$ covering K. Hence $\{\Box(U_1 \cup \cdots \cup U_n) \mid U_1, \ldots, U_n \in \mathcal{U}\}$ is an open cover of $\mathcal{K}$. By compactness of $\mathcal{K}$, there are $U_1^1, \ldots, U_{n(1)}^1, U_1^2, \ldots, U_{n(m)}^m \in \mathcal{U}$ with

$$\mathcal{K} \subseteq \Box(U_1^1 \cup \cdots \cup U_{n(1)}^1) \cup \cdots \cup \Box(U_1^m \cup \cdots \cup U_{n(m)}^m).$$

Consequently, $\bigcup\mathcal{K} \subseteq U_1^1 \cup \cdots \cup U_{n(m)}^m$. The function $\bigcup_X\colon (\mathsf{K_T})^2 X \longrightarrow \mathsf{K_T}X$ is continuous for any $\mathrm{T_0}$-space X as

$$\begin{aligned}
\textstyle\bigcup_X^{-1}(\Box U) &= \{\mathcal{K} \in (\mathsf{K_T})^2 X \mid \textstyle\bigcup\mathcal{K} \in \Box U\} \\
&= \{\mathcal{K} \in (\mathsf{K_T})^2 X \mid \textstyle\bigcup\mathcal{K} \subseteq U\} \\
&= \{\mathcal{K} \in (\mathsf{K_T})^2 X \mid \mathcal{K} \subseteq \Box U\} \\
&= \Box\Box U
\end{aligned}$$

for any open subset U of X. If $f\colon X \longrightarrow Y$ is a continuous function in **Top** then, for all $K \in (\mathsf{K_T})^2 X$, we have

$$
\begin{aligned}
\mathsf{K_T} f\Big(\bigcup{}_X(\mathcal{K})\Big) &= \uparrow f(\bigcup\mathcal{K}) = \uparrow \bigcup_{K\in\mathcal{K}} f(K) \\
&= \bigcup_{K\in\mathcal{K}} \uparrow f(K) = \bigcup_{K\in\mathcal{K}} \mathsf{K_T} f(K) \\
&= \bigcup \uparrow \{\mathsf{K_T} f(K) \mid K \in \mathcal{K}\} \\
&= \bigcup{}_Y \Big(\uparrow \mathsf{K_T} f(\mathcal{K})\Big) = \bigcup{}_Y \Big((\mathsf{K_T})^2 f(\mathcal{K})\Big)
\end{aligned}
$$

which proves that $\bigcup\colon (\mathsf{K_T})^2 \longrightarrow \mathsf{K_T}$ is a natural transformation.

Proposition 7.21 *The triple* $(\mathsf{K_T}, \uparrow, \bigcup)$ *is a monad on* **Top**.

Proof: It remains to prove the commutativity of the diagrams defining a monad. Let K be an element of $\mathsf{K_T}X$ for some T_0-space X. We calculate

$$
\begin{aligned}
\bigcup{}_X \Big(\uparrow_{\mathsf{K_T}X}(K)\Big) &= \bigcup{}_X \{K' \in \mathsf{K_T}X \mid K' \subseteq K\} \\
&= K \\
&= \bigcup{}_X \Big(\uparrow_{\mathsf{K_T}X} \{\uparrow\{x\} \mid x \in K\}\Big) \\
&= \bigcup{}_X \Big(\mathsf{K_T}\uparrow_X(K)\Big)
\end{aligned}
$$

which proves the commutativity of the two triangular diagrams. Since we also have

$$
\bigcup{}_X(\bigcup{}_{\mathsf{K_T}X}(\mathfrak{K})) = \bigcup(\bigcup\mathfrak{K}) = \bigcup(\uparrow\{\bigcup\mathcal{K} \mid \mathcal{K} \in \mathfrak{K}\}) = \bigcup{}_X(\mathsf{K_T}\bigcup{}_X(\mathfrak{K})),
$$

for all $\mathfrak{K} \in (\mathsf{K_T})^3 X$, the rectangular diagram also commutes. ∎

Lemma 7.22 *Every* $\mathsf{K_T}$*-algebra is a deflationary semilattice in* **Top**.

Proof: First note that any $\mathsf{K_T}X$ is a deflationary semilattice in **Top** since binary meet $\cup\colon \langle K, K'\rangle \longmapsto K \cup K'$ is a continuous function from $\mathsf{K_T}X \times \mathsf{K_T}X$ to $\mathsf{K_T}X$. To see this, let U be an open subset of X. Then

$$
\begin{aligned}
\cup^{-1}(\Box U) &= \{\langle K, K'\rangle \in \mathsf{K_T}X \times \mathsf{K_T}X \mid K \cup K' \subseteq U\} \\
&= \{\langle K, K'\rangle \in \mathsf{K_T}X \times \mathsf{K_T}X \mid K \subseteq U \text{ and } K' \subseteq U\} \\
&= \Box U \times \Box U
\end{aligned}
$$

which is open in $\mathsf{K_T}X \times \mathsf{K_T}X$. Hence any $\mathsf{K_T}$-algebra is a deflationary semilattice in **Top** by Proposition 5.6. ∎

However, the property of being a deflationary topological semilattice is not enough to characterize the category of $\mathsf{K_T}$-algebras.

Proposition 7.23 *A T_0-space X is a $\mathsf{K_T}$-algebra iff it has meets of non-empty compact sets and if $\bigwedge_X : K \longmapsto \bigwedge K : \mathsf{K_T}X \longrightarrow X$ is a continuous function. A continuous function between $\mathsf{K_T}$-algebras is a morphism of these algebras iff it preserves meets of non-empty compact sets.*

Proof: This proposition follows from Lemmata 4.4 and 4.6. ∎

There does not seem to be a more satisfactory characterization of the category of $\mathsf{K_T}$-algebras. It is particularly disturbing that $\mathsf{K_T}X$ still appears in the above description. Nevertheless, besides being a deflationary topological semilattice, we can show another property every $\mathsf{K_T}$-algebra must have: It must have small semilattices. We have already encountered this property at the beginning of this section. Such unital semilattices with compact Hausdorff topologies give a characterization of continuous lattices (see Theorem VI.3.4 in [GHK+80]). Note that every continuous $\wedge$-semilattice, that is every deflationary semilattice in **Dom**, has small semilattices.

Proposition 7.24 *Every $\mathsf{K_T}$-algebra is a deflationary topological semilattice with small semilattices.*

Proof: Let (X, ξ_X) be a $\mathsf{K_T}$-algebra. By Lemma 7.22, X is a deflationary topological semilattice. Therefore it remains to show that X has small semilattices. Let U be an open subset of X and let x be one of the elements of U. Then $\uparrow\{x\} \in \xi_X^{-1}(U)$ and, by continuity of ξ_X, there is an open neighbourhood U' of x such that $\xi_X(\Box U') \subseteq U$. We claim that there is a sub-semilattice of X that contains U' and is contained in U. A sub-semilattice of X that includes U' is obviously given by $S := \{x_1 \wedge \cdots \wedge x_n \mid x_1, \ldots, x_n \in U'\}$. To see that S is contained in U, let $x_1, \ldots, x_n$ be elements of U'. Then $\uparrow\{x_1, \ldots, x_n\} \in \Box U' \subseteq \xi_X^{-1}(U)$ and thus $x_1 \wedge \cdots \wedge x_n = \xi_X(\uparrow\{x_1, \ldots, x_n\}) \in U$. ∎

7.3.3 Why K_T does not give the free deflationary semilattice with small semilattices for Top

We shall prove that K_T does not give the free deflationary semilattice with small semilattices by giving an example of a deflationary topological semilattice with small semilattices which is not a K_T-algebra. Since the Scott topology on the product of two dcpo's need not be the product topology, we cannot re-use the example from Section 7.2.4.

Let us consider the unit interval with the usual (Euclidean) topology. Consider $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ as defined at the beginning of this section.

In order to show that this gives the desired example we have to show that $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ is a deflationary topological semilattice with small semilattices that is not a K_T-algebra. Note that $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ is sober, so even for the category of sober spaces the reverse of Proposition 7.24 is not true. To show sobriety, let $\mathcal{A}$ be a non-empty irreducible closed subset of $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$. Since $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ is a subspace of $\mathsf{K}_\mathsf{T}[0,1]$, the closure of $\mathcal{A}$ in $\mathsf{K}_\mathsf{T}[0,1]$ must be irreducible by Lemma 1.8. By sobriety of $\mathsf{K}_\mathsf{T}[0,1]$ there is some $K \in \mathsf{K}_\mathsf{T}[0,1]$ such that $\mathsf{cl}_{\mathsf{K}_\mathsf{T}[0,1]}\mathcal{A}$ is the closure of $\{K\}$ in $\mathsf{K}_\mathsf{T}[0,1]$. But this closure is $\{K' \in \mathsf{K}_\mathsf{T}[0,1] \mid K' \supseteq K\}$. So if $K \notin \mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$, that is if K is not finite, then all of its supersets are infinite, implying $\mathcal{A} = \mathsf{K}^\mathsf{f}_\mathsf{T}[0,1] \cap \overline{\{K\}} = \emptyset$ which contradicts $\mathcal{A}$ being non-empty. Hence K is in $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ and $\mathcal{A}$ is the closure of $\{K\}$ in $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$.

The binary meet on $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ is given by $\cup\colon \mathsf{K}^\mathsf{f}_\mathsf{T}[0,1] \times \mathsf{K}^\mathsf{f}_\mathsf{T}[0,1] \longrightarrow \mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$. This operation is continuous as it is the restriction and co-restriction of the binary meet on $\mathsf{K}_\mathsf{T}[0,1]$ whose continuity we showed above. Note further that $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ does have small semilattices.

However, $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ is not a K_T-algebra since it does not have all compact meets. The subspace $\{\{x\} \mid x \in [0,1]\}$ of $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$ consisting of all singleton subsets of $[0,1]$ is isomorphic to $[0,1]$. Hence for every compact subset K of $[0,1]$ the set $\{\{k\} \mid k \in K\}$ is compact in $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$. Its meet in $\mathsf{K}^\mathsf{f}_\mathsf{T}[0,1]$, however, would have to be the set K which of course need not be an element of $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}[0,1]$.

A similar argument works for every T_1-space X (instead of the unit interval) that has at least one non-finite compact subset.

So what goes wrong with our construction? Well, it is too big to give the free deflationary topological semilattice with small semilattices as we have just seen. We shall be able to give an alternative definition of a Smyth power space that does exclude examples of the above type.

7.3.4 Locally compact spaces

Before following that path, however, we give a subcategory of **Top** for which K_{T} does gives the free deflationary semilattice with small semilattices. Recall that **LCompS** denotes the full subcategory of **Top** whose objects are the locally compact sober spaces. The following proposition follows from the Hofmann-Mislove Theorem 7.4 and the Lawson duality [Law79].

Proposition 7.25 *If X is a locally compact sober space then $\mathsf{K}_{\mathsf{T}}X$ is a domain with respect to the specialization order. The way below relation in $\mathsf{K}_{\mathsf{T}}X$ is $K' \ll K$ iff $\mathsf{int}\, K' \supseteq K$.*

Proof: Since K_{T} preserves sobriety by Lemma 7.20, $\mathsf{K}_{\mathsf{T}}X$ is a dcpo with respect to the specialization order. If $\mathsf{int}\, K' \supseteq K$ and $K \subseteq \bigcap_{\downarrow} \mathcal{S}$ for a directed subset $\mathcal{S}$ of $\mathsf{K}_{\mathsf{T}}X$ then, because of Lemma 7.2 (i), there is $K'' \in \mathcal{S}$ with $K'' \subseteq \mathsf{int}\, K' \subseteq K'$. Therefore, $K' \ll K$. Let, on the other hand, $K' \ll K$. As $K = \bigcap_{\downarrow}\{K'' \in \mathsf{K}_{\mathsf{T}}X \mid \mathsf{int}\, K'' \supseteq K\}$ there must be $K'' \in \mathsf{K}_{\mathsf{T}}X$ with $\mathsf{int}\, K'' \supseteq K$ and $K' \supseteq K''$. So $K' \supseteq K'' \supseteq \mathsf{int}\, K'' \supseteq K$ which proves $\mathsf{int}\, K' \supseteq K$. Consequently, $\twoheaddownarrow\{K\} = \{K' \in \mathsf{K}_{\mathsf{T}}X \mid \mathsf{int}\, K' \supseteq K\}$. By local compactness of $\mathsf{K}_{\mathsf{T}}X$ we know that $\twoheaddownarrow\{K\}$ is directed. For the same reason, $\bigvee^{\uparrow} \twoheaddownarrow\{K\} = \bigcap_{\downarrow}\{K' \in \mathsf{K}_{\mathsf{T}}X \mid \mathsf{int}\, K' \supseteq K\} = K$ for all $K \in \mathsf{K}_{\mathsf{T}}X$. ∎

Remark. Note that for the last lemma we do not really need sobriety. In order to make $\mathsf{K}_\mathsf{T}X$ a continuous domain, a locally compact space X only needs the following property: If $\mathcal{S}$ is a filtered collection of non-empty compact subsets of X and if U is an open subset of X with $\bigcap_{\downarrow}\mathcal{S} \subseteq U$ then there is $K \in \mathcal{S}$ with $K \subseteq U$.

Lemma 7.26 *If X is a locally compact sober space then the Scott topology and the intrinsic topology on $\mathsf{K}_\mathsf{T}X$ coincide.*

Proof: As $\mathsf{K}_\mathsf{T}X$ is sober, its topology is coarser than the Scott topology. On the other hand, if $\mathcal{U} \subseteq \mathsf{K}_\mathsf{T}X$ is Scott-open and K is an element of $\mathcal{U}$ then, by continuity of $\mathsf{K}_\mathsf{T}X$, there is $K' \ll K$ with $K' \in \mathcal{U}$. By the above characterization of the way below relation in $\mathsf{K}_\mathsf{T}X$ we know that $\mathsf{int}\, K' \supseteq K$. Hence $K \in \Box(\mathsf{int}\, K') \subseteq \mathcal{U}$. This proves that the Scott topology for $\mathsf{K}_\mathsf{T}X$ is coarser then the topology $\mathsf{K}_\mathsf{T}X$ carries by definition. Thus the two coincide. ∎

If D is a continuous domain, then it is sober and locally compact with respect to the Scott topology. Therefore we get the following result.

Proposition 7.27 *For the category of domains,* $\mathsf{T}_\sigma \circ \mathsf{K}_\mathsf{D} = \mathsf{K}_\mathsf{T} \circ \mathsf{T}_\sigma$. ∎

Since we cannot show that $\mathsf{T}_\sigma \circ \mathsf{F}_\mathsf{D} \cong \mathsf{K}_\mathsf{T} \circ \mathsf{T}_\sigma$ for a larger subcategory of **DCPO**, we cannot transfer the results we will get for K_T to F_D. The following characterization of the category of K_T-algebras has already been published in [Sch93a].

Proposition 7.28 *The category of algebras for the monad* $(\mathsf{K}_\mathsf{T}, \uparrow, \bigcup)$ *restricted to the category* **LCompS** *of locally compact sober spaces, is the category* $\mathbf{Dom}^\wedge$ *of continuous $\wedge$-semilattices.*

Proof: By Proposition 7.25 we know that if X is a locally compact sober space then $\mathsf{K}_\mathsf{T}X$ is a domain. Furthermore, we know that $\mathsf{K}_\mathsf{T}X$ is a deflationary topological semilattice, that is a $\wedge$-semilattice with respect to the order of specialization. Because binary meet is continuous for $\mathsf{K_T}X$ and because $\mathsf{K_T}X$ and its product both are sober if X is, the binary meet function for $\mathsf{K_T}X$ is Scott-continuous. Therefore any algebra for the restricted monad $(\mathsf{K_T}, \uparrow, \bigcup)$ is a continuous $\wedge$-semilattice by Proposition 5.6. Furthermore, the topology on X is the Scott topology: Since X is sober, its topology is coarser than the Scott topology. On the other hand, if U is a Scott-open subset of X then $\xi_X^{-1}(U)$ is a Scott-open subset of $\mathsf{K_T}X$ as ξ_X is Scott-continuous. Since the Scott topology for $\mathsf{K_T}X$ coincides with the intrinsic topology for $\mathsf{K_T}X$ we get that $U = \eta_X^{-1}(\xi_X^{-1}(U))$ is open in X. Hence any $\mathsf{K_T}$-algebra is a continuous $\wedge$-semilattice with its Scott topology. Furthermore, any morphism of $\mathsf{K_T}$-algebras is a continuous function between sober spaces and therefore Scott-continuous. Also, according to Proposition 7.23, such a morphism in particular has to preserve finite meets.

On the other hand, every continuous $\wedge$-semilattice D (with its Scott topology) is a $\mathsf{K_T}$-algebra: By Lemma 7.14, D has a structure map making it an $\mathsf{F_D}$-algebra. By Proposition 7.27 the same structure map makes D a $\mathsf{K_T}$-algebra. Furthermore, any morphism of continuous $\wedge$-semilattices is a morphism of the corresponding $\mathsf{F_D}$-algebras by Lemma 7.15 and therefore also for the corresponding $\mathsf{K_T}$-algebras. ∎

Proposition 7.29 *Let X be a sober and locally compact space. If X is a deflationary topological semilattice with small semilattices then X is a domain with its Scott topology.*

Proof: As X is sober, it is a dcpo. Let x be an element of some open $U \subseteq X$. By local compactness of X, there is a compact neighbourhood K of x in U. Since X has small semilattices, there is a semilattice S in the interior of K such that $x \in \mathsf{int}\, S$. Consider $\{\downarrow\{s\} \cap K \mid s \in S\}$. This is a family of closed subsets of K (with respect to the topology K inherits from X) that has the finite intersection property: If $s_1, \ldots, s_n \in S$ then $s_1 \wedge \ldots \wedge s_n \in (\downarrow\{s_1\} \cap K) \cap \cdots \cap (\downarrow\{s_n\} \cap K)$. By compactness of K, there is some $x' \in \bigcap\{\downarrow\{s\} \cap K \mid s \in S\}$. Obviously, $x \in \mathsf{int}\, S \subseteq \uparrow\{x'\}$ which proves $x \in \mathsf{int} \uparrow\{x'\}$. As, by sobriety of X, the topology of X is coarser than the

Scott topology, $x \ll x'$ holds for all $x' \in X$ with $x \in \mathsf{int}\uparrow\{x'\}$. On the other hand, for any open neighbourhood U of X we have just shown the existence of some $x' \in U$ with $x \in \mathsf{int}\uparrow\{x'\}$ and thus with $x' \ll x$. We next show that $\{x' \in X \mid x \in \mathsf{int}\uparrow\{x'\}\}$ is directed with join x. If $x \in \mathsf{int}\uparrow\{y\}$ and $x \in \mathsf{int}\uparrow\{y'\}$ then $\mathsf{int}\uparrow\{y\} \cap \mathsf{int}\uparrow\{y'\}$ is an open neighbourhood of x and consequently we can find $y'' \in \mathsf{int}\uparrow\{y\} \cap \mathsf{int}\uparrow\{y'\}$ with $x \in \mathsf{int}\uparrow\{y''\}$. Since any open neighbourhood of x contains some x' with $x \in \mathsf{int}\uparrow\{x'\}$, the join of $\{x' \in X \mid x \in \mathsf{int}\uparrow\{x'\}\}$ has to be x. Therefore, $\{x' \in X \mid x \in \mathsf{int}\uparrow\{x'\}\}$ is a directed subset of $\twoheaddownarrow\{x\}$ with join x. This proves continuity of X. It remains to show that the Scott topology for X is coarser than the intrinsic topology for X. If $x' \ll x$ then, since $x = \bigvee^{\uparrow}\{x'' \in X \mid x \in \mathsf{int}\uparrow\{x''\}\}$, there is $x'' \in X$ with $x \in \mathsf{int}\uparrow\{x''\}$ and $x' \leq x''$. Hence any set of the form $\twoheaduparrow\{x''\}$ is open in X. As a consequence, the Scott topology is coarser than the intrinsic topology for X and thus the two must coincide. ■

Any domain is sober and locally compact. Furthermore, any continuous $\wedge$-semilattice is a deflationary topological semilattice with small semilattices. Hence $\mathbf{Dom}^{\wedge}$ is a subcategory of the category of deflationary semilattices with small semilattices in **LCompS**. Together with the preceding proposition this implies the following result.

Theorem 7.30 *The category of deflationary semilattices with small semilattices in* **LCompS** *coincides with the category of continuous $\wedge$-semilattices.* ■

Theorem 7.31 *The functor K_{T} gives the free deflationary semilattice with small semilattices for the category of locally compact sober spaces.*

Proof: By Proposition 7.28 the category of K_{T}-algebras is $\mathbf{Dom}^{\wedge}$ which, by the last theorem, is the category of deflationary semilattices with small semilattices for **LCompS**. ■

Since there are deflationary semilattices in **LCompS** that do not have small semilattices, K_{T} does not even give the free deflationary semilattice for the category of locally compact sober spaces: In [Law70], Lawson introduces a compact Hausdorff topological semilattice X that does not have small semilattices. With the help of results from [GHK+80] we can show that this example serves our needs. Interpreting this semilattice structure as meet yields an order on X. The open upper sets then give a topology on X with respect to which X is a sober topological lattice by Proposition VII.1.7 and Corollary VII.1.14. Furthermore, this new topology is just the Scott topology for X. By Exercise VII.1.18, X is locally compact with respect to the Scott topology and the intrinsic topology for X is generated by the Scott topology and the complements of Scott-compact upper sets. If we can show that X does not have small semilattices with respect to the Scott topology then we are done. Assume that this is wrong. By choice of X there is an open neighbourhood O of some $x \in X$ that does not contain a semilattice which at the same time is a neighbourhood for x. By the above characterization of the topology on X, there is a Scott-open subset U of X and a Scott-compact upper set $K \subseteq X$ with $x \in U \cap (X \setminus K) \subseteq O$. Therefore, U is a Scott neighbourhood of x and, by assumption, contains a semilattice S that also is a Scott neighbourhood for x. Hence there is a Scott-open set U' in S that contains x. But $S \cap (X \setminus K)$ is a semilattice since $X \setminus K$ is lower closed. Further, $S \cap (X \setminus K)$ contains the open set $U' \cap (X \setminus K)$. This contradicts the choice of U and x. Hence, with respect to the Scott topology, X is a deflationary topological semilattice with sober and locally compact topology that does not have small semilattices.

As the example from Section 7.3.3 shows, local compactness is necessary in the last theorem.

7.3.5 Another Smyth power construction

The second Smyth power construction for topological spaces suggested by the considerations from Section 7.2 is the following.

For a T_0-space X let $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ be the set of non-empty strongly compact and saturated subsets of X with the topology generated by the sets

$$\boxed{\mathsf{s}}U := \{K \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X \mid K \subseteq U\} = \Box U \cap \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$$

where U is an open subet of X.

This definition makes $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ a subspace of $\mathsf{K}_{\mathsf{T}}X$ for any T_0-space X.

Lemma 7.32 *If X is a sober space then so is $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$.*

Proof: We will prove the assertion by using that $\mathsf{K}_{\mathsf{T}}X$ is sober if X is. So let X be a sober space. If $\mathcal{A}$ is a non-empty irreducible closed subset of $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ then the closure of $\mathcal{A}$ in $\mathsf{K}_{\mathsf{T}}X$ is irreducible in $\mathsf{K}_{\mathsf{T}}X$. By sobriety of $\mathsf{K}_{\mathsf{T}}X$ there is $K \in \mathsf{K}_{\mathsf{T}}X$ with $\mathsf{cl}_{\mathsf{K}_{\mathsf{T}}X}\,\mathcal{A} = \mathsf{cl}_{\mathsf{K}_{\mathsf{T}}X}\{K\}$. It remains to show that K is in $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$, that is that K is strongly compact in X. Assume that this is not the case. Then we can find an open neighbourhood U of K such that K is not in $\uparrow F$ for any finite F in U. This implies $K \in \mathsf{cl}_{\mathsf{K}_{\mathsf{T}}X}\,\mathcal{A} \cap \Box U \neq \emptyset$. On the other hand, $\mathcal{A} \cap \boxed{\mathsf{s}}U = \emptyset$: If $K' \in \mathcal{A} \cap \boxed{\mathsf{s}}U$ then there is a finite set F in U with $K \subseteq K' \subseteq \uparrow F$ which contradicts $K \in \mathsf{K}_{\mathsf{T}}X \setminus \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$. Hence $\mathsf{K}_{\mathsf{T}}X \setminus \boxed{\mathsf{s}}U$ is a closed superset of $\mathcal{A}$ in $\mathsf{K}_{\mathsf{T}}X$ that does not contain $\mathsf{cl}_{\mathsf{K}_{\mathsf{T}}X}\,\mathcal{A}$ which is a contradiction. Therefore $K \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ and thus $\mathcal{A} = \mathsf{cl}_{\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}}\{K\}$. ■

The following result yields a connection between $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$ and $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$.

Proposition 7.33 *If X is a sober space then $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ is isomorphic to the sobrification of $\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$.*

Proof: If K is a stronly compact subset of a sober space X then

$$\{\uparrow\{x_1, \ldots, x_n\} \mid x_1, \ldots, x_n \in U \mid K \subseteq U\}$$

is a completely prime filter in $\Omega\,\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$. If, on the other hand, Φ is a completely prime filter in $\Omega\,\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X$ then

$$\{\boxed{\mathsf{s}}U \mid \{\uparrow\{x_1, \ldots, x_n\} \in \mathsf{K}^{\mathsf{f}}_{\mathsf{T}}X \mid x_1, \ldots, x_n \in U\} \in \Phi\}$$

is a strong filter in $\Omega\,\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$. By sobriety of X and the Hofmann-Mislove Theorem 7.4 this is sufficient to prove the assertion. ■

Note that images of strongly compact subsets under continuous functions are also strongly compact and that saturation preserves strong compactness. Hence we can make $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$ a functor by defining $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}} f \colon \mathsf{K}^{\mathsf{s}}_{\mathsf{T}} X \longrightarrow \mathsf{K}^{\mathsf{s}}_{\mathsf{T}} Y$ to be the restriction and co-restriction of $\mathsf{K}_{\mathsf{T}} f$ for any continuous $f \colon X \longrightarrow Y$. The co-restriction of the function $\uparrow_X \colon X \longrightarrow \mathsf{K}_{\mathsf{T}} X$ to $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}} X$ is also well-defined. Thus we obtain a natural transformation $id_{\mathbf{Top}} \longrightarrow \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$.

Moreover, we can define $\bigcup_X \colon (\mathsf{K}^{\mathsf{s}}_{\mathsf{T}})^2 X \longrightarrow \mathsf{K}_{\mathsf{T}} X$ by setting $\bigcup_X(\mathcal{K}) := \bigcup \mathcal{K}$ for every $\mathcal{K} \in (\mathsf{K}^{\mathsf{s}}_{\mathsf{T}})^2 X$. To prove that $\bigcup_X$ is well-defined we have to show that $\bigcup \mathcal{K}$ is strongly compact in X for all $\mathcal{K} \in (\mathsf{K}^{\mathsf{s}}_{\mathsf{T}})^2 X$. If U is an open neighbourhood of $\bigcup \mathcal{K}$ in X then $\mathcal{K}$ is a subset of $\boxdot U$. Since $\mathcal{K}$ is strongly compact there are $K_1 \ldots, K_n \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}} X$ with $\mathcal{K} \subseteq \uparrow\{K_1, \ldots, K_n\} \subseteq \boxdot U$. Hence $K_1 \cup \cdots \cup K_n$ is a strongly compact subset of U. As a consequence, we can find a finite subset F of U with $\bigcup \mathcal{K} \subseteq K_1 \cup \cdots \cup K_n \subseteq \uparrow F$. The proof that $\bigcup_X$ is continuous and that $\bigcup$ is a natural transformation is the same as for the monad $(\mathsf{K}_{\mathsf{T}}, \uparrow, \bigcup)$. Also, the proof of the following proposition is the same as for Proposition 7.21.

Proposition 7.34 *The triple* $(\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}, \uparrow, \bigcup)$ *is a monad on* **Top**. ■

The following proposition can be shown in the same way as Proposition 7.23.

Proposition 7.35 *A T_0-space X is a $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebra iff the function $\bigwedge_X \colon \mathsf{K}^{\mathsf{s}}_{\mathsf{T}} X \longrightarrow X$ defined by $\bigwedge_X(K) := \bigwedge K$ is well-defined and continuous. A continuous function $f \colon X \longrightarrow Y$ between $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebras is a morphism of these algebras iff it preserves meets of non-empty strongly compact sets.* ■

Note that the T_0-space from the example considered in Section 7.3.3 is indeed a $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebra. We can show that $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$ gives the free deflationary semilattice with small semilattices on a larger category than K_{T} does. Since the sets $\boxdot U$, where U is an open subset of X, are closed with respect to finite meets, every topological space of the form $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}} X$ has a basis of open semilattices.

Proposition 7.36 *Every $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebra is a deflationary semilattice with small semilattices.*

Proof: As for Proposition 7.24. ∎

Theorem 7.37 *The algebras for the monad $(\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}, \uparrow, \bigcup)$ restricted to the category of sober spaces are all deflationary topological semilattices with small semilattices. The morphisms of this category are morphisms of deflationary topological semilattices.*

Proof: Because of the preceding proposition we know that every $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebra is a deflationary semilattice with small semilattices in **Sob**. Now let X be a such a topological semilattice. We have to show that $\bigwedge_X : K \longmapsto \bigwedge K$ is a well-defined continuous function from $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ to X. For any $K \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ consider the set

$$\varphi_K := \{U \in \Omega X \mid K \subseteq \operatorname{int} S \subseteq S \subseteq U \text{ for some sub-semilattice } S \text{ of } X\}.$$

We claim that φ_K is a completely prime filter in ΩX. Obviously, it is an upper set in ΩX. Furthermore, let U and U' be elements of φ_K. Then there are sub-semilattices S and S' of X in U and U', respectively, with $K \subseteq \operatorname{int} S, \operatorname{int} S'$. Therefore $S \cap S'$ is a semilattice in $U \cap U'$ with $K \subseteq \operatorname{int} S \cap \operatorname{int} S' \subseteq \operatorname{int}(S \cap S')$. Moreover, if $\bigcup \mathcal{U} \in \varphi_K$ for some subset $\mathcal{U}$ of ΩX then there is a sub-semilattice S of X in $\bigcup \mathcal{U}$ with $K \subseteq \operatorname{int} S$. By strong compactness of K, there are $x_1, \ldots, x_n \in \operatorname{int} S$ with $K \subseteq \uparrow\{x_1, \ldots, x_n\}$. Hence $x_1 \wedge \ldots \wedge x_n \in S \subseteq \bigcup \mathcal{U}$. So there must be $U \in \mathcal{U}$ with $x_1 \wedge \cdots \wedge x_n \in U$. As X has small semilattices, there is a semilattice S' in U with $x_1 \wedge \cdots \wedge x_n \in \operatorname{int} S'$. Therefore $K \subseteq \uparrow\{x_1, \ldots, x_n\} \subseteq \operatorname{int} S'$ which proves $U \in \varphi_K$. Since X is sober, there is a unique $x \in X$ such that φ_K is the filter of open neighbourhoods of x. Because any open neighbourhood of x contains K we know that x is a lower bound for K. If x' is any lower bound for K in X then every open neighbourhood U of x' contains a semilattice with $x' \in \operatorname{int} S$ as X has small semilattices. Therefore, $K \subseteq \uparrow\{x'\} \subseteq \operatorname{int} S \subseteq S \subseteq U$ and consequently U is a neighbourhood of x which proves $x' \leq x$. Hence x is the greatest lower bound for K and the function $\bigwedge_X$ is well-defined.

To see that $\bigwedge_X$ is also continuous, let $K \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$ and let U be an open neighbourhood of $\bigwedge K$. Then there is a semilattice S in U with $\bigwedge K \in \mathsf{int}\, S$. We claim that $\boxdot(\mathsf{int}\, S)$ is an open neighbourhood of K such that $\bigwedge_X(\boxdot(\mathsf{int}\, S)) \subseteq U$. Obviously, $\boxdot(\mathsf{int}\, S)$ contains K. If K' is an element of $\boxdot(\mathsf{int}\, S)$ then U is an element of $\varphi_{K'}$ and therefore an open neighbourhood of $\bigwedge K'$. Hence, by Proposition 7.35, X is a $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebra.

It remains to prove that every morphism $f\colon X \longrightarrow Y$ of deflationary topological semilattices, where X an Y have small semilattices, is a morphism of the corresponding $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$-algebras. Because of Proposition 7.35 it is enough to show that f preserves meets of non-empty strongly compact sets. Since obviously $f(\bigwedge K) \leq \bigwedge f(K)$, it is sufficient to prove that $f(\bigwedge K) \geq \bigwedge f(K)$ for all $K \in \mathsf{K}^{\mathsf{s}}_{\mathsf{T}}X$. Let V be an open neighbourhood of $\bigwedge f(K)$. There is a semilattice T in V with $\bigwedge f(K) \in \mathsf{int}\, T$. As f preserves binary meets, $f^{-1}(T)$ is a semilattice. Furthermore, $K \subseteq f^{-1}(\mathsf{int}\, T) \subseteq \mathsf{int}\, f^{-1}(T)$. By strong compactness of K there are $x_1, \ldots, x_n \in \mathsf{int}\, f^{-1}(T)$ with $K \subseteq {\uparrow}\{x_1, \ldots, x_n\}$. Therefore, $f(\bigwedge K) \geq f(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n) \in T \subseteq V$. ■

Hence $\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$ gives the free deflationary semilattice with small semilattices for the category of sober spaces.

7.3.6 Unital semilattices

If one wants the semilattices resulting from the two Smyth power constructions to be unital semilattices then it is enough to add the empty set to the respective power spaces. As the specialization order for these is reverse inclusion, we thus indeed get a greatest element, that is a unit for the binary meet function.

Since any algebra of one of the monads is a retract of its power space, it also must have a greatest element. Furthermore, by Lemma 4.4 (ii), any morphism of algebras has to be a morphism of unital semilattices. The above results therefore remain true for the modified Smyth power constructions if 'semilattice' is replaced by 'unital semilattice'.

7.4 Locales

We have seen before how locales are used to give axiomatic semantics for programming languages. We have also examined how non-deterministic languages can be handled if non-determinism is considered to be angelic. The approach here is to view non-determinism as being demonic just as we did when we introduced the Smyth power construction for algebraic domains at the beginning of this chapter. So a program satisfies a given property iff each of its branches does. Once more we assume that we already know how to handle these branches using a locale A. Then any property $a \in A$ gives us a new property $\Box a$ that a non-deterministic program might (or might not) have. We say that a non-deterministic program p satisfies the property $\Box a$ iff every of its branches satisfies a. We will denote this by by $\mathsf{p} \models \Box a$. From this approach we can develop some equations that we want to hold for the new properties, that is we want the resulting equations to hold in the locale $\mathsf{P}_{\mathsf{L}}^{\sharp} A$ generated by the $\Box a$ where $a \in A$. For example, $\mathsf{p} \models \Box a$ and $\mathsf{p} \models \Box b$ hold iff every branch of p satisfies both, a and b iff $\mathsf{p} \models \Box(a \wedge b)$. Hence we want the equation $\Box a \wedge \Box b = \Box(a \wedge b)$ to hold for all a, b in A. Furthermore, $\mathsf{p} \models \bigvee^{\uparrow}_{s \in S} \Box s$ for a directed subset S of A iff there is some $s \in S$ with $\mathsf{p} \models \Box s$ iff all branches of p satisfy s. On the other hand, $\mathsf{p} \models \Box \bigvee^{\uparrow} S$ iff all branches of p satisfy $\bigvee^{\uparrow} S$. We claim that in this case there is some $s \in S$ that is satisfied by all branches of p. Assume that this is not the case. All branches of p satisfy $\bigvee^{\uparrow} S$, therefore this is a property we can observe after finite time. Since programs are supposed to be finitely branching, there can only be finitely many branches of p at that time. As S is directed, there must be a property s in S that all branches yet developed (and therefore all branches of p) must satisfy. Hence we want the equation $\bigvee^{\uparrow}_{s \in S} \Box s = \Box \bigvee^{\uparrow} S$ to hold in $\mathsf{P}_{\mathsf{L}}^{\sharp} A$ for any directed subset

S of A. Because the greatest element 1_A is supposed to be satisfied by each branch of every p, we want $1_{\mathsf{P}_\mathsf{L}^\sharp A} = \Box 1_A$. Furthermore, we also want $0_{\mathsf{P}_\mathsf{L}^\sharp A} = \Box 0_A$ for the following reason: As 0_A is a property that no branch of any p can ever satisfy, $\Box 0_A$ will never be satisfied by any p. As the same property holds for $0_{\mathsf{P}_\mathsf{L}^\sharp A}$, we want the two to be equal.

Note that in order to formulate this motivation we had to get much more involved in the kind of properties we consider and also in the kind of observations we allow than before. These ideas lead to the following definition.

Definition. For a locale A the Smyth power locale $\mathsf{P}_\mathsf{L}^\sharp A$ is given by the free frame

$$Fr\Big\langle \Box a : a \in A \quad \Big| \quad \Box 0_A = 0_{\mathsf{P}_\mathsf{L}^\sharp A}\,,\, \Box 1_A = 1_{\mathsf{P}_\mathsf{L}^\sharp A}\,,\, \Box a \wedge \Box b = \Box(a \wedge b)\ \ (a, b \in A)$$
$$\bigvee_{s \in S}^{\uparrow} \Box s = \Box \bigvee^{\uparrow} S\ \ (S \subseteq A \text{ directed})\Big\rangle.$$

We can write every element of $\mathsf{P}_\mathsf{L}^\sharp A$ in the form $\bigvee_{s\in S} \Box s$ for some subset S of A since every element of $\mathsf{P}_\mathsf{L}^\sharp A$ is the join of finite meets of generators and $\Box a \wedge \Box b = \Box(a \wedge b)$ for all $a, b \in A$. The above definition of the Smyth power locale also appears in [Rob88]; the modal operator $\Box$ is mentioned in [Win85].

As for the Hoare power construction, there is also another way to introduce non-determinism. Once more we add the command $\underline{\mathsf{or}}$ to a deterministic programming language. Again, every property a of the locale A used to interpret that deterministic language is to give us a new property of the programs we can write using the new command $\underline{\mathsf{or}}$. Because of the view of non-determinism described above we want $\mathsf{p}\ \underline{\mathsf{or}}\ \mathsf{q} \models \Box a$ to hold iff both, $\mathsf{p} \models \Box a$ and $\mathsf{q} \models \Box a$ hold. This is the case iff $\langle p, q\rangle \models \Box a \otimes \Box a$ in $\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A$ where $\langle p, q\rangle$ is the point of $\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A$ given by $\langle p, q\rangle'(\Box a \otimes \Box b) = 1_{\mathcal{2}}$ iff $\mathsf{p} \models \Box a$ and $\mathsf{q} \models \Box b$. Thus we obtain a connection between $\mathsf{P}_\mathsf{L}^\sharp A$ and its product $\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A$. From our considerations of the Hoare power construction we would expect $\Box a \longmapsto \Box a \otimes \Box a$ to define a frame homomorphism from $\mathsf{P}_\mathsf{L}^\sharp A$ to $\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A$. We shall see that this is indeed the case.

There is also a motivation for the definition of the Smyth power locale from our studies of the Smyth power constructions for topological spaces. Above we have seen that the equation $\Box U \cap \Box U' = \Box(U \cap U')$ holds for all $U, U' \in \Omega X$ where X is an arbitrary T_0-space. Furthermore, $\Box\emptyset = \emptyset$ and $\Box X = \mathsf{K}_\mathsf{T} X$. For any directed subset $\mathcal{U}$ of ΩX we have seen that $\Box \bigcup^{\uparrow} \mathcal{U} = \bigcup^{\uparrow}\{\Box U \mid U \in \mathcal{U}\}$. Note that the corresponding equations also hold for $\boxed{\mathsf{s}}$ and $\mathsf{K}^\mathsf{s}_\mathsf{T} X$. As these are the only equations that can be shown to hold for all T_0-spaces we get the same definition for a Smyth power locale based on the first of the two Smyth power constructions from the last section. The reader should note that although $\mathsf{K}^\mathsf{s}_\mathsf{T} X$ has less points than $\mathsf{K}_\mathsf{T} X$ we cannot prove more equations of the above type for the open set lattice of $\mathsf{K}^\mathsf{s}_\mathsf{T} X$. Therefore there is only one Smyth power locale construction. We will see later why this construction corresponds to K_T rather than $\mathsf{K}^\mathsf{s}_\mathsf{T}$.

7.4.1 The monad

As is the case for the power constructions considered so far, we can easily extend $\mathsf{P}^\sharp_\mathsf{L}$ to an endofunctor on **Loc**. For any morphism $f\colon A \longrightarrow B$ in **Loc** we define $\mathsf{P}^\sharp_\mathsf{L} f\colon \mathsf{P}^\sharp_\mathsf{L} A \longrightarrow \mathsf{P}^\sharp_\mathsf{L} B$ by setting $(\mathsf{P}^\sharp_\mathsf{L} f)'(\Box\, b) := \Box\, f'(b)$. Since f' preserves directed joins and finite meets, this does define a continuous function between the locales $\mathsf{P}^\sharp_\mathsf{L} A$ and $\mathsf{P}^\sharp_\mathsf{L} B$. Also, $\mathsf{P}^\sharp_\mathsf{L}$ is a functor: Obviously, $\mathsf{P}^\sharp_\mathsf{L}$ preserves identities. If $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are continuous functions between locales then for all $c \in C$ we have

$$(\mathsf{P}^\sharp_\mathsf{L} f)'((\mathsf{P}^\sharp_\mathsf{L} g)'(\Box\, c)) = (\mathsf{P}^\sharp_\mathsf{L} f)'(\Box\, g'(c)) = \Box\, f'(g'(c)) = (\mathsf{P}^\sharp_\mathsf{L}(g \circ f))'(\Box\, c).$$

Hence $(\mathsf{P}^\sharp_\mathsf{L}(f \circ g))'$ and $(\mathsf{P}^\sharp_\mathsf{L} f \circ \mathsf{P}^\sharp_\mathsf{L} g)'$ are two frame homomorphisms agreeing on the generators of $\mathsf{P}^\sharp_\mathsf{L} C$ and therefore must coincide.

In order to give a natural transformation $\eta\colon id_{\mathbf{Loc}} \longrightarrow \mathsf{P}^\sharp_\mathsf{L}$, for any locale A we set $\eta'_A(\Box\, a) := a$ which obviously defines a morphism in **Loc**. In order to see that we have thus given a natural transformation as claimed, note that for every $f\colon A \longrightarrow B$ in **Loc** we get $\eta'_A((\mathsf{P}^\sharp_\mathsf{L} f)'(\Box\, b)) = \eta'_A(\Box\, f'(b)) = f'(b) = f'(\eta'_B(\Box\, b))$ for all $b \in B$.

We still have to find a natural transformation $\mu\colon (\mathsf{P}^\sharp_\mathsf{L})^2 \longrightarrow \mathsf{P}^\sharp_\mathsf{L}$ to obtain a monad. By setting $\mu'_A(\Box\, a) := \Box\,\Box\, a$ we define a continuous function from $(\mathsf{P}^\sharp_\mathsf{L})^2 A$ to $\mathsf{P}^\sharp_\mathsf{L} A$ as, for example,

$$\mu'_A(\Box(a \wedge b)) = \Box\,\Box(a \wedge b) = \Box(\Box\, a \wedge \Box\, b) = \Box\,\Box\, a \wedge \Box\,\Box\, b.$$

The other equations needed can be shown analogously. This defines a natural transformation as desired since for all $f\colon A \longrightarrow B$ in **Loc** and all $b \in B$ we have

$$\begin{aligned}
((\mathsf{P}^\sharp_\mathsf{L})^2 f)'(\mu'_B(\Box\, b)) &= ((\mathsf{P}^\sharp_\mathsf{L})^2 f)'(\Box\,\Box\, b) \\
&= \Box(\mathsf{P}^\sharp_\mathsf{L} f)'(\Box\, b) \\
&= \Box\,\Box\, f'(b) \\
&= \mu'_A(\Box\, f'(b)) \\
&= \mu'_A((\mathsf{P}^\sharp_\mathsf{L} f)'(\Box\, b)).
\end{aligned}$$

The remaining diagrams defining a monad can also easily be shown to commute. We shall only give one more example for this. For $a \in A$ we have $\eta'_{\mathsf{P}^\sharp_\mathsf{L} A}(\mu'_A(\Box\, a)) = \eta'_{\mathsf{P}^\sharp_\mathsf{L} A}(\Box\,\Box\, a) = \Box\, a$ proving $\mu_A \circ \eta_{\mathsf{P}^\sharp_\mathsf{L} A} = id_{\mathsf{P}^\sharp_\mathsf{L} A}$.

Proposition 7.38 *The triple* $(\mathsf{P}^\sharp_\mathsf{L}, \eta, \mu)$ *is a monad on* **Loc**. ■

7.4.2 Power space versus power locale

For the Hoare power construction we could show that $\mathsf{Sp}\mathsf{P}^\flat_\mathsf{L}\Omega X \cong \mathsf{P}^\flat_\mathsf{T} X$ for any T_0-space X. We will now see that the above Smyth power construction for locales corresponds to the first of the two Smyth power constructions we introduced for topological spaces.

What are the points of $\mathsf{P}^\sharp_\mathsf{L} A$? They are given by frame homomorphisms from $\mathsf{P}^\sharp_\mathsf{L} A$ to *2*, that is functions from A to *2* that preserve directed joins, finite meets (including the empty meet) and 1. Such a function is uniquely determined by the inverse image of 1_2 which has to be a Scott-open filter not containing 0_A. Hence the points of $\mathsf{P}_{\mathsf{L}}^{\sharp} A$ are the Scott-open filters of $A \setminus \{0_A\}$.

Thus, if we start with a sober space X then, by the Hofmann-Mislove Theorem 7.4, the points of $\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X$ correspond to the elements of $\mathsf{K}_{\mathsf{T}} X$. Furthermore, this yields a homeomorphism $\kappa_X \colon \mathsf{K}_{\mathsf{T}} X \longrightarrow \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X$. The set $\Box U := \{K \in \mathsf{K}_{\mathsf{T}} X \mid K \subseteq U\}$, for some open subset U of X, corresponds to the set

$$
\begin{aligned}
\{p\colon 2 \longrightarrow \mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X \mid \exists K \subseteq U \,.\, p'(\Box U') = 1_2 \text{ iff } K \subseteq U'\} \\
= \{p \in \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X \mid p'(\Box U) = 1_2\} \\
= \{p \in \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X \mid p \models \Box U\}.
\end{aligned}
$$

Moreover, if $f \colon X \longrightarrow Y$ is a continuous function then for each $K \in \mathsf{K}_{\mathsf{T}} X$ we have $\Big(\kappa_Y(\mathsf{K}_{\mathsf{T}} f(K))\Big)'(\Box V) = \Big(\kappa_Y(\uparrow f(K))\Big)'(\Box V) = 1_2$ iff V is a neighbourhood of $f(K)$. On the other hand,

$$
\begin{aligned}
\Big(\mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega f(\kappa_X(K))\Big)'(\Box V) &= (\kappa_X(K))'\Big((\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega f)'(\Box V)\Big) \\
&= (\kappa_X(K))'\Big(\Box \Omega f(V)\Big) \\
&= (\kappa_X(K))'(\Box f^{-1}(V)) \\
&= 1_2
\end{aligned}
$$

iff $f^{-1}(V)$ is a neighbourhood of K. Therefore $\mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega f \circ \kappa_X = \kappa_Y \circ \mathsf{K}_{\mathsf{T}} f$.

These considerations yield the following result.

Proposition 7.39 *There is a natural isomorphism* $\mathsf{K}_{\mathsf{T}} \longrightarrow \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega$ *for the functors* $\mathsf{K}_{\mathsf{T}}, \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega \colon \mathbf{Sob} \longrightarrow \mathbf{Sob}$. ∎

So, if X is a sober space such that $\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X$ is spatial then $\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X \cong \Omega\,\mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega X \cong \Omega\,\mathsf{K}_{\mathsf{T}} X$. On the other hand, if A is a spatial locale, then $\mathsf{K}_{\mathsf{T}}\mathsf{Sp} A \cong \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp}\Omega\,\mathsf{Sp} A \cong \mathsf{Sp}\mathsf{P}_{\mathsf{L}}^{\sharp} A$.

7.4.3 The algebras for the monad

In this section we will study the category of algebras for the monad $(\mathsf{P}_{\mathsf{L}}^{\sharp}, \eta, \mu)$. First of all note that if $\xi\colon \mathsf{P}_{\mathsf{L}}^{\sharp}A \longrightarrow A$ is a structure map for a locale A then ξ must be the upper adjoint for η_A. Recall that this is the case if ξ' is the lower adjoint for η'_A in the usual sense. As $\eta'_A \circ \xi' = id'_A$ it remains to show that $\xi' \circ \eta'_A \leq id'_{\mathsf{P}_{\mathsf{L}}^{\sharp}A}$. Let a be an arbitrary element of A and let $\bigvee_{i \in I} \Box\, a_i := \xi'(a)$. Then $\xi'(\eta'_A(\Box\, a)) = \bigvee_{i \in I} \Box\, a_i$. Since $a = \eta'_A(\xi'(\eta'_A(\Box\, a))) = \eta'_A(\bigvee_{i \in I} \Box\, a_i) = \bigvee_{i \in I} a_i$ we may conclude that $a_i \leq a$ for all $i \in I$. Therefore, $\Box\, a_i \leq \Box\, a$ for all $i \in I$ implying $\bigvee_{i \in I} \Box\, a_i \leq \Box\, a$. Consequently, $\xi' \circ \eta'_A \leq id'_{\mathsf{P}_{\mathsf{L}}^{\sharp}A}$ holds. Hence there is at most one structure map for any locale A which we refer to as ξ_A from now on.

Lemma 7.40 *Every locale of the form $\mathsf{P}_{\mathsf{L}}^{\sharp}A$ is a deflationary localic semilattice. The semilattice structure $v\colon \mathsf{P}_{\mathsf{L}}^{\sharp}A \otimes \mathsf{P}_{\mathsf{L}}^{\sharp}A \longrightarrow \mathsf{P}_{\mathsf{L}}^{\sharp}A$ is defined by $v'(\Box\, a) := \Box\, a \otimes \Box\, a$.*

Proof: Obviously, $v'(1_{\mathsf{P}_{\mathsf{L}}^{\sharp}A}) = v'(\Box\, 1_A) = \Box\, 1_A \otimes \Box\, 1_A = 1_{\mathsf{P}_{\mathsf{L}}^{\sharp}A} \otimes 1_{\mathsf{P}_{\mathsf{L}}^{\sharp}A} = 1_{\mathsf{P}_{\mathsf{L}}^{\sharp}A \otimes \mathsf{P}_{\mathsf{L}}^{\sharp}A}$ and $v'(0_{\mathsf{P}_{\mathsf{L}}^{\sharp}A}) = v'(\Box\, 0_A) = \Box\, 0_A \otimes \Box\, 0_A = 0_{\mathsf{P}_{\mathsf{L}}^{\sharp}A \otimes \mathsf{P}_{\mathsf{L}}^{\sharp}A}$. For $a, a' \in A$ we have

$$\begin{aligned} v'(\Box(a \wedge a')) &= \Box(a \wedge a') \otimes \Box(a \wedge a') \\ &= (\Box\, a \wedge \Box\, a') \otimes (\Box\, a \wedge \Box\, a') \\ &= (\Box\, a \otimes \Box\, a) \wedge (\Box\, a' \otimes \Box\, a') \\ &= v'(\Box\, a) \;\wedge\; v'(\Box\, a'). \end{aligned}$$

Let S be a directed subset of A. Then

$$\begin{aligned} v'(\Box \textstyle\bigvee^{\uparrow} S) &= (\Box(\textstyle\bigvee^{\uparrow} S)) \otimes (\Box(\textstyle\bigvee^{\uparrow} S)) \\ &= (\bigvee^{\uparrow}_{s \in S} \Box\, s) \otimes (\bigvee^{\uparrow}_{s \in S} \Box\, s) \\ &= \bigvee^{\uparrow}_{s \in S} \bigvee^{\uparrow}_{s' \in S} (\Box\, s \otimes \Box\, s') \\ &= \bigvee^{\uparrow}_{s \in S} (\Box\, s \otimes \Box\, s) \\ &= \bigvee^{\uparrow}_{s \in S} v'(\Box\, s). \end{aligned}$$

Hence v' preserves the relations defining $\mathsf{P}_\mathsf{L}^\sharp A$ and thus defines a continuous function $v\colon \mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A \longrightarrow \mathsf{P}_\mathsf{L}^\sharp A$.

This function is idempotent since

$$\Delta'_{\mathsf{P}_\mathsf{L}^\sharp A}(v'(a)) = \Delta'_{\mathsf{P}_\mathsf{L}^\sharp A}(\Box\, a \otimes \Box\, a) = \Box\, a \wedge \Box\, a = \Box\, a$$

for all $a \in A$. Similarly, we can show associativity and symmetry of v. Therefore it remains to prove $\Delta_{\mathsf{P}_\mathsf{L}^\sharp A} \circ v \geq id_{\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A}$, that is $v' \circ \Delta'_{\mathsf{P}_\mathsf{L}^\sharp A} \leq id'_{\mathsf{P}_\mathsf{L}^\sharp A \otimes \mathsf{P}_\mathsf{L}^\sharp A}$ for the corresponding frame homomorphisms. The calculation

$$v'(\Delta'_{\mathsf{P}_\mathsf{L}^\sharp A}(\Box\, a \otimes \Box\, b)) = v'(\Box\, a \wedge \Box\, b) = v'(\Box(a \wedge b)) = \Box(a \wedge b) \otimes \Box(a \otimes b) \leq \Box\, a \otimes \Box\, b$$

for all a and b in A suffices to prove the desired result. ■

The semilattice structure from the last lemma is closely related to the semilattice structure P. Johnstone defines for the Vietoris power locale [Joh82a].

Proposition 7.41 *The category of $\mathsf{P}_\mathsf{L}^\sharp$-algebras is a subcategory of* $\mathbf{Loc}^\wedge$, *the category of deflationary semilattices in* $\mathbf{Loc}$.

Proof: Because of the last lemma as well as because of Lemma 5.5 we may conclude that every $\mathsf{P}_\mathsf{L}^\sharp$-algebra is a deflationary localic semilattice. Now for morphisms. Let $f\colon A \longrightarrow B$ be a morphism of the $\mathsf{P}_\mathsf{L}^\sharp$-algebras (A, ξ_A) and (B, ξ_B). First of all we show that $\mathsf{P}_\mathsf{L}^\sharp f$ is a morphism of the deflationary semilattices $\mathsf{P}_\mathsf{L}^\sharp A$ and $\mathsf{P}_\mathsf{L}^\sharp B$. Let the semilattice structures from Lemma 7.40 for $\mathsf{P}_\mathsf{L}^\sharp A$ and $\mathsf{P}_\mathsf{L}^\sharp B$ be denoted by v and w, respectively. For all $b \in B$ we have

$$\begin{aligned}
v'((\mathsf{P}_\mathsf{L}^\sharp f)'(\Box\, b)) &= v'(\Box\, f'(b)) \\
&= \Box\, f'(b) \otimes \Box\, f'(b) \\
&= (\mathsf{P}_\mathsf{L}^\sharp f)'(\Box\, b) \otimes (\mathsf{P}_\mathsf{L}^\sharp f)'(\Box\, b) \\
&= (\mathsf{P}_\mathsf{L}^\sharp f \otimes \mathsf{P}_\mathsf{L}^\sharp f)'(\Box\, b \otimes \Box\, b) \\
&= (\mathsf{P}_\mathsf{L}^\sharp f \otimes \mathsf{P}_\mathsf{L}^\sharp f)'(w'(\Box\, b)).
\end{aligned}$$

By Lemma 5.5 we know that the semilattice structures for A and B are given by $\xi_A \circ v \circ (\eta_A \otimes \eta_A)$ and $\xi_B \circ w \circ (\eta_B \otimes \eta_B)$, respectively. Hence

$$\begin{array}{lll} & f \circ (\xi_A \circ v \circ (\eta_A \otimes \eta_A)) & \\ = & \xi_B \circ \mathsf{P}_\mathsf{L}^\sharp f \circ v \circ (\eta_A \otimes \eta_A) & (f \text{ morphism of } \mathsf{P}_\mathsf{L}^\sharp\text{-algebras}) \\ = & \xi_B \circ w \circ (\mathsf{P}_\mathsf{L}^\sharp f \otimes \mathsf{P}_\mathsf{L}^\sharp f) \circ (\eta_A \otimes \eta_A) & (\mathsf{P}_\mathsf{L}^\sharp f \text{ morphism of } \mathsf{P}_\mathsf{L}^\sharp\text{-algebras}) \\ = & \xi_B \circ w \circ ((\mathsf{P}_\mathsf{L}^\sharp f \circ \eta_A) \otimes (\mathsf{P}_\mathsf{L}^\sharp f \circ \eta_A)) & \\ = & \xi_B \circ w \circ ((\eta_B \circ f) \otimes (\eta_B \circ f)) & (\text{Naturality of } \eta) \\ = & \xi_B \circ w \circ (\eta_B \otimes \eta_B) \circ (f \otimes f). & \blacksquare \end{array}$$

For T_0-spaces we have seen that every algebra for one of the two Smyth power functors has an additional property besides being a deflationary semilattice. Such an algebra has small semilattices, that is any open set is the union of the interiors of the semilattices it contains. We were able to deduce this from the fact that any space of the form $\mathsf{K}_\mathsf{T}X$ or $\mathsf{K}_\mathsf{T}^\mathsf{s}X$ has a basis of open semilattices. A subset S of a $\wedge$-semilattice is a sub-semilattice iff $\wedge_n(S \times \cdots \times S) \subseteq S$ or $S \times \cdots \times S \subseteq \wedge_n^{-1}(S)$ for all $n \in \mathbb{N}$.

If A is a locale then for any element a of A we get that $\Box\, a$ is a semilattice (with respect to the semilattice structure of Lemma 7.40) in the above sense: For all $n \in \mathbb{N}$ we have $v'_n(\Box\, a) = \Box\, a \otimes \cdots \otimes \Box\, a$. Put differently, this means that we have

$$\mathsf{a} = \bigvee\{\mathsf{b} \in \mathsf{P}_\mathsf{L}^\sharp A \mid \forall n \in \mathbb{N}\,.\,\mathsf{b} \otimes \cdots \otimes \mathsf{b} \leq v'_n(\mathsf{a})\}$$

for all $\mathsf{a} \in \mathsf{P}_\mathsf{L}^\sharp A$.

Definition. We say that a deflationary localic semilattice (A, v) has *small semilattices* if, for all $a \in A$,

$$a = \bigvee\{b \in A \mid \forall n \in \mathbb{N}\,.\,b \otimes \cdots \otimes b \leq v'_n(a)\}.$$

The following lemma tells us that this definition indeed captures our intentions.

Proposition 7.42 *Let (A, v) be a deflationary localic semilattice with small semilattices. Then the deflationary topological semilattice* $\mathsf{Sp}A$ *has small semilattices.*

Proof: Because of Lemma 5.2, $\mathsf{Sp}A$ is a deflationary topological semilattice. The semilattice structure is induced by $\mathsf{Sp}v\colon \mathsf{Sp}(A \otimes A) \longrightarrow \mathsf{Sp}A$. By Proposition 2.4, $\mathsf{Sp}(A \otimes A)$ and $\mathsf{Sp}A \times \mathsf{Sp}A$ are isomorphic. Abusing our notation slightly, let $\mathsf{Sp}v$ denote the semilattice structure on $\mathsf{Sp}A$. Recall that the open sets in $\mathsf{Sp}A$ are the sets of the form $U_a := \{p \in \mathsf{Sp}A \mid p \models a\}$ where $a \in A$. So let $p \in \mathsf{Sp}A$ be an element of some U_a. We have to show that there is a semilattice in U_a that is a neighbourhood of p. Since $a = \bigvee\{b \in A \mid \forall n \in \mathbb{N} \,.\, b \otimes \cdots \otimes b \leq v'_n(a)\}$ there is some $b \in A$ with $b \otimes \cdots \otimes b \leq v'_n(a)$ for all $n \in \mathbb{N}$ such that $p \models b$. Let

$$S := \bigcup_{n \in \mathbb{N}} \{p_1 \wedge \cdots \wedge p_n \mid p_1, \ldots, p_n \in U_a\} = \bigcup_{n \in \mathbb{N}} (\mathsf{Sp}v_n)(U_a \times \cdots \times U_a).$$

Obviously, S is a semilattice that contains U_b and therefore is a neighbourhood of p. It remains to prove that S is a subset of U_a. Since $b \otimes \cdots \otimes b \leq v'_n(a)$ we have $U_b \times \cdots \times U_b \subseteq (\mathsf{Sp}v_n)^{-1}(U_a)$ which proves $\mathsf{Sp}v_n(U_b \times \cdots \times U_b) \subseteq U_a$ for all $n \in \mathbb{N}$. ∎

Lemma 7.43 *Every $\mathsf{P}_{\mathsf{L}}^{\sharp}$-algebra has small semilattices.*

Proof: Let (A, ξ_A) be a -algebra. By Lemma 7.40 and Proposition 7.41 both, $\mathsf{P}_{\mathsf{L}}^{\sharp}A$ and A, are deflationary semilattices with semilattice structures, say w and v, respectively. Furthermore, ξ_A is a morphism of the $\mathsf{P}_{\mathsf{L}}^{\sharp}$-algebras $(\mathsf{P}_{\mathsf{L}}^{\sharp}A, \mu_A)$ and (A, ξ_A) and therefore is a semilattice morphism from $(\mathsf{P}_{\mathsf{L}}^{\sharp}A, w)$ to (A, v), also by Proposition 7.41. For all $a \in A$ we have

$$\begin{aligned}
a &= \eta'_A(\xi'_A(a)) \\
&= \eta'_A(\textstyle\bigvee\{\mathsf{a} \in \mathsf{P}_{\mathsf{L}}^{\sharp}A \mid \forall n \in \mathbb{N} \,.\, \mathsf{a} \otimes \cdots \otimes \mathsf{a} \leq w'_n(\xi'_A(a))\}) \\
&= \textstyle\bigvee\{\eta'_A(\mathsf{a}) \mid \forall n \in \mathbb{N} \,.\, \mathsf{a} \otimes \cdots \otimes \mathsf{a} \leq (\xi_A \otimes \cdots \otimes \xi_A)'(v'_n(a))\} \\
&\leq \textstyle\bigvee\{\eta'_A(\mathsf{a}) \mid \forall n \in \mathbb{N} \,.\, (\eta_A \otimes \cdots \otimes \eta_A)'(\mathsf{a} \otimes \cdots \otimes \mathsf{a}) \\
&\qquad\qquad \leq (\eta_A \otimes \cdots \otimes \eta_A)'(\xi_A \otimes \cdots \otimes \xi_A)'(v'_n(a)) = v'_n(a)\} \\
&\leq \textstyle\bigvee\{b \in A \mid \forall n \in N \,.\, b \otimes \cdots \otimes b \leq v'_n(a)\}.
\end{aligned}$$

If $b \otimes \cdots \otimes b \leq v'_n(a)$ for all $n \in \mathbb{N}$ then $b \leq a$. Therefore, we are done. ∎

We cannot show in general, however, that every deflationary localic semilattice with small semilattices is an algebra for the monad. Neither can we give a counterexample. Since frames that are presented via generators and relations are not easy to handle, the search for a counterexample might be quite a tedious task.

Problem. Is there a deflationary semilattice with small semilattices in **Loc** that is not a $\mathsf{P}_{\mathsf{L}}^{\sharp}$-algebra?

Note that we cannot re-use our example for topological spaces (see Section 7.3.3) since a topological semilattice in general does not give rise to a localic semilattice. Hence we only have the following result which follows from Lemma 4.6.

Proposition 7.44 *A deflationary localic semilattice with small semilattices A is a $\mathsf{P}_{\mathsf{L}}^{\sharp}$-algebra iff $\eta_A : A \longrightarrow \mathsf{P}_{\mathsf{L}}^{\sharp} A$ has an upper adjoint ξ_A in* **Loc** *with $\xi_A \circ \eta_A = id_A$.* ■

7.4.4 Locally compact locales

From our studies of locally compact spaces it follows that, restricted to the category **LCompL** of locally compact locales, $\mathsf{P}_{\mathsf{L}}^{\sharp}$ does give the free deflationary semilattice with small semilattices.

Proposition 7.45 *If A is a deflationary semilattice with small semilattices in the category of locally compact locales then it is isomorphic to the open set lattice of a domain.*

Proof: By Proposition 7.42, $\mathsf{Sp}A$ is a deflationary topological semilattice with small semilattices. Furthermore, it is a locally compact sober space. Hence, by Theorem 7.30, $\mathsf{Sp}A$ is a domain with its Scott topology. Since A is isomorphic to $\Omega\,\mathsf{Sp}A$, we are done. ■

Proposition 7.46 *If A is a locally compact locale so is $\mathsf{P}_{\mathsf{L}}^{\sharp} A$.*

Proof: We will show that $\Box a \ll \Box b$ in $\mathsf{P}^{\sharp}_{\mathsf{L}}A$ if $a \ll b$ in A. Suppose that $\mathcal{S}$ is a directed subset of $\mathsf{P}^{\sharp}_{\mathsf{L}}A$ with $\Box b \leq \bigvee^{\uparrow}\mathcal{S}$. Let $\mathcal{S}' := \{\Box c \in \mathsf{P}^{\sharp}_{\mathsf{L}}A \mid \forall c' \ll c.\exists s \in S\,.\,\Box c' \leq s\}$. Then $\mathcal{S}'$ is a lower set of generators for $\mathsf{P}^{\sharp}_{\mathsf{L}}A$ that is closed with respect to directed joins: If $\mathcal{T}$ is a directed subset of $\mathcal{S}'$ then for all $c' \ll \bigvee^{\uparrow}\{c \in A \mid \Box c \in \mathcal{T}\}$ there is $\Box c \in \mathcal{T}$ with $c' \leq c$. Therefore $\Box c' \leq s$ for some $s \in S$ which proves that $\Box \bigvee^{\uparrow}\{c \in A \mid \Box c \in \mathcal{T}\} \in \mathcal{S}'$. Furthermore, the join of $\mathcal{S}'$ is the join of $\mathcal{S}$. Consequently, $\mathcal{S}'$ is a C-ideal in the sense of Theorem 1.11. Hence $\Box b \in \mathcal{S}'$ which proves $\Box a \leq s$ for some $s \in \mathcal{S}$. Thus $\{\Box a \in \mathsf{P}^{\sharp}_{\mathsf{L}}A \mid a \ll b\}$ is a directed subset of $\twoheaddownarrow\{\Box b\}$ with join $\Box b$. Therefore, $\twoheaddownarrow\{\Box b\}$ is directed with join $\Box b$. Since $\mathsf{a} = \bigvee\{\Box b \mid \Box b \leq \mathsf{a}\}$ and $\twoheaddownarrow\{\mathsf{a}\} \supseteq \{\twoheaddownarrow\{\Box b\} \mid \Box b \leq \mathsf{a}\}$ for every $\mathsf{a} \in \mathsf{P}^{\sharp}_{\mathsf{L}}A$, we are done. ∎

For a locally compact locale A we know that $\mathsf{P}^{\sharp}_{\mathsf{L}}A \cong \Omega\,\mathsf{Sp}\mathsf{P}^{\sharp}_{\mathsf{L}}\Omega\,\mathsf{Sp}A \cong \Omega\,\mathsf{K}_{\mathsf{T}}\mathsf{Sp}A$ by Proposition 7.39. Furthermore, this isomorphism is natural $\mathsf{P}^{\sharp}_{\mathsf{L}} \longrightarrow \Omega\,\mathsf{K}_{\mathsf{T}}\mathsf{Sp}$; we will denote it by ι. Note that the components of ι are defined by $\iota'_A(\Box U_a) = \Box a$ where $U_a := \{p \in \mathsf{Sp}A \mid p \models a\}$ denotes the open subset of $\Omega\,\mathsf{Sp}A$ that is generated by a. In order to characterize the category of algebras for the monad $(\mathsf{P}^{\sharp}_{\mathsf{L}}, \eta, \mu)$ restricted to **LCompL** we try to find an equivalent monad which is built from $(\mathsf{K}_{\mathsf{T}}, \uparrow, \bigcup)$. We already have a natural isomorphism $\iota\colon \mathsf{P}^{\sharp}_{\mathsf{L}} \longrightarrow \Omega\,\mathsf{K}_{\mathsf{T}}\mathsf{Sp}$.

In addition, $\Omega\!\uparrow\!\mathsf{Sp}\colon \Omega\,\mathsf{Sp} \longrightarrow \Omega\,\mathsf{K}_{\mathsf{T}}\mathsf{Sp}$ is a natural transformation. There is a natural isomorphism α from the identity of the full subcategory of spatial locales to $\Omega\,\mathsf{Sp}$ (see Chapter 2). Recall that the components of α are defined by $\alpha_A(U_a) := a$ for all $a \in A$. Thus we get a natural transformation $\overline{\eta} := (\Omega\!\uparrow\!\mathsf{Sp}) \bullet \alpha\colon id_{\mathbf{LCompL}} \longrightarrow \Omega\,\mathsf{K}_{\mathsf{T}}\mathsf{Sp}$. We claim that $\iota \bullet \eta = \Omega\!\uparrow\!\mathsf{Sp}\bullet\alpha$. Let A be a locally compact locale. For every $a \in A$ we calculate

$$\begin{aligned}
\alpha'_A\Big((\Omega\!\uparrow_{\mathsf{Sp}\,A})'(\Box U_a)\Big) &= \alpha'_A(\uparrow^{-1}_{\mathsf{Sp}\,A}(\Box U_a)) \\
&= \alpha'_A(U_a) \\
&= a \\
&= \eta'_A(\Box a) \\
&= \eta'_A(\iota'_A(\Box U_a)).
\end{aligned}$$

We still have to find a natural transformation $(\Omega\,\mathsf{K_T Sp})^2 \longrightarrow \Omega\,\mathsf{K_T Sp}$ in order to obtain a monad for the functor $\Omega\,\mathsf{K_T Sp}$. Let $\gamma\colon \mathsf{Sp}\Omega \longrightarrow id_{\mathbf{Sob}}$ be the natural isomorphism we introduced in Chapter 2. Since $\Omega\,\mathsf{K_T}\gamma\,\mathsf{K_T Sp}\colon \Omega\,\mathsf{K_T Sp}\Omega\,\mathsf{K_T Sp} \longrightarrow \Omega\,(\mathsf{K_T})^2\,\mathsf{Sp}$ and $\Omega \bigcup \mathsf{Sp}\colon \Omega\,(\mathsf{K_T})^2\,\mathsf{Sp} \longrightarrow \Omega\,\mathsf{K_T Sp}$ are natural, their composite $\overline{\mu} := \Omega(\bigcup \bullet(\mathsf{K_T}\gamma\,\mathsf{K_T}))\,\mathsf{Sp}$ is a natural transformation as needed. We want to show that

$$\iota \circ \mu = \ (\Omega \circ (\bigcup \bullet (\mathsf{K_T} \circ \gamma \circ \mathsf{K_T})) \circ \mathsf{Sp}) \bullet (\iota \circ \iota)$$

since, by Proposition 4.1, this implies that $(\Omega\,\mathsf{K_T Sp}, \overline{\eta}, \overline{\mu})$ is a monad that is equivalent to $(\mathsf{P}^\sharp_\mathsf{L}, \eta, \mu)$. We achieve this by verifying that for any locale A we have

$$\iota_A \circ \mu_A = \Omega \textstyle\bigcup_{\mathsf{Sp}\,A} \circ \Omega\,\mathsf{K_T}\gamma_{\mathsf{K_T Sp}\,A} \circ \iota_{\Omega\,\mathsf{K_T Sp}\,A} \circ (\mathsf{P}^\sharp_\mathsf{L}\iota_A).$$

This computation is a bit messy since we have to keep track of all the natural transformations involved but it is not difficult. Recall that $\gamma_X^{-1}(V) = U_V$ for any open subset V of some topological space X. Let a be an arbitrary element of some locale A. Then $(\iota_A \circ \mu_A)'(\Box U_a) = \mu'_A(\iota'_A(\Box U_a)) = \mu'_A(\Box\, a) = \Box\,\Box\, a$. On the other hand we have

$$\begin{aligned}
& (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'\Big(\iota'_{\Omega\,\mathsf{K_T Sp}\,A}\Big((\Omega\,\mathsf{K_T}\gamma_{\mathsf{K_T Sp}\,A})'((\Omega \textstyle\bigcup_{\mathsf{Sp}\,A})'(\Box U_a))\Big)\Big) \\
= \ & (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'\Big(\iota'_{\Omega\,\mathsf{K_T Sp}\,A}\Big((\Omega\,\mathsf{K_T}\gamma_{\mathsf{K_T Sp}\,A})' \textstyle\bigcup^{-1}_{\mathsf{Sp}\,A}(\Box U_a)\Big)\Big) \\
= \ & (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'\Big(\iota'_{\Omega\,\mathsf{K_T Sp}\,A}\Big((\mathsf{K_T}\gamma_{\mathsf{K_T Sp}\,A})^{-1}(\Box\Box U_a)\Big)\Big) \\
= \ & (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'\Big(\iota'_{\Omega\,\mathsf{K_T Sp}\,A}(\Box\gamma^{-1}_{\mathsf{K_T Sp}\,A}(\Box U_a))\Big) \\
= \ & (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'\Big(\iota'_{\Omega\,\mathsf{K_T Sp}\,A}(\Box U_{\Box U_a})\Big) \\
= \ & (\mathsf{P}^\sharp_\mathsf{L}\iota_A)'(\Box\,\Box U_a) \\
= \ & \Box\,\iota'_A(\Box U_a) \\
= \ & \Box\,\Box\, a.
\end{aligned}$$

Hence we have shown the following proposition.

Proposition 7.47 *The monad* $(\mathsf{P}^\sharp_\mathsf{L}, \eta, \mu)$ *is equivalent to the monad* $(\Omega\,\mathsf{K_T Sp}, \overline{\eta}, \overline{\mu})$ *on* **LCompL**. ■

Theorem 7.48 *The category of algebras for the monad* $(\mathsf{P}_\mathsf{L}^\sharp, \eta, \mu)$ *on the category* **LCompL** *of locally compact locales is the category of deflationary semilattices that have small semilattices in* **LCompL**.

Proof: Let A be a deflationary semilattice with small semilattices in **LCompL**. By Proposition 7.42, $\mathsf{Sp}A$ is a deflationary semilattice with small semilattices in the category of locally compact sober spaces. Hence, by Theorem 7.31, there is a structure map $\bigwedge_{\mathsf{Sp}\,A}\colon \mathsf{K}_\mathsf{T}\mathsf{Sp}A \longrightarrow \mathsf{Sp}A$ making $\mathsf{Sp}A$ a K_T-algebra. Then $(\Omega\,\mathsf{Sp}A, \Omega \bigwedge_{\mathsf{Sp}\,A})$ is an algebra for the monad $(\Omega\,\mathsf{K}_\mathsf{T}\mathsf{Sp}, \overline{\eta}, \overline{\mu})$. By Proposition 4.2 this implies that $(\Omega\,\mathsf{Sp}A, \Omega \bigwedge_{\mathsf{Sp}\,A} \circ \iota_A)$ is a $\mathsf{P}_\mathsf{L}^\sharp$-algebra. Since $\Omega\,\mathsf{Sp}A$ and A are (naturally) isomorphic, we can indeed find a structure map for A. Analogously we can show that every morphism of deflationary semilattices with small semilattice is a morphism of the corresponding $\mathsf{P}_\mathsf{L}^\sharp$-algebras. ∎

The proof we gave for this characterization of $\mathsf{P}_\mathsf{L}^\sharp$-algebras in the category of locally compact locales is not 'localic' in nature. Hence it does not tell us why we could not show an analogous characterization for arbitrary locales. However, the only other proof we were able to come up with is to consider open set lattices of locally compact sober spaces and then more or less repeat the proof for topological spaces. We consider this proof even less expressive than the one we gave. That we have not been able to find a truly 'pointless' proof is quite unsatisfactory.

7.4.5 Unital semilattices

If we want to make every Smyth power locale a unital semilattice in **Loc** then we drop the equation $\Box\, 0_A = 0_{\mathsf{P}_\mathsf{L}^\sharp A}$ from the above definition. We thus get a slightly different Smyth power construction, say $\dot{\mathsf{P}}_\mathsf{L}^\sharp$. Obviously, we also get a monad for $\dot{\mathsf{P}}_\mathsf{L}^\sharp$. To see that every locale of the form $\dot{\mathsf{P}}_\mathsf{L}^\sharp A$ is a unital deflationary semilattice, note that $\dot{\mathsf{P}}_\mathsf{L}^\sharp A$ carries a semilattice structure similar to the one we have for $\mathsf{P}_\mathsf{L}^\sharp A$. We obtain this semilattice structure $v\colon \dot{\mathsf{P}}_\mathsf{L}^\sharp A \otimes \dot{\mathsf{P}}_\mathsf{L}^\sharp A \longrightarrow \dot{\mathsf{P}}_\mathsf{L}^\sharp A$ by setting $v'(\Box\, a) := \Box\, a \otimes \Box\, a$ for all $a \in A$. To show that this definition makes $\dot{\mathsf{P}}_{\mathsf{L}}^{\sharp}A$ a deflationary semilattice we can proceed in the same way as we did for $\mathsf{P}_{\mathsf{L}}^{\sharp}A$. By Proposition 5.8 it is sufficient to show that $o'(\Box a) := 1_{\mathit{2}}$ defines a continuous function from $\mathit{2}$ to $\dot{\mathsf{P}}_{\mathsf{L}}^{\sharp}A$ in order to prove that v has a unit. Since the element $0_{\dot{\mathsf{P}}_{\mathsf{L}}^{\sharp}A}$ is freely adjoint to $\mathsf{P}_{\mathsf{L}}^{\sharp}A$, the above definition makes o the unit for the semilattice structure for $\dot{\mathsf{P}}_{\mathsf{L}}^{\sharp}A$. For locally compact locales we can also prove that every unital deflationary semilattice with small semilattices is a $\dot{\mathsf{P}}_{\mathsf{L}}^{\sharp}$-algebra.

Appendix.

Name	Objects
$\mathbf{C}^{\mathbf{SLat}}$	semilattices in $\mathbf{C}$
$\mathbf{C}^{\vee}$	inflationary semilattices in $\mathbf{C}$
$\mathbf{C}^{\wedge}$	deflationary semilattices in $\mathbf{C}$

Table 1: Subcategories of an order-enriched category $\mathbf{C}$; in each case the morphisms are all semilattice homomorphisms.

Name	Objects	Morphisms: Scott-continuous functions that additionally preserve
DCPO	directed complete partial orders	
$\mathbf{DCPO}^{\vee}$	dcpo's that have binary joins = up-complete $\vee$-semilattices	binary joins
$\mathbf{DCPO}^{\wedge}$	dcpo's that have binary meets = $\wedge$-continuous $\wedge$-semilattices	binary meets
Dom	continuous dcpo's = domains	
$\mathbf{Dom}^{\vee}$	continuous $\vee$-semilattices	binary joins
$\mathbf{Dom}^{\wedge}$	continuous $\wedge$-semilattices	binary meets
AlgD	algebraic dcpo's	
$\mathbf{AlgD}^{\vee}$	algebraic $\vee$-semilattices	binary joins
$\mathbf{AlgD}^{\wedge}$	algebraic $\wedge$-semilattices	binary meets

Table 2: Subcategories of **DCPO**

Name	Objects
Top	T_0-spaces
Sob	sober spaces
LCompT	locally compact T_0-spaces
LCompS	locally compact sober spaces

Table 3: Full Subcategories of **Top**

Name	Objects
Loc	locales
LCompL	locally compact locales

Table 4: Full subcategories of **Loc**

Name	Operation
$\Omega_{ne}\,X$	non-empty open subsets of a topological space X
$\mathsf{C}D$	compact elements of an algebraic dcpo D
$\mathsf{K}X$	non-empty compact saturated subsets of a T_0-space X ordered by reverse inclusion
$\mathsf{K}_{\mathsf{D}}D$	non-empty compact upper subsets of a dcpo D ordered by reverse inclusion
$\mathsf{K}^{\mathsf{s}}_{\mathsf{D}}D$	non-empty strongly compact upper subsets of a dcpo D ordered by reverse inclusion
$\mathsf{P}S$	power set of a set S
$\mathsf{P}_{\mathsf{fin}}S$	finite subsets of a set S
$\mathsf{P}_{\mathsf{fne}}S$	finite non-empty subsets of a set S
$\mathsf{P}_L P$	lower subsets of a poset P ordered by inclusion
$\mathsf{P}_U P$	upper subsets of a poset P ordered by reverse inclusion

Table 5: Operators

Name	From to	Operator
Idl	$\mathbf{Pos} \longrightarrow \mathbf{AlgD}$	ideal completion
Ω	$\mathbf{Top} \longrightarrow \mathbf{Loc}$	$X \longmapsto$ the open set lattice of X
Sp	$\mathbf{Loc} \longrightarrow \mathbf{Top}$	$A \longmapsto$ the spatialization of A
Pt	$\mathbf{Loc} \longrightarrow \mathbf{DCPO}$	$A \longmapsto$ the set of points for A
		with the specialization order
A	$\mathbf{Top} \longrightarrow \mathbf{DCPO}^{\vee}$	$X \longmapsto (\{A \subseteq X \mid \emptyset \neq A \text{ closed}\}, \subseteq)$
$\mathsf{A_D}$	$\mathbf{DCPO} \longrightarrow \mathbf{DCPO}$	$D \longmapsto (\{A \subseteq D \mid \emptyset \neq A \text{ Scott-closed}\}, \subseteq)$
T_σ	$\mathbf{DCPO} \longrightarrow \mathbf{Top}$	$D \longmapsto$ the set D with the Scott topology
T_ω	$\mathbf{DCPO}^{\vee} \longrightarrow \mathbf{Top}$	$L \longmapsto$ the set L with the weak topology

Table 6: Functors

Name	Category	Operator
$\mathsf{P}^\flat_{\mathsf{A}} \cong \mathsf{A}_{\mathsf{D}}$	**AlgD**	$D \longmapsto \mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^\flat)$
$\mathsf{P}^\flat_{\mathsf{D}}$	**DCPO**	$\mathsf{P}^\flat_{\mathsf{D}} = \mathsf{A}_{\mathsf{D}}$
$\mathsf{P}^\flat_{\mathsf{T}}$	**Top**	$\mathsf{P}^\flat_{\mathsf{T}} = \mathsf{T}_\omega \mathsf{A}$
$\mathsf{P}^\flat_{\mathsf{L}}$	**Loc**	$A \longmapsto Fr\langle \diamond a : a \in A \mid \diamond 1_A = 1_{\mathsf{P}^\flat_{\mathsf{L}} A}$
		$\bigvee_{s \in S} \diamond s = \diamond \bigvee S \ (S \subseteq A)\rangle$

Table 7: Hoare power constructions

Name	Category	Algebras
$\mathsf{P}^\flat_{\mathsf{A}} \cong \mathsf{A}_{\mathsf{D}}$	**AlgD**	$\mathbf{AlgD}^\vee$
$\mathsf{P}^\flat_{\mathsf{D}}$	**DCPO**	$\mathbf{DCPO}^\vee$
	Dom	$\mathbf{Dom}^\vee$
$\mathsf{P}^\flat_{\mathsf{T}}$	**Top**	$\mathbf{Top}^\vee \cap \mathbf{Sob}$
	LCompT	$\mathbf{Top}^\vee \cap \mathbf{LCompS}$
$\mathsf{P}^\flat_{\mathsf{L}}$	**Loc**	$\mathbf{Loc}^\vee$
	LCompL	$\mathbf{Loc}^\vee \cap \mathbf{LCompL}$

Table 8: Algebras for Hoare power constructions

Name	Category	Operator
$\mathsf{P}^\sharp_{\mathsf{A}}$	**AlgD**	$D \longmapsto \mathsf{Idl}(\mathsf{P}_{\mathsf{fne}}\mathsf{C}D, \leq^\sharp)$
$\mathsf{K}_{\mathsf{D}}(\cong \mathsf{F}_{\mathsf{D}} \cong \mathsf{F}^{\mathsf{s}}_{\mathsf{D}})$	**Dom**	$D \longmapsto (\{K \subseteq D \mid \emptyset \neq K$ saturated and Scott-compact$\}, \supseteq)$
F_{D}	**DCPO**	$D \longmapsto \{\varphi \subseteq \Omega_{ne}\, D \mid \varphi$ Scott-open filter$\}$
$\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$	**DCPO**	$D \longmapsto \{\varphi \subseteq \Omega_{ne}\, D \mid \varphi$ strong filter$\}$
K_{T}	**Top**	$X \longmapsto (\{K \subseteq X \mid \emptyset \neq K$ saturated and compact$\}, \square$-topology$)$
$\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$	**Top**	$X \longmapsto (\{K \subseteq X \mid \emptyset \neq K$ saturated and strongly compact$\}, \boxed{\mathsf{s}}$-topology$)$
$\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$	**Top**	$X \longmapsto \{\uparrow\{x_1, \ldots, x_n\} \mid x_1, \ldots, x_n \in X\}$ as a subspace of $\mathsf{K}_{\mathsf{T}}X$
$\mathsf{P}^\sharp_{\mathsf{L}}$	**Loc**	$A \longmapsto Fr\langle \square\, a : a \in A \;\mid\; \square\, 0_A = 0_{\mathsf{P}^\sharp_{\mathsf{L}} A},\ \square\, 1_A = 1_{\mathsf{P}^\sharp_{\mathsf{L}} A},$ $\square\, a \wedge \square\, b = \square(a \wedge b)\ (a, b \in A),$ $\bigvee^{\uparrow}_{s \in S} \square\, s = \square \bigvee^{\uparrow} S\ (S \subseteq A \text{ directed})\rangle$

Table 9: Smyth power constructions

Name	Category	Algebras
$\mathsf{P}^\sharp_{\mathsf{A}}$	**AlgD**	$\mathbf{AlgD}^{\wedge}$
$\mathsf{K}_{\mathsf{D}}(\cong \mathsf{F}_{\mathsf{D}} \cong \mathsf{F}^{\mathsf{s}}_{\mathsf{D}})$	**Dom**	$\mathbf{Dom}^{\wedge}$
F_{D}	**DCPO**	$\subset \mathbf{DCPO}^{\wedge}$
$\mathsf{F}^{\mathsf{s}}_{\mathsf{D}}$	**DCPO**	$\subset \mathbf{DCPO}^{\wedge}$
K_{T}	**Top**	$\subset \mathbf{Top}^{\wedge}_{ssl}$
	LCompS	$\mathbf{Top}^{\wedge}_{ssl} \cap \mathbf{LCompS} \cong \mathbf{Dom}^{\wedge}$
$\mathsf{K}^{\mathsf{s}}_{\mathsf{T}}$	**Sob**	$\mathbf{Top}^{\wedge}_{ssl} \cap \mathbf{Sob}$
$\mathsf{K}^{\mathsf{f}}_{\mathsf{T}}$	**Top**	$\mathbf{Top}^{\wedge}_{ssl}$
$\mathsf{P}^\flat_{\mathsf{L}}$	**Loc**	$\subseteq \mathbf{Loc}^{\wedge}_{ssl}$
	LCompL	$\mathbf{Loc}^{\wedge}_{ssl} \cap \mathbf{LCompL}$

Table 10: Algebras for Smyth power constructions; $\mathbf{C}^{\wedge}_{ssl}$ is the full subcategory of semilattices with small semilattices of $\mathbf{C}^{\wedge}$

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