

Forty-five minutes

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

11/11/18

Time: 14.00

Marking Scheme Included

Do not publish

Please answer all TWO Questions

This is a CLOSED book examination

The use of electronic calculators is not permitted.

1. a) Consider the following function:

$$f: \mathbb{Z} \longrightarrow \{n \in \mathbb{Z} \mid n \text{ is odd}\} \cup \{0\}$$

$$x \longmapsto x \cdot (x \bmod 2).$$

Is this function injective? Is it surjective? Justify your answers and indicate which properties you are using. (5 marks)

Model answer and marking scheme for Q1a

This function is not injective. All even numbers are mapped to 0, so in particular $f0 = 0 = f2$, but 0 and 2 are not equal.

This function is surjective. Assume that we have an odd number n in \mathbb{Z} . Then there exists a number k in \mathbb{Z} such that $n = 2k + 1$. We calculate

$$\begin{aligned} fn &= f(2k + 1) && \text{def } k \\ &= (2k + 1) \cdot ((2k + 1) \bmod 2) && \text{def } f \\ &= (2k + 1) \cdot 1 && \text{def mod} \\ &= 2k + 1 && \text{1 unit for mult} \\ &= n && \text{def } k. \end{aligned}$$

One mark each for the two correct answers, one for a counterexample to injectivity and two for the proof of surjectivity.

For the counterexample to injectivity we need some kind of explicit statement that gives an idea for how to find specific numbers that provide a counterexample. To get both marks for the proof of surjectivity I expect there to be a sensible attempt to justify the statement that does not contain any false claims. The model answer has more steps than I expect a typical student answer to have, but I would like to see a little more than just ' $fn = n$ for odd n '.

- b) Recall the max operation which returns the largest number in a finite set, so for example $\max\{1, e, \pi, 10, \sqrt{2}\} = 10$. Consider the binary operation on complex numbers:

$$(a + bi) \circledast (a' + b'i) = \max\{a, b\} + \max\{a', b'\}i$$

For example we have

$$(1 + 2i) \circledast (\pi + 4i) = 2 + 4i.$$

Is this operation associative? Is it commutative?

(5 marks)

Model answer and marking scheme for Q1b

The operation is not associative. Playing a little with the expressions one obtains makes it fairly easy to pick up a counterexample.

$$\begin{aligned} (0 \circledast (1 + i)) \circledast (2 + 2i) &= (0 + i) \circledast (2 + 2i) && \text{def } \circledast \\ &= 1 + 2i && \text{def } \circledast \end{aligned}$$

whereas

$$\begin{aligned} 0 \circledast ((1 + i) \circledast (2 + 2i)) &= 0 \circledast (1 + 2i) && \text{def } \circledast \\ &= 0 + 2i. && \text{def } \circledast \end{aligned}$$

Since the two results are different we have found a counterexample for associativity.

The operation is not commutative. Finding a counterexample is quite easy, for example

$$0 \circledast (1 + i) = 0 + i \neq 1 = (1 + i) \circledast 0.$$

One mark each for the correct answer, two marks for the counterexample for associativity, one for the counterexample to commutativity. *I expect the students to struggle a bit to write up their argument. If you can understand what they're trying to say then please give them both marks even if the write-up is not that good, but if the write-up contains wrong statements then please withhold one of the marks.*

2. a) Show

$$P \wedge (\neg P \vee Q) \equiv P \wedge Q$$

in the Boolean semantics by using truth tables.

(3 marks)

Model answer and marking scheme for Q2a

We construct a truth table to compute the interpretations of the formulas $P \wedge (\neg P \vee Q)$ and $P \wedge Q$ for the all possible valuations of P and Q .

P	Q	$\neg P$	$\neg P \vee Q$	$P \wedge (\neg P \vee Q)$	$P \wedge Q$
1	1	0	1	1	1
1	0	0	0	0	0
0	1	1	1	0	0
0	0	1	1	0	0

Since the truth values in the last two columns are the same, the two formulas are semantically equivalent.

**Two marks for correct truth table; one mark for small mistake.
One mark for final answer and explanation.**

b) Give a brief explanation of **one** of the following.

(2 marks)

- i) contradiction
- ii) contrapositive law

Model answer and marking scheme for Q2b

Instructions. Answers will obviously vary. Give 1 mark for each underlined aspects if they appear in their answers and what is written makes sense and is not complete non-sense.

- i) A contradiction is a (propositional) formula if its interpretation is 0 in all valuations (or not satisfiable).
- ii) The (fundamental) semantic equivalence $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$.

c) Consider the following propositional formula.

$$(P \vee Q) \rightarrow (\neg(P \rightarrow Q) \vee \neg P)$$

- i) Use our CNF algorithm to transform the formula into conjunctive normal form. (2 marks)
- ii) Simplify the conjunctive normal form as far as possible using our fundamental semantic equivalences. (3 marks)

Justify all the steps in your derivations and simplifications.

Model answer and marking scheme for Q2c

i) Derivation using our CNF algorithm:

$$\begin{aligned}
 & (P \vee Q) \rightarrow (\neg(P \rightarrow Q) \vee \neg P) \\
 & \equiv \neg(P \vee Q) \vee \neg(\neg P \vee Q) \vee \neg P && \text{Step 1/elim } \rightarrow, \text{ flattening} \\
 & \equiv (\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg Q) \vee \neg P && \text{Step 2/De Morgan} \\
 & \equiv (\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee \neg P && \text{Step 3/double negation} \\
 & \equiv (\neg P \vee P \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg P) && \text{Step 4/distributivity} \\
 & \quad \wedge (\neg Q \vee P \vee \neg P) \wedge (\neg Q \vee \neg Q \vee \neg P)
 \end{aligned}$$

Give .5 mark for each type of rule used correctly and with correct justifications (4 × .5 = 2 marks).

ii) Simplifying this we get:

$$\begin{aligned}
 & (\top \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg Q \vee \top) \wedge (\neg Q \vee \neg Q \vee \neg P) && \text{Excluded middle } \times 2 \\
 & \equiv \top \wedge (\neg P \vee \neg Q) \wedge \top \wedge (\neg Q \vee \neg P) && A \vee \top \equiv \top \times 2, \text{ commutativity,} \\
 & && \text{idempotency } \times 2 \\
 & \equiv (\neg P \vee \neg Q) \wedge (\neg Q \vee \neg P) && A \wedge \top \equiv A \times 2 \\
 & \equiv \neg P \vee \neg Q && \text{commutativity, idempotency}
 \end{aligned}$$

Give .5 mark for each type of rule used correctly and with correct justifications (6 × .5 = 3 marks).

Only perfect answer should receive full marks! This means 4.5/5 is rounded down to 4 marks. *If a student has given one derivation for both parts then mark similarly giving .5 mark for each type of rule used correctly and with correct justifications (10 × .5 = 5 marks). If students apply simplification steps early and/or don't follow our algorithm, similarly give .5 mark for each type of rule used correctly and with correct justification. In this case the derivation may be shorter and fewer types of rules were applied leading to fewer marks even though the final answer is correct.*