Cartesian Closed Categories

Andrea Schalk · November 16, 2010

THIS IS A supplementary part of the notes on category theory as provided by Harold Simmons' 'An Introduction to Category Theory'. It explains the notion of a cartesian closed category, which is merely a special example of an adjunction.

Cartesian closed categories are precisely what is required to model the simply-typed λ-calculus (or indeed intuitionistic logic with implication). This is explained in the last lecture on category theory but that material is not examinable and there will be no handout for it.[1]

It is worth stating the following very general ideas (they go much beyond the simply-typed λ-calculus):

- types are modelled by objects and
- derivations (and, ultimately, judgements) are modelled by arrows.

For types, this means that

- to each atomic type or type variable we have assigned an object in the category and
- for each type constructor (this is just the arrow in the case of the simply-typed λ-calculus) there is a construction on objects that models it. Without going into detail here it makes sense to demand that this construction is *functorial*.

For atomic types we may have to demand some additional properties, depending on what constants there are in our calculus.

Contexts are modelled by *products*, that is, we form the product of all the objects interpreting the types that are mentioned in the context (honouring repetitions). The empty context is therefore modelled by the terminal object.

The aim is now to *inductively define* an arrow for each derivation. The 'type' of the arrow is given by information in its final judgement: Its source is the interpretation of the context and its target the interpretation of the type of the derived λ-term.

How this works will be explained in the lecture mentioned above. Note that once one has such an assignment the following questions become of interest:

- If there are two derivations for the same judgement, can we say anything about the corresponding arrows?
- If two terms are related by
  - α-equivalence,
  - β-reduction

  what, if anything, can we say about the arrows interpreting their derivations?

[1] Writing this all out such that all the details are correct is quite difficult—all the published versions that I know of have a flaw or two.

You should pause to think about what you would expect to happen.

Note that if you take a derivation in the simply-typed λ-calculus and remove all the terms you get a *derivation in the sequent calculus formulation of intuitionistic logic* (with implication the only connective). One can understand the λ-terms constructed along the way as 'proof terms': They encode the derivation in question. Hence by modelling this λ-calculus we can also model intuitionistic logic.

A lot of what we want is already described in Section 5.2.2 of the book, albeit in the special case of the category **Set**.

In order to model the simply-typed λ-calculus as outlined above we need products to model the contexts. Since products are only defined up to unique isomorphism we need something a bit stronger to have a proper *assignment* for our interpretation of types: We have to assume that we have chosen a particular product in all cases.

A category $\mathbb{C}$ with *selected products* is a category with

- a chosen terminal object and
- for each pair of objects A and B a chosen specific object $A \times B$ (with projections π_l and π_r) which is a product of the two.

From Section 3.3.3 of the notes we know that in this situation given an object B in $\mathbb{C}$ there is a functor $- \times B \colon \mathbb{C} \longrightarrow \mathbb{C}$ which maps objects A of $\mathbb{C}$ to $A \times B$, and arrows $f \colon A \longrightarrow A'$ to $f \times 1_B \colon A \times B \longrightarrow A' \times B$.

Once we have this we can define what else is needed: As per the above sketch of interpreting derivations it is clear that we require a construction that models the arrow formation on types.

To then be able to inductively define the interpretation of derivations, and for that interpretation to have good properties, it is not enough to merely assume that we have an assignment $\Rightarrow$ on objects for this purpose. It turns out that what is wanted here is the following.

Definition 1. A category $\mathbb{C}$ with designated products is **cartesian closed** (a *ccc* if for every object B of $\mathbb{C}$ the functor $- \times B \colon \mathbb{C} \longrightarrow \mathbb{C}$ has a right adjoint.

It is fairly standard to say that an object B is **exponentiable** if the functor $(- \times B)$ has a right adjoint, so a category is cartesian closed if and only if all its objects are exponentiable.

If such an adjoint exists we typically use $B \Rightarrow -$ to refer to it. Given some object C in $\mathbb{C}$, the object $B \Rightarrow C$ is often referred to as the *internal function space* or *internal hom-object* or, the *internal arrow object of B and C*. When we look at how $B \Rightarrow -$ acts on arrows we frequently write $1_B \Rightarrow e$ for $(B \Rightarrow -)e$ where $e \colon C' \longrightarrow C$.

The remainder of the notes look at this definition in a bit more detail.

The general theory of adjunctions applies, so in particular we know that for an exponentiable B there is a natural isomorphism with components

$$\phi_{A,C} \colon \mathbb{C}[A \times B, C] \longrightarrow \mathbb{C}[A, B \Rightarrow C],$$

which we call *currying*.

Exercise 1. Which two functors are connected by the natural isomorphism ϕ? If you find this hard then read on a bit.

The 'typical' example of a cartesian closed category is **Set**. Here for sets B and C the object $B \Rightarrow C$ is the set of all functions from B to C. But this is merely the (covariant version of) the hom-functor as introduced in Section 3.2! We are in a fortunate situation here since we need to define a *set* $B \Rightarrow C$, and the set of arrows gives us back a set. Clearly we cannot apply the same trick for any category other than **Set**. Nonetheless there are many cases where $B \Rightarrow C$ has a close connection with the set of arrows from B to C, but this has to be turned into an object of the category in question. This is the origin of the notion of 'internalizing the hom-set'.

Now the components of ϕ *are* currying and uncurrying as you have met them in your lectures on λ-calculi:

Given $f\colon A \times B \longrightarrow C$ we define the function $\phi_{A,C}(f)$ with source A and target $B \Rightarrow C$ by setting

$$\phi_{A,C}(f)(a)\colon b \longmapsto f(a, b).$$

So each $\phi_{A,C}(f)(a)$ becomes a function from B to C as required. You should now be able to write down a definition of uncurrying.

We look at an alternative description of an adjunction, and this is a case where the 'co-free' version of an adjunction seems to work better than the free one (mostly because we already have a functor that we desire to be a left adjoint). Here is the data required:

- A functor $- \times B\colon \mathbb{C} \longrightarrow \mathbb{C}$,
- an assignment on objects $C \longmapsto B \Rightarrow C$,
- a family of arrows (indexed by objects C of $\mathbb{C}$)

  $$\mathsf{eval}_C\colon (B \Rightarrow C) \times B \longrightarrow C$$

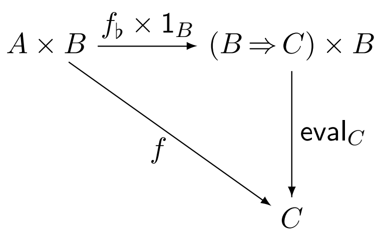
  known as 'evaluation arrows' with the property that for all

  $$f\colon A \times B \longrightarrow C$$

  there exists a unique

  $$f_\flat\colon A \longrightarrow B \Rightarrow C$$

  such that the following diagram commutes.



For the category **Set**, the eval_C are evaluation functions in that given

a function $f\colon B \longrightarrow C$ and an element b of B

we set

$$\mathsf{eval}_C(f, b) = f(b).$$

Exercise 2. Prove that the category **Set** is cartesian closed. You may want to consider doing it in more than one way.

Unfortunately there aren't a lot of simple examples of cartesian closed categories. While it is often possible to equip a function space with the structure required to turn it into an object of the category under consideration, in general this does *not* lead to a right adjoint for the product functor.

Categories of algebras are good examples for this. One can then look for a *left adjoint* for the function space construction instead; the result is typically know as *a tensor product*[2] and one gets what are known as *symmetric monoidal closed categories*.

Examples that do work are:

- Categories of partially ordered sets, for example the category of posets. The function space is given the 'pointwise order', which means that a function $f\colon P \longrightarrow Q$ is below some function g between the same posets if and only if for every $p \in P$ it is the case that $f(p) \leq g(p)$. This does provide a right adjoint for the product functor.

- Some categories of topological spaces. However, for this to work on has to restrict oneself to *locally compact topological spaces*, which requires too much detail regarding the general theory of topological spaces to make it a suitable example for this course.

- Categories of monoid actions (see Chapter 6 of the notes). But this is hardly a simple example!

- The category of functors from a given (small) category to the category of sets.

- The category of (small) categories. A hom-object $\mathbb{C} \Rightarrow \mathbb{D}$ is given by the functor category of all functors from $\mathbb{C}$ to $\mathbb{D}$.

Exercise 3. It would be a good idea at this stage to work out the details of another cartesian closed category. If the above examples all seem daunting you might want to caracterize which posets (viewed as categories) are cartesian closed. Be warned, however, that the answer may not look very meaningful unless you have studied (at least) Boolean algebras.

There is one more thing to be discussed in these notes. In a cartesian closed category we have a great number of pairs of adjoint functors, namely one for each object. In some sense all these adjunctions 'behave in the same way'.

Instead of looking at the functors $(- \times B)$ we can certainly turn the B into another variable and look at the functor $- \times -\colon \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$.

We make a general observation about functors with two arguments, also known as *bifunctors*. Given categories $\mathbb{C}$ and $\mathbb{D}$ we can form their product (in the category of categories) $\mathbb{C} \times \mathbb{D}$, as described in Example 1.9 of the notes. The objects of $\mathbb{C} \times \mathbb{D}$ are pairs (C, D), where C is an object of $\mathbb{C}$ and D one of $\mathbb{D}$. Morphisms, identities and composition are also defined component-wise. So if we have a functor whose source is the product of two categories $\mathbb{C} \times \mathbb{D}$ we can think of it as a functor with two arguments, one from $\mathbb{C}$ and one from $\mathbb{D}$.

Now let $F\colon \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{E}$ be such a functor 'of two arguments'. We can define two families of derived functors, one with components $\mathbb{C} \longrightarrow \mathbb{E}$ and one with components $\mathbb{D} \longrightarrow \mathbb{E}$ by 'making one argument constant'. For D in $\mathbb{D}$ let $F(-, D)$ be the functor $\mathbb{C} \longrightarrow \mathbb{E}$ which maps an object C to $F(C, D)$ and an arrow $f\colon C \longrightarrow C'$ to $F(f, 1_D)\colon F(C, D) \longrightarrow F(C', D)$. We can define $F(C, -)$ for C an object of $\mathbb{C}$ in a similar manner.

Exercise 4. Let $F, G\colon \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{E}$ be two functors. Show that a family of arrows with components $\alpha_{C,D}\colon F(C \times D) \longrightarrow G(C \times D)$ (where C and D are objects of $\mathbb{C}$ and $\mathbb{D}$ respectively) is a natural transformation from F to G if and only if '*it is natural in both components separately*', that is

[2]This is a notion that is strictly weaker than a categorical product.

- given an object D of $\mathbb{D}$ we have that $\alpha_{-,D}$ defines a natural transformation from $F(-,D)\colon \mathbb{C} \longrightarrow \mathbb{E}$ to $G(-,D)\colon \mathbb{C} \longrightarrow \mathbb{E}$ and
- given an object C of $\mathbb{C}$ we have that $\alpha_{C,-}$ is a natural transformation from $F(C,-)$ to $G(C,-)$.

We know that for every object B of a ccc $\mathbb{C}$ the functor $(-\times B)$ has a right adjoint. What does that mean for the functor of two arguments $(-\times -)$? The answer to that question is that such a functor has good properties which we describe below. Instead of doing so in the restricted setting of cccs we generalize the problem.

Definition 2. A functor $F\colon \mathbb{C}\times\mathbb{D} \longrightarrow \mathbb{E}$ *is a left adjoint of a parametrized adjunction* if all the functors

$$F(-,D)\colon \mathbb{D} \longrightarrow \mathbb{E}$$

have a right adjoint

$$^{D}G\colon \mathbb{E} \longrightarrow \mathbb{C}.$$

Before we look at parametrized adjunctions we summarize some of the material explained in Chapter 5.3 of the notes as a reminder.

Recall the hom-functors in Chapter 2.4.1 of the notes. Given an object A of some category $\mathbb{C}$ we can define functors

$$\mathbb{C}[A,-]\colon \mathbb{C} \longrightarrow \mathbf{Set} \qquad \text{and} \qquad \mathbb{C}[-,B]\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

The first one maps an object B of $\mathbb{C}$ to the set of arrows from A to B, $\mathbb{C}[A,B]$ (which is also the result of applying the second functor to A). An arrow $g\colon B \longrightarrow B'$ is mapped to the function

$$\mathbb{C}[A,B] \longrightarrow \mathbb{C}[A,B']$$

that acts on some $e \in \mathbb{C}[A,B]$ by mapping it to the composite $g\circ e$.

Exercise 5. These functors are described in more detail in Section 3.2 of the notes, starting on page 73. If you don't feel comfortable with them yet, reread that part of the notes and do any proofs omitted there to establish that they are indeed functors.

We can now put these two functors together and turn them into one 'of two variables'. In other words, we have a functor $\mathbb{C}[-,-]\colon \mathbb{C}^{\mathrm{op}}\times\mathbb{C} \longrightarrow \mathbf{Set}$.

Exercise 6. Define this functor and show that it is one—this is Exercise 3.2.5 of the notes.

Let $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ be left adjoint to $G\colon \mathbb{D} \longrightarrow \mathbb{C}$. When describing the adjunction according to Definition 5.1 of the notes we need to have these two-variable hom-functors in order to make precise what is meant by naturality in this context (you were encouraged to formulate this in Exercise 1).

In particular, in Section 5.3 of the notes it is explained how to define two functors

$$\mathbb{C}^{\mathrm{op}}\times\mathbb{D} \longrightarrow \mathbf{Set},$$

namely by composing the hom-functors

$$\mathbb{D}[-,-]\colon \mathbb{D}^{\mathrm{op}}\times\mathbb{D} \longrightarrow \mathbf{Set} \qquad \text{and} \qquad \mathbb{C}[-,-]\colon \mathbb{C}^{\mathrm{op}}\times\mathbb{C} \longrightarrow \mathbf{Set}$$

with $F^{\mathrm{op}}\times 1_{\mathbb{D}}$ and $1_{\mathbb{C}^{\mathrm{op}}}\times G$ respectively to get

$$\mathbb{D}[F-,-],\ \mathbb{C}[-,G-]\colon \mathbb{C}^{\mathrm{op}}\times\mathbb{D} \longrightarrow \mathbf{Set}.$$

Then the family of isomorphisms with components

$$\phi_{C,D}\colon \mathbb{D}[FC, D] \longrightarrow \mathbb{C}[C, GD]$$

is a natural isomorphism between these two functors.

Exercise 7. Let $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ be left adjoint to $G\colon \mathbb{D} \longrightarrow \mathbb{C}$, and let ϕ be the corresponding natural isomorphism with components

$$\phi_{C,D}\colon \mathbb{D}[FC, D] \longrightarrow \mathbb{C}[C, GD].$$

Show that the action of G on arrows is uniquely determined by ϕ. Now assume that you are given assignments on objects F from $\mathbb{C}$ to $\mathbb{D}$ and G in the opposite direction. What condition does ϕ have to satisfy so that we get a functor when defining G on arrows by using this uniqueness condition? Show that this condition can be formulated such that if we use this definition of G then $\phi_{C,-}$ provides a natural transformation from $\mathbb{D}[FC, -]$ to $\mathbb{C}[C, G-]$. Now derive similar statements for F.

Hint: For $g\colon D \longrightarrow D'$ we get $Gg = \phi_{GD,D'}(g \circ \phi^{-1}_{GD,D}(1_{GD}))$.

The following result states what we can say about parametrized adjunctions.

Theorem 3. *Let $F\colon \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{E}$ be the left adjoint of a parametrized adjunction with ${}^{D}G\colon \mathbb{E} \longrightarrow \mathbb{C}$ giving the right adjoints for $F(-, D)$ and the natural isomorphism from $\mathbb{E}[F(- \times D), -]$ to $\mathbb{C}[-, {}^{D}G-]$ having components*

$${}^{D}\phi_{C,E}\colon \mathbb{E}[F(C \times D), E] \longrightarrow \mathbb{C}[C, {}^{D}GE].$$

Then there is a unique functor $G\colon \mathbb{D}^{\mathrm{op}} \times \mathbb{E} \longrightarrow \mathbb{C}$ such that

- *$G(D, E) = {}^{D}GE$;*
- *for the two functors*

$$\mathbb{E}[F(-,-),-],\ \mathbb{C}[-, G(-,-)]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{D}^{\mathrm{op}} \times \mathbb{E} \longrightarrow \mathbf{Set}$$

we have a natural isomorphism ψ with components $\psi_{C,D,E} = {}^{D}\phi_{C,E}$.

Proof. We begin by making an observation regarding the purported natural isomorphism ψ. First of all we know that all the components of ψ are indeed isomorphisms. Secondly by Exercise 4 naturality of ψ is equivalent to it being natural in all three components separately. We already know that it is natural in the components from $\mathbb{C}$ and $\mathbb{E}$, but with respect to the functors $F(-, D)$ and ${}^{D}G$. This is expressed by the following square, which commutes for all objects D of $\mathbb{D}$ and arrows[3] $f\colon C' \longrightarrow C$ in $\mathbb{C}$ and $h\colon E \longrightarrow E'$ in $\mathbb{E}$.

$$\begin{array}{ccc}
\mathbb{E}[F(C,D),E] & \xrightarrow{\ {}^{D}\phi_{C,E}\ } & \mathbb{C}[C, {}^{D}GE] \\
{\scriptstyle h \circ - \circ F(f, 1_D)}\ \Big\downarrow & & \Big\downarrow\ {\scriptstyle {}^{D}Gh \circ - \circ f} \\
\mathbb{E}[F(C',D),E'] & \xrightarrow{\ {}^{D}\phi_{C',E'}\ } & \mathbb{C}[C', {}^{D}GE']
\end{array}$$

[3] Note that $\mathbb{E}[F(-,-),-]$, $\mathbb{C}[-, {}^{D}G]$, and $\mathbb{C}[-, G(-,-)]$ are contravariant in their $\mathbb{C}$ and $\mathbb{D}$ arguments. We prefer to describe arrows in $\mathbb{D}$ rather than in $\mathbb{D}^{\mathrm{op}}$, but it should be noted that the arrows then compose in the opposite way of what might be expected.

Put into equations, we know that for all $e \in \mathbb{E}[F(C,D),E]$ we have

$$ {}^D\phi_{C',E}(e \circ F(f, 1_D)) = {}^D\phi_{C,E}(e) \circ f $$

and

$$ {}^D\phi_{C,E'}(h \circ e) = {}^DGh \circ {}^D\phi_{C,E}(e). $$

On objects we define $G(D,E) = {}^DGE$ as suggested by the statement of the theorem. This implies naturality of ψ in the $\mathbb{C}$ component, which we achieve by letting $E = E'$ and $h = 1_E$ in the above diagram.

In order to define G on arrows[4] we look at one of the requirements, namely naturality for ψ in the $\mathbb{D}$ and $\mathbb{E}$ components. This is expressed by the following square, where $g\colon D' \longrightarrow D$ in $\mathbb{D}$ and $h\colon E \longrightarrow E'$ in $\mathbb{E}$, and C is an arbitrary object from $\mathbb{C}$. Any G we define must satisfy

$$
\begin{array}{ccc}
\mathbb{E}[F(C,D),E] & \xrightarrow{\ \psi_{C,D,E} = {}^D\phi_{C,E}\ } & \mathbb{C}[C, G(D,E)] \\
{\scriptstyle h \circ - \circ F(1_C, g)} \big\downarrow & & \big\downarrow {\scriptstyle G(g,h) \circ -} \\
\mathbb{E}[F(C,D'),E'] & \xrightarrow{\ \psi_{C,D',E'} = {}^{D'}\phi_{C,E'}\ } & \mathbb{C}[C, G(D',E')]
\end{array}
$$

By setting $C = G(D,E) = {}^DGE$ we may consider

$$ {}^D\phi^{-1}_{{}^DGE,E}(1_{{}^DGE}) \in \mathbb{E}[F(C,D),E] $$

and chase that through the diagram. Across the top and right we get

$$ G(g,h) \circ {}^D\phi_{{}^DGE,E}({}^D\phi^{-1}_{{}^DGE,E}(1_{{}^DGE})) = G(g,h) \circ 1_{{}^DGE} = G(g,h). $$

Along the left and bottom we get

$$ {}^{D'}\phi_{{}^DGE,E'}(h \circ {}^D\phi^{-1}_{{}^DGE,E}(1_{{}^DGE}) \circ F(1_C, g)) $$

and so any functor G with the desired properties must satisfy

$$ G(g,h) = {}^{D'}\phi_{{}^DGE,E'}(h \circ {}^D\phi^{-1}_{{}^DGE,E}(1_{{}^DGE}) \circ F(1_C, g)) $$

which shows that such a G is indeed uniquely determined if it exists, and we use this as the definition.

From now on we drop subscripts from the natural transformation ϕ. They do not add any information, and can indeed be restored from the given context. This makes the presentation much less cluttered, and it is easier to follow the equations. If you want a version of the same proofs *with* all indices talk to me.

So we know that G has to satisfy the equation

$$ G(g,h) = {}^{D'}\phi(h \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g)). $$

We note that by Exercise 7 for all objects D of $\mathbb{D}$ and h as before we have

$$ {}^DGh = {}^D\phi(h \circ {}^D\phi^{-1}(1_{{}^DGD})). $$

It remains to show that G as defined is a functor and that ψ is natural in its $\mathbb{D}$ and $\mathbb{E}$ components. For the $\mathbb{E}$ component this amounts to showing

[4] A simpler version of doing precisely the same thing is the subject of Exercise 7, so doing that first will help with understanding the proof that follows.

that the above diagram commutes for $D = D'$ and $h = 1_D$. But that then is a direct consequence of the naturality of ${}^D\phi$ in $\mathbb{E}$ since

$$G(1_D, h) = {}^D\phi(h \circ {}^D\phi^{-1}(1_{{}^DGE})) = {}^DGh.$$

For naturality in the $\mathbb{D}$ component, let $g\colon D' \longrightarrow D$ as before. Then for arbitrary objects C of $\mathbb{C}$ and E of $\mathbb{E}$ and $e \in \mathbb{E}[F(C, D), E]$

$$\begin{array}{lll}
G(g, 1_E)({}^D\phi(e)) & & \\
\quad = {}^{D'}\phi\left(1_E \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g)\right)({}^D\phi(e)) & & \text{Def } G \\
\quad = {}^{D'}\phi\left({}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g) \circ F({}^D\phi(e), 1_D)\right) & & \text{Nat } {}^{D'}\phi \\
\quad = {}^{D'}\phi\left({}^D\phi^{-1}(1_{{}^DGE}) \circ F({}^D\phi(e), g)\right) & & \text{Funct } F \\
\quad = {}^{D'}\phi\left({}^D\phi^{-1}(1_{{}^DGE}) \circ F({}^D\phi(e), 1_D) \circ F(1_C, g)\right) & & \text{Funct } F \\
\quad = {}^{D'}\phi\left({}^D\phi^{-1}(1_{{}^DGE} \circ {}^D\phi(e)) \circ F(1_C, g)\right) & & \text{Nat } {}^D\phi^{-1} \\
\quad = {}^{D'}\phi\left({}^D\phi^{-1}({}^D\phi(e)) \circ F(1_C, g)\right) & & \text{Id} \\
\quad = {}^{D'}\phi(e \circ F(1_C, g)) & & \text{Inv}
\end{array}$$

as desired.

It remains to show that by defining G through this equation we have indeed defined a functor. The identity on an object (D, E) of $\mathbb{D}^{\mathrm{op}} \times E$ is $(1_D, 1_E)$, and applying G to that we get

$$\begin{array}{rcll}
G(1_D, 1_E) & = & {}^D\phi\left(1_E \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, 1_D)\right) & \text{Def } G \\
& = & {}^D\phi\left({}^D\phi^{-1}(1_{{}^DGE}) \circ 1_{F(C,D)}\right) & \text{Funct } F \\
& = & {}^D\phi({}^D\phi^{-1}(1_{{}^DGE})) & \text{Id} \\
& = & 1_{{}^DGE} & \text{Inv} \\
& = & 1_{G(D,E)}. & \text{Def } G
\end{array}$$

Now assume that we have $g\colon D' \longrightarrow D$ and $g'\colon D'' \longrightarrow D'$ in $\mathbb{D}$ as well as $h\colon E \longrightarrow E'$ and $h\colon E' \longrightarrow E''$ in $\mathbb{E}$. Then

$$\begin{array}{lll}
G(g', h') \circ G(g, h) & & \\
\quad = G(g', h') \circ {}^{D'}\phi\left(h \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g)\right) & & \text{Def } G \\
\quad = {}^{D''}\phi\left(h' \circ h \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g) \circ F(1_C, g')\right) & & \text{Nat } \psi \\
\quad = {}^{D''}\phi\left((h' \circ h) \circ {}^D\phi^{-1}(1_{{}^DGE}) \circ F(1_C, g \circ g')\right) & & \text{Funct } F \\
\quad = G(g \circ g', h' \circ h) & & \text{Def } G
\end{array}$$

This completes the proof. □

So we now know that in a cartesian closed category the functor

$$- \times -\colon \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

is the left adjoint in a parametrized adjunction, hence there is a functor of two arguments formed by the individual right adjoints

$$- \Rightarrow -\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbb{C}.$$

Currying and uncurrying are natural transformations between the functors

$$\mathbb{C}[- \times -, -], \mathbb{C}[-, - \Rightarrow -]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{Set},$$

and they are natural in all three arguments.