CS3191 Section 2

Small Games

Andrea Schalk

Department of Computer Science, University of Manchester

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For this section we assume that a game is given *via* its normal form, which is a matrix for 2-person games.

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We will see some of the material repeatedly, first for 2-person zero-sum games, where it makes the most sense and is easiest to grasp, then in its generalization to 2-person (non zero-sum games) and ultimately for more than 2 players.

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The mathematical answer to this is the following: A good strategy for me is one where even if the other Player does his worst, the damage he can do is limited.

Question. Is this really the best thing to do?

Question. Will a winning strategy be recognized as ‘good’ under this definition?

2-person zero sum games

A Camping Holiday

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Prize!

A chocolate bar to him or her who can dig up the obscure references for these names!

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Last year, lots of people got Amelia, and one person got Scottie, the year before nobody got him. He really is obscure.

A Camping Holiday

Assume there is a couple heading for a camping holiday in the American Rockies.

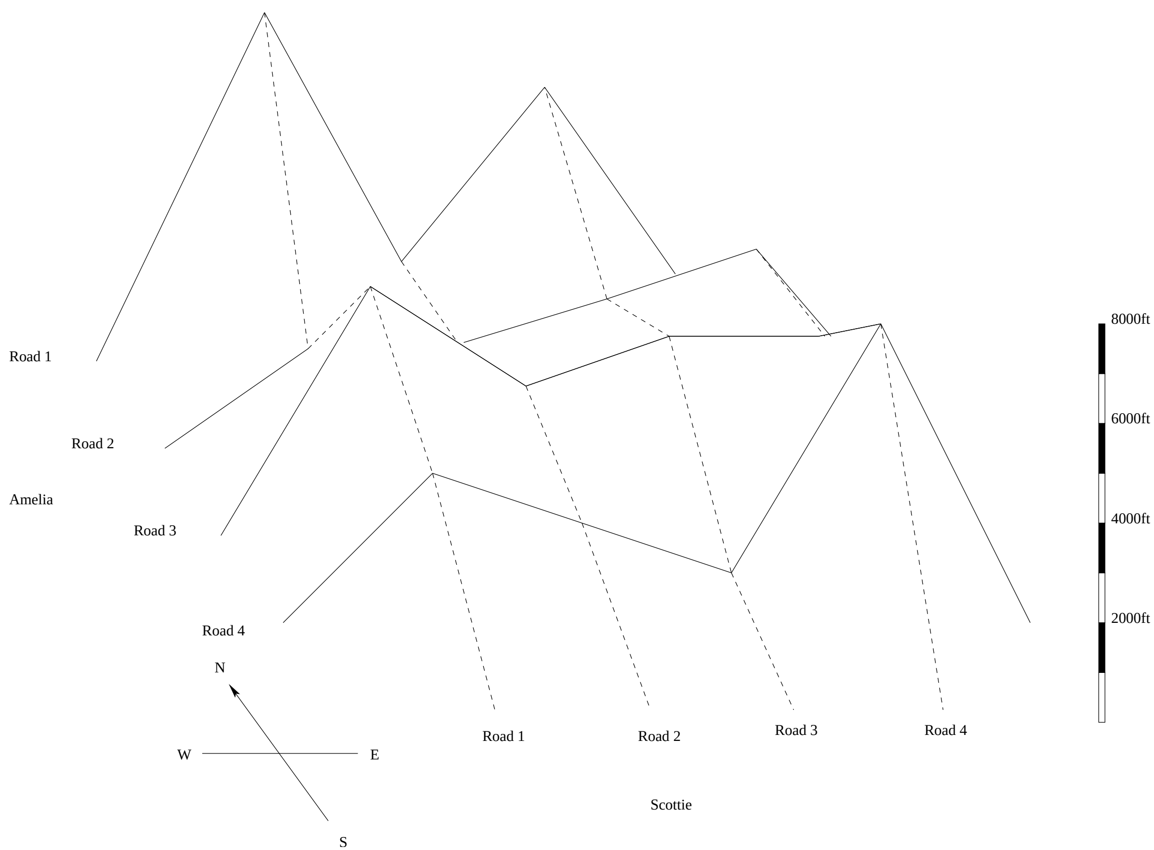
Amelia: She likes staying high up at night because it's cooler and there are fewer mosquitoes.

Scottie: He doesn't like the thin air at altitude and would rather be staying in the valleys.

The area they're in forms a natural grid of forestry roads. They decide that Amelia will choose one of the east-west roads while Scottie chooses one of the north-south ones, and they will camp at the intersection of the two.

A Camping Holiday

The landscape they're in looks like this.



A Camping Holiday

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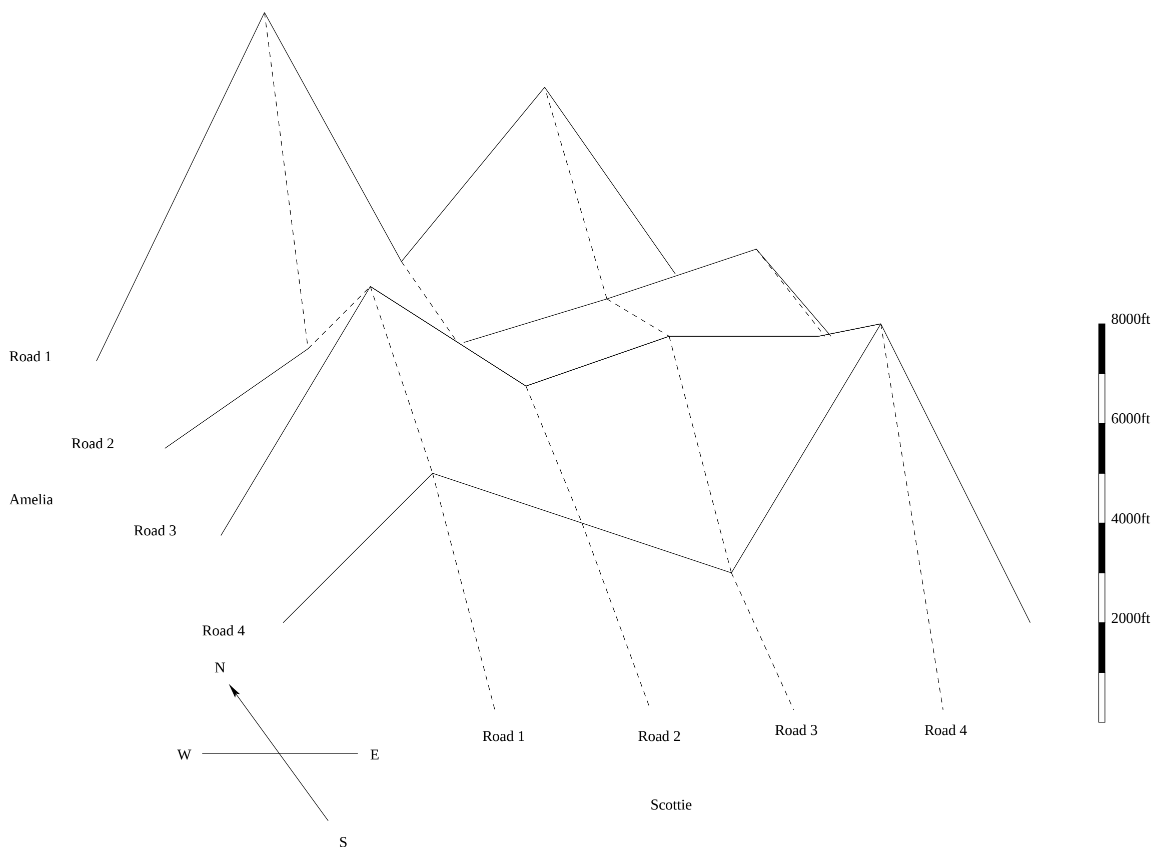

The heights in thousand feet of the various intersections can also be given as a matrix.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	7	1	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

If she choses her road 1, they may end up camping at a nice and lofty 7000 feet.

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

If she choses her road 1, they may end up camping at a nice 7000 feet. On the other hand, Scottie might push her as low as 1000 feet.

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie				
		1	2	3	4	
Amelia	1	7	2	5	1	1
	2	2	2	3	4	
	3	5	3	4	4	
	4	3	2	1	6	

She performs a worst-case analysis by calculating for each of her choices the lowest altitude Scottie might push her to.

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie				
		1	2	3	4	
Amelia	1	7	2	5	1	1
	2	2	2	3	4	2
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	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1

She performs a worst-case analysis by calculating for each of her choices the lowest altitude Scottie might push her to.

If she chooses her road 3 then the worst Scottie can do to her is to push her down to 3000 feet.

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie				
		1	2	3	4	
Amelia	1	7	2	5	1	1
	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1

She performs a worst-case analysis by calculating for each of her choices the lowest altitude Scottie might push her to.

If she chooses her road 3 then the worst Scottie can do to her is to push her down to 3000 feet. So this seems a sensible choice for her to make.

Amelia's choice

From Amelia's point of view, the situation looks like this.

		Scottie				min of row
		1	2	3	4	
Amelia	1	7	2	5	1	1
	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1

She performs a worst-case analysis by calculating for each of her choices the lowest altitude Scottie might push her to.

If she chooses her road 3 then the worst Scottie can do to her is to push her down to 3000 feet. So this seems a sensible choice for her to make.

Amelia has found the minimum in each row and chosen the largest of those.

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	1	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

If he chooses his road 1, they may end up camping at a reasonable 2000 feet.

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6

If he chooses his road 1, they may end up camping at a reasonable 2000 feet. On the other hand, Amelia might push him all the way up to 7000 feet.

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7			

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7	3		

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

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From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7	3	5	

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

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From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7	3	5	6

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		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7	3	5	6

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

If he chooses his road 2 then the worst Amelia can do to him is to push him up to 3000 feet.

Scottie's choice

From Scottie's point of view, the situation looks like this.

		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
		7	3	5	6

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

If he chooses his road 2 then the worst Amelia can do to him is to push him up to 3000 feet. So this seems a sensible choice for him to make.

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		Scottie			
		1	2	3	4
Amelia	1	7	2	5	1
	2	2	2	3	4
	3	5	3	4	4
	4	3	2	1	6
	max of col	7	3	5	6

He performs a worst-case analysis by calculating for each of his choices the highest altitude Amelia might push him to.

If he chooses his road 2 then the worst Amelia can do to him is to push him up to 3000 feet. So this seems a sensible choice for him to make.

Scottie has found the maximum in each column and chosen the least of those.

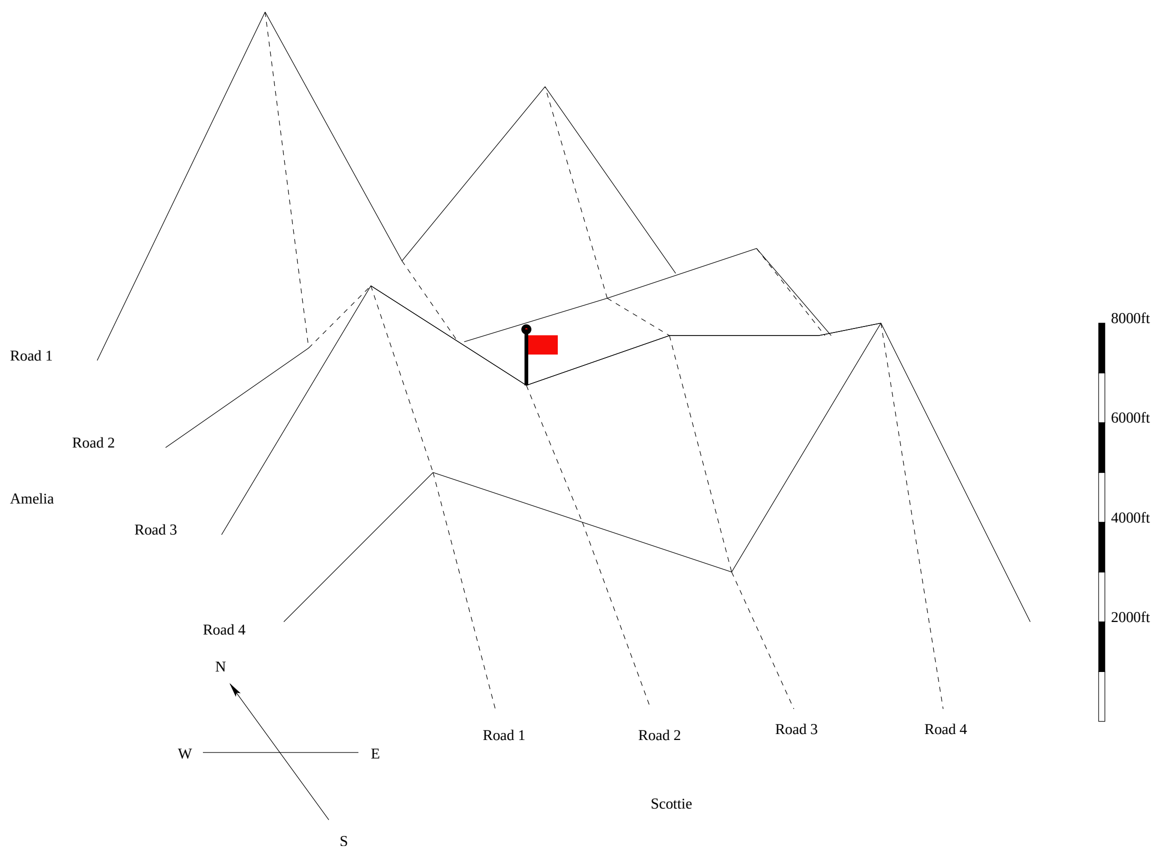
Their joint choice

We can summarize what Amelia and Scottie have done in the following table.

		Scottie				
		1	2	3	4	min of row
	1	7	2	5	1	1
Amelia	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	$3\backslash 3$

Their joint choice

If they both go with this choice they will camp here.



Their joint choice

		Scottie				
		1	2	3	4	min of row
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	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	$3\backslash 3$

Question. What happens if Amelia switches from her road while Scottie doesn't?

Their joint choice

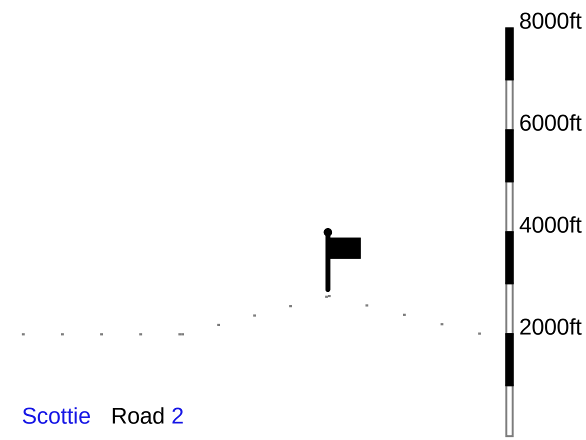
		Scottie				
		1	2	3	4	min of row
	1	7	2	5	1	1
Amelia	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	$3 \backslash 3$

If Amelia changes her mind while Scottie doesn't she will be worse off.

Their joint choice

		Scottie				
		1	2	3	4	min of row
	1	7	2	5	1	1
Amelia	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	$3 \backslash 3$

Scottie sticking with his choice (which amounts to choosing a column in the matrix) leaves her with the following picture, and she has made the best out of that situation. She gets the maximal value in Scottie's chosen column.



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		Scottie				
		1	2	3	4	min of row
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	4	3	2	1	6	1
	max of col.	7	3	5	6	$3\backslash 3$

Question. What if she keeps her choice while Scottie changes his?

Their joint choice

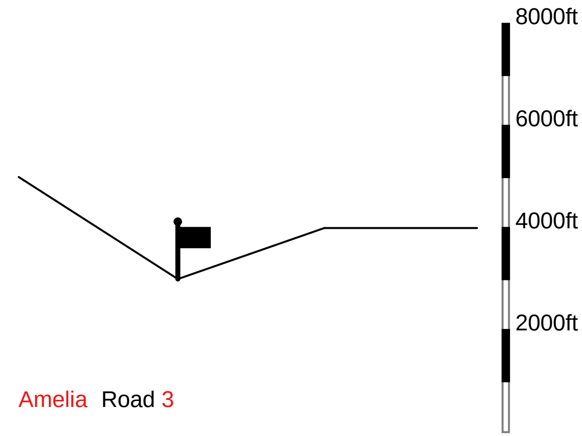
		Scottie				
		1	2	3	4	min of row
	1	7	2	5	1	1
Amelia	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	$3 \backslash 3$

Similarly if Scottie changes his mind while Amelia sticks with her choice he will be worse off.

Their joint choice

		Scottie				
		1	2	3	4	min of row
	1	7	2	5	1	1
Amelia	2	2	2	3	4	2
	3	5	3	4	4	3
	4	3	2	1	6	1
	max of col.	7	3	5	6	3\3

Amelia sticking with her choice (which amounts to choosing a row in the matrix) leaves him with the this picture, and he has made the best out of that situation. He gets the minimal value in Amelia's chosen column.



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So if one of them unilaterally changes his or her decision, they will be punished for it.

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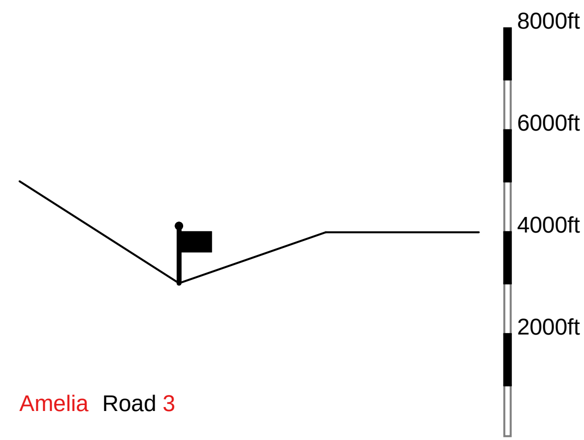
So if one of them unilaterally changes his or her decision, they will be punished for it. Note that from Scottie's point of view, they are in a valley



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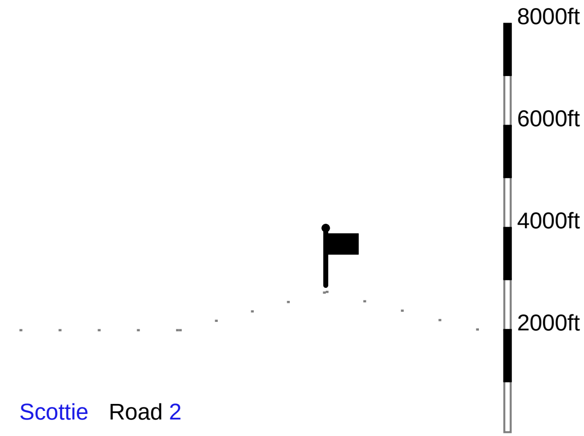
So if one of them unilaterally changes his or her decision, they will be punished for it. Note that from Scottie's point of view, they are in a valley while from Amelia's perspective, they are on a mountain top.



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So if one of them unilaterally changes his or her decision, they will be punished for it. Note that from Scottie's point of view, they are in a valley while from Amelia's perspective, they are on a mountain top. Such points are known as saddle points.

Generalization

We can view Scottie and Amelia's problem as the same one facing two players playing a matrix game: We view the matrix given the heights as the pay-off matrix for Amelia, who is trying to maximize the altitude. Scottie's pay-off matrix is the negative of hers, since his wishes are the opposite of hers.

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How can we generalize what Scottie and Amelia have done to reach a solution to their predicament?

Generalization

We can view Scottie and Amelia's problem as the same one facing two players playing a matrix game: We view the matrix given the heights as the pay-off matrix for Amelia, who is trying to maximize the altitude. Scottie's pay-off matrix is the negative of hers, since his wishes are the opposite of hers.

Assume we are given a matrix with elements $a_{i,j}$, with m rows and n columns, where i indicates the row and j indicate the column.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

We say that this is a $(m \times n)$ matrix.

Generalization

Amelia's chosen road gives her the maximum in Scottie's chosen column.

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Scottie’s chosen road gives him the minimum in Amelia’s chosen row.

Generalization

Amelia's chosen road gives her the maximum in Scottie's chosen column.

Scottie's chosen road gives him the minimum in Amelia's chosen row.

They have reached a point of balance, where both sides can be reasonably happy with the outcome, and where both sides will be worse off if one of them unilaterally changes his or her mind.

Equilibrium point

We turn these ideas into a formal definition.

Equilibrium point

Definition 4 *Let G be 2-person zero-sum game with strategies 1, 2, $\dots n$ for Player 1 and 1, 2, $\dots m$ for Player 2. Let $(a_{i,j})$ be the pay-off matrix for G. We say that (i', j') is an* ***equilibrium point for G*** *if the corresponding matrix entry*

$$a_{i',j'}$$

is maximal in its column and minimal in its row.

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$$a_{i',j'}$$

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In the camping holiday game, $(3, 2)$ is an equilibrium point.

$$\begin{vmatrix} 7 & 2 & 5 & 1 \\ 2 & 2 & 3 & 4 \\ 5 & 3 & 4 & 4 \\ 3 & 2 & 1 & 6 \end{vmatrix}$$

Equilibrium point

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The name 'equilibrium' reflects the idea that this defines point of balance: If either side unilaterally changes its mind and switches away from an equilibrium point, they get a worse pay-off. We can think of this as a punishment for causing a disturbance.

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In the camping holiday game, $(3, 2)$ is an equilibrium point.

The name 'equilibrium' reflects the idea that this defines point of balance: If either side unilaterally changes its mind and switches away from an equilibrium point, they get a worse pay-off. We can think of this as a punishment for causing a disturbance.

Sometimes these points are also known as Nash equilibria, after the founder of game theory.

Existence of equilibrium points

Question. Do all 2-person zero-sum games have equilibrium points?

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Consider our matrix for Paper-Stone-Scissors.

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

Existence of equilibrium points

Question. Do all 2-person zero-sum games have equilibrium points?

Consider our matrix for Paper-Stone-Scissors.

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

Then no entry is maximal in its column and minimal in its row at the same time.

Generalization–Amelia

But when we originally looked at the considerations Amelia might make, she appears to have looked for something different:

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Amelia calculated the worst thing that could happen to her for any of her choices i, where $1 \leq i \leq m$, that is, for a given i she calculated

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$$\min_{1 \leq j \leq n} a_{i,j}.$$

Then she made the best choice under the circumstances, that is she picked the option which gave her

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j}.$$

Here i is minimal in its row.

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Similarly for Scottie's considerations:

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Scottie, on the other hand, first computed, for a fixed column, that is for a fixed j, the maximum of the $a_{i,j}$, that is

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Generalization–Scottie

Similarly for Scottie's considerations:

Scottie, on the other hand, first computed, for a fixed column, that is for a fixed j, the maximum of the $a_{i,j}$, that is

$$\max_{1 \leq i \leq m} a_{i,j}.$$

Then he made the best choice under the circumstance, that is he picked the option which gave him

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j}.$$

His j is maximal in its column.

Existence of equilibrium points

When we looked at the considerations for Amelia and Scottie, she was interested in

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j},$$

Existence of equilibrium points

When we looked at the considerations for Amelia and Scottie,

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For Paper-Stone-Scissors,

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$$\max_{1 \le i \le m} \min_{1 \le j \le n} a_{i,j} = \max\{-1, -1, -1\} = -1,$$

while $$\min_{1 \le j \le n} \max_{1 \le i \le m} a_{i,j} = \min\{1, 1, 1\} = 1,$$

so the two do not have to agree.

Criteria for existence of eq pts

It turns out that the equality of these two numbers is closely related to the existence of equilibrium points.

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Proposition 2.1 *Let (i', j') be an equilibrium point for a 2-person zero-sum game with m strategies for Player 1 (rows) and n strategies for Player 2 (columns). Then it is the case that*

$$a_{i',j'} = \min_{1 \le j \le n} \max_{1 \le i \le m} a_{i,j} = \max_{1 \le i \le m} \min_{1 \le j \le n} a_{i,j}.$$

If on the other hand

$$\min_{1 \le j \le n} \max_{1 \le i \le m} a_{i,j} = \max_{1 \le i \le m} \min_{1 \le j \le n} a_{i,j},$$

then the game has an equilibrium point.

Proof–I

Let us first assume that the game has an equilibrium point at (i', j').

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$$a_{i',j'} = \max_{1 \le i \le m} a_{i,j'} \ge \min_{1 \le j \le n} \max_{1 \le i \le m} a_{i,j}.$$

Since it is also the case that $a_{i',j'}$ is the minimum of its row,

$$a_{i',j'} = \min_{1 \le j \le n} a_{i',j} \le \max_{1 \le i \le m} \min_{1 \le j \le n} a_{i,j}.$$

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Hence $$\max_{1\le i\le m} \min_{1\le j\le n} a_{i,j} \ge a_{i',j'} \ge \min_{1\le j\le n} \max_{1\le i\le m} a_{i,j}.$$

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Hence $$\max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{i,j} \geq a_{i',j'} \geq \min_{1\leq j\leq n} \max_{1\leq i\leq m} a_{i,j}.$$

Since we know from Exercise 9 (b) that

$$\max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{i,j} \leq \min_{1\leq j\leq n} \max_{1\leq i\leq m} a_{i,j},$$

we are done with this direction.

Proof–II

Let us now assume that

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j} = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j} = x.$$

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We can now choose i' in $\{1,\ldots,m\}$ such that $\min_{1\leq j\leq n} a_{i',j} = x$ and j' in $\{1,\ldots,n\}$ such that $\max_{1\leq i\leq m} a_{i,j'} = x$.

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We claim that (i',j') is an equilibrium point. We note that

$$x = \min_{1\leq j\leq n} a_{i',j} \leq a_{i',j'} \leq \max_{1\leq i\leq m} a_{i,j'} = x$$

and so all these numbers are equal.

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$$x = \min_{1\leq j\leq n} a_{i',j} \leq a_{i',j'} \leq \max_{1\leq i\leq m} a_{i,j'} = x$$

and so all these numbers are equal.

In particular $\min_{1\leq j\leq n} a_{i',j} = a_{i',j'}$ says that $a_{i',j'}$ is minimal in its row and $\max_{1\leq i\leq m} a_{i,j'} = a_{i',j'}$ says that $a_{i',j'}$ is maximal in its column, so we are done.

Several equilibrium points

This proposition tells us something important about the case where a game might have more than one equilibrium points:

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Question. Can you see what this means if a game has several equilibrium points?

Several equilibrium points

This proposition tells us something important about the case where a game might have more than one equilibrium points:

All equilibrium points for a game lead to the same pay-off for both players, namely

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j} = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j}$$

for Player 1, and its negative for Player 2.

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We call this the value of the game—it tells us what pay-off both players can expect when playing this game.

Hence all equilibrium points are ‘equally good’ for both players.

Note that even when we announce our intention of playing an equilibrium point strategy, our opponent cannot take advantage of this! As Player 1 we will still secure the value of the game as our pay-off.

Equilibrium points–a new perspective

There is another way we can think about equilibrium points, namely as the minimal pay-off that a player can secure for him-or herself.

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Proposition 2.2 *A 2-person zero-sum game has an equilibrium point if and only if there exists a value* $v \in \mathbb{R}$ *such that*

- v *is the highest value such that Player 1 can ensure a pay-off of at least* v*;*
- v *is the smallest value such that Player 2 can ensure that she will not have to pay out more than* $-v$*.*

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There is a value for 2-person zero-sum games which gives a guarantee for the pay-off in the worst case for both players

Proof–I

If the game has an equilibrium point then by Proposition 2.1 the resulting pay-off is equal to

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j},$$

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which is the highest value such that Player 1 can ensure at least pay-off v.

By the same Proposition this value is also equal to

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j},$$

which results in the smallest pay-out, $-v$, that Player 2 can ensure.

Proof–II

If, on the other hand, there is a value v satisfying the proposition then we can argue as follows.

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But then

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j} = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j},$$

and by Proposition 2.1 this is sufficient to ensure the existence of an equilibrium point.

When are eq points guaranteed to exist

We can guarantee the existence of equilibrium points for certain kinds of 2-person zero-sum games.

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Proposition 2.3 *Every 2-person zero-sum game of perfect information has at least one equilibrium point.*

Proof

The proof of this result is similar to that of Theorem 1.6. Instead of putting the outcome as 1, 0 or -1 at the root of a game tree, we now use the value of the game for the same purpose.

Proof

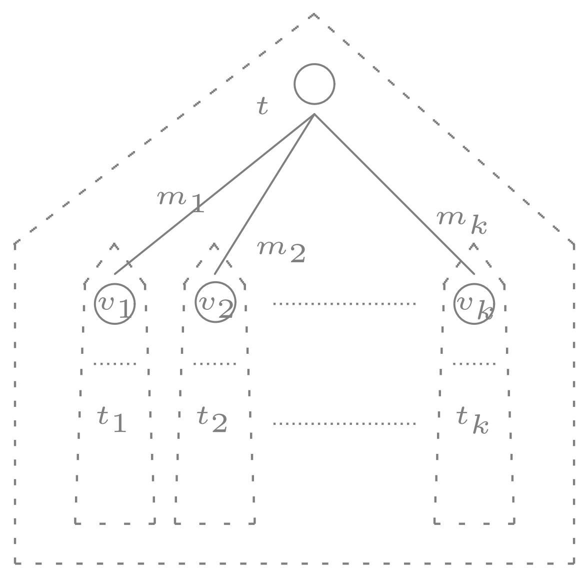
Base case. Clearly this works for games of height 0.

Proof

Induction hypothesis. Assume that every game tree of height at most n has a value v as described.

Proof

Induction step. We assume we have a game tree of height $n+1$. After one move we reach a game tree of height n, whose root we assume to be labelled with its value.



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Player 1 makes the first move. Then Player 1 can ensure that his pay-off is the maximum

$$v = \max_{1 \leq i \leq k} v_i$$

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of the values v_i of the sub-games reached after the first move. Player 2, on the other hand, can ensure that her pay-out is no worse than $-v$: No matter which first move Player 1 chooses, the worst case for Player 2 is that where she has to pay out $-v$. Hence v is indeed the value of the overall game.

Proof

Induction step. We assume we have a game tree of height $n + 1$. After one move we reach a game tree of height n, whose root we assume to be labelled with its value.

Player 2 makes the first move. Player 2 will be looking for the *least*

$$v = \min_{1 \leq i \leq k} v_i$$

of the values labelling the sub-games. The argument that this v is the value of the game is similar to the one before, only that the roles of Player 1 and Player 2 are reversed.

Proof

Induction step. We assume we have a game tree of height $n + 1$. After one move we reach a game tree of height n, whose root we assume to be labelled with its value.

Question. Which case is still missing?

Proof

Induction step. We assume we have a game tree of height $n + 1$. After one move we reach a game tree of height n, whose root we assume to be labelled with its value.

Chance makes the first move. Then the highest pay-off Player 1 can hope for is the expected pay-off

$$\sum_{1 \leq i \leq k} q_i v_i,$$

which is calculated by taking the probability q_i that a particular move m_i occurs times the value v_i of the subsequent game, and summing those up over all possible first moves. But that is precisely the least pay-out that Player 2 can expect.

General (non-cooperative) games

Normal forms

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But the possible combinations that occur when each player j picks one of his possible strategies from $\{1, 2, \ldots, n_j\}$ is the product

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to the real numbers. For Player j that function is typically called p_j.

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$$p_j : \prod_{1 \leq j \leq l} \{1, \dots, n_j\} \longrightarrow \mathbb{R}.$$

Example: Camping holiday

So how does our notion of matrix game fit together with this normal form?

Example: Camping holiday

Question. For the 2-person zero-sum game in the camping holiday example, what is the space of all strategies, and how do you calculate the pay-off function for each player? Can you generalize this to any 2-person zero-sum game?

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- Pay-off function for Player 2:

Example: Camping holiday

Question. For the 2-person zero-sum game in the camping holiday example, what is the space of all strategies, and how do you calculate the pay-off function for each player? Can you generalize this to any 2-person zero-sum game?

- List of players: 1, 2.
- List of their strategies:
  - Player 1: 1, 2, 3, 4;
  - Player 2: 1, 2, 3, 4.
- Pay-off function for Player 1: $p_1(i, j) = a_{i,j}$,

  where $(a_{i,j})$ is the matrix given for that game;
- Pay-off function for Player 2: $p_2(i, j) = -a_{i,j}$.

Example: 3-person Morra

Consider the following three person game. Each player pays an ante of one. On a signal, all the players hold up one or two fingers. If the number of fingers held up is divisible by 3, Player 3 gets the pot. If the remainder when dividing is 1, Player 1 gets it, otherwise Player 2 is the lucky one.

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Question. Do you think that this game is likely to be 'fair', in the sense of giving all the players an even chance to win? Which player would you prefer to be?

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Each player has two strategies: Holding up one finger or holding up two fingers. We number them as 1 and 2 (in that order).

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Each player has two strategies. Hence the space of all strategies is

$$\prod_{j=1}^{3}\{1,2\} = \{1,2\} \times \{1,2\} \times \{1,2\}$$

$$= \{(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (2,1,1),\ (2,1,2),\ (2,2,1),\ (2,2,2)\}$$

It has $2 \times 2 \times 2 = 8$ elements.

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$(2,1,2)$	-1	2	-1
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Definition: equilibrium point

With the definition of normal form, the generalized definition of equilibrium point is straight-forward.

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Definition 6 *Let G be a (non-cooperative) game in normal form with l players. Then a choice of strategies for each player,*

$$(i'_1, \ldots, i'_l) \in \prod_{j=1}^{l} \{1, \ldots, n_j\}$$

gives an equilibrium point for the game if it is the case that for all $1 \leq j \leq l$ and for all $1 \leq i \leq n_j$

$$p_j(i'_1, \ldots, i'_{j-1}, i'_j, i'_{j+1}, \ldots, i'_l) \geq p_j(i'_1, \ldots, i'_{j-1}, i, i'_{j+1}, \ldots, i'_l).$$

In other words: If Player j changes away from his choice, strategy i'_j, then his pay-off can only decrease (or at best stay the same).

Comparison with the old notion

Clearly for a 2-person zero-sum game, the new notion of equilibrium point agrees with the old one.

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- maximal in its column means that if Player 1 changes his mind then this amounts to choosing another value in the same column, so his pay-off can only decrease;
- minimal in its row means that if Player 2 changes her mind then this amounts to choosing another value in the same row, so her pay-off can only decrease.

Example

We check the 3-person version of Two-Finger Morra for equilibrium points.

	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
$(2,1,2)$	-1	2	-1
$(2,2,1)$	-1	2	-1
$(2,2,2)$	-1	-1	2

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	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
$(2,1,2)$	-1	2	-1
$(2,2,1)$	-1	2	-1
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$(1,1,1)$ is not an equilibrium point because

Example

	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
$(2,1,2)$	-1	2	-1
$(2,2,1)$	-1	2	-1
$(2,2,2)$	-1	-1	2

$(1,1,1)$ is not an equilibrium point because if Player 1 changes his mind he is better off.

Example

	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
$(2,1,2)$	-1	2	-1
$(2,2,1)$	-1	2	-1
$(2,2,2)$	-1	-1	2

Similarly $(1,1,2)$ is not an equilibrium point because if Player 3 changes his mind, he is better off.

Example

	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
$(2,1,2)$	-1	2	-1
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For $(1,2,1)$ we are in the following situation:

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$(2,1,1)$	2	-1	-1
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For $(1,2,1)$ we are in the following situation:

- If Player 1 changes his mind, his pay-off decreases from 2 to -1.

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$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
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$(1,2,2)$	-1	2	-1
$(2,1,1)$	2	-1	-1
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For $(1,2,1)$ we are in the following situation:

- If Player 1 changes his mind, his pay-off decreases from 2 to -1.
- If Player 2 changes her mind, her pay-off stays at -1.

Example

	p_1	p_2	p_3
$(1,1,1)$	-1	-1	2
$(1,1,2)$	2	-1	-1
$(1,2,1)$	2	-1	-1
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$(1, 2, 1)$ is an equilibrium point. Similarly we can show that $(1, 2, 2)$ is an equilibrium point.

	p_1	p_2	p_3
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$(1, 2, 1)$ is an equilibrium point. Similarly we can show that $(1, 2, 2)$ is an equilibrium point. These are the only equilibrium points.

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Note, however, that the players who do not win at any given equilibrium point can change strategies unilaterally without actually being worse off—they just suffer the same loss of -1.

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Several equilibrium points

Note that if our games are not 2-person zero-sum then the pay-off for two different equilibrium points need not agree.

Consider the game given by the following matrix.

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- if the column player sticks to her strategy 1 then changing from strategy 1 to strategy 2 reduces the pay-off and
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Hence the notion of equilibrium point becomes problematic if we are not dealing with 2-person zero-sum games.

More problems with equilibria

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In other words, the idea that equilibrium points provide a solution for general games has to be taken with a grain of salt, and every such 'solution' should be studied carefully to check whether it really makes sense.

However, this is still the best theory of games in existence!

Are equilibria any use?

Consider the game given by the following matrix.

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It has two equilibrium points at $(1,2)$ and $(2,1)$. Player 1 prefers $(1,2)$ which gives him a pay-off of 15, while Player 2 prefers $(2,1)$ which gives him the same amount.

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Which equilibrium should they settle on? Once they have reached one equilibrium point, a player switching strategies unilaterally risks a pay-off of -20.

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Which equilibrium should they settle on?

Wouldn't it make sense to settle on $(10,10)$ as a compromise? But then either player would be tempted to switch strategies for the higher pay-off of 15.

The Prisoner's Dilemma

Fred and Joe have been arrested. The police has the problem that while they are certain the two have committed armed robbery, there is no any evidence of that—the best they can only get them for possession of their guns, which carries a sentence of 2 years.

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What the police doesn't emphasize is that if they both confess, they'll be put away for 8 years.

The following pay-off matrix represents this situation:

		Joe	
		talk	don't talk
Fred	talk	$(-8,-8)$	$(0,-10)$
	don't talk	$(-10,0)$	$(-2,-2)$

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		Joe	
		talk	don't talk
	talk	$(-8, -8)$	$(0, -10)$
Fred			
	don't talk	$(-10, 0)$	$(-2, -2)$

Question. What would you do in the place of Joe or Fred? Why?

The Prisoner's Dilemma

		Joe	
		talk	don't talk
Fred	talk	$(-8, -8)$	$(0, -10)$
	don't talk	$(-10, 0)$	$(-2, -2)$

This game has only one equilibrium point at $(\text{talk}, \text{talk})$, since for both, Joe and Fred, the situation will get worse if they shift away from that strategy.

The Prisoner's Dilemma

		Joe	
		talk	don't talk
Fred	talk	$(-8, -8)$	$(0, -10)$
	don't talk	$(-10, 0)$	$(-2, -2)$

This game has only one equilibrium point at $(\text{talk}, \text{talk})$, since for both, Joe and Fred, the situation will get worse if they shift away from that strategy.

Hence from the game theory point of view, the 'solution' to this game is for each of them to talk and spend 8 years in prison.

The Prisoner's Dilemma

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Hence from the game theory point of view, the 'solution' to this game is for each of them to talk and spend 8 years in prison.

Clearly, this is not a particularly good outcome. If one takes their collective situation into account, it is very clear that what they should both do is to remain silent (much to the police's regret!).

Prisoner's Dilemma-type situations

Question. Can you think of other situations which are similar to the Prisoner's Dilemma? By that I mean situations where two players have to cooperate for the mutual good, but are tempted by better outcomes if they betray their playing partner.

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We will return to situations of this kind in the section on Game Models.

20 Prisoners

Douglas R. Hofstadter is probably best known for writing *Gödel, Escher, Bach*, a book that tries to draw out connections between art, music and mathematics *via* self-reference. He is a computer scientist with a particular interest in how the mind works. He wrote a column in the *Scientifi c American* for a while, from which the following example is taken.

20 Prisoners

Douglas R. Hofstadter once sent a postcard to twenty of his friends.

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'. . . Each of you is to give me a single letter: 'C' or 'D' standing for 'cooperate' or 'defect'. This will be used as your move in a Prisoner's Dilemma with *each* of the nineteen other players. The pay-off matrix I am using for the Prisoner's Dilemma is given in the diagram.

		Player B	
		C	D
Player A	C	$(3, 3)$	$(0, 5)$
	D	$(5, 0)$	$(1, 1)$

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Thus if everyone sends in 'C', everyone will get \$57, while if everyone sends in 'D', everyone will get \$19. You can't lose! And, of course, anyone who sends in a 'D' will get at least as much as everyone else will. If, for example, 11 people send in 'C' and 9 send in 'D', then the 11 C-ers will get \$3 apiece for each of the other C-ers, (making \$30), and zero for the D-ers. So C-ers will get \$30 each. The D-ers, by contrast, will pick up \$5 apiece for each of the C-ers, making \$55, and \$1 each for the other D-ers, making \$8, for a grand total of \$63.

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. . . You are not aiming at maximizing the total number of dollars *Scientifi c American* shells out, only maximizing the number that come to you!

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. . . I want all answers by telephone (call collect, please) the day you receive this letter.

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It is to be understood (it almost goes without saying, but not quite) that you are not to try to get in touch with and consult with others who you guess have been asked to participate. In fact, please consult with no one at all. The purpose is to see what people will do on their own, in isolation. . . .'

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Question. What would you do if you received such a letter? Why?

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Hofstadter had hoped for twenty 'C's.

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Question. Can you think of a way in which a clever person could have convinced himself that everybody should go for ‘C’?

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Question. How many people do you think chose ‘C’, how many ‘D’?

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In fact, he got 14 ‘D’s and 6 ‘C’s, so the ‘defectors’ each received $43 while the ‘cooperators’ had to make do with $15 each.

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Question. What would game theory have to tell these people? What would have happened if they had all applied this theory?

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Question. What would game theory have to tell these people? What would have happened if they had all applied this theory? On the other hand, how do you think this sort of game would develop if it were played repeatedly, say once a week over a year?

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- when congestion occurs (which is detected by a time-out occurring when waiting for an acknowledgement), reset the threshold value to half the current batch size and start over (with a batch size of one packet).

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The idea of the algorithm is to adapt the batch size to what is currently supportable within the network, and it is fair since everybody follows the same rules, thus trying to split resources evenly between all users.

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For an individual, however, it is tempting to employ a 'greedier' algorithm than the above slow start: For example jump straight back to the new threshold value (rather than starting with a batch size of 1). (It doesn't make sense to go for a bigger size since in all likelihood, that will just result in more time-outs.)

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For an individual, however, it is tempting to employ a ‘greedier’ algorithm than the above slow start: For example jump straight back to the new threshold value (rather than starting with a batch size of 1).

But if everybody does that, the network will become very congested.

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This is a multi-player prisoner’s dilemma situation: The community is best off if everybody exercises restraint. A small number of people, however, could get away with getting a better return than the average (and, indeed, the slow-start algorithm strategy does not lead to an equilibrium even when everybody employs it).

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This is a multi-player prisoner's dilemma situation: The community is best off if everybody exercises restraint. A small number of people, however, could get away with getting a better return than the average (and, indeed, the slow-start algorithm strategy does not lead to an equilibrium even when everybody employs it). But the more people cheat, the worse the situation gets for everybody.

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If two (or more) players try this then their packets will collide and they have to try again—but if they stick with this then they will block the channel until all but one gives up.

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However, this is still the best theory of games in existence! In other words, there are no better answers.

Generalizing strategies

In order to deal with the problem that some games don't have equilibrium points (for example Paper-Stone-Scissors) we will now

generalize the notion of strategy.

Lack of equilibrium points

As we have seen with the example of Paper-Stone-Scissors,

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

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If we are looking for a procedure which helps us decide which strategy to play under the assumption that we may play a game many times then it makes sense not to stick to merely one strategy (in particular if the game has no equilibrium points).

Instead, we can assign a probability to each available strategy.

Mixed strategies–idea

Giving a probability to each available strategy means we are now playing tuples, where the

- the first component gives the probability for the first strategy,
- the second component gives the probability for the second strategy,

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- the last (lth) component gives the probability for the lth strategy.

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If we play in accord with this strategy, we proceed as follows: Before the game, conduct an experiment which produces three different outcomes with the probabilities $1/2$, $1/6$ and $1/3$ respectively.

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If we play in accord with this strategy, we proceed as follows: Before the game, conduct an experiment which produces three different outcomes with the probabilities $1/2$, $1/6$ and $1/3$ respectively. We then play, for that one game, the strategy that is given by the outcome of the experiment.

Mixed strategy: Definition

We turn this idea into a formal definition.

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Definition 7 *A* *mixed strategy* *for a player consists of assigning a probability to each of the player's strategies so that these probabilities add up to one. If the player has strategies numbered* $\{1, \ldots, m\}$ *then we represent a mixed strategy by an* m*-tuple*

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where q_i *is the probability that strategy* i *will be employed, and such that* $q_1 + q_2 + \cdots + q_m = 1$.

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where q_i *is the probability that strategy* i *will be employed, and such that* $q_1 + q_2 + \cdots + q_m = 1$.

Note that while there are finitely many pure strategies for each player there are infinitely many mixed strategies, provided the player has at least two pure ones.

Pure strategies as mixed ones

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The mixed strategy $(1, 0, \ldots, 0)$ amounts to playing the pure strategy 1. Similarly, by weighing a specific strategy with 1 while weighing all the others with 0, we can obtain all the pure strategies as mixed ones.

Occasionally we use the number i to refer to the mixed strategy which has a 1 in the ith component and 0s everywhere else.

Pay-offs for mixed strategies

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In a 2-person game, with strategies $1, 2, \ldots, m$ for Player 1 and strategies $1, 2, \ldots, n$ for Player 2, a mixed strategy for Player 1 is an m-tuple

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Playing this strategy against Player 2's pure strategy j will lead to the expected pay-off

$$q_1 p_1(1, j) + q_2 p_1(2, j) + \cdots + q_m p_1(m, j)$$

for Player 1.

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$$(q_1, q_2, \dots, q_m) \qquad\qquad (r_1, r_2, \dots, r_n)$$

for Player 1, for Player 2.

The expected pay-off for Player 1 is then

$$\begin{aligned}
& q_1 r_1 p_1(1,1) + q_1 r_2 p_1(1,2) + \cdots + q_1 r_n p_1(1,n) \\
+ \quad & q_2 r_1 p_1(2,1) + q_2 r_2 p_1(2,2) + \cdots + q_2 r_n p_1(2,n) \\
+ \quad & \cdots \\
+ \quad & q_m r_1 p_1(m,1) + q_m r_2 p_1(m,2) + \cdots + q_m r_n p_1(m,n) \\
= \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} q_i r_j p_1(i,j).
\end{aligned}$$

Worst case analysis

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$$\max_{s \in S_1} \min_{t \in S_2} p_1(s, t),$$

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For a 2-person zero-sum game that is the same as having to pay at most the following to Player 1

$$\min_{t \in S_2} \max_{s \in S_1} p_1(s,t).$$

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We have seen that for pure strategies, these values do not have to coincide. When we consider mixed strategies this changes! (See Proposition 2.6.)

Equilibrium Points

It makes sense to generalize the definition of equilibrium points to mixed strategies.

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Definition 8 *Let G be a non-cooperative game in normal form. A tuple of mixed strategies, one for each player, $(s_1, \ldots s_l)$ is an* equilibrium point *of G if for all $1 \leq j \leq l$ and all elements s of the set of mixed strategies for Player j it is the case that*

$$p_j(s_1, \ldots, s_{j-1}, s_j, s_{j+1}, \ldots, s_l) \geq p_j(s_1, \ldots, s_{j-1}, s, s_{j+1}, \ldots, s_l).$$

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Again, a player can only worsen his situation by unilaterally moving away from an equilibrium point.

Infinitely many possibilities

There is a clear problem with our definition of equilibrium point: There are infinitely many ways of moving away from an equilibrium points, so we have to perform infinitely many checks to make sure that something is an equilibrium point!

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Proposition 2.4 *A tuple of mixed strategies* $(s_1, \ldots s_l)$ *is an equilibrium point for a non-cooperative game if and only if for all* $1 \leq j \leq l$ *and all pure strategies* $k \in \{1, 2, \ldots, n_j\}$ *for Player* j *it is the case that*

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So in fact finitely many checks suffice!

Proof

Clearly

$$p_j(s_1, \ldots, s_{j-1}, s_j, s_{j+1}, \ldots, s_l) \geq p_j(s_1, \ldots, s_{j-1}, k, s_{j+1}, \ldots, s_l)$$

is a necessary condition for being an equilibrium point—the inequality is just a special case for the one given in the definition.

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as required.

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We first calculate the expected pay-off resulting from playing these strategies against each other.

$$\begin{aligned}
p_1((1/3,1/3,1/3),(1/3,1/3,1/3)) = & \\
& (1/9 \times 0 + 1/9 \times (-1) + 1/9 \times 1) + (1/9 \times 1 + 1/9 \times 0 + 1/9 \times (-1)) \\
& +(1/9 \times (-1) + 1/9 \times 1 + 1/9 \times 0) \\
= & \quad 3 \times (1/9 \times 1) + 3 \times (1/9 \times (-1)) \\
= & \quad 0.
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By Proposition 2.4, we have indeed found an equilibrium point.

Existence of eq points

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Note that Nash's Theorem does not lead to an algorithm for finding mixed strategy equilibrium points. In fact, no such algorithm is known for games with 3 or more players.

Value of 2-person zero-sum games

We established earlier that for 2-person zero-sum games, all equilibrium points of pure strategies lead to the same pay-off for both players.

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Proposition 2.6 *For a 2-person zero-sum game all equilibrium points, whether they consist of pure or mixed strategies, lead to the same pay-off, which we call the value of the game.*

So the value still works for mixed strategies!

Proof

One possible proof of this result is similar to that of Proposition 2.1, just using the definition of the pay-off function for mixed strategies.

Proof

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Since Player 2 changing away from (s',t') to (s',t) can only decrease her pay-off, and thus increase that for Player 1,

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So

$$p_1(s, t) \geq p_1(s', t) \geq p_1(s', t') \geq p_1(s, t') \geq p_1(s, t)$$

and all these numbers must be equal, in particular $p_1(s, t) = p_1(s', t')$.

Some facts

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- If s and s' are optimal strategies for Player 1, then so is

$$\lambda s + (1 - \lambda)s'$$

  for all $0 \leq \lambda \leq 1$. Here λs is the mixed strategy we obtain by multiplying all entries in s by the number λ. Similarly for Player 2.

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  - combinations of those as suggested above.

Finding equilibria in 2-person zero-sum games

Methods for solving games

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$$a_{i,j} \geq a_{i',j}.$$

We say that a strategy j *for Player 2* dominates *another such strategy* j' *for the same player if it is the case that for all strategies* $1 \leq i \leq m$ *for Player 1*

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In other words, a strategy is dominated by another if it is outperformed by it. Since the numbers in the matrix mean different things to the different players, the inequality changes from $\leq$ to $\geq$ when moving from Player 1 to Player 2.

Example ctd

Having removed Player 2's strategy 3 we are left with the following matrix.

$$\begin{vmatrix} 1 & -1 \\ -1 & 1 \\ -3 & -2 \end{vmatrix}$$

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But now Player 1's strategy 3 is dominated by his strategy 1 (or his strategy 2).

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$$\begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & 3 \\ -3 & -2 & 4 \end{vmatrix}$$

But the original game had 3 strategies for each player! We can get optimal strategies for it from the ones above by setting the component of all discarded strategies to 0, giving

$(1/2, 1/2, 0)$ for Player 1 and

$(1/2, 1/2, 0)$ for Player 2.

Working the algorithm

Very often we can apply this algorithm to make the game smaller, typically until we get a (2×2)-matrix. We will see in a little while how these can be solved.

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However, it should be noted that when we reduce a matrix *via* dominance arguments then we may lose some solutions. Clearly this is no problem if we are merely interested in finding one solution.

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For many matrices, however, dominance arguments do not apply unless we make our definition a bit more generous: Sometimes, no strategy is dominated by one single other, but only by a combination of other strategies.

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Definition 10 *A pure strategy i for Player 1 is dominated by a mixed strategy $(q_1, \ldots, q_m)$ for the same player if for all pure strategies j of Player 2 it is the case that the expected pay-off for strategy i played against strategy j is less than or equal to the pay-off for strategy $(q_1, \ldots, q_m)$ against strategy j. In other words, for all $1 \leq j \leq n$:*

$$a_{i,j} \leq q_1 a_{1,j} + q_2 a_{2,j} + \cdots + q_m a_{m,j}.$$

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It is possible to generalize this definition to having mixed strategies dominated by other mixed strategies, but that is not useful in reducing the size of matrices.

Example

No strategy for Player 1 is dominated by any other.

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To show this we have to demonstrate that there is a $0 \leq \lambda \leq 1$ such that

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$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

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which is equivalent to finding such a λ such as both of the following are true:

$$\begin{aligned} 0 &\leq -\lambda + 2(1-\lambda) = 2 - 3\lambda \\ 0 &\leq 2\lambda - (1-\lambda) = 3\lambda - 1. \end{aligned}$$

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So we are looking for one $0 \leq \lambda \leq 1$ such that both

$$\begin{aligned} 0 &\leq -\lambda + 2(1-\lambda) = 2 - 3\lambda \\ 0 &\leq 2\lambda - (1-\lambda) = 3\lambda - 1 \end{aligned}$$

The former is equivalent to $\lambda \leq \frac{2}{3}$ and the latter to $\lambda \geq \frac{1}{3}$, so $\lambda = 1/3$ will do the job.

Example

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No strategy for Player 1 is dominated by any other. However, his strategy 3 looks weaker than the others. And, indeed, we can show that it is dominated by a mixed strategy based on the other two.

Hence Player 1's strategy 3 is dominated by his mixed strategy $(1/3, 2/3, 0)$ and we may discard it.

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Proposition 2.7 *Let G' be a game that results from the 2-person zero-sum game G by removing a strategy i for Player 1 which is dominated by some mixed strategy $(q_1, \ldots, q_m)$ with $q_i = 0$. If $(q'_1, \ldots, q'_{i-1}, q'_{i+1}, \ldots, q'_m)$ is an optimal strategy for Player 1 in the game G' then $(q'_1, , \ldots, q'_{i-1}, 0, q'_{i+1}, \ldots, q'_m)$ is an optimal strategy for Player 1 in the game G.*
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In other words, dominance works!

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An analogous result holds for Player 2.

The proof of this result is lengthy but not too difficult—it consists of making calculations regarding pay-off functions. Since it is of no particular interest we omit it here.

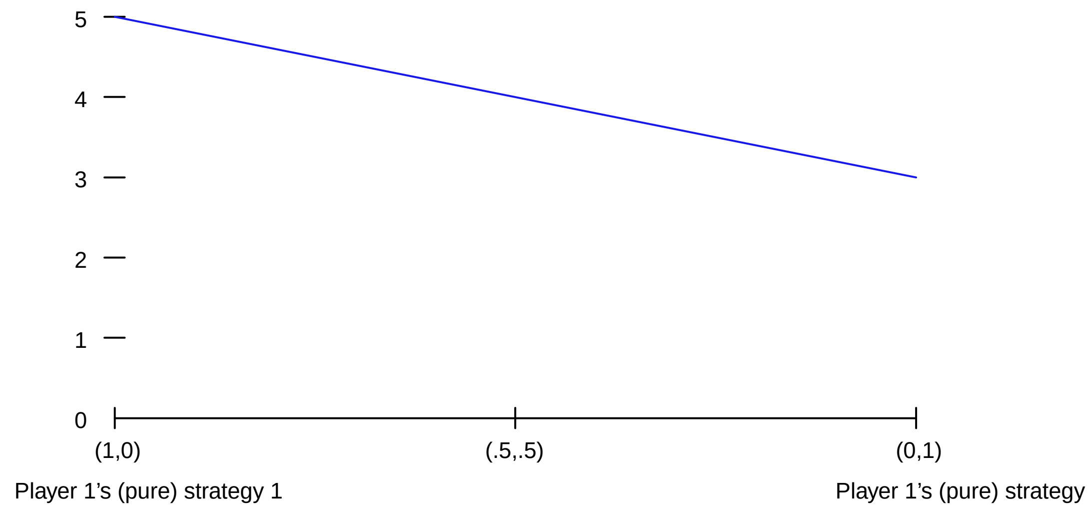
Solving (2×2) games

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$

Solving (2×2) games

If Player 2 sticks to her strategy 1, Player 1 can get a pay-off somewhere from 5 to 3:

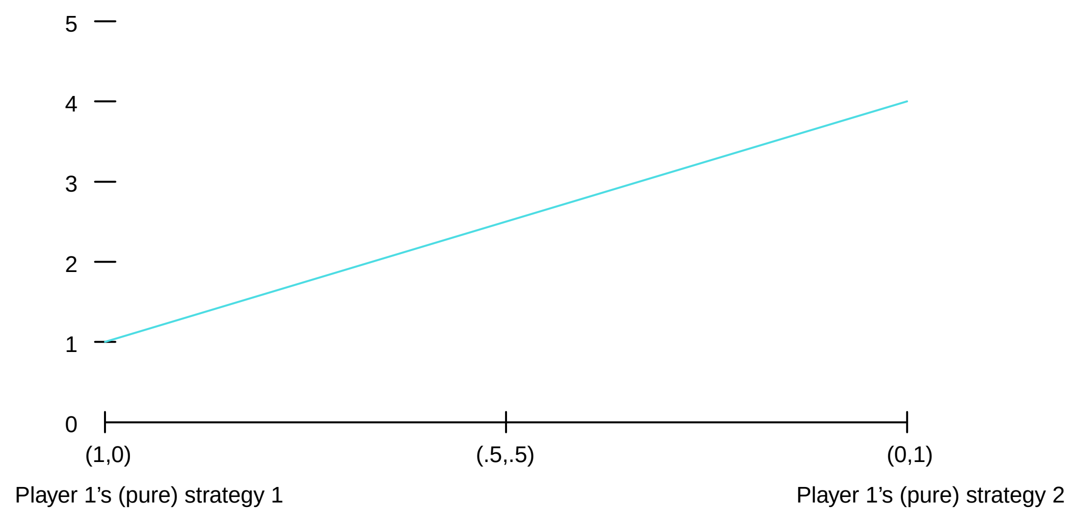
$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



Solving (2×2) games

If Player 2 sticks to her strategy 2, Player 1 can get a pay-off somewhere from 1 to 4:

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



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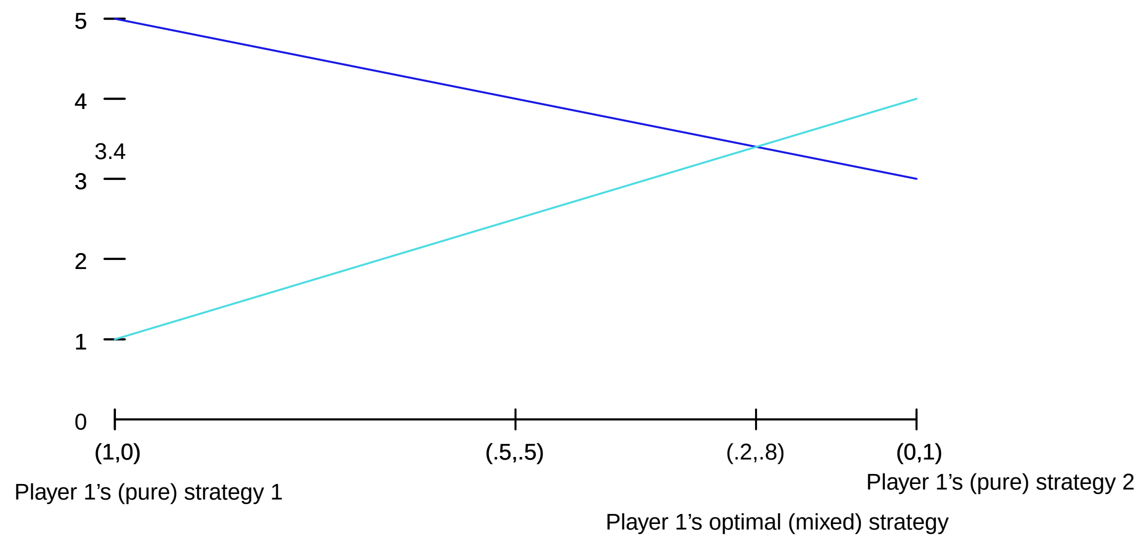

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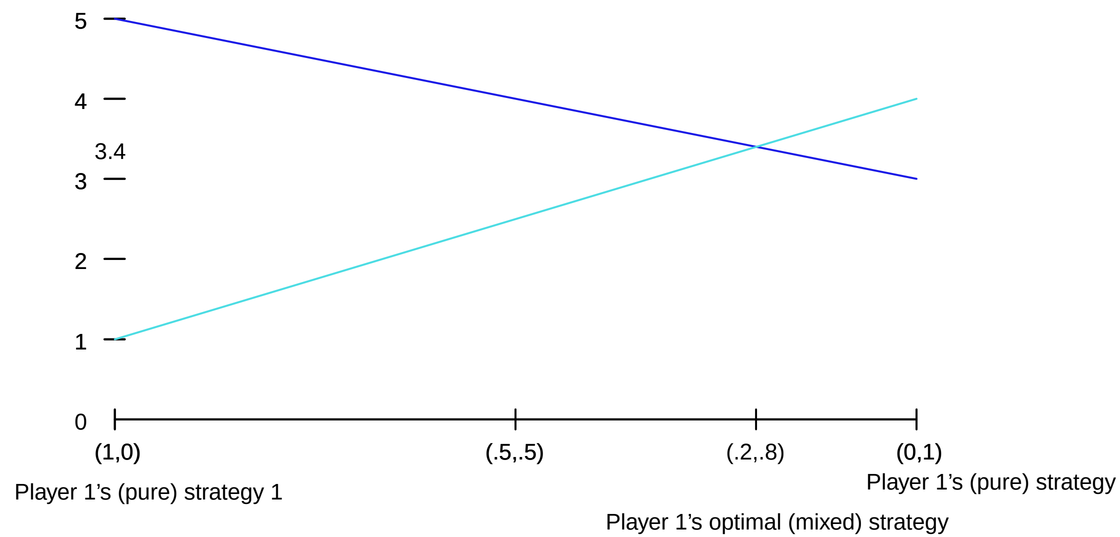
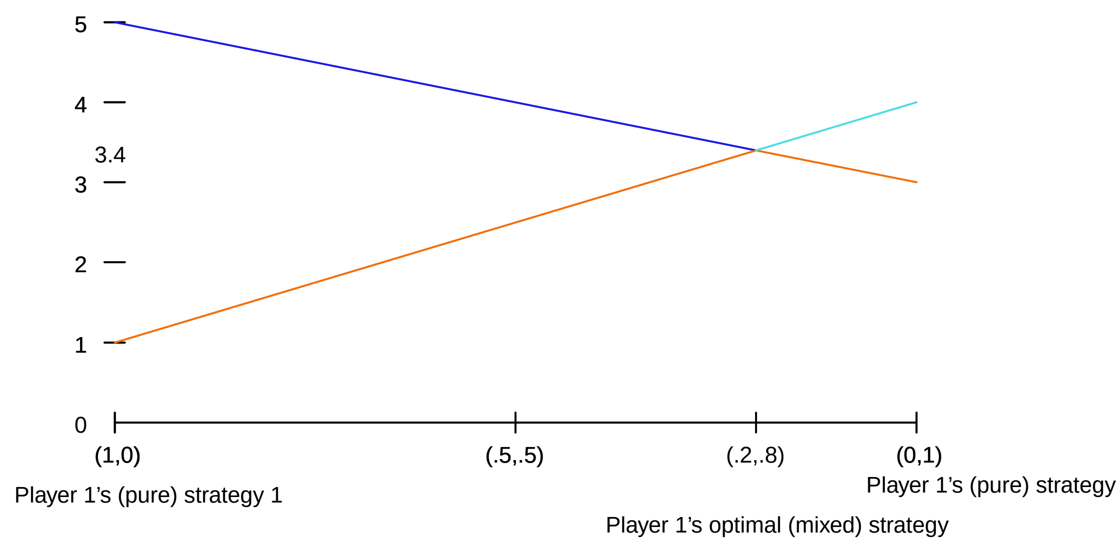


For Player 1, the minimum pay-off he can guarantee for himself is given by the point-wise minimum of the two lines.

Solving (2×2) games

Minimum guaranteed pay-off for Player 1: orange line.

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



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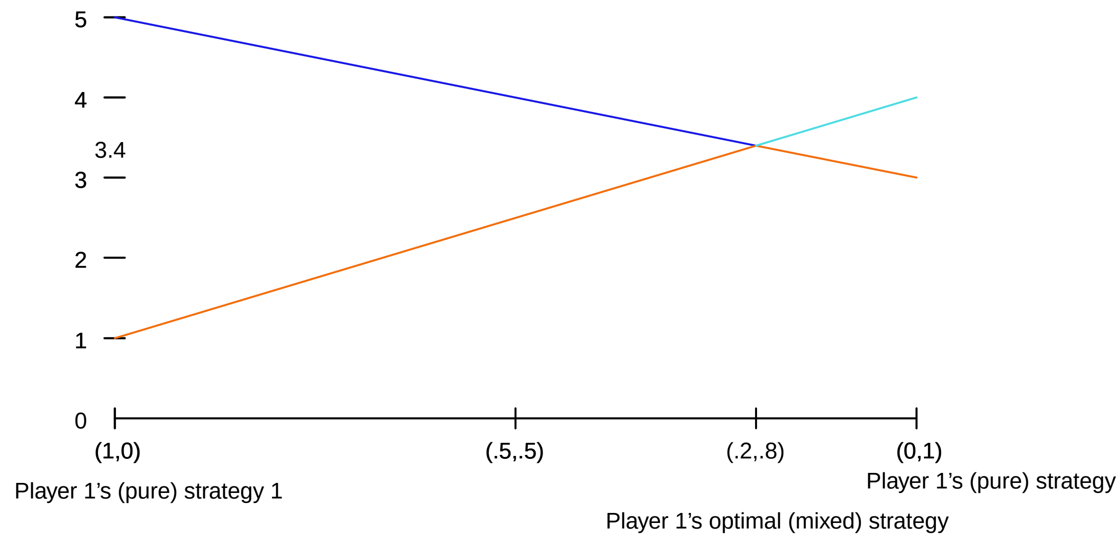
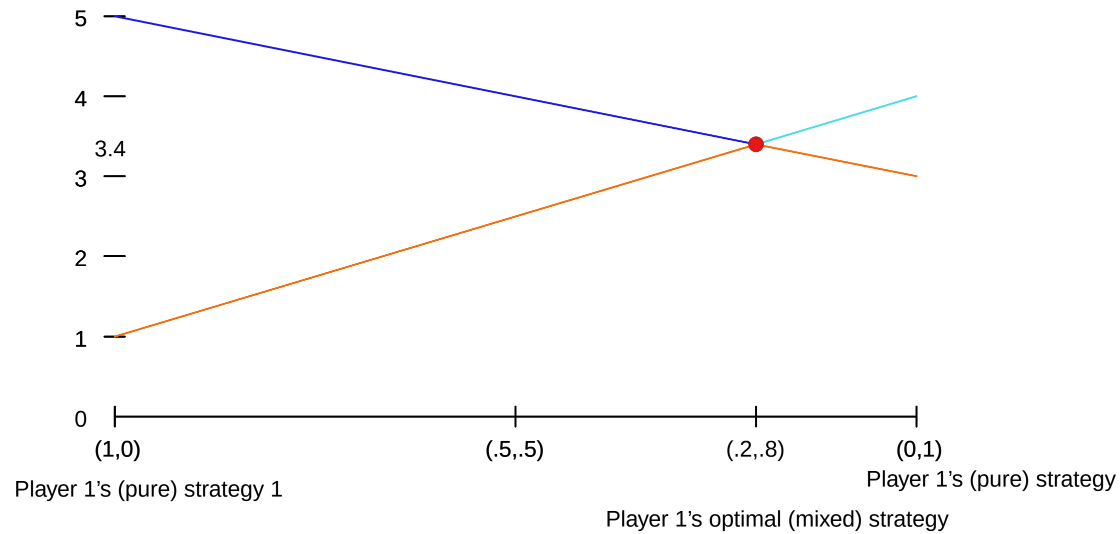


And the best Player 1 can do under those circumstances is to go for the maximum on the line, that is the point where the two original lines intersect.

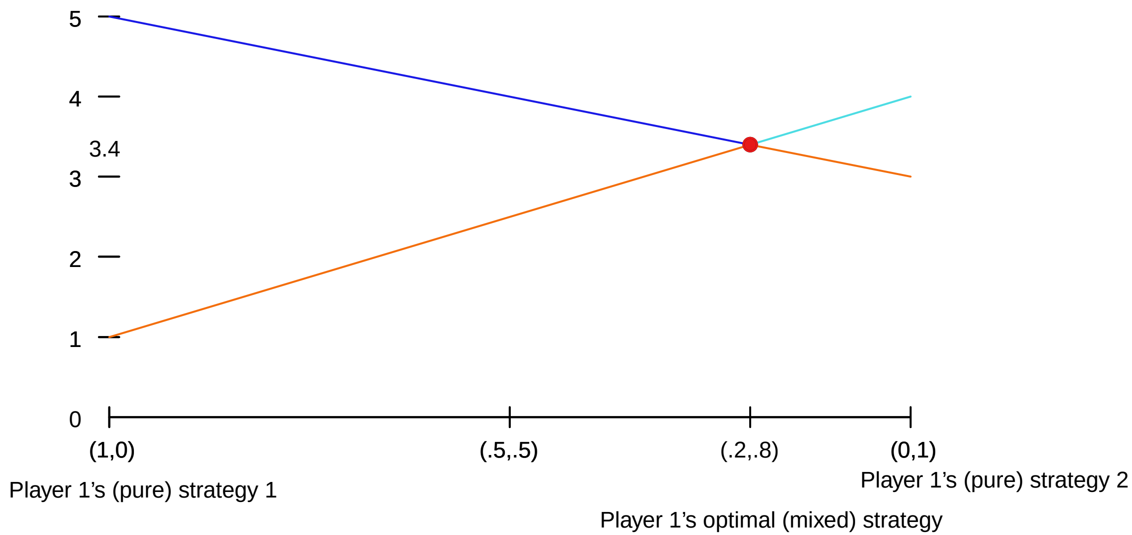
Solving (2×2) games

Maximum over the minimum guaranteed pay-offs for Player 1: red dot.

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



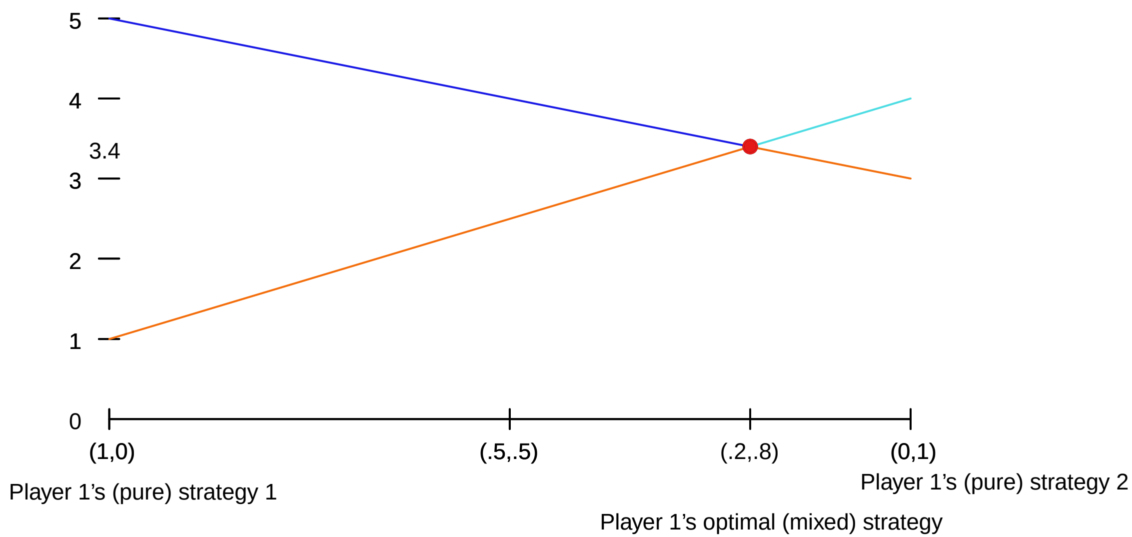
Intersection of two lines



The first line is given by

$$y = -2x + 5$$

Intersection of two lines



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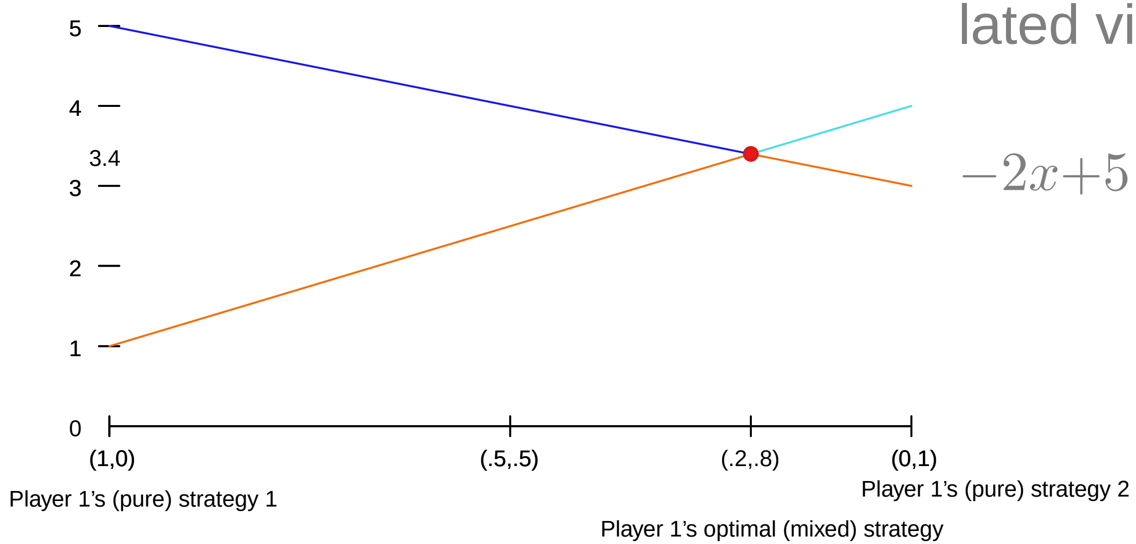

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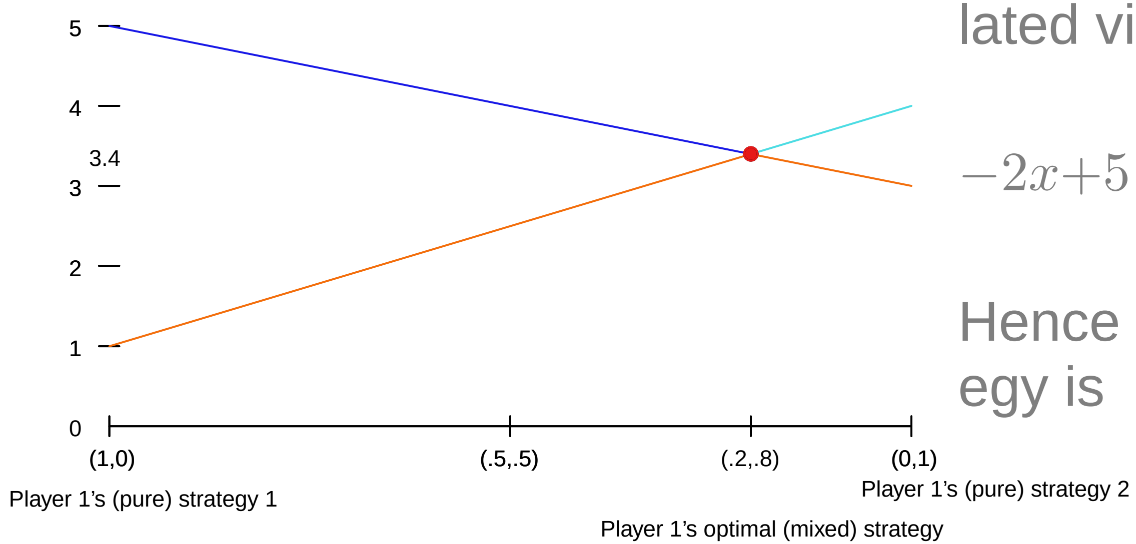

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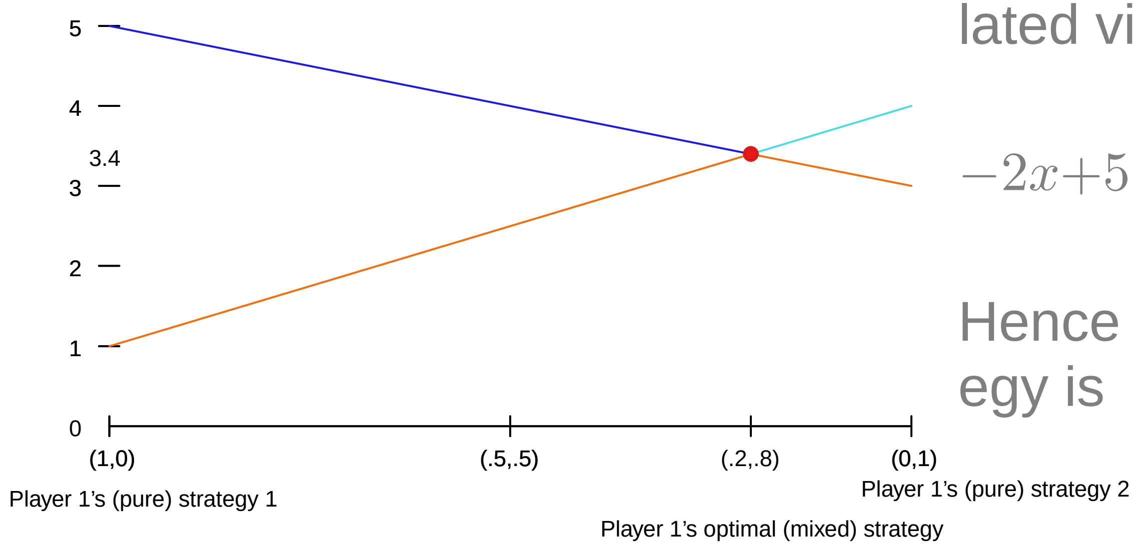

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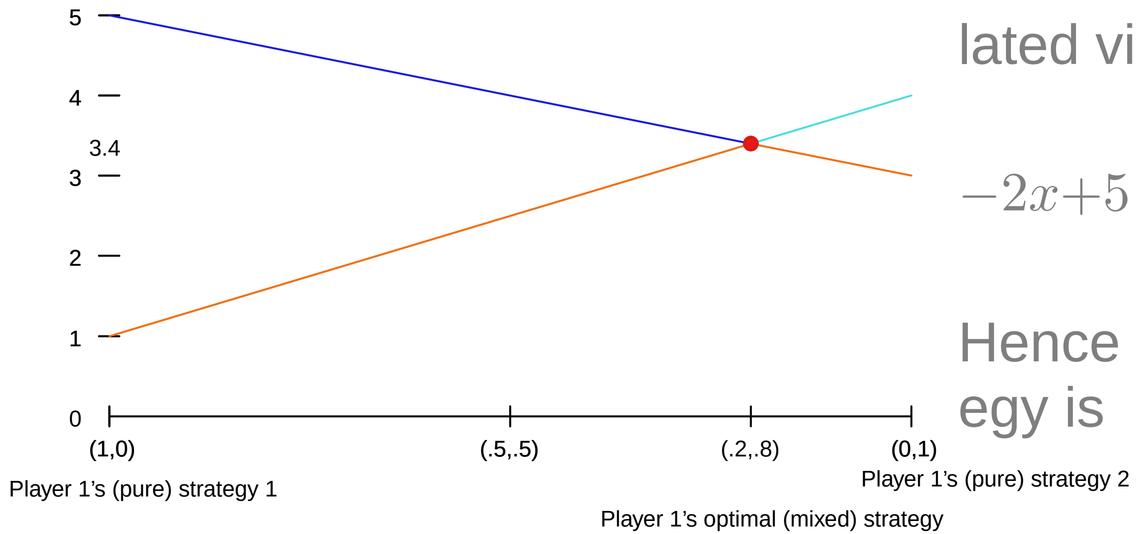

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The value of the game is the y-value of the intersection given by

$$-2\frac{4}{5} + 5 = 3.4$$

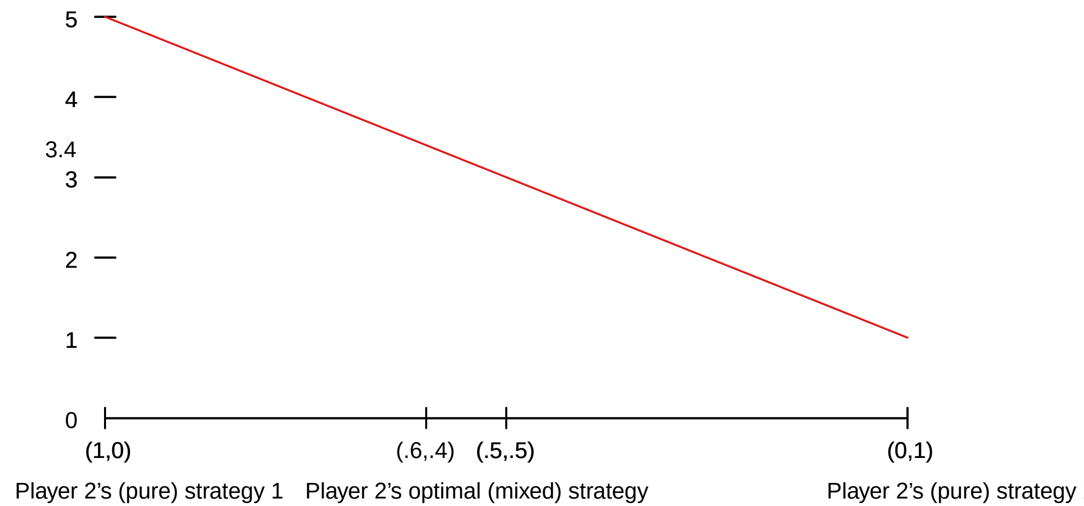
Player 2

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$

Player 2

If Player 1 sticks to his strategy 1, Player 2 can get a pay-off somewhere from 5 to 1:

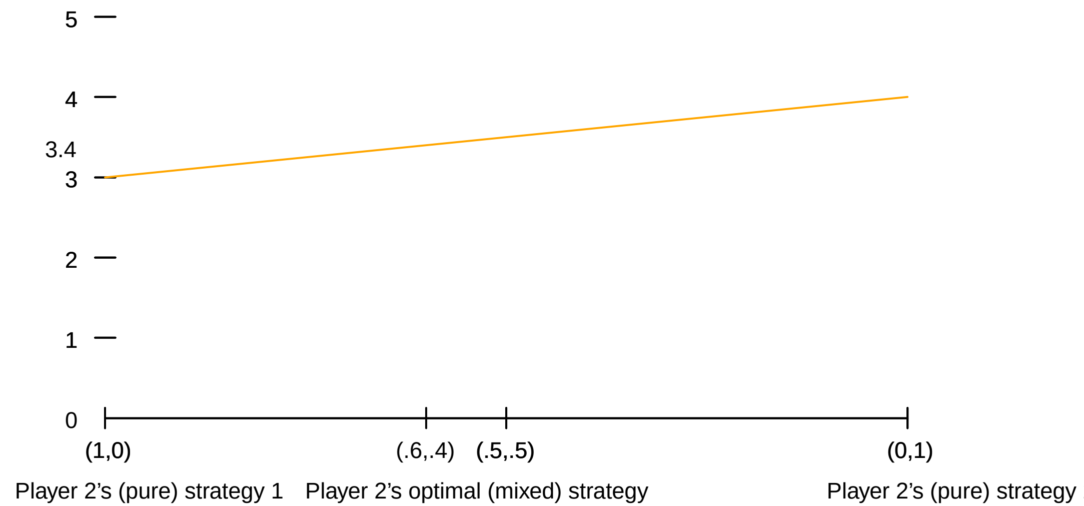
$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



Player 2

If Player 1 sticks to his strategy 2, Player 2 can get a pay-off somewhere from 3 to 4:

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



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If we superimpose the two images we get:

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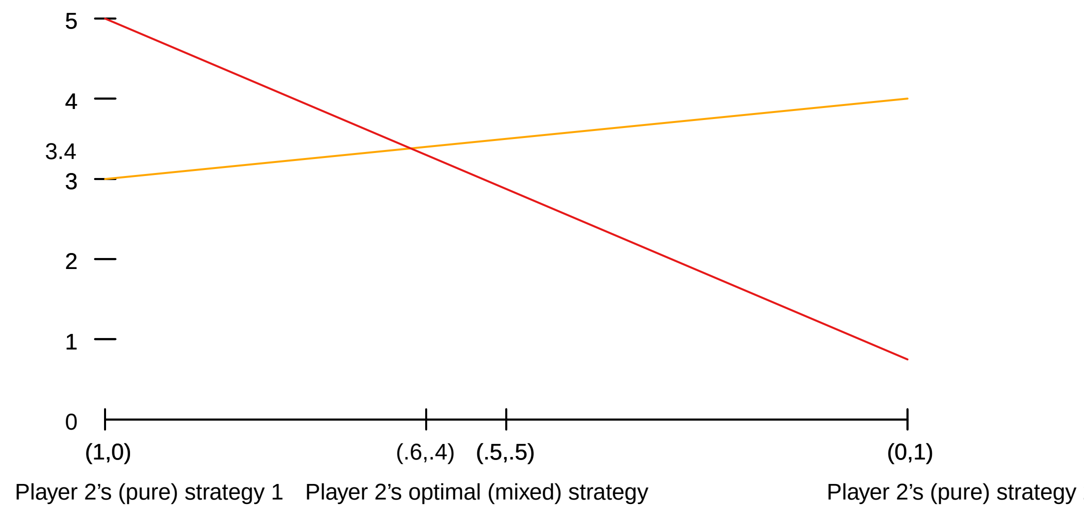

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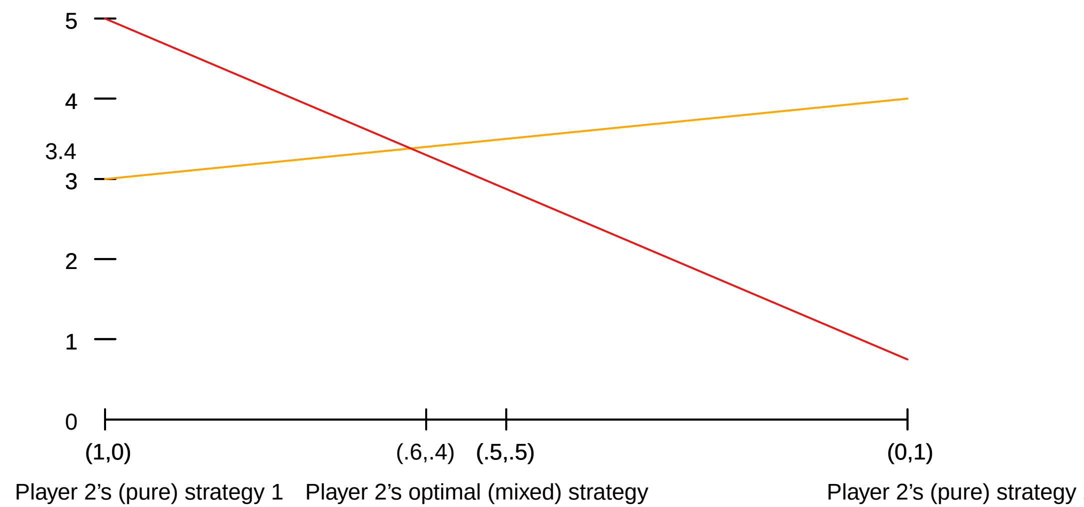
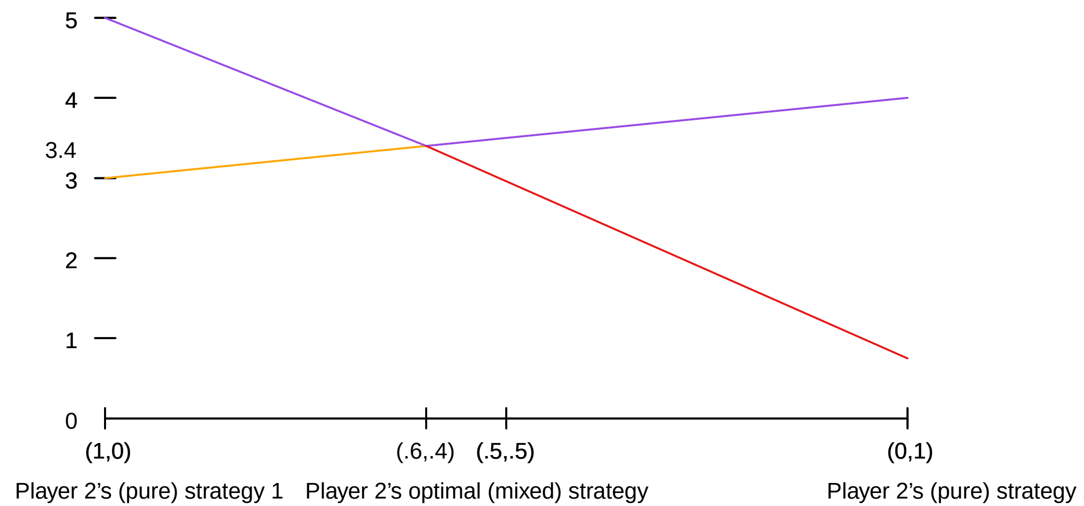


For Player 2, the maximum pay-out she may have to lose is given by the point-wise maximum of the two lines.

Player 2

Worst case for Player 2: violet line.

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



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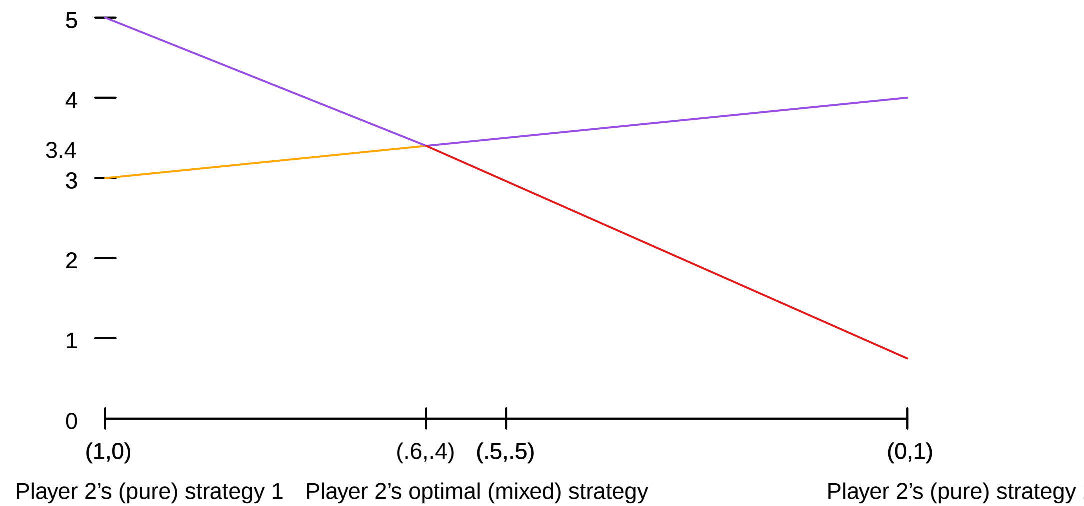


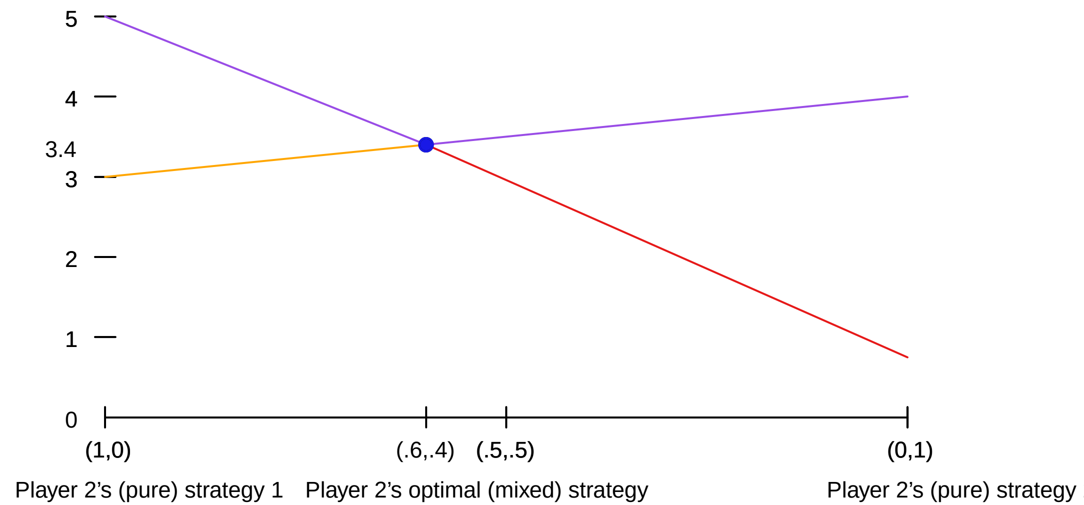
$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$

And the best Player 2 can do under those circumstances is to go for the minimum on the line, that is the point where the two original lines intersect.

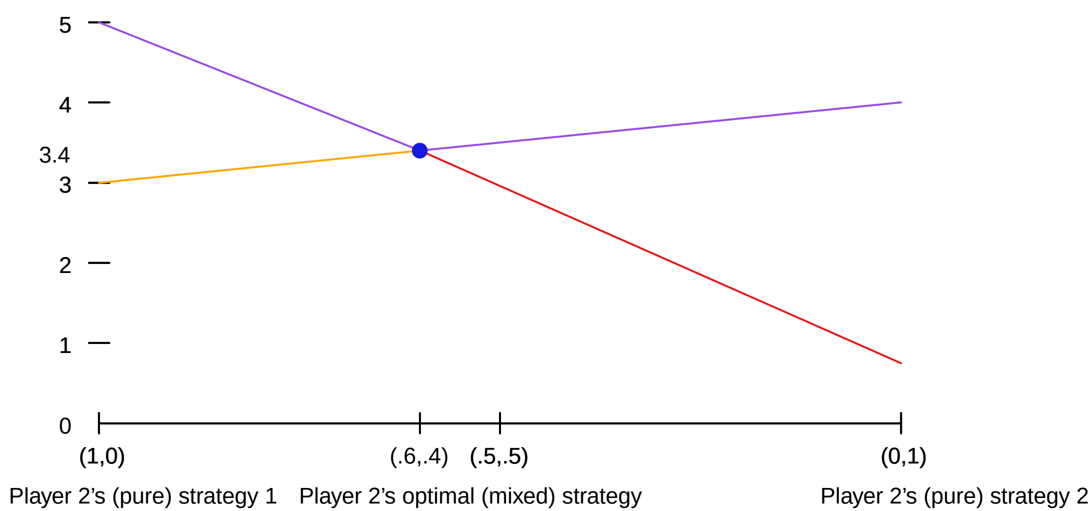
Player 2

Minimum over the maximum pay-out for Player 2: blue dot.

$$\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix}$$



Intersection of two lines



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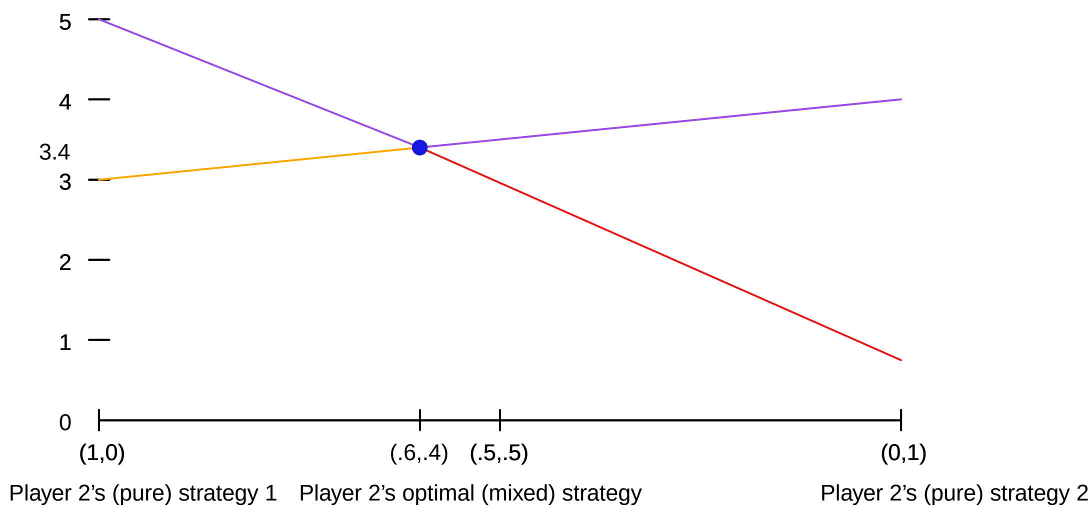

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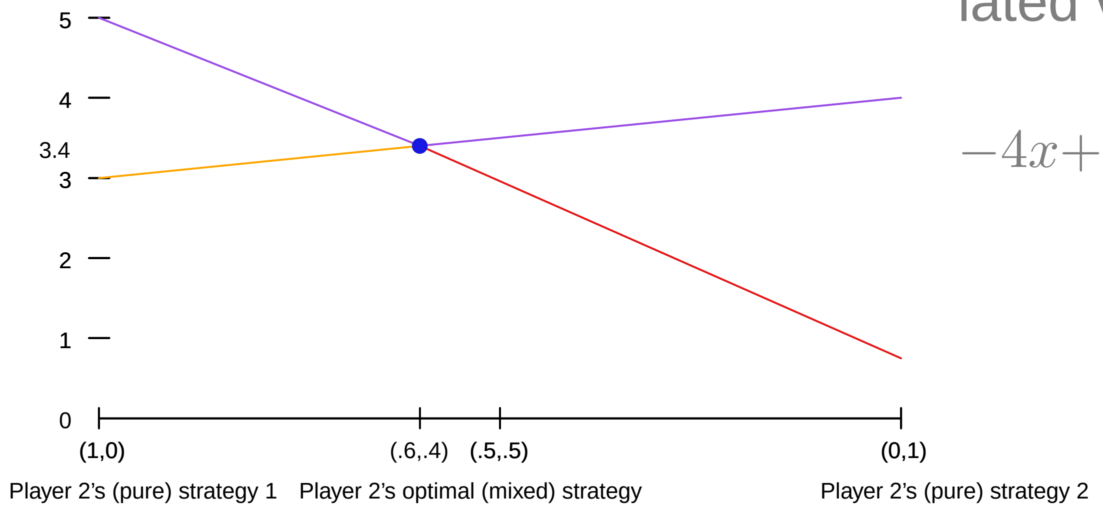

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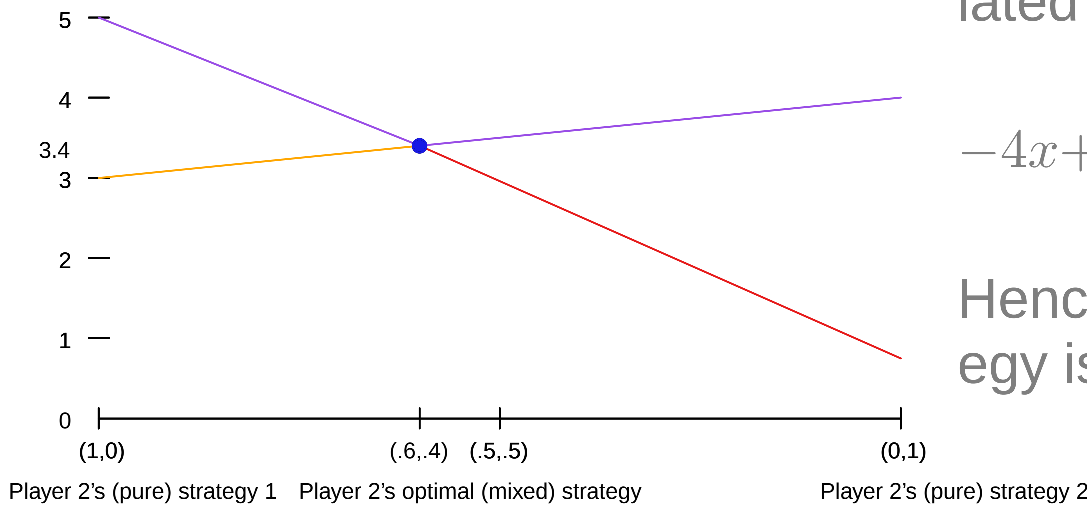

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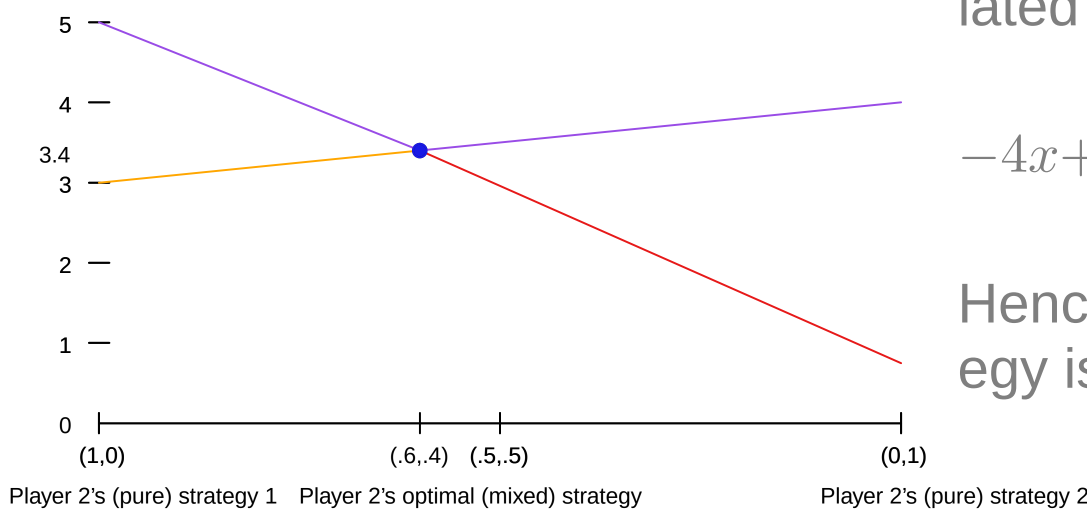

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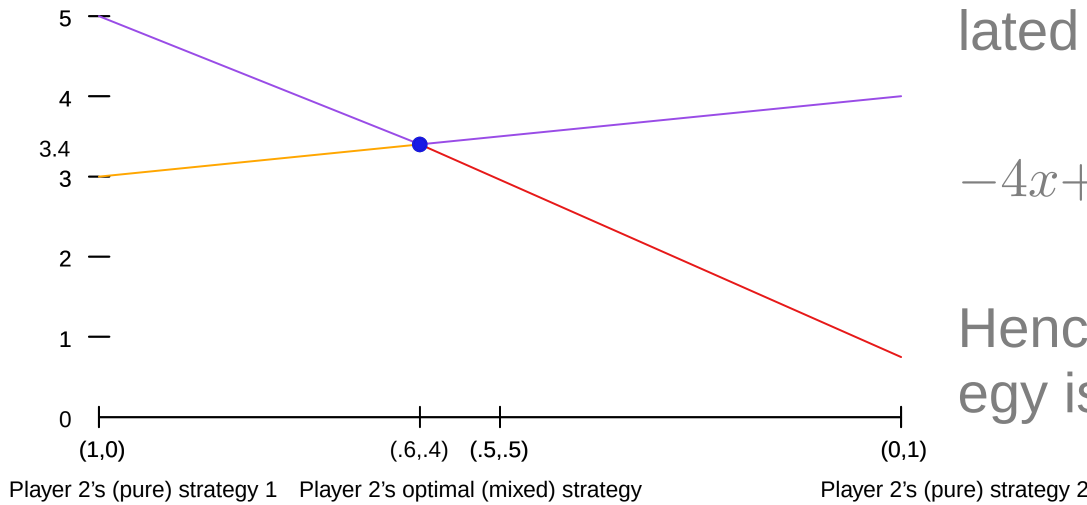

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Simplified Poker

The game

Simplified Poker is played with two players and a deck of three cards labelled J, Q and K.

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Each of the two players has to pay one unit to enter a game (the ante). They then get one card each from the deck. There are 6 possible deals, 3 cards that Player 1 might get, and Player 2 will then get one of the two remaining cards, making $3 \times 2 = 6$ possibilities.

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The players then have the choice between either betting one unit or passing. The game ends when

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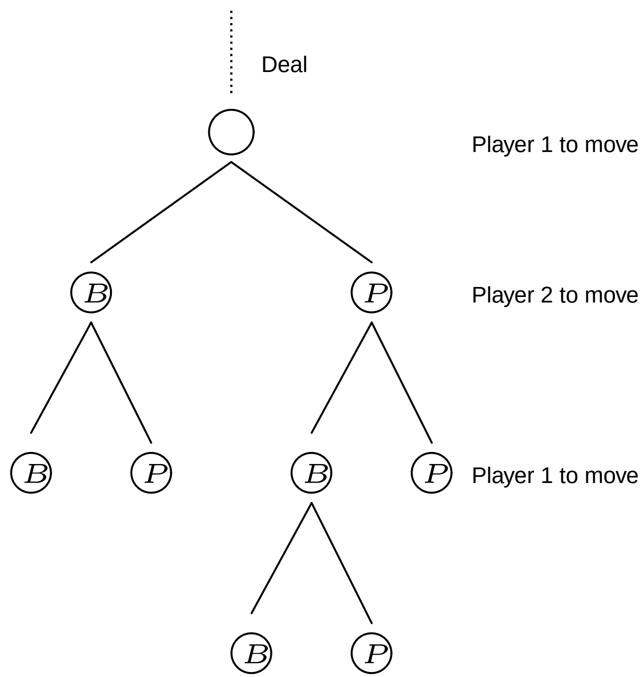
- either a player passes after the other has bet, in which case the better takes the money on the table (the pot),
- or there are two successive passes or bets, in which case the player with the higher card (K beats Q beats J) wins the pot.

The game

After the deal, the following actions can occur, where P means to pass and B means to bet:



Simplified Poker is played with two players and a deck of three cards labelled J, Q and K.

Each of the players pays one unit; they get one card each. The players then have the choice between either betting one unit or passing. The game ends when

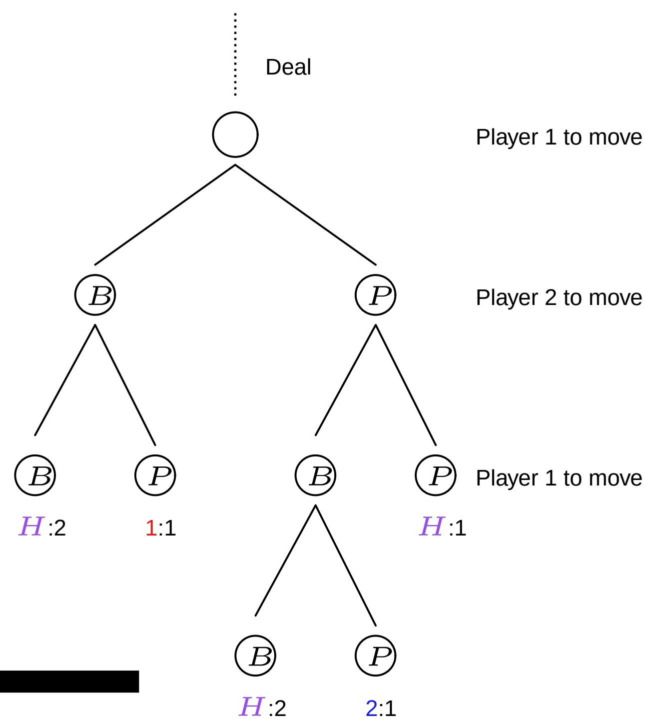
- either a player passes after the other has bet, in which case the better takes the money on the table (the pot),
- or there are two successive passes or bets, in which case the player with the higher card (K beats Q beats J) wins the pot.

The game

The winner is given as

- 1: Player 1;
- 2: Player 2;
- H: higher card.

The amount the winner gets is the number after the colon.



Simplified Poker is played with two players and a deck of three cards labelled J, Q and K.

Each of the players pays one unit; they get one card each. The players then have the choice between either betting one unit or passing. The game ends when

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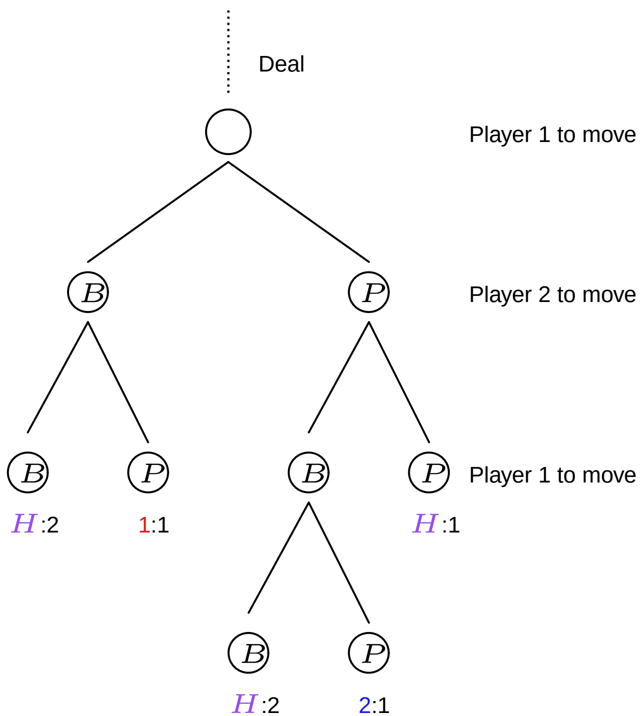
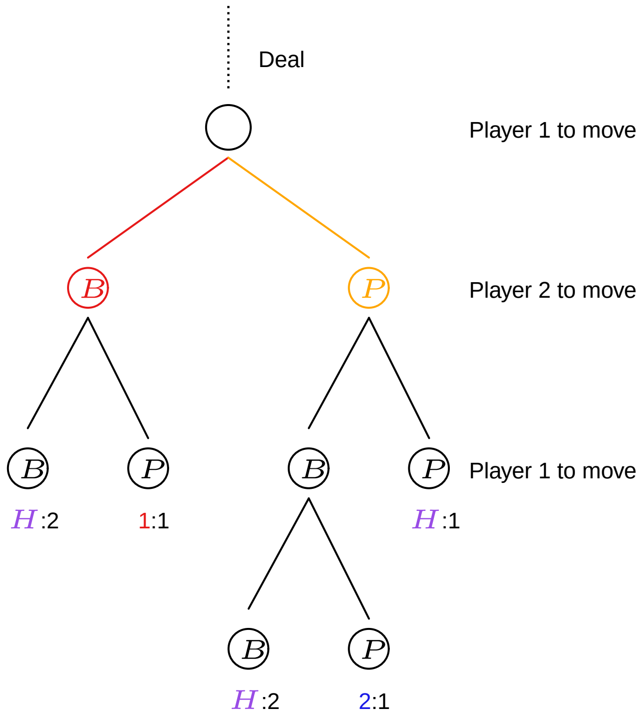
The strategies: Player 1

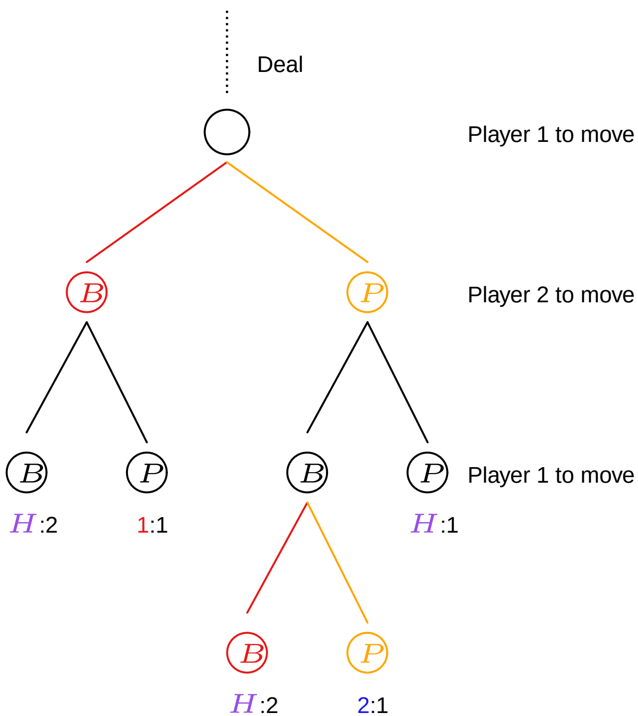


The strategies: Player 1



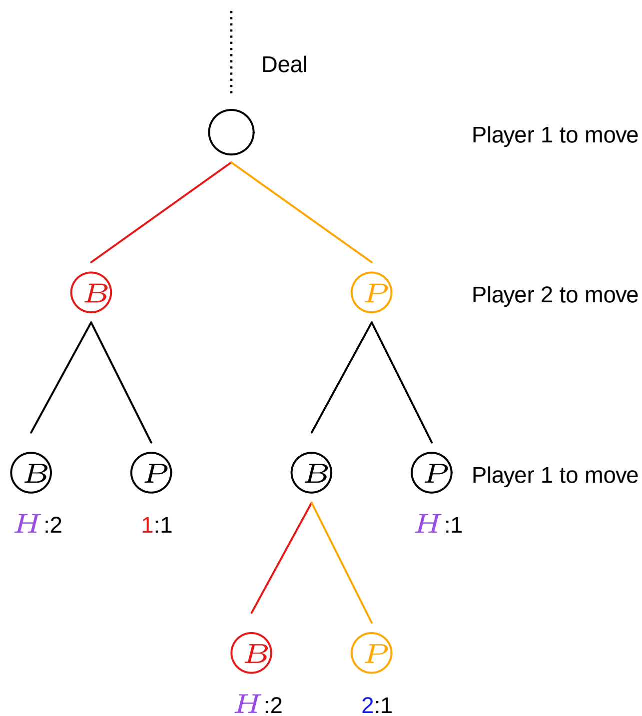
Player 1 has one of three cards, and for each of these possibilities he has to decide whether he wants to bet or to pass.

The strategies: Player 1



Player 1 has one of three cards, and for each of these possibilities he has to decide whether he wants to bet or to pass. If he has passed on the first move, he must make a further decision as to whether to bet or to pass.

The strategies: Player 1



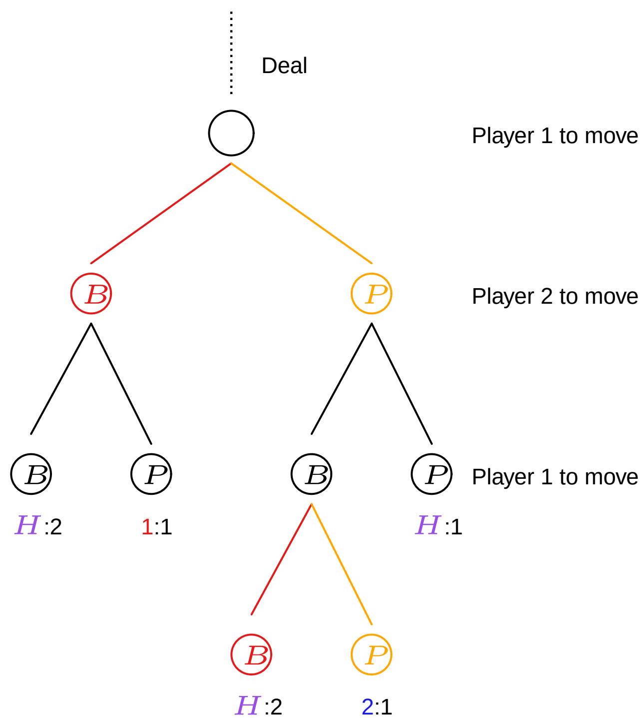
Player 1 has one of three cards, and for each of these possibilities he has to decide whether he wants to bet or to pass. If he has passed on the first move, he must make a further decision as to whether to bet or to pass. We give Player 1's choices as B, PB and PP.

The strategies: Player 1

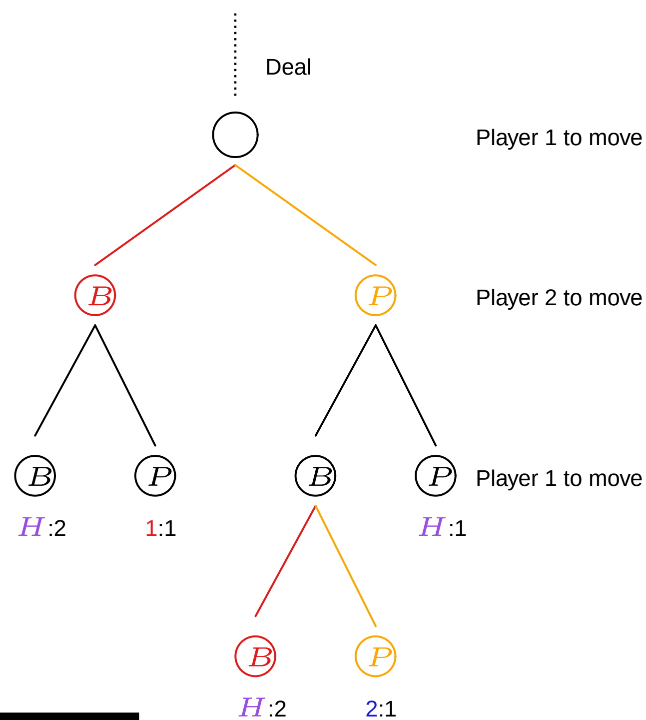
We may record any of his strategies using a table, where for example

J	Q	K
PB	PP	B

means that if Player 1 has the Jack (J) he will pass in the first round and bet in the second (if they get that far), if he has the Queen (Q) he will always pass, and if he has the King (K), he will bet in the first round.



The strategies: Player 1



We may record any of his strategies using a table, for example

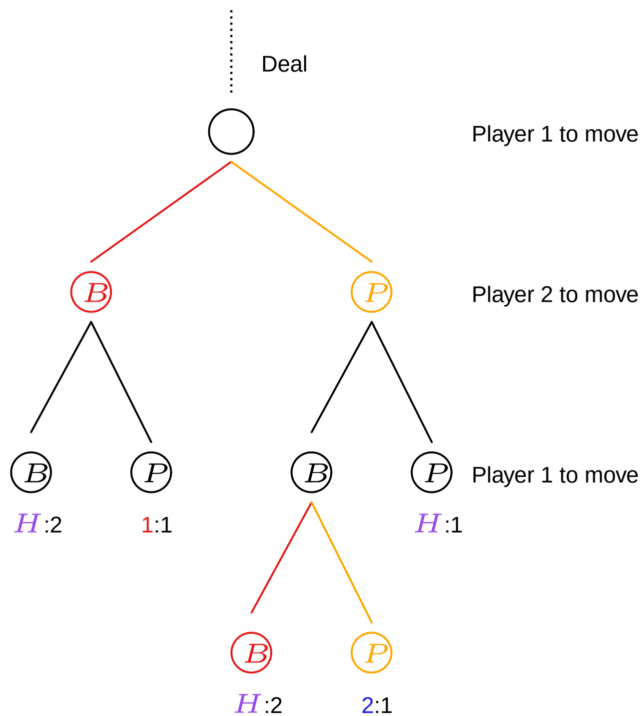
J	Q	K
PB	PP	B

But we do not really need a table to gather all the information: After all, the order J, Q, K is constant. Hence we can use a triple

$$(PB, PP, B)$$

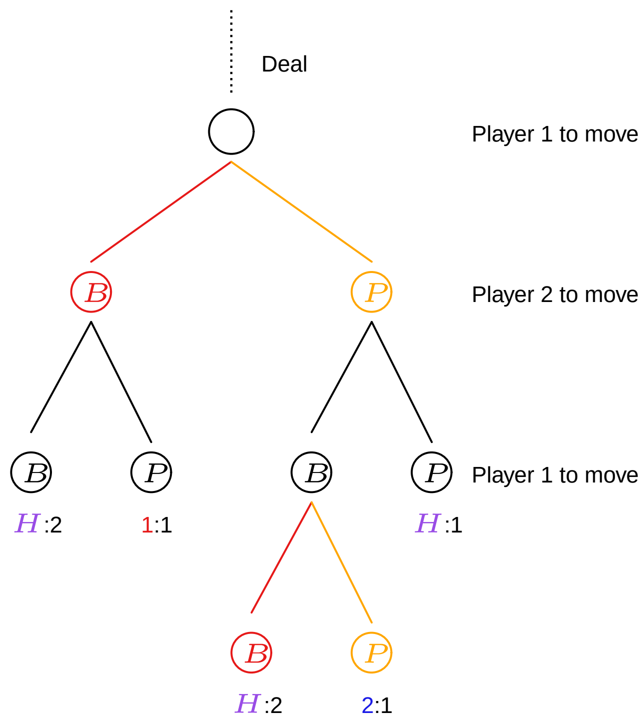
to denote the same information.

The strategies: Player 1



We may record any of his strategies using a triple.

The strategies: Player 1



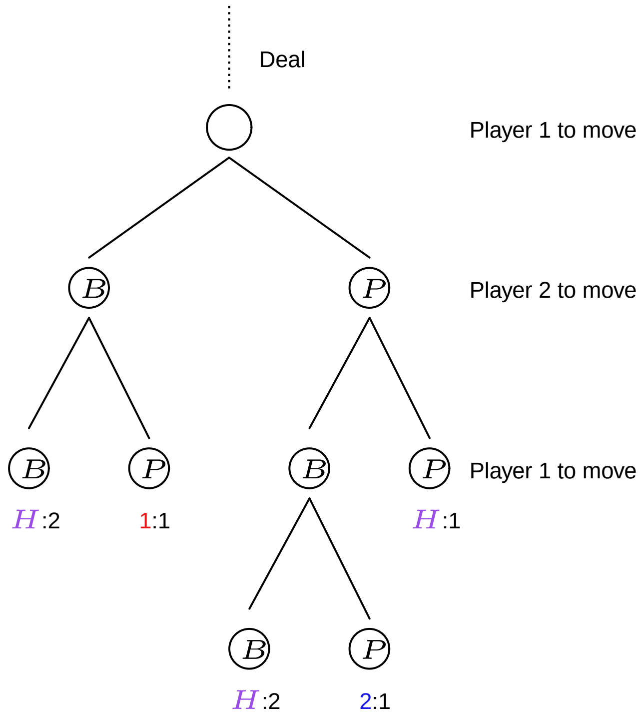
We may record any of his strategies using a triple.

Since Player 1 has three choices, PP, PB, and B, for each of the three cards he might possibly be dealt, and so for each entry in the triple, he has

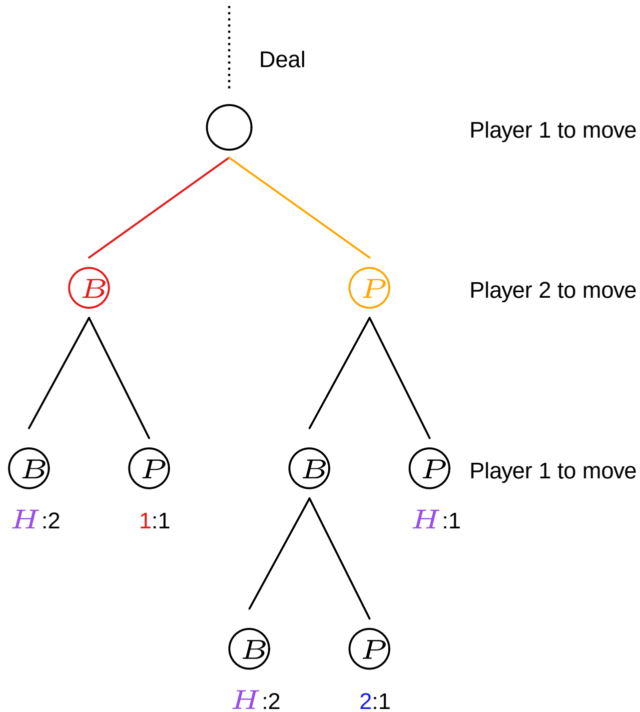
$$3 \times 3 \times 3 = 3^3 = 27$$

(pure) strategies.

The strategies: Player 2

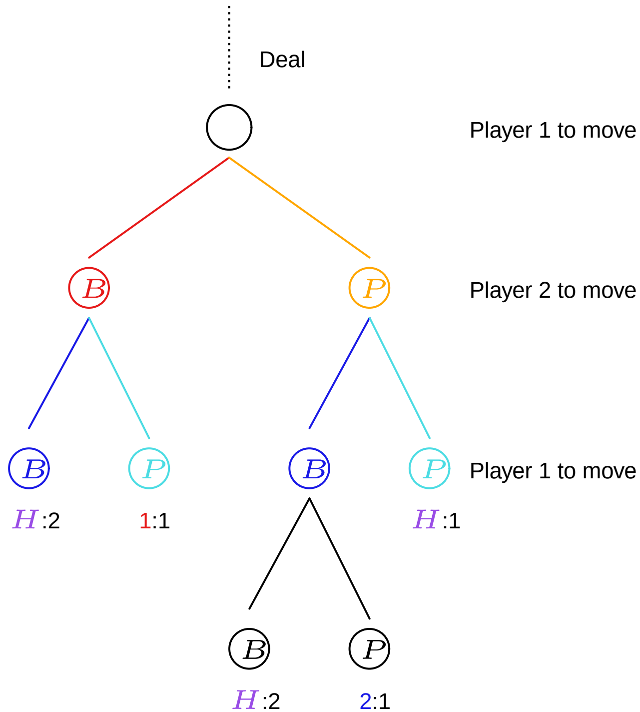


The strategies: Player 2



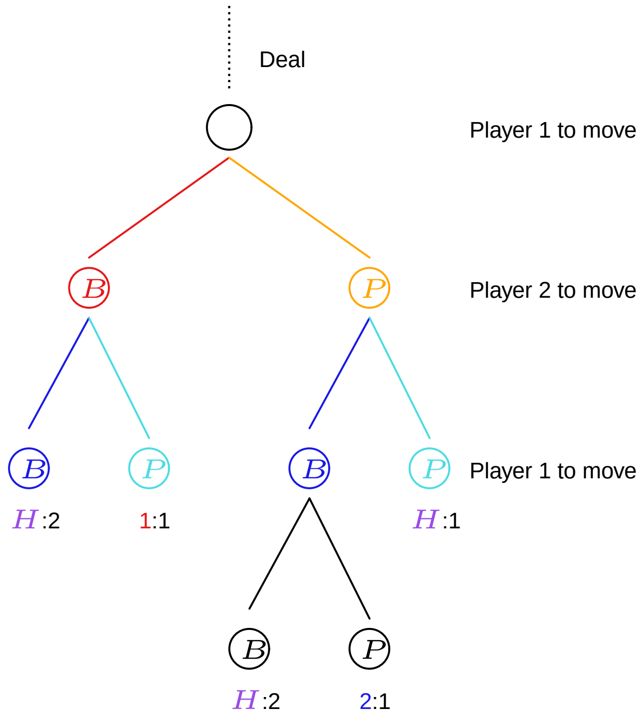
Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

The strategies: Player 2



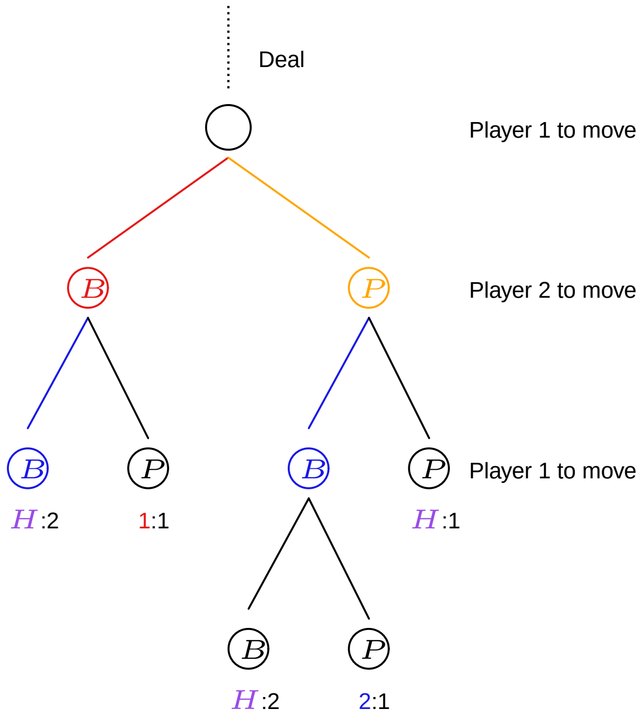
Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed. For each of these possibilities she has to decide whether she wants to bet or to pass.

The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed. For each of these possibilities she has to decide whether she wants to bet or to pass. We give Player 2's choices as B and P.

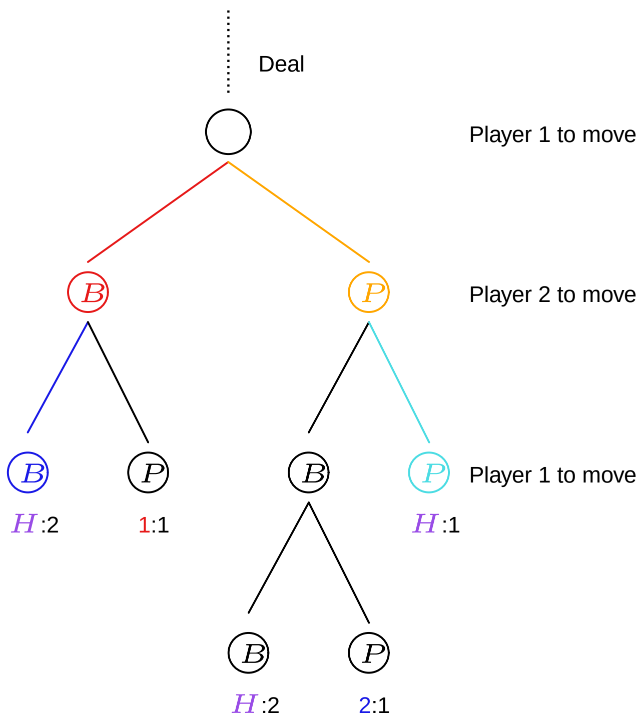
The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

We use $B|B$ to mean that she will bet no matter what Player 1 has done,

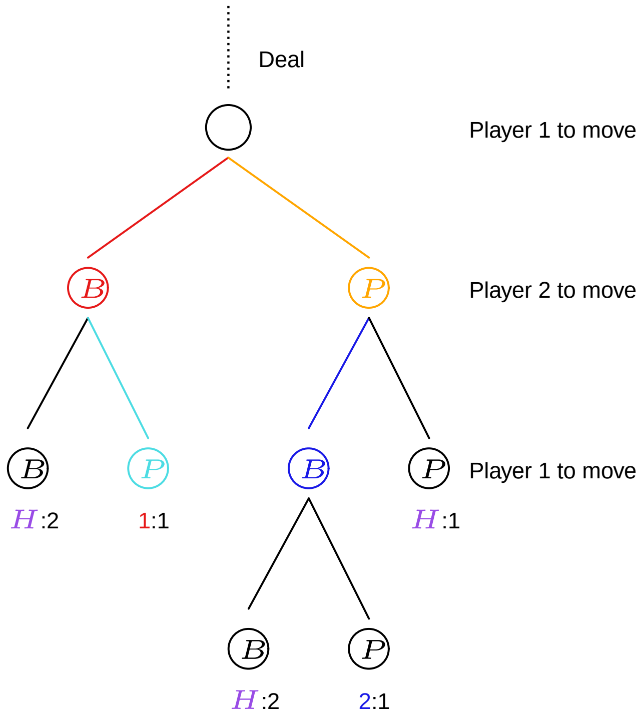
The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

We use $B|B$ to mean that she will bet no matter what Player 1 has done, $B|P$ to mean that she will bet if Player 1 has bet and pass if Player 1 has passed,

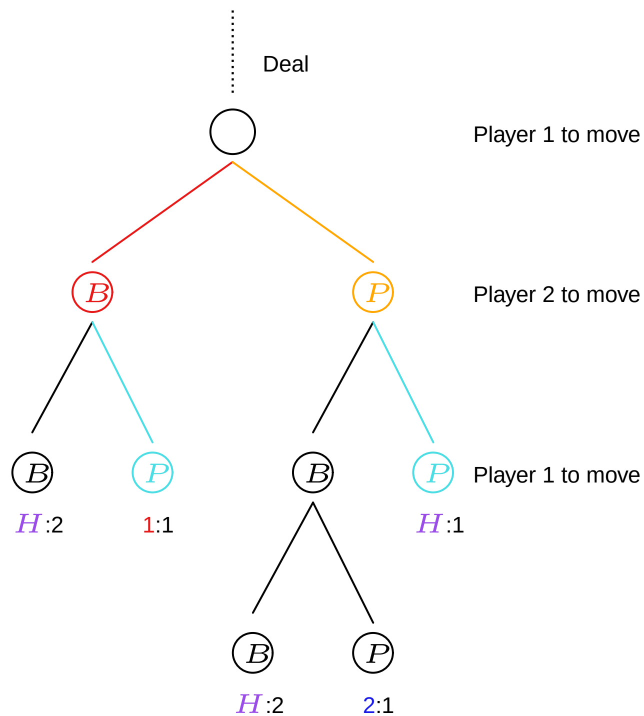
The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

We use $B|B$ to mean that she will bet no matter what Player 1 has done, $B|P$ to mean that she will bet if Player 1 has bet and pass if Player 1 has passed, similarly for $P|B$

The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

We use $B|B$ to mean that she will bet no matter what Player 1 has done, $B|P$ to mean that she will bet if Player 1 has bet and pass if Player 1 has passed, similarly for $P|B$ and $P|P$.

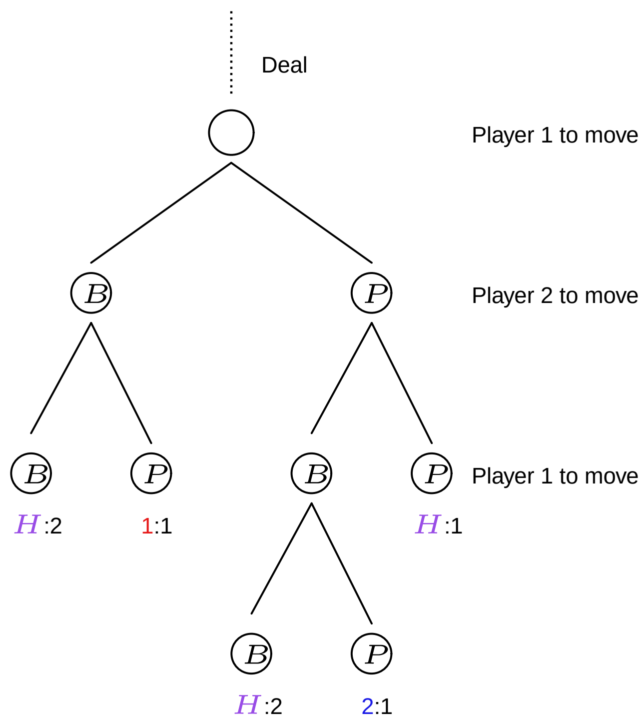
The strategies: Player 2

Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

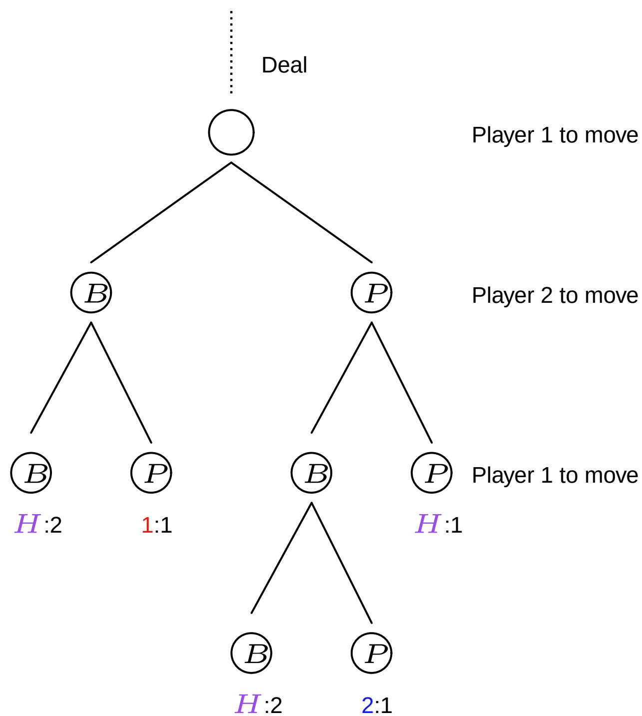
Again we can use triples, where for example

$$(P|B, P|P, B|P)$$

means that if Player 2 has the Jack (J) she will pass if Player 2 has bet and bet if he has passed, if she has the Q she will always pass, and if she has the K, she will bet if Player 1 has bet and pass if Player 1 has passed.



The strategies: Player 2



Player 2 has one of three cards, and by the time it is her move Player 1 will have either bet of passed.

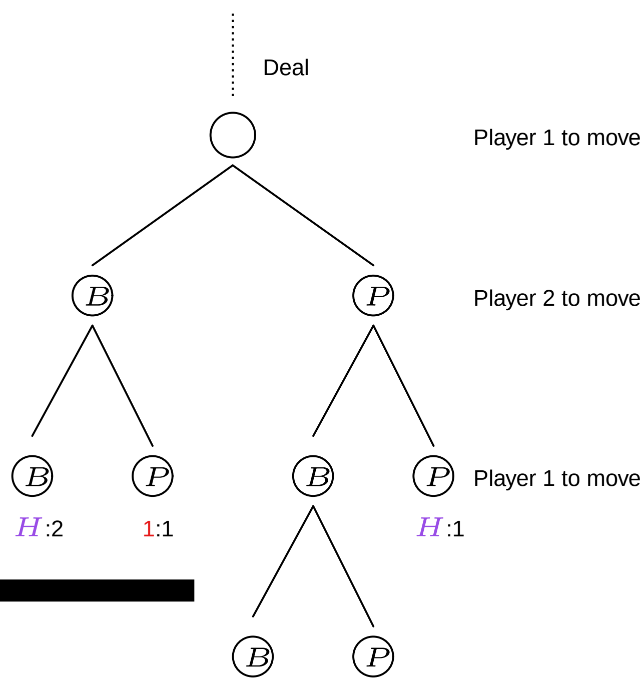
Since Player 2 has four choices, $P|P$, $P|B$, $B|P$, and $B|B$ for each of the three cards she might possibly be dealt, she has

$$4 \times 4 \times 4 = 4^3 = 64$$

(pure) strategies.

Removing strategies: Player 2

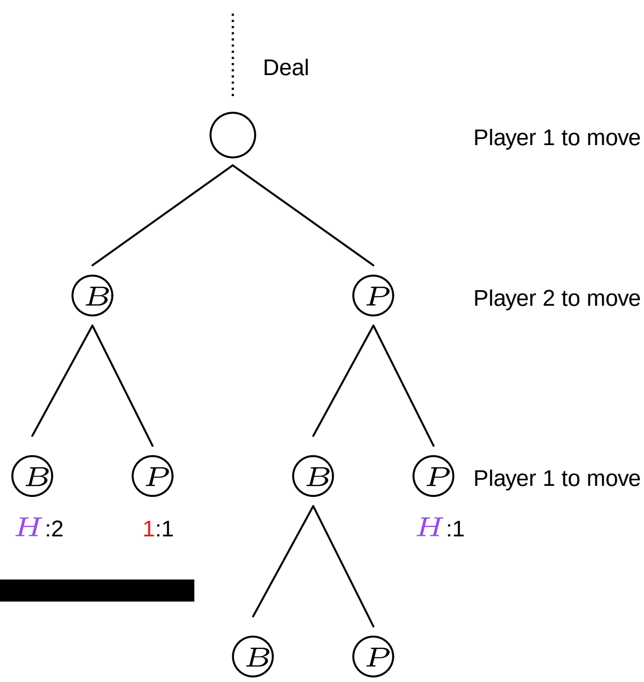
Clearly this game is far too big for us to supply a matrix ‘by hand’. We will therefore employ dominance arguments to make the game smaller.



Removing strategies: Player 2

Clearly this game is far too big for us to supply a matrix ‘by hand’. We will therefore employ dominance arguments to make the game smaller.

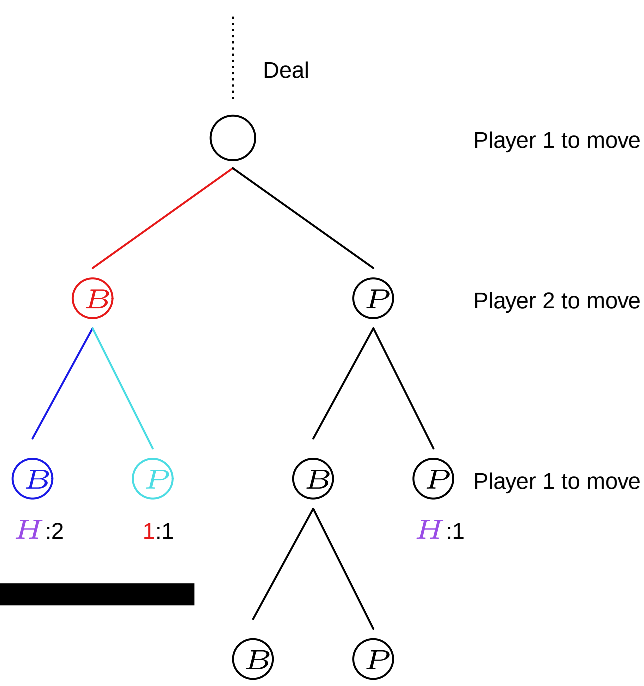
As we have seen, dominated strategies are less successful than others. We can check for this without having a matrix, just be comparing pay-offs.



Removing strategies: Player 2

We start by removing obviously stupid strategies for Player 2.

Assume that Player 1 has bet.

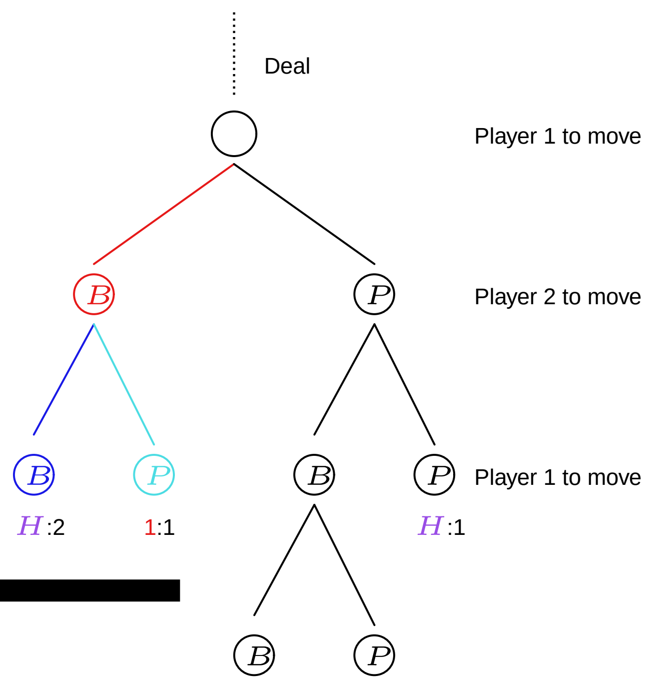


Removing strategies: Player 2

We start by removing obviously stupid strategies for Player 2.

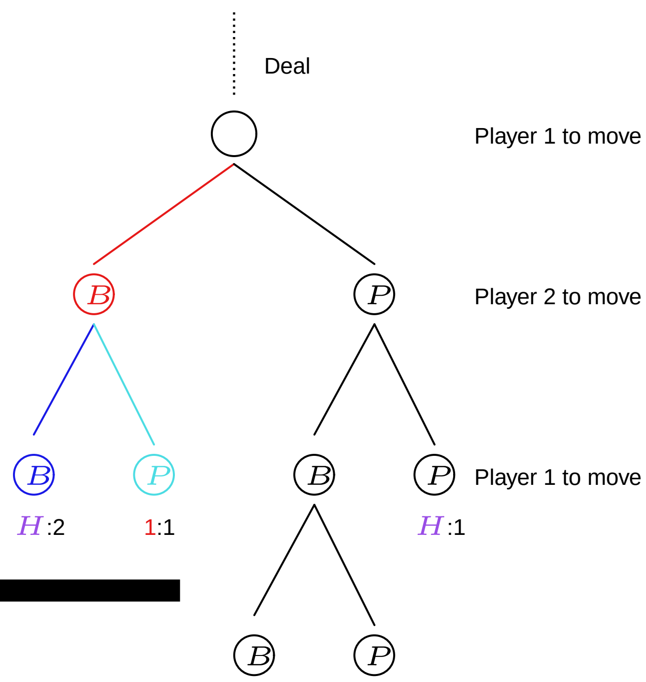
Assume that Player 1 has bet.

Player 2 can choose to bet or pass in turn. If she passes, then no matter what card she has, she will lose 1. If she bets then she will lose 2 if she has the lower card and win 2 if she has the higher card.



Removing strategies: Player 2

We start by removing obviously stupid strategies for Player 2.



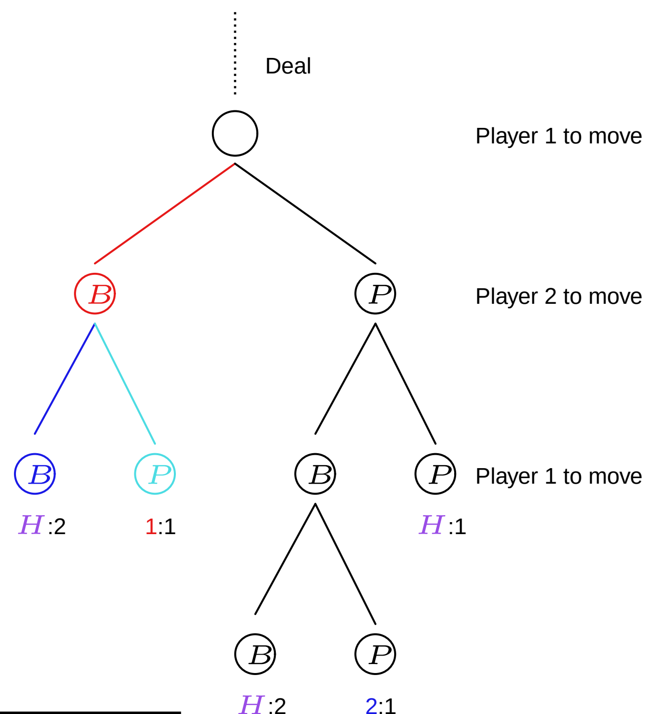
Assume that Player 1 has bet.

If Player 2 passes, she will lose 1. If she bets then she will lose 2 if she has the lower card and win 2 if she has the higher card.

If she has the J, this turns into a choice between losing 1 and losing 2, so the sensible thing to do is to pass and only lose 1. Hence if she has the J, she should never play a strategy of the form $B|$. In other words, the first component of her strategy should be $P|B$ or $P|P$, leaving two possibilities.

Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.



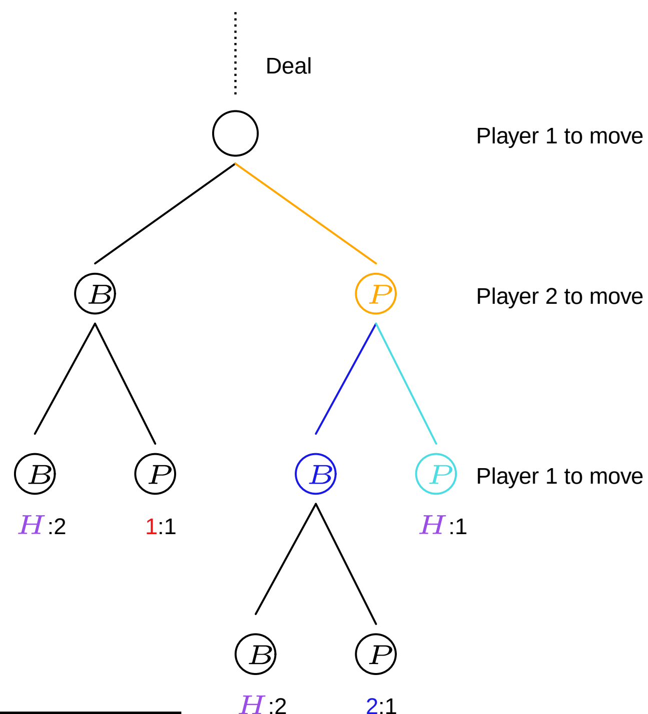
Assume that Player 1 has bet.

If Player 2 passes, she will lose 1. If she bets then she will lose 2 if she has the lower card and win 2 if she has the higher card.

If she has the K, this turns into a choice between losing 1 and winning 2, so the sensible thing to do is to bet and win 2. Hence if she has the K, she should always play a strategy of the form $B|\ $.

Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.



Assume that Player 1 has bet.

If Player 2 passes, she will lose 1. If she bets then she will lose 2 if she has the lower card and win 2 if she has the higher card.

If she has the K, this turns into a choice between losing 1 and winning 2, so the sensible thing to do is to bet and win 2. Hence if she has the K, she should always play a strategy of the form $B|$ .If she has the K, she should always play a strategy of the form $B|$.

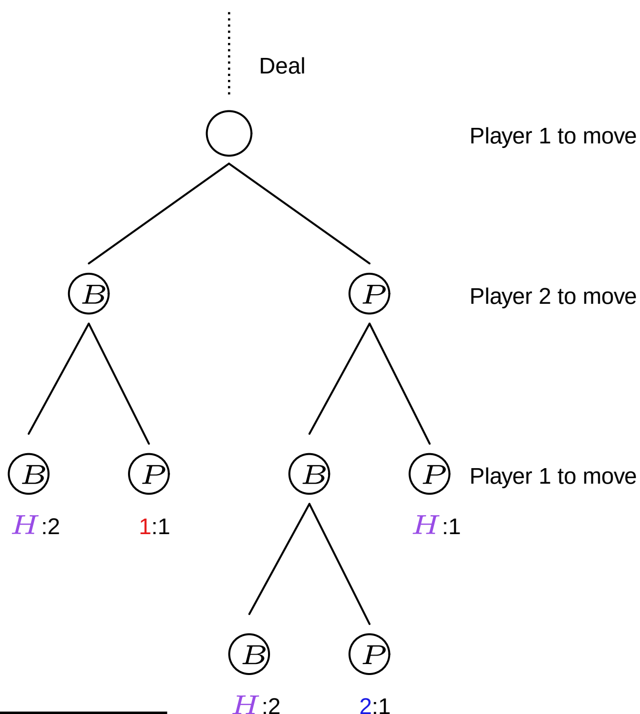
Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.

The third component of her strategy should be $B|B$ or $B|P$.

Assume that Player 1 has bet.

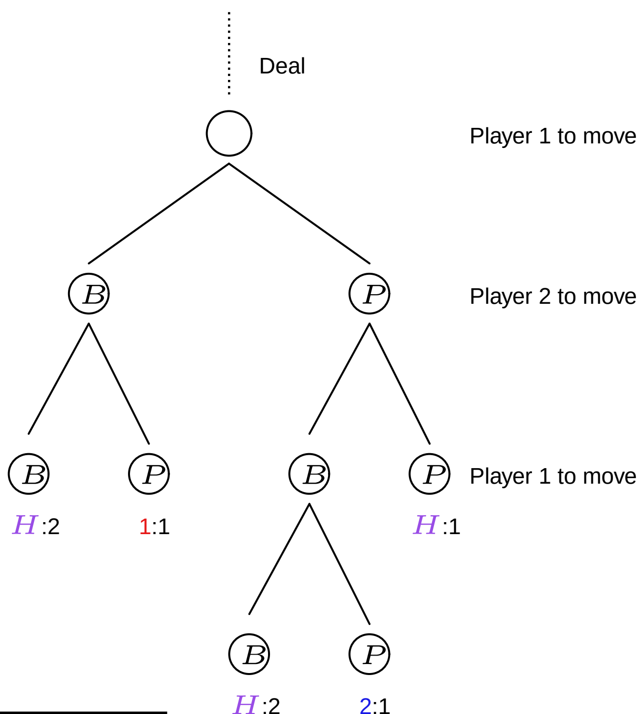
Assume that Player 1 has passed and Player 2 has the K.



Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.

The third component of her strategy should be $B|B$ or $B|P$.



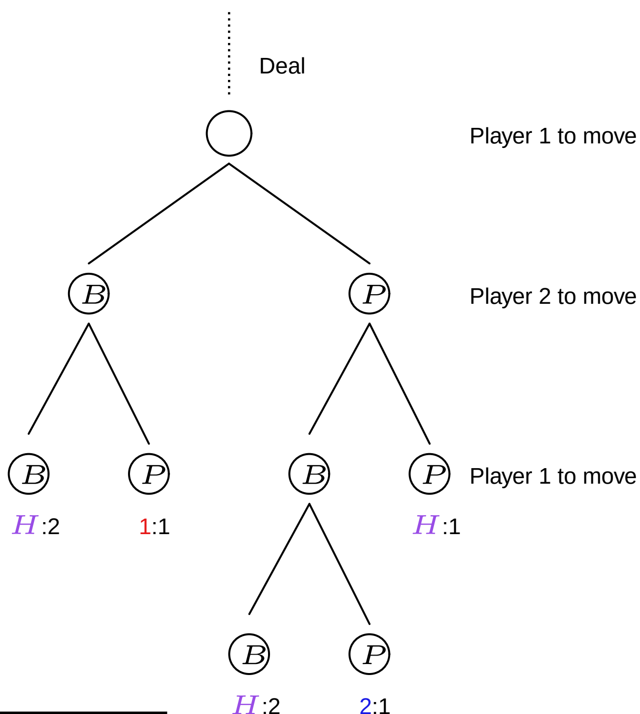
Assume that Player 1 has passed and Player 2 has the K.

She should bet, since she may win 2, but is definitely guaranteed to win 1, which is the best she can hope for if she passes.

Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.

The third component of her strategy should be $B|B$ or $B|P$.



Assume that Player 1 has passed and Player 2 has the K.

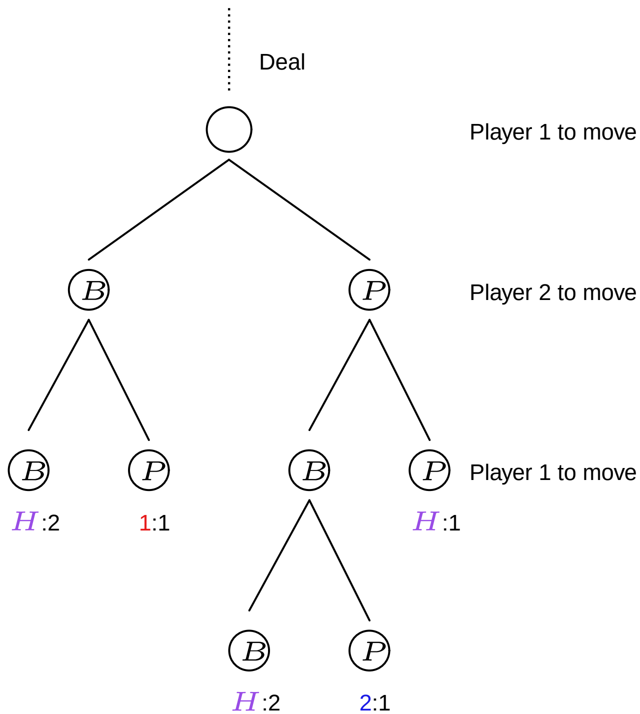
She should bet, since she may win 2, but is definitely guaranteed to win 1, which is the best she can hope for if she passes.

So Player 2's strategy when she has a K should be $B|B$, leaving just one possibility for the third component of her strategy.

Removing strategies: Player 2

The first component of her strategy should be $P|B$ or $P|P$.

The third component of her strategy should be $B|B$.

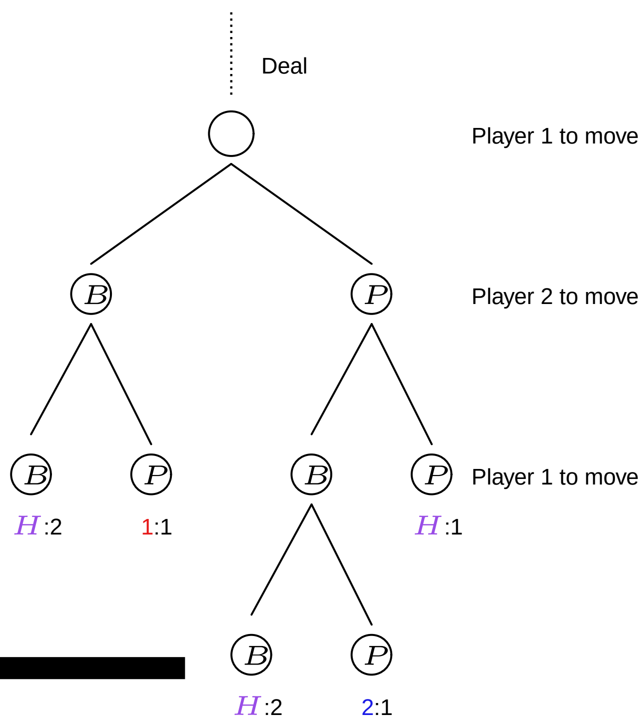


Hence we have reduced the number of her strategies to

$$2 \times 4 \times 1 = 8.$$

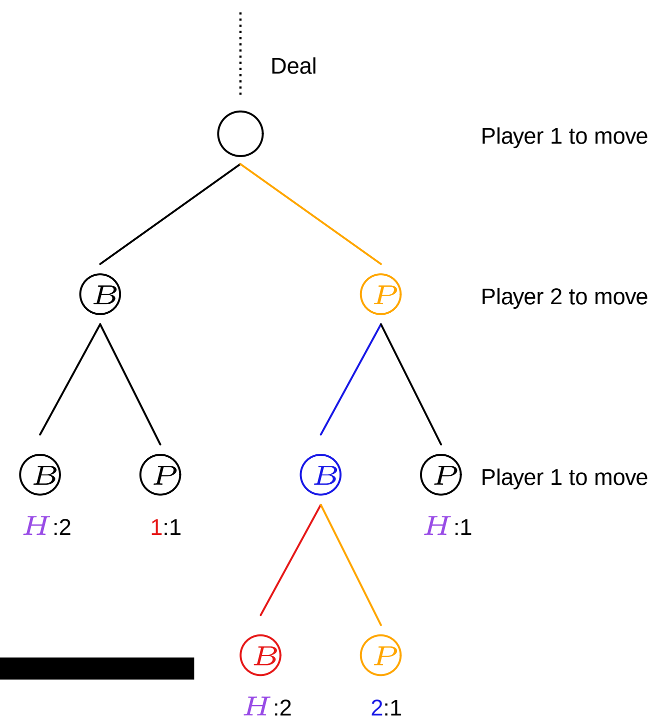
Removing strategies: Player 1

We continue by removing obviously stupid strategies for Player 1.



Removing strategies: Player 1

We continue by removing obviously stupid strategies for Player 1.

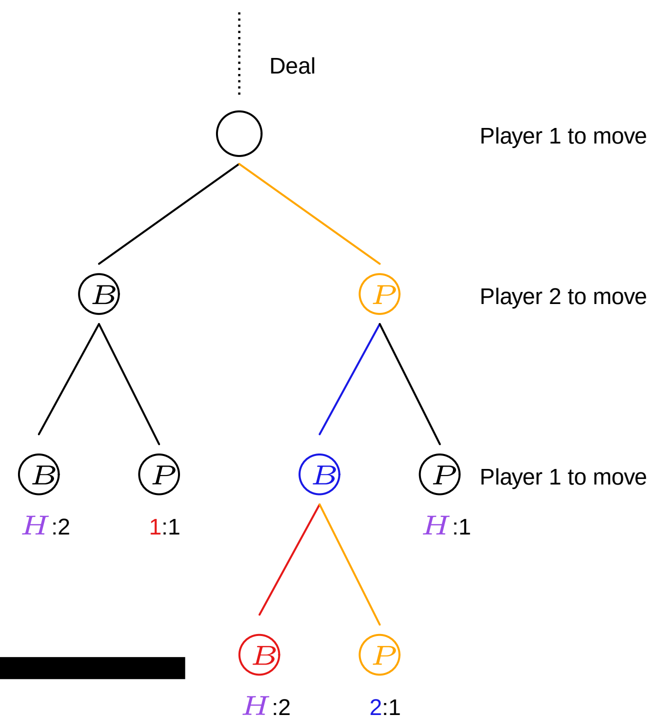


Assume that Player 1 has passed, and that Player 2 has bet.

Player 1 can choose to bet or pass in turn. If he passes, then no matter what card he has, he will lose 1. If he bets then he will lose 2 if he has the lower card and win 2 if he has the higher card.

Removing strategies: Player 1

We continue by removing obviously stupid strategies for Player 1.



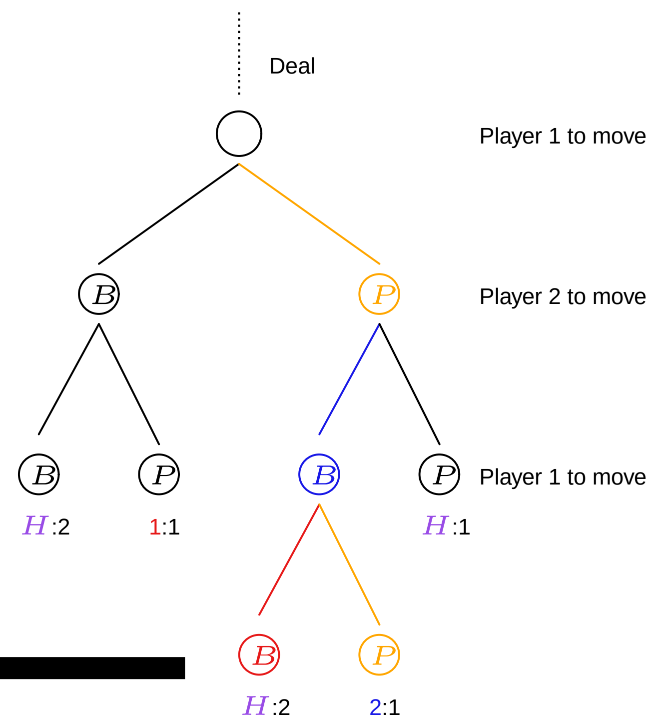
Assume that Player 1 has passed, and that Player 2 has bet.

Player 1 can choose to bet or pass in turn. If he passes, then no matter what card he has, he will lose 1. If he bets then he will lose 2 if he has the lower card and win 2 if he has the higher card.

If he has the J, this turns into a choice between losing 1 and losing 2, so the sensible thing to do is to pass and only lose 1. Hence if he has the J, he should never play a strategy of the form PB.

Removing strategies: Player 1

We continue by removing obviously stupid strategies for Player 1.



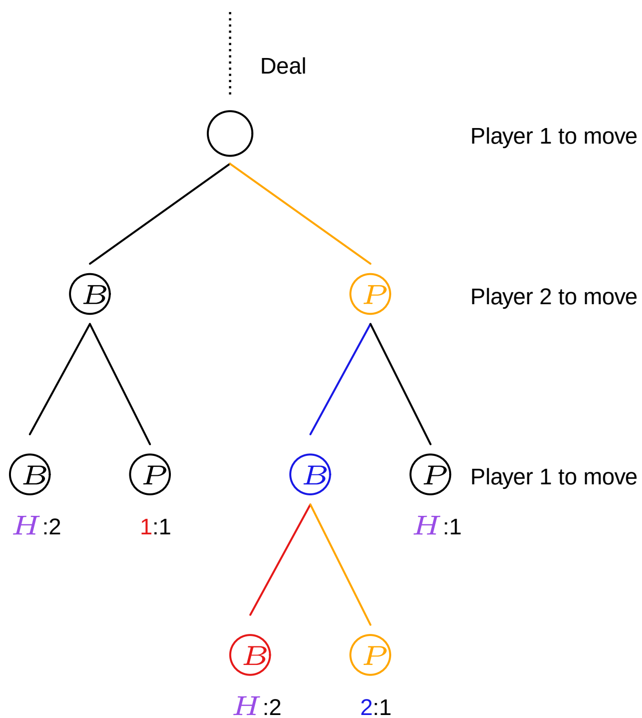
Assume that Player 1 has passed, and that Player 2 has bet.

Player 1 can choose to bet or pass in turn. If he passes, then no matter what card he has, he will lose 1. If he bets then he will lose 2 if he has the lower card and win 2 if he has the higher card.

If he has the J, this turns into a choice between losing 1 and losing 2, so the sensible thing to do is to pass and only lose 1. Hence if he has the J, he should never play a strategy of the form PB. In other words, the first component of his strategy should be B or PP, leaving two possibilities.

Removing strategies: Player 1

The first component of his strategy should be B or PP.



Assume that Player 1 has passed, and that Player 2 has bet.

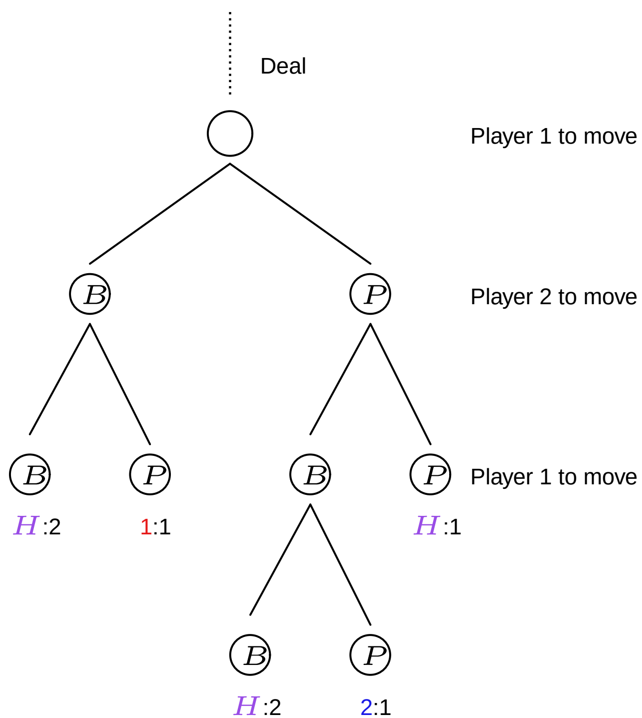
Player 1 can choose to bet or pass in turn. If he passes, then no matter what card he has, he will lose 1. If he bets then he will lose 2 if he has the lower card and win 2 if he has the higher card.

If he has the K, this turns into a choice between losing 1 and winning 2, so the sensible thing to do is to bet and win 2. Hence if he has the K, he should never play a strategy of the form PP. In other words, the last component of his strategy should be B or PB, leaving two possibilities.

Removing strategies: Player 1

The first component of his strategy should be B or PP.

The last component of his strategy should be B or PB.

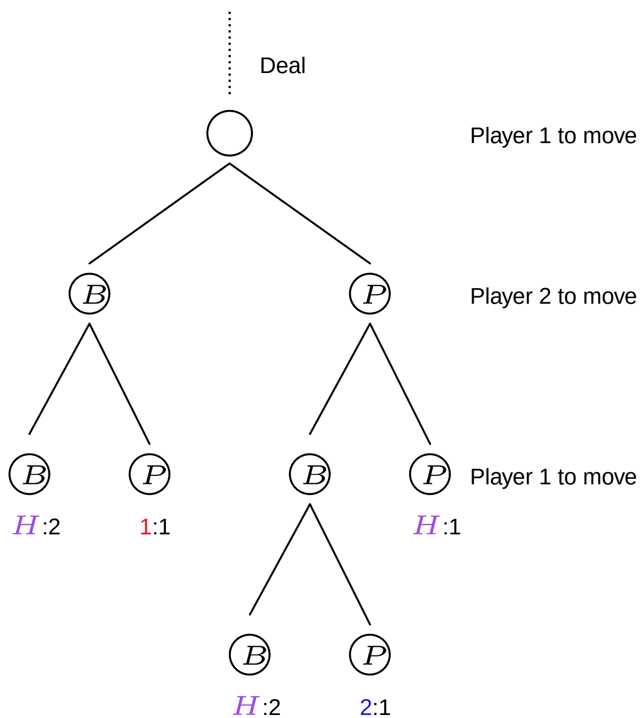


Hence we have reduced the number of his strategies to

$$2 \times 3 \times 2 = 12.$$

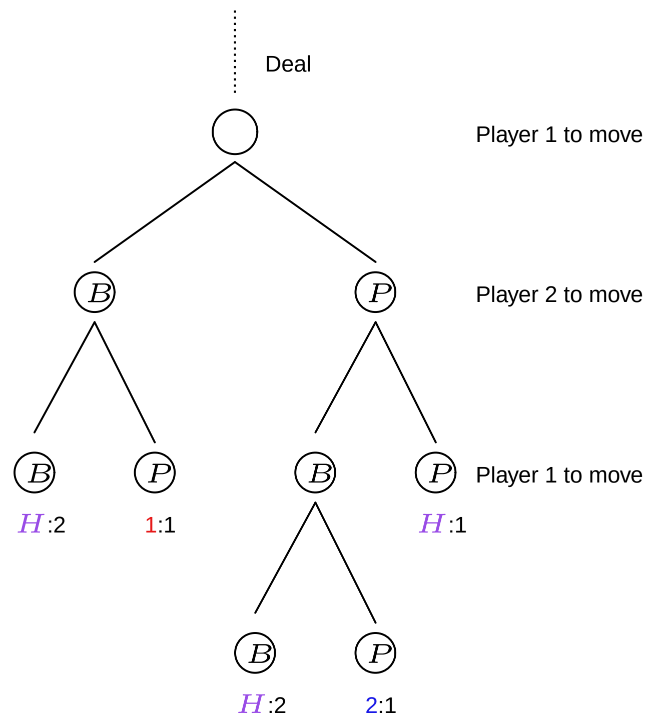
Removing more strategies: Player 1

So far we have removed ‘obviously stupid’ strategies without making any assumptions about what the other player might do. Now we will make use of the fact that the other player is also rational and will not play a dominated strategy.



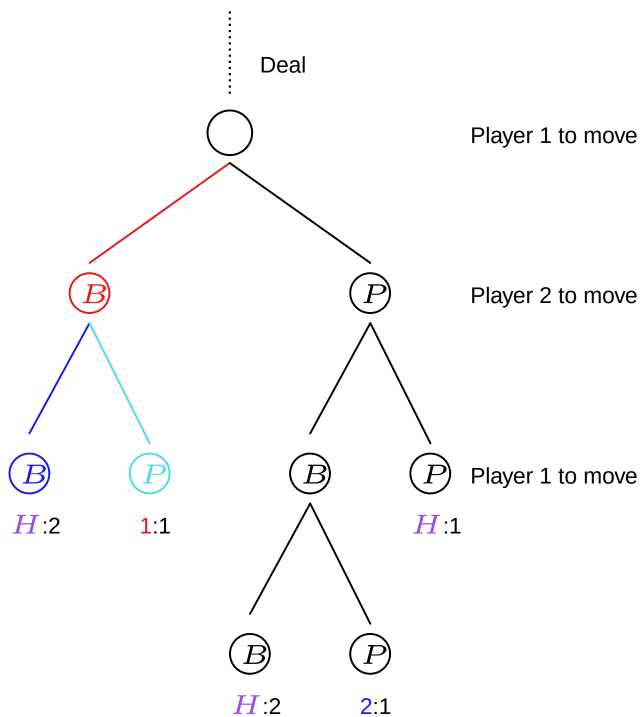
Removing more strategies: Player 1

Assume Player 1 has got the Q.



Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.

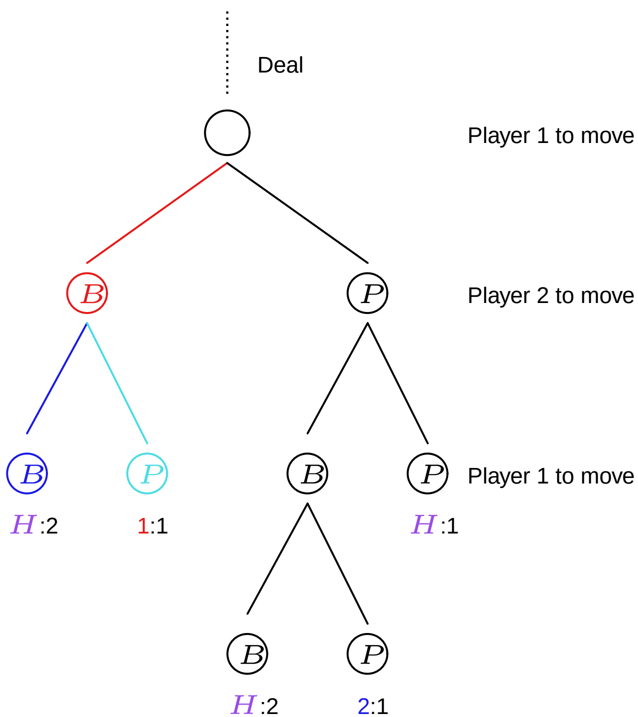


Assume Player 1 has got the Q.

Then if he bets on the first move, the following possibilities can arise:

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



Assume Player 1 has got the Q.

Then if he bets on the first move, the following possibilities can arise:

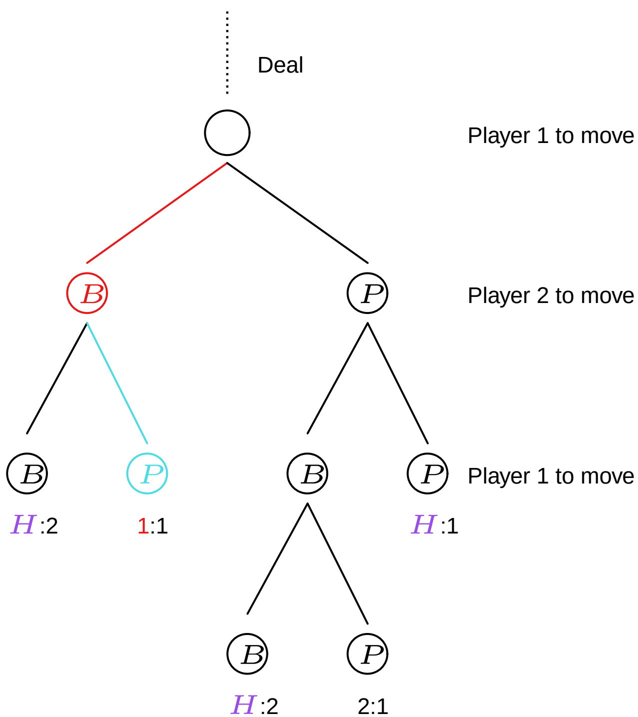
Player 2 has J. Player 2 has K.

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.

Assume Player 1 has got the Q.

Then if he bets on the first move, the following possibilities can arise:



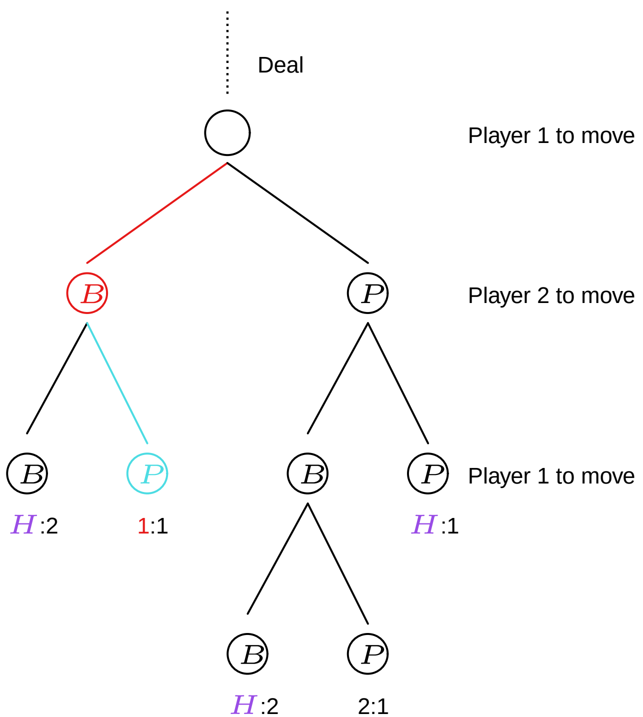
Player 2 has J.

Then Player 2 will pass (we have ruled out all her other strategies).

Player 2 has K.

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



Assume Player 1 has got the Q.

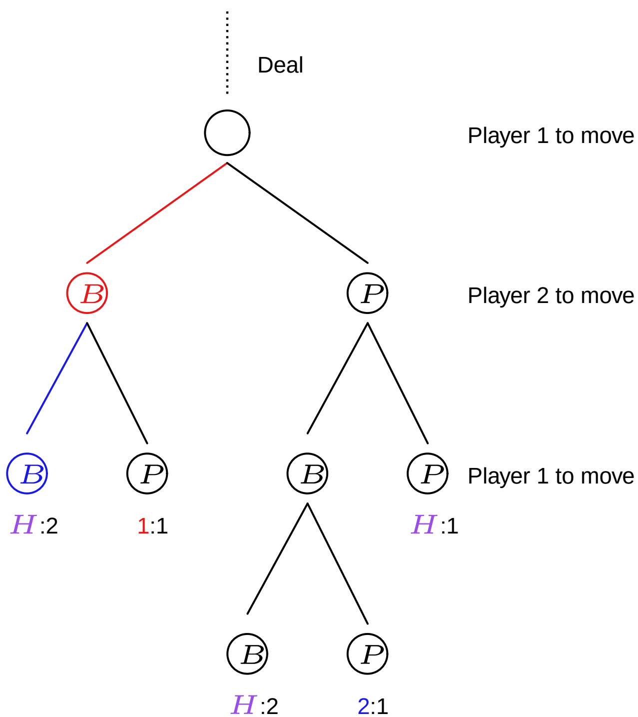
Then if he bets on the first move, the following possibilities can arise:

Player 2 has J. Player 2 has K.

Player 1 wins 1.

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



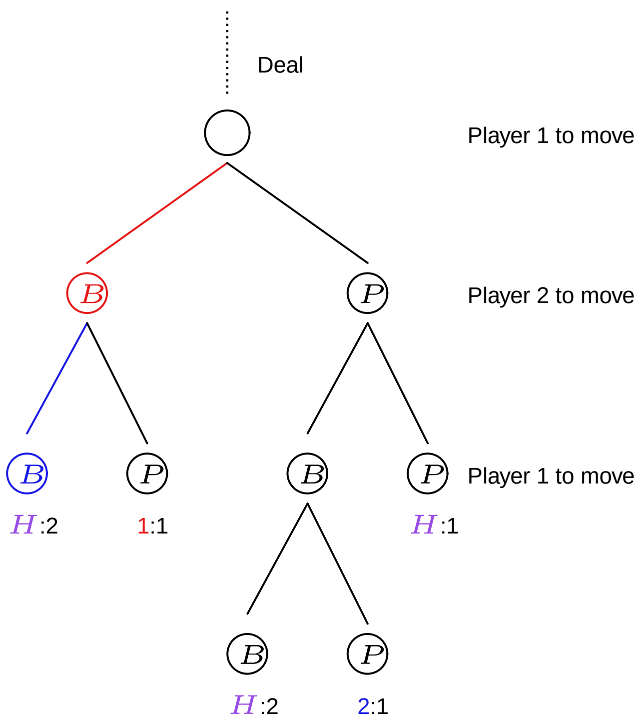
Assume Player 1 has got the Q.

Then if he bets on the first move, the following possibilities can arise:

Player 2 has J.	Player 2 has K.
Player 1 wins 1.	Then Player 2 will bet (we have ruled out all her other strategies).

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



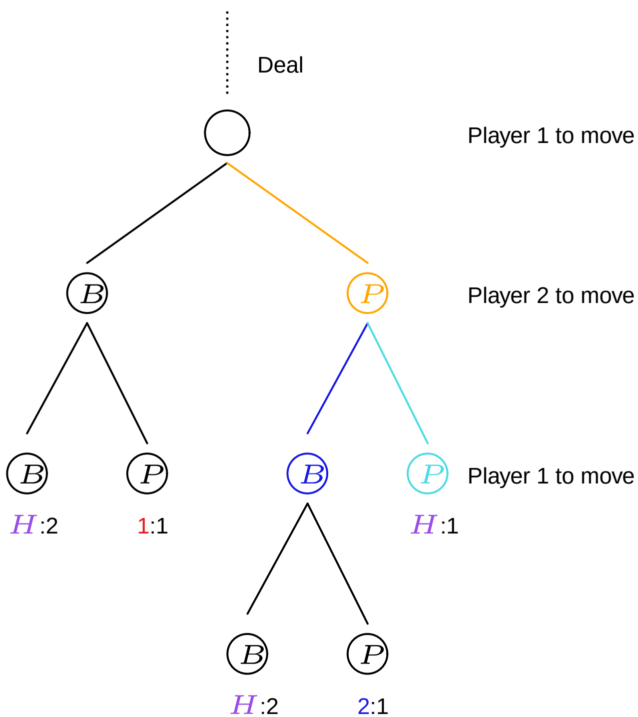
Assume Player 1 has got the Q.

Then if he bets on the first move, the following possibilities can arise:

Player 2 has J.	Player 2 has K.
Player 1 wins 1.	Player 1 looses 2.

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



Assume Player 1 has got the Q.

He bets on the first move:

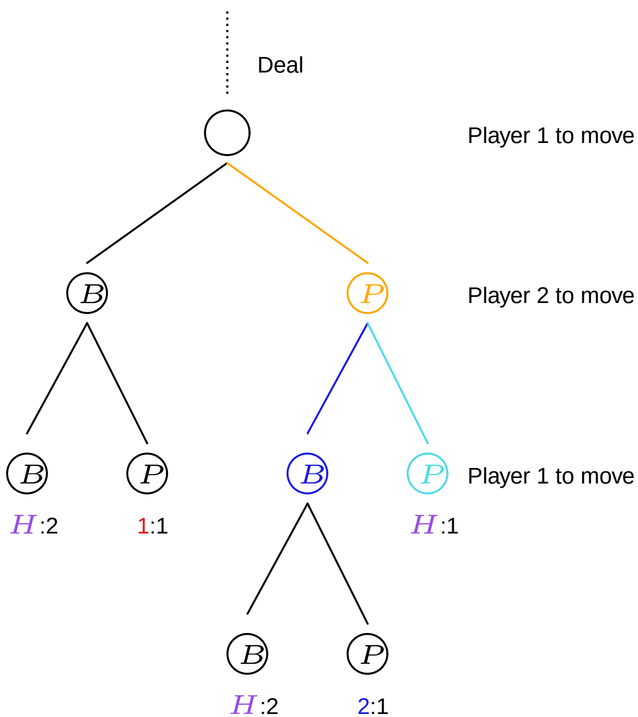
Player 2 has J. Player 2 has K.

Player 1 wins 1. Player 1 looses 2.

If he passes on the first move, and bets on the third if possible:

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



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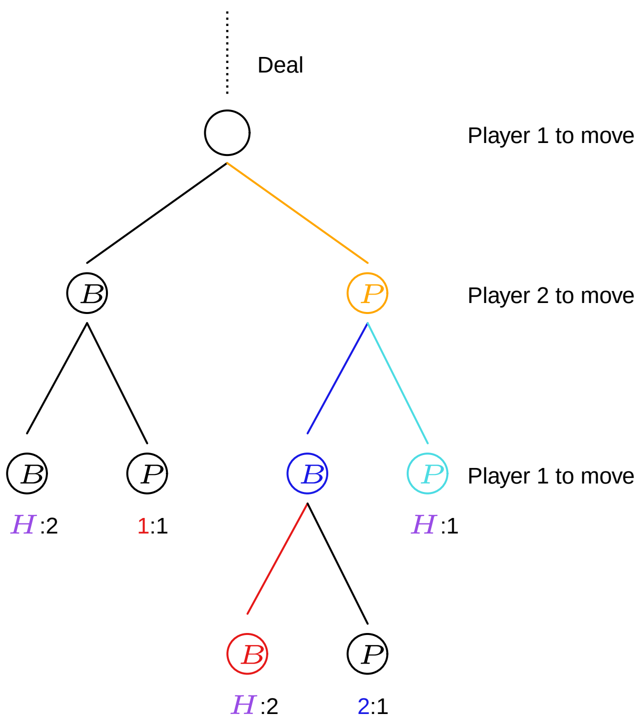


Assume Player 1 has got the Q.

He bets on the first move:

Player 2 has J. Player 2 has K.

Player 1 wins 1. Player 1 looses 2.

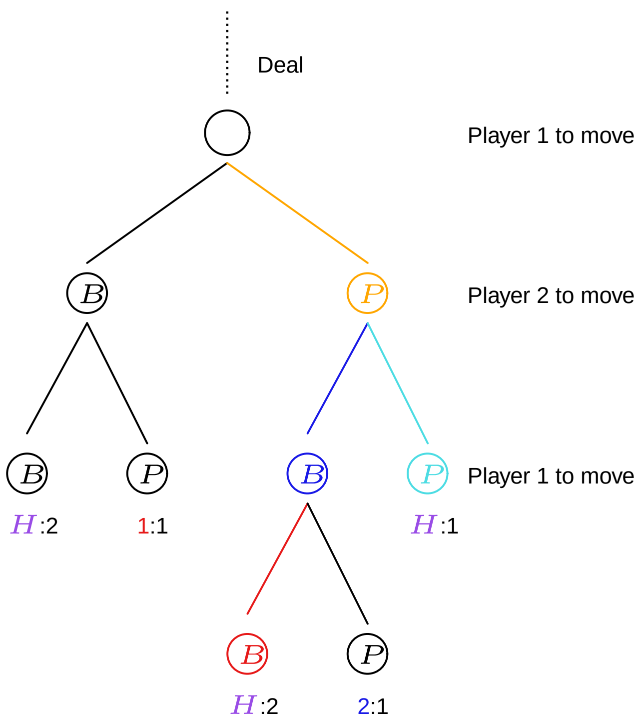
If he passes on the first move, and bets on the third if possible:

Player 2 has J. Player 2 has K.

Then Player 2 may bet, in which case Player 1 will bet and win 2, or she may pass, in which case Player 1 will win 1.

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



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Player 1 wins 1 or 2.

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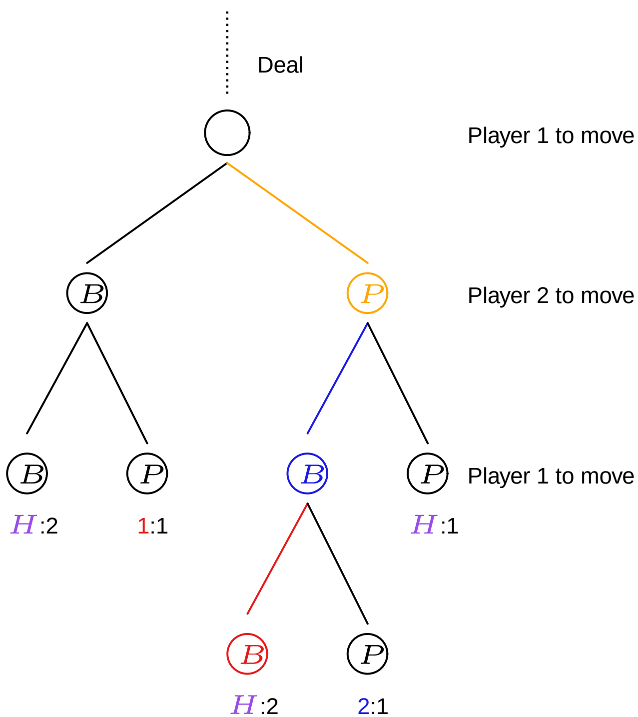


Assume Player 1 has got the Q.

He bets on the first move:

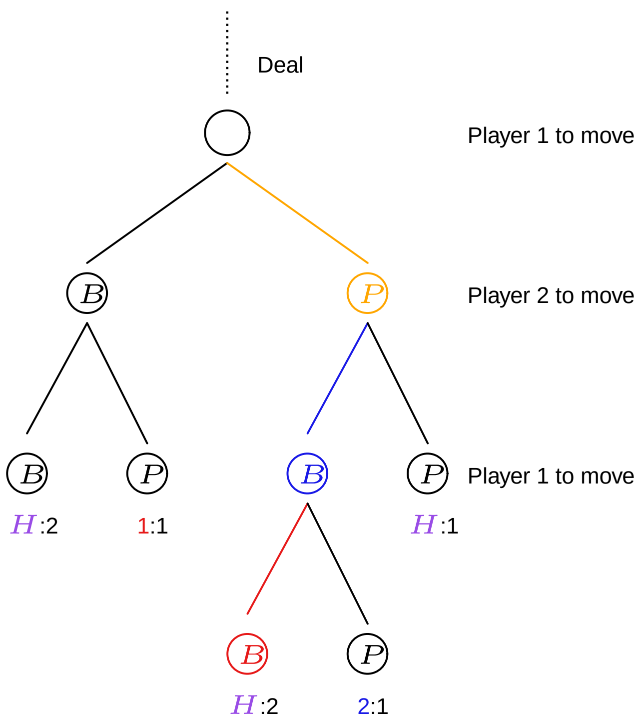
Player 2 has J.	Player 2 has K.
Player 1 wins 1.	Player 1 looses 2.

If he passes on the first move, and bets on the third if possible:

Player 2 has J.	Player 2 has K.
Player 1 wins 1 or 2.	Then Player 2 will bet (we have ruled out all her other strategies).

Removing more strategies: Player 1

We compare Player 1's strategy B with his strategy PB.



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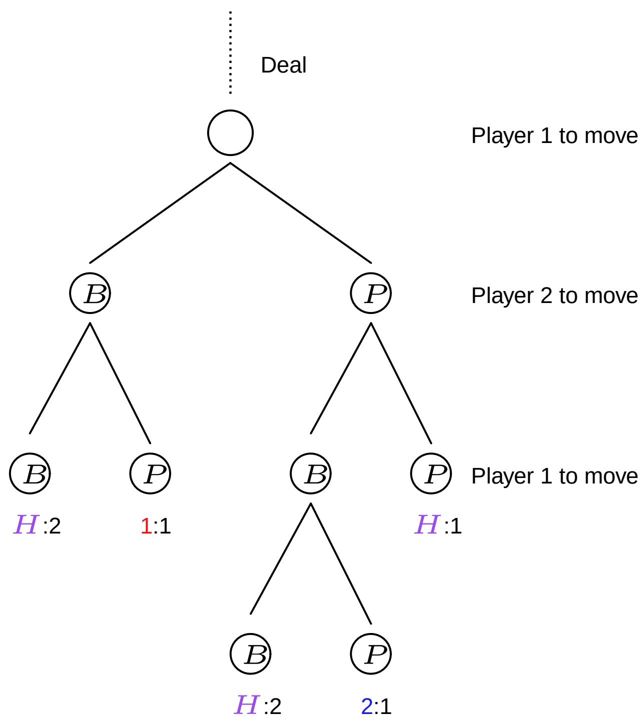

Assume Player 1 has got the Q.

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Player 2 has J. Player 2 has K.

Player 1 wins 1. Player 1 looses 2.

If he passes on the first move, and bets on the third if possible:

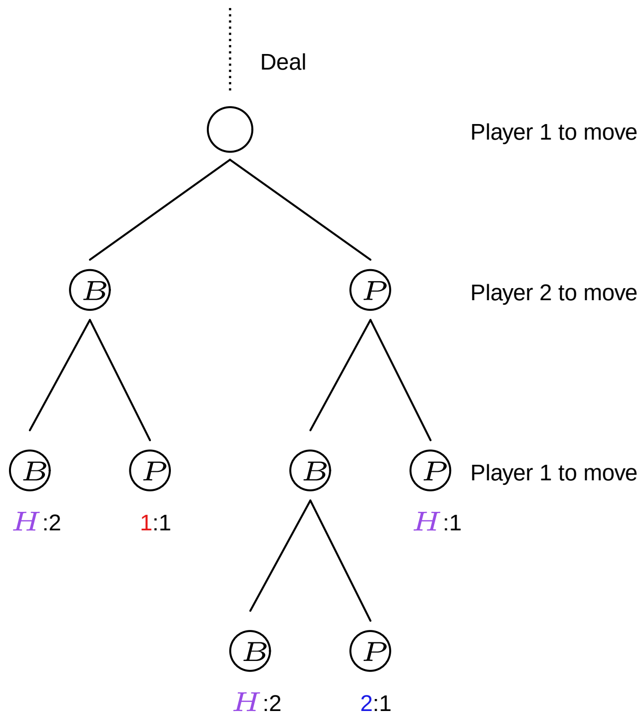
Player 2 has J. Player 2 has K.

Player 1 wins 1 or 2. Player 1 looses 2.

Hence if Player 1 has the Q then his strategy PB dominates his strategy B.

Removing more strategies: Player 1

The second component of his strategy should be PP or PB.

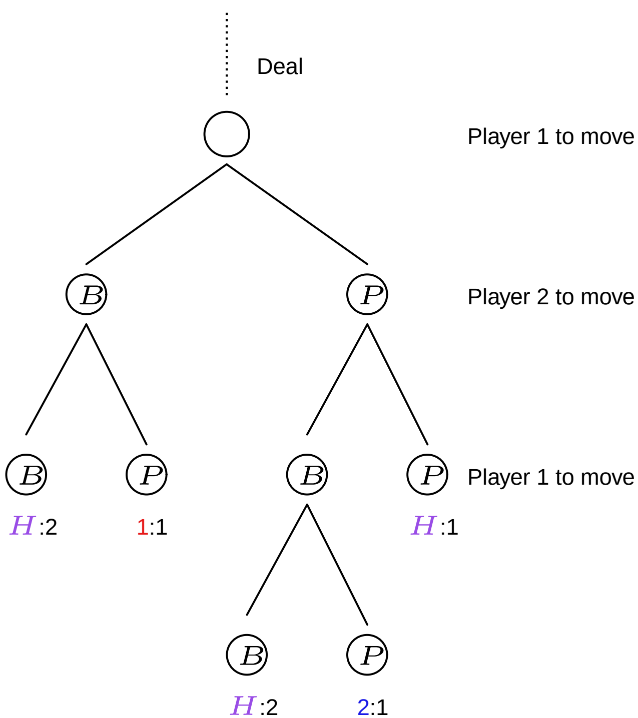


We rule out B in the middle coordinate, reducing the number of strategies to

$$2 \times 2 \times 2 = 8.$$

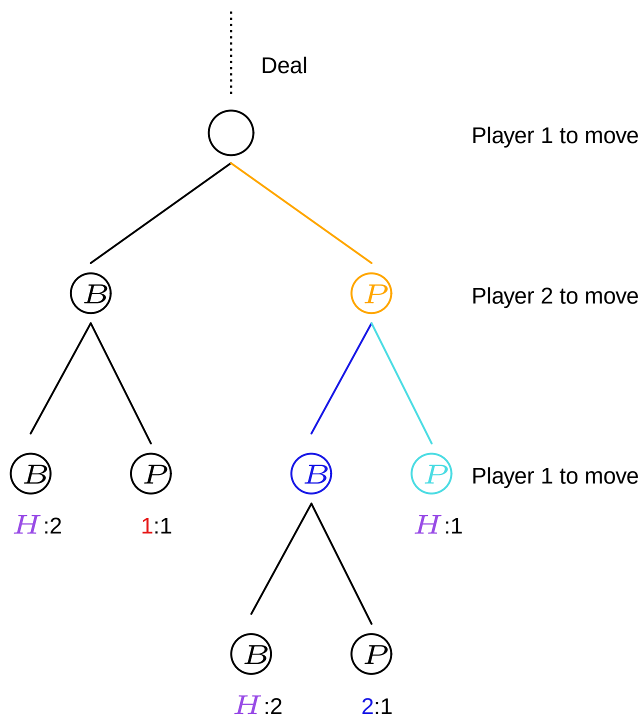
Removing more strategies: Player 2

Again we make use of the fact that we have discarded some strategies for the other player, here Player 1.



Removing more strategies: Player 2

Assume Player 2 has got the Q, and that Player 1 has passed on the first move.

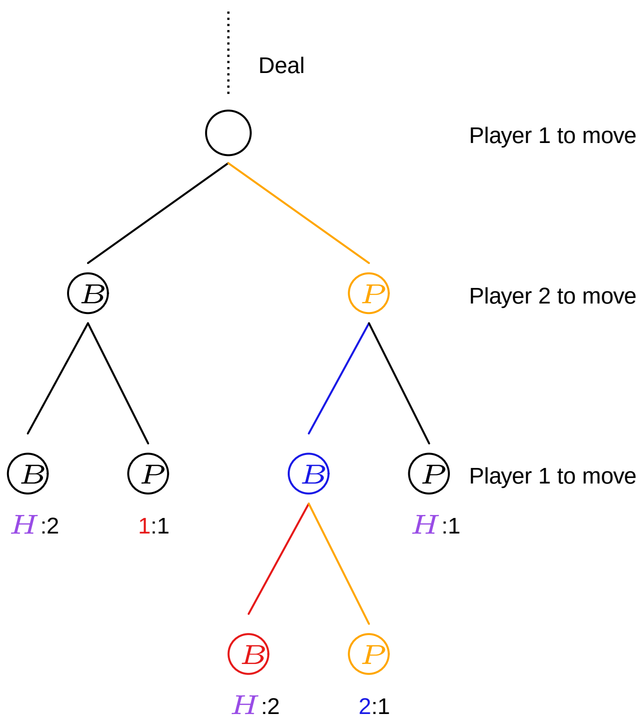


Removing more strategies: Player 2

We compare Player 2's strategies of the form $|B$ with those of the form $|B$.

Assume Player 2 has got the Q, and that Player 1 has passed on the first move.

Then if Player 2 bets on the second move:



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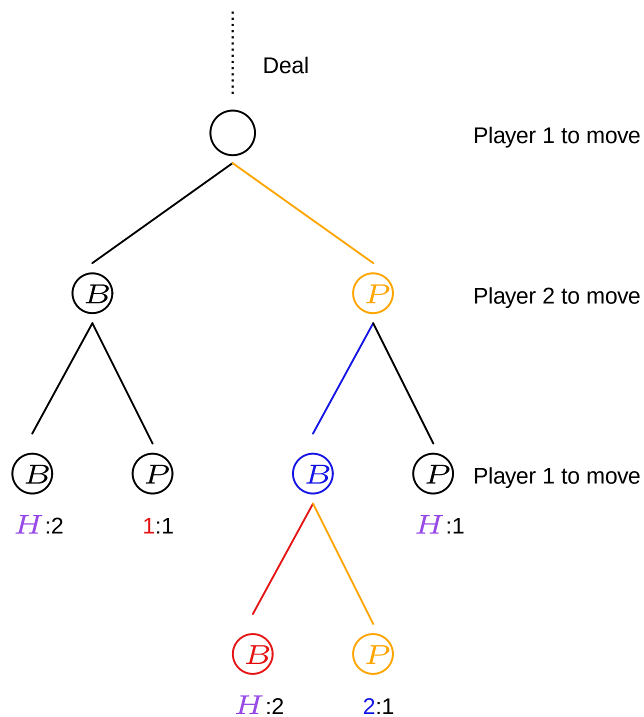

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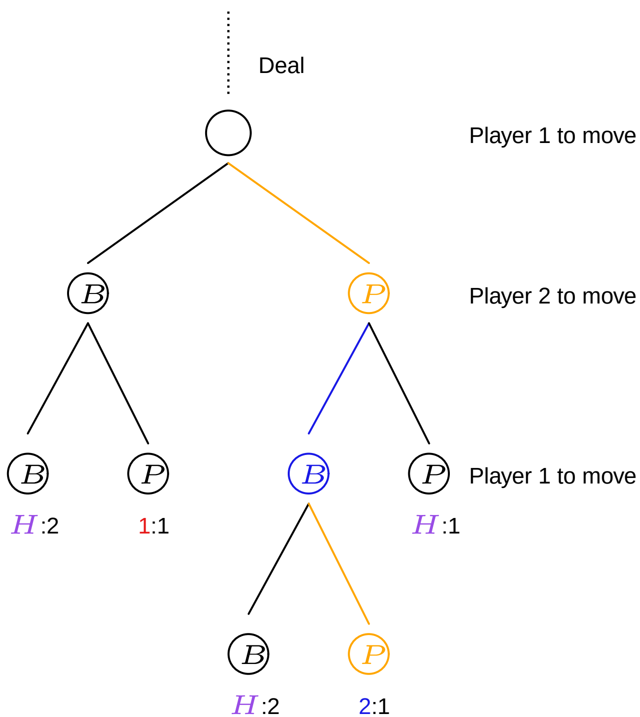

Assume Player 2 has got the Q, and that Player 1 has passed on the first move.

Then if Player 2 bets on the second move:

Player 1 has J. Player 1 has K.

Then Player 1 will pass (we have ruled out all his other strategies).

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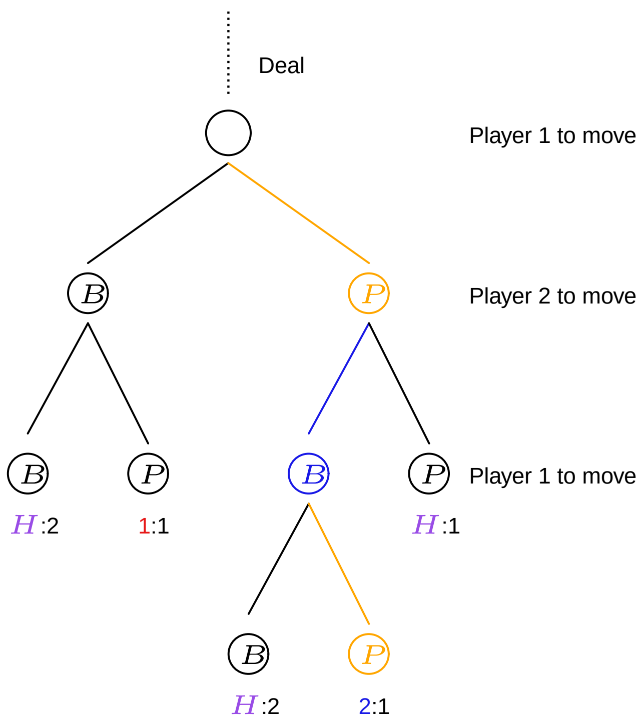

Assume Player 2 has got the Q, and that Player 1 has passed on the first move.

Then if Player 2 bets on the second move:

Player 1 has J. Player 1 has K.

Player 2 wins 1.

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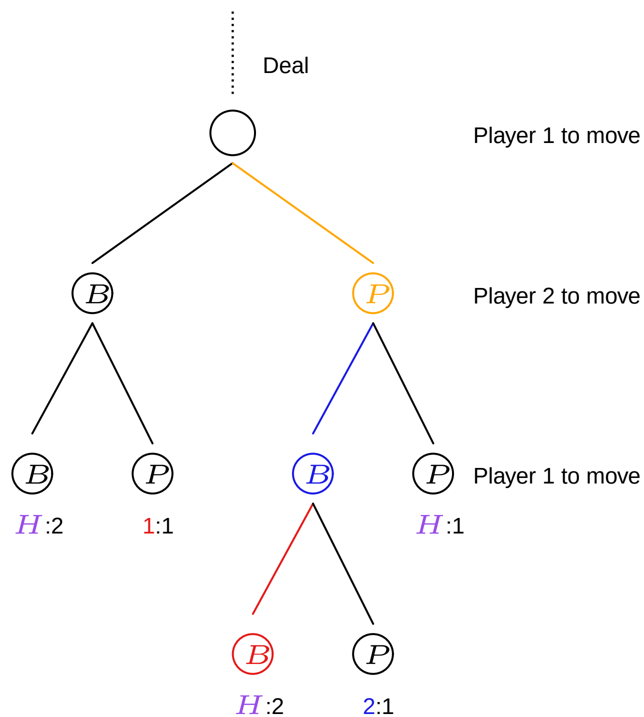

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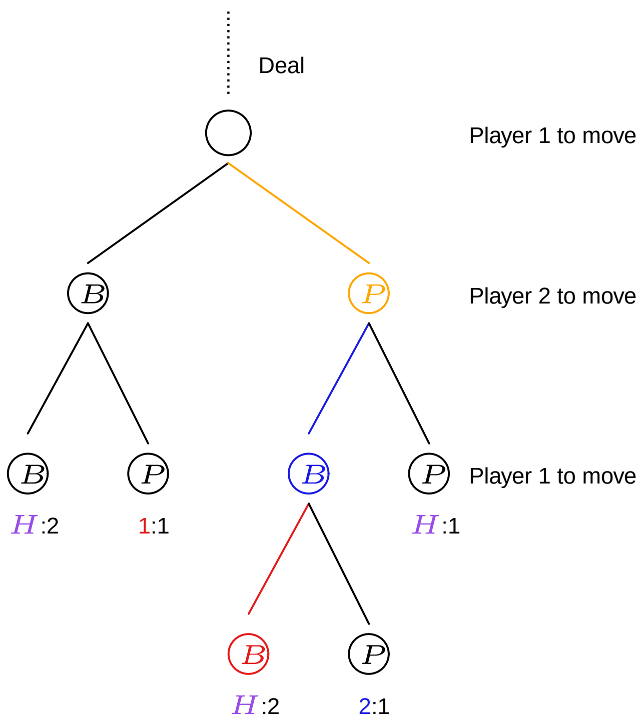

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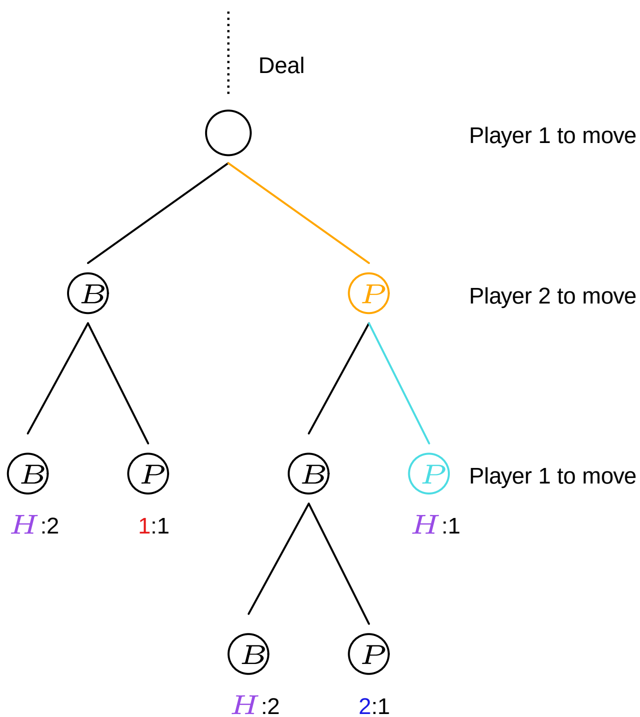

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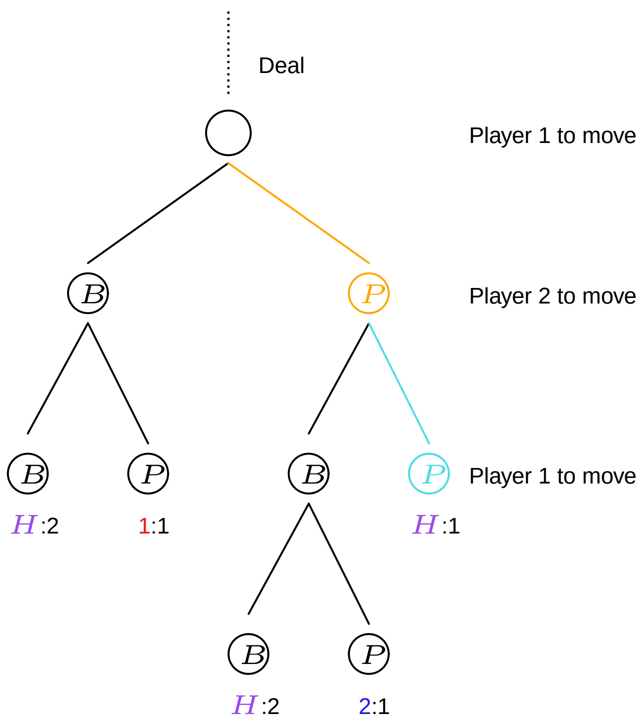

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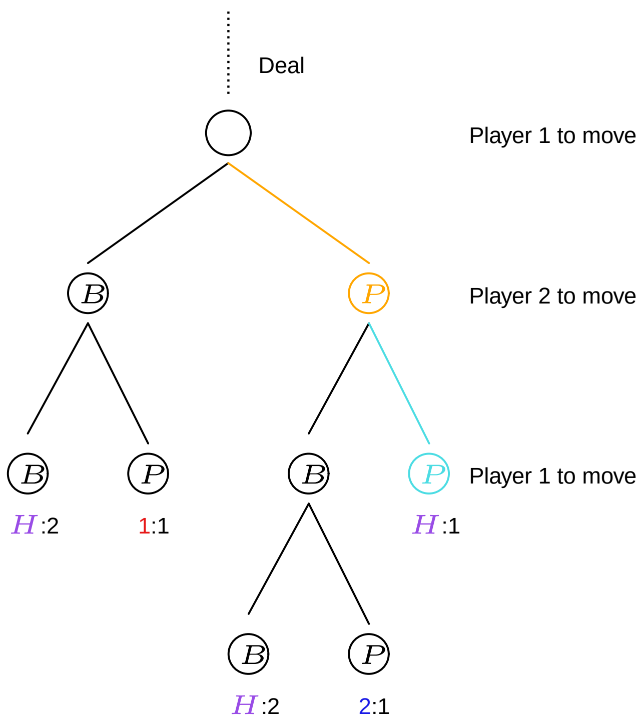

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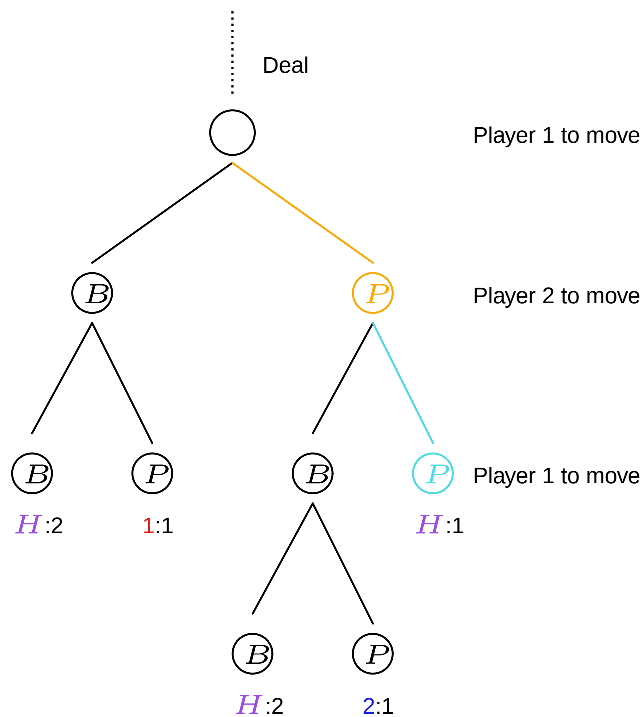

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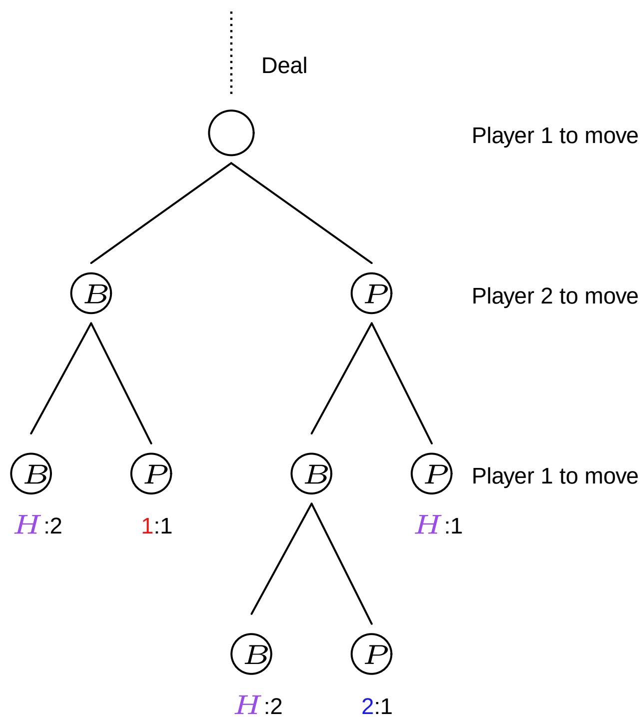

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Player 2 wins 1.	Player 2 looses 2.

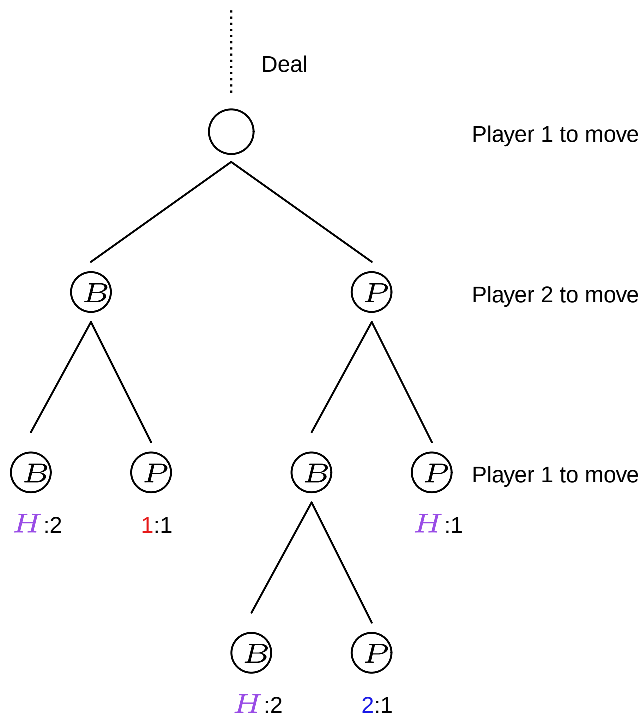
If she passes on the second move:

Player 1 has J.	Player 1 has K.
Player 2 wins 1.	Player 2 looses 1.

Hence her strategy $B|B$ is dominated by her strategy $B|P$, and her strategy $P|B$ is dominated by her strategy $P|P$.

Removing more strategies: Player 2

The second component of her strategy should be $B|P$ or $P|P$.



Hence her strategy $B|B$ is dominated by her strategy $B|P$, and her strategy $P|B$ is dominated by her strategy $P|P$.

So Player 2 should always pass when faced with a pass and holding a Q.

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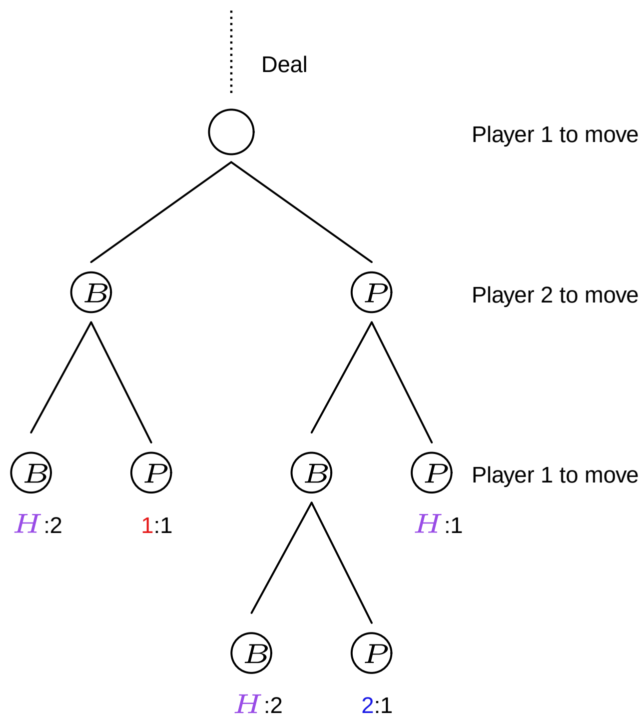

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So Player 2 should always pass when faced with a pass and holding a Q.

We have therefore reduced her strategies to

$$2 \times 2 \times 1 = 4.$$

Remaining strategies

Player 1:

Player 2:

Remaining strategies

Player 1: For component

one: J**:** PP, B;

two: Q**:** PP, PB;

three: K**:** PB, B.

Player 2:

Remaining strategies

Player 1: For component

one: J: PP, B;

two: Q: PP, PB;

three: K: PB, B.

So he has these strategies:

$$(PP, PP, PB)$$
$$(PP, PP, B)$$
$$(PP, PB, PB)$$
$$(PP, PB, BB)$$
$$(B, PP, PB)$$
$$(B, PP, B)$$
$$(B, PB, PB)$$
$$(B, PB, B)$$

Player 2:

Remaining strategies

Player 1: For component

one: $\mathbf{J}$**:** PP, B;

two: $\mathbf{Q}$**:** PP, PB;

three: $\mathbf{K}$**:** PB, B.

So he has these strategies:

$$(PP, PP, PB)$$
$$(PP, PP, B)$$
$$(PP, PB, PB)$$
$$(PP, PB, BB)$$
$$(B, PP, PB)$$
$$(B, PP, B)$$
$$(B, PB, PB)$$
$$(B, PB, B)$$

Player 2: For component

one: $\mathbf{J}$**:** $P|P$, $P|B$;

two: $\mathbf{Q}$**:** $P|P$, $B|P$;

three: $\mathbf{K}$**:** $B|B$.

Remaining strategies

Player 1: For component

one: J: PP, B;

two: Q: PP, PB;

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Calculacing pay-offs

The game is now small enough to be converted into matrix form.

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To calculate the pay-off (for Player 1) when playing 2 strategies against each other, we need to do the following:

Calculacing pay-offs

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Example: (PP, PP, PB) against $(P|P, P|P, B|B)$. We have to calculate a result for each possible deal, and weigh each of those by $1/6$ before adding them up.

Calculacing pay-offs

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Example: (PP, PP, PB) against $(P|P, P|P, B|B)$.

$(\mathbf{J}, \mathbf{Q})$**:** The moves played are P, P and Player 1 gets -1.

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Example: (PP, PP, PB) against $(P|P, P|P, B|B)$.

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- $(\mathbf{J}, \mathbf{Q})$: The moves played are P, P and Player 1 gets -1.
- $(\mathbf{J}, \mathbf{K})$: The moves played are P, B, P and Player 1 gets -1.
- $(\mathbf{Q}, \mathbf{J})$: The moves played are P, P and Player 1 gets 1.
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- $(\mathbf{K}, \mathbf{J})$: The moves played are P, P and Player 1 gets 1.
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Example: (PP, PP, PB) against $(P|P, P|P, B|B)$.

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$(\mathbf{K}, \mathbf{J})$: The moves played are P, P and Player 1 gets 1.

$(\mathbf{K}, \mathbf{Q})$: The moves played are P, P and Player 1 gets 1.

Hence the expected pay-off when playing (PP, PP, PB) against $(P|P, P|P, B|B)$ is

$$(1/6\times(-1))+(1/6\times(-1))+(1/6\times1)+(1/6\times(-1))+(1/6\times1)+(1/6\times1) = 0.$$

Matrix form

We give the full matrix, but to make it easier to compare the entries, we multiply all of them by 6. So the true game matrix is $1/6$ times the one given below.

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	$(P\|P,\ P\|P,\ B\|B)$	$(P\|P,\ B\|P,\ B\|B)$	$(P\|B,\ P\|P,\ B\|B)$	$(P\|B,\ B\|P,\ B\|B)$
(PP, PP, PB)	0	0	-1	-1
(PP, PP, B)	0	1	-2	-1
(PP, PB, PB)	-1	-1	1	1
(PP, PB, BB)	-1	0	0	1
(B, PP, PB)	1	-2	0	-3
(B, PP, B)	1	-1	-1	-3
(B, PB, PB)	0	-3	2	-1
(B, PB, B)	0	-2	1	-1

Optimal strategies: Player 1

One can verify that the following 12 mixed strategies are optimal for Player 1 for this matrix (and thus for Simplified Poker).

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$2/3$	(PP, PP, PB)	$+$	$1/3$	(PP, PB, PB)			
$1/2$	(PP, PP, PB)	$+$	$1/2$	(PP, PB, B)	$+$	$1/6$	(B, PP, PB)
$5/9$	(PP, PP, PB)	$+$	$1/3$	(PP, PB, B)	$+$	$1/9$	(B, PB, PB)
$1/2$	(PP, PP, PB)	$+$	$1/3$	(PP, PB, B)	$+$	$1/6$	(B, PB, B)
$2/5$	(PP, PP, B)	$+$	$7/15$	(PP, PB, PB)	$+$	$2/15$	(B, PP, PB)
$1/3$	(PP, PP, B)	$+$	$1/2$	(PP, PB, PB)	$+$	$1/6$	(B, PP, B)
$1/2$	(PP, PP, B)	$+$	$1/3$	(PP, PB, PB)	$+$	$1/6$	(B, PB, PB)
$4/9$	(PP, PP, B)	$+$	$1/3$	(PP, PB, PB)	$+$	$2/9$	(B, PB, B)
$1/6$	(PP, PP, B)	$+$	$7/12$	(PP, PB, B)	$+$	$1/4$	(B, PP, PB)
$5/12$	(PP, PP, B)	$+$	$1/3$	(PP, PB, B)	$+$	$1/4$	(B, PB, PB)
$1/3$	(PP, PP, B)	$+$	$1/3$	(PP, PB, B)	$+$	$1/3$	(B, PB, PB)
$2/3$	(PP, PB, B)	$+$	$1/3$	(B, PP, B)			

Optimal strategies: Player 2

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$$1/3\,(P|P,P|P,B|B) \quad + \quad 1/3\,(P|P,B|P,B|B) \quad + \quad 1/3\,(P|B,P|P,B|B)$$

$$2/3\,(P|P,P|P,B|B) \quad + \quad 1/3\,(P|B,B|P,B|B).$$

What is this good for?

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If one assumes that both, the ante as well as the amount players can place on a bet, are real numbers, then one can play with these parameters and see their effect on the optimal strategies. Some of these variants rule out bluffing. See Jones for a discussion of this.

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  In ordinary Poker (as opposed to our ‘baby’ version here), underbidding is usually more varied. When people have a reasonably good hand, they will almost inevitably bet fairly highly on it, assuming erroneously that their having an above-average hand must mean that nobody else can have one (let alone one which beats theirs). While our example is too simple to show this effect, we get at least its shadow.

Lessons to be learned

When looking at ordinary Poker it is commonly observed that the average player will

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The lesson of this section is that game theory can indeed help us to improve our game at the kinds of games people play as a pastime. If the actual game is too large to analyse, looking at simplified versions can produce useful guidelines.

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- The solution to a game is given by equilibrium points.

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- Every (non-cooperative) game has at least one equilibrium point.
- In 2-person zero-sum games, all equilibrium points lead to the same pay-off, the value of the game. The value is the least pay-off that Player 1 can guarantee for himself, while Player 2 can ensure that it is the highest amount she may have to pay to Player 1. In such a game, it makes sense to talk of equilibrium strategies as optimal for the respective player. If the game is one of perfect information, then an equilibrium point consisting of pure strategies exist.

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- In 2-person zero-sum games, all equilibrium points lead to the same pay-off, the value of the game. In such a game, it makes sense to talk of equilibrium strategies as optimal for the respective player. If the game is one of perfect information, then an equilibrium point consisting of pure strategies exist.
- In practice, we can make a game matrix smaller by removing dominated strategy, and we can solve (2×2) matrices.